Min-Max Control of LQ Systems under the H_∞ Norm Constraint

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Abstract: Satisfactory optimal control of linear systems under the existence of disturbance is studied. The satisfactory optimal control minimizes a quadratic maximal-valued objective functional subject to the H_∞ norm constraint. We obtain Riccati-type equations, with which the satisfactory optimal control is calculated in a state-feedback-like form.

Keywords: optimal control, min-max control, H_∞ norm constraint, Riccati equation, convex optimization, multiobjective optimization

1. Introduction

As a control problem under the existence of disturbance, we study the satisfactory optimal control problem which minimizes an objective function under a satisfaction condition that the value of another performance index is less than or equal to a specified value for any disturbance. The objective function can be a maximal value function with respect to the disturbance, which leads to one of min-max type problems. Such a problem has already been studied for static systems by Shimizu et al. [10, 11], yet no studies are known for dynamic systems. The most difficult matter of this type of problem is the existence of normally nondifferentiable functionals.

This paper first formulates the satisfactory optimal control problem as a satisfaction optimization problem in a Hilbert space and then analyzes the problem through the application of nondifferentiable programming [3]. Since the satisfaction optimization problem is a generalization of a min-max problem, we must handle nondifferentiability associated with maximal value functions. One of the most useful tools for this is the concept of generalized gradients [2, 3]. We provide a necessary optimality condition of the Kuhn–Tucker type for the satisfaction optimization problem by using the generalized gradients. In this analysis, the representation of the generalized gradient of a maximal value functional [6] plays an important role. The analytic result for the satisfaction optimization problem in Hilbert space is then applied to the satisfactory optimal control problem. As a result, we obtain an actual representation of the necessary optimality condition in time domain by use of Hamiltonian functions.

Following this general nonlinear theory, we next consider the linear-quadratic case. For linear systems, the H_∞ norm constraint can be an actual example of the satisfaction condition, which requires the uniform norm of a transfer operator from disturbances to controlled outputs to be less than a specified value. Such a satisfaction condition has been studied extensively as the H_∞ control problem. In addition, the satisfactory optimal control problem sets a quadratic objective functional to be optimized at the upper level. Among related

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works, the mixed H_2/H_∞ control problem is widely studied [1, 5, 9], where one minimizes the expected value of a quadratic objective functional under the H_∞ norm constraint. In contrast, our paper deals with the case where the objective functional is defined as a maximal value functional with respect to the disturbance. That is, the whole problem is formulated as a min-max control problem under the H_∞ norm constraint. The satisfactory optimal control is then given in a form of state-feedback-like control law with Riccati type equations. This is not a state feedback in an ordinary sense, but the optimal control is given realtime with the state of an imaginary system.

2. Nondifferentiable Optimization Theory in Hilbert Space

2.1. Nondifferentiable nonlinear programming problems. Let X be a Hilbert space, D a nonempty open subset of X, and f a functional defined on D. The generalized directional derivative and the generalized gradient, introduced by Clarke [2], are defined as follows: The generalized directional derivative of f at $x \in D$ in the direction $s \in X$ is given by

$$f^\circ(x;s) \triangleq \limsup_{\substack{\xi \to x \\ \alpha \downarrow 0}} \frac{f(\xi+\alpha s) - f(\xi)}{\alpha},$$

and the generalized gradient of f at $x \in D$ is then given by

$$\partial^\circ f(x) \triangleq \{\, z \in X \mid f^\circ(x;s) \geqq \langle z, s\rangle \quad \forall s \in X \,\}.$$

Locally Lipschitz functionals have the following properties [2, 3].

Lemma 2.1. *Let f be Lipschitz near $x \in D$. Then, i) for any direction $s \in X$ there exists the generalized directional derivative $f^\circ(x;s)$ of f at x, and the functional $f^\circ(x;\cdot)$ is continuous and sublinear on X; ii) the generalized gradient $\partial^\circ f(x)$ is a nonempty, weak-compact, convex subset of X; iii) $f^\circ(x;s) = \max_{z \in \partial^\circ f(x)} \langle z, s\rangle$ for all $s \in X$.*

Continuously Fréchet differentiable functionals have the following properties.

Lemma 2.2. *Let f be continuously Fréchet differentiable near $x \in D$. Then, i) f is Lipschitz near x; ii) $f^\circ(x;s) = f'(x;s) = \langle \nabla f(x), s\rangle$ for all $s \in X$; iii) $\partial^\circ f(x) = \{\, \nabla f(x) \,\}$.*

Let us consider the following inequality-constrained optimization problem in Hilbert space:

$$\min_x f(x) \tag{2.1a}$$

$$\text{subj. to} \quad \boldsymbol{g}(x) \leqq \boldsymbol{0}, \tag{2.1b}$$

where $f : X \to R$ and $\boldsymbol{g} : X \to R^m$. Assuming the local Lipschitz continuity of f and g_i, $i = 1, 2, \ldots, m$, let us derive an optimality condition for problem (2.1).

(Cottle's constraint qualification) Let $\boldsymbol{g}$ be Lipschitz near $x^o \in X$. The functional $\boldsymbol{g}$ is said to satisfy Cottle's constraint qualification at x^o if there exists $s \in X$ satisfying $g_i^\circ(x^o;s) < 0$ for all $i \in I(x^o)$, where $I(x^o)$ is an index set defined by $I(x^o) \triangleq \{\, i \mid g_i(x^o) = 0 \,\}$.

When g_i, $i \in I(x^o)$, are convex, the following Slater's condition is known as a stronger constraint qualification.

(Slater's constraint qualification) Let g_i, $i \in I(x^o)$, be convex on X. The functional $\boldsymbol{g}$ is said to satisfy Slater's constraint qualification at x^o if there exists $x \in X$ satisfying $g_i(x) < 0$ for all $i \in I(x^o)$.

A linearization lemma holds under Cottle's constraint qualification [3, 11]. Applying a generalized Farkas lemma, we obtain the following Kuhn–Tucker type necessary optimality condition.

Theorem 2.1. *Let f and $\boldsymbol{g}$ be Lipschitz near $x^o \in X$, and let $\boldsymbol{g}$ satisfy Cottle's constraint qualification at x^o. If x^o is a locally optimal solution of problem (2.1), then there exists $\boldsymbol{\lambda} \in R^m$ such that*

$$0 \in \partial^\circ f(x^o) + \sum_{i=1}^{m} \lambda_i \partial^\circ g_i(x^o), \tag{2.2a}$$

$$\boldsymbol{g}(x^o) \leqq \mathbf{0}, \qquad \langle \boldsymbol{\lambda}, \boldsymbol{g}(x^o) \rangle = 0, \qquad \boldsymbol{\lambda} \geqq \mathbf{0}. \tag{2.2b}$$

2.2. Properties of maximal value functionals. Let D be a nonempty open subset of X, Y a nonempty metric space, let $f : D \times Y \to R$, and let us make the following assumptions:
(Assumption 2.1) Y is compact;
(Assumption 2.2) f is continuous on $D \times Y$;
(Assumption 2.3) f is partially Fréchet differentiable on D for arbitrarily fixed $y \in Y$, and $\nabla_x f : D \times Y \to X$ is continuous on $D \times Y$.

We define a maximal value functional $F : D \to R$ by

$$F(x) \triangleq \max_{y \in Y} f(x, y) \tag{2.3}$$

and the set of maximum solutions by

$$P(x) \triangleq \arg\max_{y \in Y} f(x, y) = \{\, y \in Y \mid f(x, y) = F(x) \,\}. \tag{2.4}$$

Under Assumptions 2.1 $\sim$ 2.3, $P(x)$ is nonempty and compact for any $x \in D$ and F is Lipschitz near each point of D [6].

The next theorem gives the representation of the directional derivative, the generalized directional derivative and the generalized gradient of F.

Theorem 2.2. *Let Assumptions 2.1 $\sim$ 2.3 be satisfied. Then*
i) At each $x \in D$ there exist the directional derivative $F'(x; s)$ and the generalized directional derivative $F^\circ(x; s)$ of F in any direction $s \in X$, which are given by

$$F'(x; s) = F^\circ(x; s) = \max_{y \in P(x)} \langle \nabla_x f(x, y), s \rangle; \tag{2.5}$$

ii) At each $x \in D$ there exists the generalized gradient $\partial^\circ F(x)$ of F, which is given by

$$\partial^\circ F(x) = \overline{\mathrm{co}}\, \{\, \nabla_x f(x, y) \mid y \in P(x) \,\}. \tag{2.6}$$

Proof. i) Similar to the proof of Danskin's theorem [4, 2] in R^n.

ii) The local Lipschitz continuity of F ensures the existence of $\partial^\circ F(x)$. Let $Z(x) \triangleq \{\, \nabla_x f(x, y) \mid y \in P(x) \,\}$, which is compact by the compactness of $P(x)$ and the continuity of $\nabla_x f$ on $\{x\} \times Y$. Hence $\overline{\mathrm{co}}\, Z(x)$ is also compact by Mazur's theorem.

Let $z \in \mathrm{co}\, Z(x)$. Then there exist a natural number k, vectors $y^i \in P(x)$, $i = 1, 2, \ldots, k$, and scalars $\lambda^i \in R$, $i = 1, 2, \ldots, k$, such that

$$z = \sum_{i=1}^{k} \lambda^i \nabla_x f(x, y^i); \qquad \sum_{i=1}^{k} \lambda^i = 1; \qquad \lambda^i \geqq 0, \quad i = 1, 2, \ldots, k.$$

Taking account of the result

$$\langle \nabla_x f(x, y), s \rangle \leqq F^\circ(x; s) \quad \forall y \in P(x)$$

that immediately follows from assertion i, we have

$$\langle z, s\rangle = \sum_{i=1}^{k} \lambda^i \langle \nabla_x f(x, y^i), s\rangle \leqq \sum_{i=1}^{k} \lambda^i F^\circ(x; s) = F^\circ(x; s)$$

for each $s \in X$, which implies $z \in \partial^\circ F(x)$. Hence $\mathrm{co}\, Z(x) \subset \partial^\circ F(x)$. Since $\partial^\circ F(x)$ is closed, it follows that $\overline{\mathrm{co}}\, Z(x) \subset \partial^\circ F(x)$.

Now let $z \in \partial^\circ F(x)$. Then, using assertion i, we have

$$\max_{u \in \overline{\mathrm{co}} Z(x)} \langle u, s\rangle \geqq \max_{u \in Z(x)} \langle u, s\rangle = \max_{y \in P(x)} \langle \nabla_x f(x, y), s\rangle = F^\circ(x; s) \geqq \langle z, s\rangle \quad \forall s \in X. \tag{2.7}$$

Then $z \in \overline{\mathrm{co}}\, Z(x)$, since otherwise, by the strict separation theorem of convex sets [7, 8], there exist $p \in X$ and $\alpha \in R$ such that

$$\langle u, p\rangle < \alpha \quad \forall u \in \overline{\mathrm{co}}\, Z(x), \qquad \langle z, p\rangle > \alpha,$$

or equivalently, there exists $p \in X$ satisfying

$$\max_{u \in \overline{\mathrm{co}} Z(x)} \langle u, p\rangle < \langle z, p\rangle,$$

which contradicts (2.7). ∎

3. Satisfactory Optimal Control Problems

Let us first consider the following satisfactory optimal control problem:

$$\min_{\boldsymbol{u} \in U} \max_{\boldsymbol{v} \in V} \int_0^\infty f_0\big(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{v}(t)\big)\, dt \tag{3.1a}$$

$$\text{subj. to} \quad \dot{\boldsymbol{x}}(t) = \boldsymbol{f}\big(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{v}(t)\big), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{3.1b}$$

$$\max_{\boldsymbol{v} \in V} \int_0^\infty g\big(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{v}(t)\big)\, dt \leqq 0, \tag{3.1c}$$

where $\boldsymbol{x}(t) \in R^n$ is a state vector, $\boldsymbol{u}(t) \in R^r$ is a control vector, $\boldsymbol{v}(t) \in R^q$ is a disturbance vector, and let $f_0 : R^n \times R^r \times R^q \to R$, $\boldsymbol{f} : R^n \times R^r \times R^q \to R^n$ and $g : R^n \times R^r \times R^q \to R$ be continuously differentiable.

This is a problem to seek a solution of the min-max problem (3.1a) subject to the satisfaction condition (3.1c). We consider the classes of admissible $\boldsymbol{u}$'s and $\boldsymbol{v}$'s to be the Hilbert spaces $L_2([0, \infty), R^r)$ and $L_2([0, \infty), R^q)$, which will be denoted by U and V, respectively. We assume that, for any $\boldsymbol{u} \in U$ and $\boldsymbol{v} \in V$, equation (3.1b) has a unique continuous solution $\boldsymbol{x}(\boldsymbol{u}, \boldsymbol{v})$, whose value at t will be denoted by $\boldsymbol{x}(t; \boldsymbol{u}, \boldsymbol{v})$, and integrals (3.1a,c) have finite values.

Define the functionals $\phi : U \times V \to R$ and $\omega : U \times V \to R$ by

$$\phi[\boldsymbol{u}, \boldsymbol{v}] \triangleq \int_0^\infty f_0\big(\boldsymbol{x}(t; \boldsymbol{u}, \boldsymbol{v}), \boldsymbol{u}(t), \boldsymbol{v}(t)\big)\, dt, \tag{3.2}$$

$$\omega[\boldsymbol{u}, \boldsymbol{v}] \triangleq \int_0^\infty g\big(\boldsymbol{x}(t; \boldsymbol{u}, \boldsymbol{v}), \boldsymbol{u}(t), \boldsymbol{v}(t)\big)\, dt. \tag{3.3}$$

Problem (3.1) is then reformulated as a min-max optimization problem with a satisfaction condition:

$$\min_{\boldsymbol{u} \in U} \max_{\boldsymbol{v} \in V} \phi[\boldsymbol{u}, \boldsymbol{v}] \tag{3.4a}$$

$$\text{subj. to} \quad \max_{\boldsymbol{v} \in V} \omega[\boldsymbol{u}, \boldsymbol{v}] \leqq 0. \tag{3.4b}$$

This type of problem has been studied in [10, 11].

Let us further define the maximal value functionals $\Phi : U \to R$ and $\Omega : U \to R$ by

$$\Phi[\boldsymbol{u}] \triangleq \max_{\boldsymbol{v}\in V} \phi[\boldsymbol{u}, \boldsymbol{v}] = \max_{\boldsymbol{v}\in V} \int_0^\infty f_0\big(\boldsymbol{x}(t; \boldsymbol{u}, \boldsymbol{v}), \boldsymbol{u}(t), \boldsymbol{v}(t)\big)\, dt, \tag{3.5}$$

$$\Omega[\boldsymbol{u}] \triangleq \max_{\boldsymbol{v}\in V} \omega[\boldsymbol{u}, \boldsymbol{v}] = \max_{\boldsymbol{v}\in V} \int_0^\infty g\big(\boldsymbol{x}(t; \boldsymbol{u}, \boldsymbol{v}), \boldsymbol{u}(t), \boldsymbol{v}(t)\big)\, dt. \tag{3.6}$$

Then problem (3.4) is written as

$$\min_{\boldsymbol{u}\in U} \Phi[\boldsymbol{u}] \tag{3.7a}$$

$$\text{subj. to} \quad \Omega[\boldsymbol{u}] \leqq 0. \tag{3.7b}$$

We make the following assumptions:
(Assumption 3.1) V is compact;
(Assumption 3.2) ϕ and ω are continuous on $U \times V$;
(Assumption 3.3) ϕ and ω are partially Fréchet differentiable on U for arbitrarily fixed $\boldsymbol{v} \in V$, and $\nabla_{\boldsymbol{u}}\phi : U \times V \to U$ and $\nabla_{\boldsymbol{u}}\omega : U \times V \to U$ are continuous on $U \times V$.

Let us define the sets of maximum solutions by

$$P_1[\boldsymbol{u}] \triangleq \arg\max_{\boldsymbol{v}\in V} \phi[\boldsymbol{u}, \boldsymbol{v}], \qquad P_2[\boldsymbol{u}] \triangleq \arg\max_{\boldsymbol{v}\in V} \omega[\boldsymbol{u}, \boldsymbol{v}].$$

Under Assumptions 3.1 $\sim$ 3.3, $P_1[\boldsymbol{u}]$ and $P_2[\boldsymbol{u}]$ are nonempty and compact for any $\boldsymbol{u} \in U$, and Φ and Ω are Lipschitz near each point of U.

We assume Cottle's constraint qualification to derive a Kuhn-Tucker type optimality condition. As a direct consequence of Theorem 2.1, we have the following corollary for problem (3.7) (i.e., problem(3.4)).

Corollary 3.1. *Let Assumptions 3.1 $\sim$ 3.3 be satisfied, and let Ω satisfy Cottle's constraint qualification at $\boldsymbol{u}^o \in U$. If $\boldsymbol{u}^o$ is a locally optimal solution of problem (3.7), then there exists a scalar λ such that*

$$\boldsymbol{0} \in \partial^\circ \Phi[\boldsymbol{u}^o] + \lambda \partial^\circ \Omega[\boldsymbol{u}^o], \tag{3.8a}$$

$$\Omega[\boldsymbol{u}^o] \leqq 0, \qquad \lambda\Omega[\boldsymbol{u}^o] = 0, \qquad \lambda \geqq 0. \tag{3.8b}$$

The generalized gradients of Φ and Ω are given as follows by assertion ii of Theorem 2.2 and Theorem A of Appendix.

Lemma 3.1. *Let Assumptions 3.1 $\sim$ 3.3 be satisfied. The generalized gradient $\partial^\circ\Phi[\boldsymbol{u}]$ of the functional Φ, defined by (3.5), at $\boldsymbol{u}$ is given by*

$$\partial^\circ \Phi[\boldsymbol{u}] = \overline{\text{co}}\,\{\, \nabla_{\boldsymbol{u}} H_1\big(\boldsymbol{x}(\boldsymbol{u}, \boldsymbol{v}^*), \boldsymbol{u}, \boldsymbol{v}^*, \boldsymbol{\psi}_1\big) \mid \boldsymbol{v}^* \in P_1[\boldsymbol{u}] \,\}, \tag{3.9}$$

where

$$H_1(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}^*, \boldsymbol{\psi}_1) \triangleq f_0(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}^*) + \boldsymbol{\psi}_1^T \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}^*),$$

and $\boldsymbol{\psi}_1$ is a continuous R^n-valued function defined on $[0, \infty)$ by

$$\dot{\boldsymbol{\psi}}_1(t) = -H_{1\boldsymbol{x}}\big(\boldsymbol{x}(t; \boldsymbol{u}, \boldsymbol{v}^*), \boldsymbol{u}(t), \boldsymbol{v}^*(t), \boldsymbol{\psi}_1(t)\big)^T, \quad \lim_{t\to\infty} \boldsymbol{\psi}_1(t) = \boldsymbol{0}. \tag{3.10}$$

Note that $\boldsymbol{v}^* \in P_1[\boldsymbol{u}]$ satisfies

$$\nabla_{\boldsymbol{v}} \phi[\boldsymbol{u}, \boldsymbol{v}^*](t) = \nabla_{\boldsymbol{v}} H_1\big(\boldsymbol{x}(t; \boldsymbol{u}, \boldsymbol{v}^*), \boldsymbol{u}(t), \boldsymbol{v}^*(t), \boldsymbol{\psi}_1(t)\big) = \boldsymbol{0}. \tag{3.11}$$

(Remark 3.1) The notation $\nabla_{\boldsymbol{u}} H_1\big(\boldsymbol{x}(\boldsymbol{u}, \boldsymbol{v}^*), \boldsymbol{u}, \boldsymbol{v}^*, \boldsymbol{\psi}_1\big)$ denotes the trajectory of $\nabla_{\boldsymbol{u}} H_1\big(\boldsymbol{x}(t; \boldsymbol{u}, \boldsymbol{v}^*), \boldsymbol{u}(t), \boldsymbol{v}^*(t), \boldsymbol{\psi}_1(t)\big)$ for $t \in [0, \infty)$.

Lemma 3.2. *Let Assumptions 3.1 $\sim$ 3.3 be satisfied. The generalized gradient $\partial^\circ \Omega[\boldsymbol{u}]$ of the functional Ω, defined by (3.6), at $\boldsymbol{u}$ is given by*

$$\partial^\circ \Omega[\boldsymbol{u}] = \overline{\mathrm{co}}\,\{\, \nabla_{\boldsymbol{u}} H_2\big(\boldsymbol{x}(\boldsymbol{u}, \boldsymbol{v}^{**}), \boldsymbol{u}, \boldsymbol{v}^{**}, \boldsymbol{\psi}_2\big) \mid \boldsymbol{v}^{**} \in P_2[\boldsymbol{u}] \,\}, \tag{3.12}$$

where

$$H_2(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}^{**}, \boldsymbol{\psi}_2) \triangleq g(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}^{**}) + \boldsymbol{\psi}_2^T \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}^{**}),$$

and $\boldsymbol{\psi}_2$ and $\boldsymbol{v}^{**} \in P_2[\boldsymbol{u}]$ satisfy

$$\dot{\boldsymbol{\psi}}_2(t) = -H_{2\boldsymbol{x}}\big(\boldsymbol{x}(t; \boldsymbol{u}, \boldsymbol{v}^{**}), \boldsymbol{u}(t), \boldsymbol{v}^{**}(t), \boldsymbol{\psi}_2(t)\big)^T, \quad \lim_{t\to\infty} \boldsymbol{\psi}_2(t) = \boldsymbol{0}, \tag{3.13}$$

$$\nabla_{\boldsymbol{v}} H_2\big(\boldsymbol{x}(t; \boldsymbol{u}, \boldsymbol{v}^{**}), \boldsymbol{u}(t), \boldsymbol{v}^{**}(t), \boldsymbol{\psi}_2(t)\big) = \boldsymbol{0}, \tag{3.14}$$

respectively.

The next theorem immediately follows from Corollary 3.1, Lemmas 3.1 and 3.2.

Theorem 3.1. *Let Assumptions 3.1 $\sim$ 3.3 be satisfied, and let Ω satisfy Cottle's constraint qualification at $\boldsymbol{u}^o \in U$. If $\boldsymbol{u}^o$ is a locally optimal solution of the satisfactory optimal control problem (3.1), then there exists a scalar λ such that*

$$\begin{aligned} \boldsymbol{0} \in\ & \overline{\mathrm{co}}\,\{\, \nabla_{\boldsymbol{u}} H_1\big(\boldsymbol{x}(\boldsymbol{u}^o, \boldsymbol{v}^*), \boldsymbol{u}^o, \boldsymbol{v}^*, \boldsymbol{\psi}_1\big) \mid \boldsymbol{v}^* \in P_1[\boldsymbol{u}^o] \,\} \\ & + \lambda\, \overline{\mathrm{co}}\,\{\, \nabla_{\boldsymbol{u}} H_2\big(\boldsymbol{x}(\boldsymbol{u}^o, \boldsymbol{v}^{**}), \boldsymbol{u}^o, \boldsymbol{v}^{**}, \boldsymbol{\psi}_2\big) \mid \boldsymbol{v}^{**} \in P_2[\boldsymbol{u}^o] \,\}, \end{aligned} \tag{3.15a}$$

$$\omega[\boldsymbol{u}^o, \boldsymbol{v}^{**}] \leqq 0, \quad \lambda\omega[\boldsymbol{u}^o, \boldsymbol{v}^{**}] = 0, \quad \lambda \geqq 0 \quad \textit{where} \ \ \boldsymbol{v}^{**} \in P_2[\boldsymbol{u}^o]. \tag{3.15b}$$

4. Linear-Quadratic Case under the H_∞ Norm Constraint

In this section, we formulate the satisfactory optimal control problem for an LQ system and derive its optimality conditions. Consider the following linear system

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B_1\boldsymbol{u}(t) + B_2\boldsymbol{v}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{4.1a}$$

$$\boldsymbol{z}(t) = C\boldsymbol{x}(t), \tag{4.1b}$$

where $\boldsymbol{z}(t) \in R^l$ denotes a performance output vector.

Let us consider the H_∞ norm condition for plant (4.1) as a satisfaction condition. The relationship between $\boldsymbol{z}$ and $\boldsymbol{v}$ is affine, i.e., $\boldsymbol{z} = T_{vz}\boldsymbol{v} + \boldsymbol{d}$, where both T_{vz} and $\boldsymbol{d}$ depend on $\boldsymbol{u}$. The H_∞ norm condition is such that the uniform norm $\|T_{vz}\|_\infty$ of the bounded linear operator T_{vz} representing a transfer property from $\boldsymbol{v}$ to $\boldsymbol{z}$ is less than or equal to a specified value γ. The uniform norm $\|T_{vz}\|_\infty$ is defined by

$$\|T_{vz}\|_\infty = \sup_{\boldsymbol{v}\neq\boldsymbol{0}} \frac{\|T_{vz}\boldsymbol{v}\|_2}{\|\boldsymbol{v}\|_2}, \tag{4.2}$$

where $\|\cdot\|_2$ denotes an L_2-norm. When the relationship of $\boldsymbol{z}$ to $\boldsymbol{v}$ is linear, which especially holds if $\boldsymbol{u}$ is given in the form of linear state feedback, the satisfaction condition $\|T_{vz}\|_\infty \leqq \gamma$ is equivalently expressed as

$$\sup_{\boldsymbol{v}\neq\boldsymbol{0}} \frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{v}\|_2} \leqq \gamma, \tag{4.3}$$

i.e.,

$$\sup_{\boldsymbol{v}\in V} \int_0^\infty \boldsymbol{z}(t)^T\boldsymbol{z}(t) - \gamma^2\boldsymbol{v}(t)^T\boldsymbol{v}(t)\, dt \leqq 0. \tag{4.4}$$

Meanwhile, we consider a quadratic objective function of min-max type. Then the satisfactory optimal control problem for plant (4.1) is formulated as follows:

$$\min_{\boldsymbol{u}\in U}\max_{\boldsymbol{v}\in V}\int_0^\infty \boldsymbol{x}(t)^T Q\boldsymbol{x}(t)+\boldsymbol{u}(t)^T R_1\boldsymbol{u}(t)-\boldsymbol{v}(t)^T R_2\boldsymbol{v}(t)\,dt \tag{4.5a}$$

$$\text{subj. to}\quad \dot{\boldsymbol{x}}(t)=A\boldsymbol{x}(t)+B_1\boldsymbol{u}(t)+B_2\boldsymbol{v}(t),\quad \boldsymbol{x}(0)=\boldsymbol{x}_0, \tag{4.5b}$$

$$\max_{\boldsymbol{v}\in V}\int_0^\infty \boldsymbol{z}(t)^T\boldsymbol{z}(t)-\gamma^2\boldsymbol{v}(t)^T\boldsymbol{v}(t)\,dt \leqq 0 \tag{4.5c}$$

$$\text{subj. to}\quad \dot{\boldsymbol{x}}(t)=A\boldsymbol{x}(t)+B_1\boldsymbol{u}(t)+B_2\boldsymbol{v}(t),\quad \boldsymbol{x}(0)=\boldsymbol{0}, \tag{4.5d}$$

$$\boldsymbol{z}(t)=C\boldsymbol{x}(t), \tag{4.5e}$$

where Q is a positive semidefinite symmetric matrix, R_1 and R_2 are positive definite symmetric matrices. For given $\boldsymbol{u}$ and $\boldsymbol{v}$, $\boldsymbol{x}(t;\boldsymbol{u},\boldsymbol{v};\boldsymbol{x}_0)$ and $\boldsymbol{x}(t;\boldsymbol{u},\boldsymbol{v};\boldsymbol{0})$ denote unique solutions of (4.5b) and (4.5d), respectively. Similarly, let us denote the output $\boldsymbol{z}$ of (4.5e) by $\boldsymbol{z}(t;\boldsymbol{u},\boldsymbol{v})$, i.e., $\boldsymbol{z}(t;\boldsymbol{u},\boldsymbol{v})=C\boldsymbol{x}(t;\boldsymbol{u},\boldsymbol{v};\boldsymbol{0})$. Let

$$\phi[\boldsymbol{u},\boldsymbol{v}]\triangleq\int_0^\infty \boldsymbol{x}(t;\boldsymbol{u},\boldsymbol{v};\boldsymbol{x}_0)^T Q\boldsymbol{x}(t;\boldsymbol{u},\boldsymbol{v};\boldsymbol{x}_0)+\boldsymbol{u}(t)^T R_1\boldsymbol{u}(t)-\boldsymbol{v}(t)^T R_2\boldsymbol{v}(t)\,dt, \tag{4.6a}$$

$$\omega[\boldsymbol{u},\boldsymbol{v}]\triangleq\int_0^\infty \boldsymbol{z}(t;\boldsymbol{u},\boldsymbol{v})^T\boldsymbol{z}(t;\boldsymbol{u},\boldsymbol{v})-\gamma^2\boldsymbol{v}(t)^T\boldsymbol{v}(t)\,dt. \tag{4.6b}$$

It is obvious that the functionals ϕ and ω are respectively strictly convex and convex w.r.t. $\boldsymbol{u}$ for any fixed $\boldsymbol{v}$. On the other hand, the following lemma holds.

Lemma 4.1. *For given control $\boldsymbol{u}$ or control law, the functional ω is concave w.r.t. $\boldsymbol{v}$ if the control or the controller is admissible in the sense that the H_∞ norm condition $\|T_{vz}\|_\infty\leqq\gamma$ is satisfied. Furthermore, for any control or controller satisfying $\|T_{vz}\|_\infty<\gamma$, the functional ω is strictly concave.*

Proof. Using the affine relation $\boldsymbol{z}=T_{vz}\boldsymbol{v}+\boldsymbol{d}$, we have $\omega[\boldsymbol{u},\boldsymbol{v}]=\langle\boldsymbol{v},(T_{vz}^*T_{vz}-\gamma^2 I)\boldsymbol{v}\rangle+2\langle T_{vz}\boldsymbol{v},\boldsymbol{d}\rangle+\langle\boldsymbol{d},\boldsymbol{d}\rangle$, where T_{vz}^* denotes the adjoint of T_{vz}. It is easy to show that $T_{vz}^*T_{vz}-\gamma^2 I\leqq O$ $(<O)$ if and only if $\|T_{vz}\|_\infty\leqq\gamma$ $(<\gamma)$. ∎

We also assume that Q and R_2 are given so that the functional ϕ is strictly concave w.r.t. $\boldsymbol{v}$.

We define max-valued functionals as follows:

$$\begin{aligned}\Phi[\boldsymbol{u}]\triangleq\max_{\boldsymbol{v}\in V}\phi[\boldsymbol{u},\boldsymbol{v}]=\max_{\boldsymbol{v}\in V}\int_0^\infty &\boldsymbol{x}(t;\boldsymbol{u},\boldsymbol{v};\boldsymbol{x}_0)^T Q\boldsymbol{x}(t;\boldsymbol{u},\boldsymbol{v};\boldsymbol{x}_0)\\ &+\boldsymbol{u}(t)^T R_1\boldsymbol{u}(t)-\boldsymbol{v}(t)^T R_2\boldsymbol{v}(t)\,dt,\end{aligned} \tag{4.7a}$$

$$\Omega[\boldsymbol{u}]\triangleq\max_{\boldsymbol{v}\in V}\omega[\boldsymbol{u},\boldsymbol{v}]=\max_{\boldsymbol{v}\in V}\int_0^\infty \boldsymbol{z}(t;\boldsymbol{u},\boldsymbol{v})^T\boldsymbol{z}(t;\boldsymbol{u},\boldsymbol{v})-\gamma^2\boldsymbol{v}(t)^T\boldsymbol{v}(t)\,dt. \tag{4.7b}$$

Then problem (4.5) becomes the following convex optimization problem:

$$\min_{\boldsymbol{u}\in U}\Phi[\boldsymbol{u}] \tag{4.8a}$$

$$\text{subj. to}\quad \Omega[\boldsymbol{u}]\leqq 0. \tag{4.8b}$$

Note that the (strict) convexity of Φ and Ω follows from that of ϕ and ω w.r.t. $\boldsymbol{u}$ and the nonemptiness of $P_1[\boldsymbol{u}]$ and $P_2[\boldsymbol{u}]$.

Assumptions 3.2 and 3.3 automatically hold for ϕ and ω defined by (4.6), and Assumption 3.1 can be shown to be unnecessary for problem (4.5).

Lemma 4.2. *Let $\boldsymbol{u}\in U$ be such that $\phi[\boldsymbol{u},\boldsymbol{v}]$ is strictly concave w.r.t. $\boldsymbol{v}$. Then the*

gradient $\nabla\Phi[\boldsymbol{u}]$ of the functional Φ defined by (4.7a) exists at $\boldsymbol{u}$ and is given by

$$\nabla\Phi[\boldsymbol{u}] = \nabla_{\boldsymbol{u}} H_1\big(\boldsymbol{x}(\boldsymbol{u}, \boldsymbol{v}^*; \boldsymbol{x}_0), \boldsymbol{u}, \boldsymbol{v}^*, \boldsymbol{\psi}_1\big), \tag{4.9}$$

where

$$H_1(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}^*, \boldsymbol{\psi}_1) \triangleq \boldsymbol{x}^T Q\boldsymbol{x} + \boldsymbol{u}^T R_1 \boldsymbol{u} - \boldsymbol{v}^{*T} R_2 \boldsymbol{v}^* + \boldsymbol{\psi}_1^T (A\boldsymbol{x} + B_1\boldsymbol{u} + B_2\boldsymbol{v}^*), \tag{4.10}$$

$\boldsymbol{v}^*$ is a unique maximizing disturbance of $\phi[\boldsymbol{u}, \cdot\,]$, which is given by

$$\boldsymbol{v}^*(t) = \frac{1}{2} R_2^{-1} B_2^T \boldsymbol{\psi}_1(t), \tag{4.11}$$

and $\boldsymbol{\psi}_1$ is a continuous R^n-valued function defined by

$$\dot{\boldsymbol{\psi}}_1(t) = -H_{1\boldsymbol{x}}\big(\boldsymbol{x}(t; \boldsymbol{u}, \boldsymbol{v}^*; \boldsymbol{x}_0), \boldsymbol{u}(t), \boldsymbol{v}^*(t), \boldsymbol{\psi}_1(t)\big)^T, \quad \lim_{t\to\infty} \boldsymbol{\psi}_1(t) = \boldsymbol{0}. \tag{4.12}$$

Lemma 4.3. *Let $\boldsymbol{u} \in U$ be admissible in the sense that $\|T_{vz}\|_\infty \leqq \gamma$. Then the gradient $\nabla\Omega[\boldsymbol{u}]$ of the functional Ω defined by (4.7b) exists at $\boldsymbol{u}$ and is given by*

$$\nabla\Omega[\boldsymbol{u}] = \nabla_{\boldsymbol{u}} H_2\big(\boldsymbol{x}(\boldsymbol{u}, \boldsymbol{v}^{**}; \boldsymbol{0}), \boldsymbol{u}, \boldsymbol{v}^{**}, \boldsymbol{\psi}_2\big), \tag{4.13}$$

where

$$H_2(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}^{**}, \boldsymbol{\psi}_2) \triangleq \boldsymbol{x}^T C^T C\boldsymbol{x} - \gamma \boldsymbol{v}^{**T} \boldsymbol{v}^{**} + \boldsymbol{\psi}_2^T (A\boldsymbol{x} + B_1\boldsymbol{u} + B_2\boldsymbol{v}^{**}), \tag{4.14}$$

$\boldsymbol{v}^{**}$ is a unique maximizing disturbance of $\omega[\boldsymbol{u}, \cdot\,]$, which is given by

$$\boldsymbol{v}^{**}(t) = \frac{1}{2\gamma^2} B_2^T \boldsymbol{\psi}_2(t), \tag{4.15}$$

and $\boldsymbol{\psi}_2$ is a continuous R^n-valued function defined by

$$\dot{\boldsymbol{\psi}}_2(t) = -H_{2\boldsymbol{x}}\big(\boldsymbol{x}(t; \boldsymbol{u}, \boldsymbol{v}^{**}; \boldsymbol{0}), \boldsymbol{u}(t), \boldsymbol{v}^{**}(t), \boldsymbol{\psi}_2(t)\big)^T, \quad \lim_{t\to\infty} \boldsymbol{\psi}_2(t) = \boldsymbol{0}. \tag{4.16}$$

Since problem (4.8) is a convex programming problem, we assume Slater's condition, which corresponds to the following assumption:
(Assumption 4.1) The aspiration level γ in the H_∞ norm condition (4.3) is given so that there exists a control or control law satisfying $\|T_{vz}\|_\infty < \gamma$.

We have necessary optimality conditions for problem (4.5) as follows.

Theorem 4.1. *(Necessary optimality condition for problem (4.5)) Let Assumption 4.1 be satisfied, and let Q and R_2 be given so that $\phi[\boldsymbol{u}^o, \boldsymbol{v}]$ is strictly concave w.r.t. $\boldsymbol{v}$ for $\boldsymbol{u}^o \in U$. If $\boldsymbol{u}^o$ is a satisfactory optimal control of problem (4.5), then there exists a scalar λ such that*

$$2R_1\boldsymbol{u}^o(t) + B_1^T \boldsymbol{\psi}_1(t) + \lambda B_1^T \boldsymbol{\psi}_2(t) = \boldsymbol{0}, \tag{4.17a}$$

$$\int_0^\infty \boldsymbol{x}^{**}(t)^T C^T C \boldsymbol{x}^{**}(t) - \gamma^2 \boldsymbol{v}^{**}(t)^T \boldsymbol{v}^{**}\, dt \leqq 0, \tag{4.17b}$$

$$\lambda \int_0^\infty \boldsymbol{x}^{**}(t)^T C^T C \boldsymbol{x}^{**}(t) - \gamma^2 \boldsymbol{v}^{**}(t)^T \boldsymbol{v}^{**}\, dt = 0, \tag{4.17c}$$

$$\lambda \geqq 0. \tag{4.17d}$$

Here, $\boldsymbol{x}^*(t)$, $\boldsymbol{\psi}_1(t)$, $\boldsymbol{x}^{**}(t)$, $\boldsymbol{\psi}_2(t)$ are the solutions of the following differential equations:

$$\dot{\boldsymbol{x}}^*(t) = A\boldsymbol{x}^*(t) + B_1\boldsymbol{u}^o(t) + B_2\boldsymbol{v}^*(t), \quad \boldsymbol{x}^*(0) = \boldsymbol{x}_0, \tag{4.18a}$$

$$\dot{\boldsymbol{\psi}}_1(t) = -A^T \boldsymbol{\psi}_1(t) - 2Q\boldsymbol{x}^*(t), \quad \lim_{t\to\infty} \boldsymbol{\psi}_1(t) = \boldsymbol{0}, \tag{4.18b}$$

$$\dot{\boldsymbol{x}}^{**}(t) = A\boldsymbol{x}^{**}(t) + B_1\boldsymbol{u}^o(t) + B_2\boldsymbol{v}^{**}(t), \quad \boldsymbol{x}^{**}(0) = \boldsymbol{0}, \tag{4.19a}$$

$$\dot{\boldsymbol{\psi}}_2(t) = -A^T \boldsymbol{\psi}_2(t) - 2C^T C\boldsymbol{x}^{**}(t), \quad \lim_{t\to\infty} \boldsymbol{\psi}_2(t) = \boldsymbol{0}, \tag{4.19b}$$

where

$$\boldsymbol{v}^*(t) = \frac{1}{2}R_2^{-1}B_2^T\boldsymbol{\psi}_1(t), \tag{4.20}$$

$$\boldsymbol{v}^{**}(t) = \frac{1}{2\gamma^2}B_2^T\boldsymbol{\psi}_2(t). \tag{4.21}$$

Proof. The existence of $\lambda \in R$ satisfying

$$\nabla_{\boldsymbol{u}}H_1\big(\boldsymbol{x}(\boldsymbol{u}^o,\boldsymbol{v}^*;\boldsymbol{x}_0),\boldsymbol{u}^o,\boldsymbol{v}^*,\boldsymbol{\psi}_1\big) + \lambda\nabla_{\boldsymbol{u}}H_2\big(\boldsymbol{x}(\boldsymbol{u}^o,\boldsymbol{v}^{**};\boldsymbol{0}),\boldsymbol{u}^o,\boldsymbol{v}^{**},\boldsymbol{\psi}_2\big) = \boldsymbol{0}, \tag{4.22a}$$

$$\omega[\boldsymbol{u}^o,\boldsymbol{v}^{**}] \leqq 0, \qquad \lambda\omega[\boldsymbol{u}^o,\boldsymbol{v}^{**}] = 0, \qquad \lambda \geqq 0 \tag{4.22b}$$

is an immediate consequence of Theorem 3.1 since $P_1[\boldsymbol{u}]$ and $P_2[\boldsymbol{u}]$ are singletons. (4.17a) is then obtained by substituting the definitions of H_1 and H_2 into (4.22a). Substituting the definition of ω into (4.22b), we have

$$\int_0^\infty \boldsymbol{z}(t;\boldsymbol{u}^o,\boldsymbol{v}^{**})^T\boldsymbol{z}(t;\boldsymbol{u}^o,\boldsymbol{v}^{**}) - \gamma^2\boldsymbol{v}^{**}(t)^T\boldsymbol{v}^{**}(t)\,dt \leqq 0,$$

$$\lambda\int_0^\infty \boldsymbol{z}(t;\boldsymbol{u}^o,\boldsymbol{v}^{**})^T\boldsymbol{z}(t;\boldsymbol{u}^o,\boldsymbol{v}^{**}) - \gamma^2\boldsymbol{v}^{**}(t)^T\boldsymbol{v}^{**}(t)\,dt = 0,$$

$$\lambda \geqq 0.$$

By (4.5e)

$$\boldsymbol{z}(t;\boldsymbol{u}^o,\boldsymbol{v}^{**}) = C\boldsymbol{x}(t;\boldsymbol{u}^o,\boldsymbol{v}^{**};\boldsymbol{0}) = C\boldsymbol{x}^{**}(t).$$

Substituting this into the above equations, we have (4.17b)–(4.17d). (4.18a) and (4.19a) are the plant equations themselves. (4.18b) and (4.19b) are obtained by calculating (4.12) and (4.16) with $\boldsymbol{u} = \boldsymbol{u}^o$. ∎

5. Calculation of the Satisfactory Optimal Control Input

We shall compute a satisfactory optimal control $\boldsymbol{u}^o$ using the necessary optimality conditions in Theorem 4.1. First, it follows from (4.17a) that

$$\boldsymbol{u}^o(t) = -\frac{1}{2}R_1^{-1}\big(B_1^T\boldsymbol{\psi}_1(t) + \lambda B_1^T\boldsymbol{\psi}_2(t)\big). \tag{5.1}$$

Substituting (4.20) and (5.1) into (4.18), we have

$$\dot{\boldsymbol{x}}^*(t) = A\boldsymbol{x}^*(t) + \frac{1}{2}(-B_1R_1^{-1}B_1^T + B_2R_2^{-1}B_2^T)\boldsymbol{\psi}_1(t) - \frac{1}{2}\lambda B_1R_1^{-1}B_1^T\boldsymbol{\psi}_2(t), \quad \boldsymbol{x}^*(0) = \boldsymbol{x}_0, \tag{5.2a}$$

$$\dot{\boldsymbol{\psi}}_1(t) = -2Q\boldsymbol{x}^*(t) - A^T\boldsymbol{\psi}_1(t), \quad \lim_{t\to\infty}\boldsymbol{\psi}_1(t) = \boldsymbol{0}. \tag{5.2b}$$

Similarly, substituting (4.21) and (5.1) into (4.19), we have

$$\dot{\boldsymbol{x}}^{**}(t) = A\boldsymbol{x}^{**}(t) - \frac{1}{2}B_1R_1^{-1}B_1^T\boldsymbol{\psi}_1(t) + \left(-\frac{1}{2}\lambda B_1R_1^{-1}B_1^T + \frac{1}{2\gamma^2}B_2B_2^T\right)\boldsymbol{\psi}_2(t),\ \boldsymbol{x}^{**}(0) = \boldsymbol{0}, \tag{5.3a}$$

$$\dot{\boldsymbol{\psi}}_2(t) = -2C^TC\boldsymbol{x}^{**}(t) - A^T\boldsymbol{\psi}_2(t), \quad \lim_{t\to\infty}\boldsymbol{\psi}_2(t) = \boldsymbol{0}. \tag{5.3b}$$

(5.2) and (5.3) can be bundled as follows:

$$\frac{d}{dt}\begin{bmatrix}\boldsymbol{x}^*\\ \boldsymbol{x}^{**}\\ \boldsymbol{\psi}_1\\ \boldsymbol{\psi}_2\end{bmatrix} = \begin{bmatrix} A & O & \frac{1}{2}(-B_1R_1^{-1}B_1^T + B_2R_2^{-1}B_2^T)\\ O & A & -\frac{1}{2}B_1R_1^{-1}B_1^T\\ -2Q & O & -A^T\\ O & -2C^TC & O\end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{2}\lambda B_1 R_1^{-1} B_1^T \\ -\frac{1}{2}\lambda B_1 R_1^{-1} B_1^T + \frac{1}{2\gamma^2} B_2 B_2^T \\ O \\ -A^T \end{bmatrix} \begin{bmatrix} \boldsymbol{x}^* \\ \boldsymbol{x}^{**} \\ \boldsymbol{\psi}_1 \\ \boldsymbol{\psi}_2 \end{bmatrix},$$

$$\begin{bmatrix} \boldsymbol{x}^*(0) \\ \boldsymbol{x}^{**}(0) \\ \lim\limits_{t\to\infty} \boldsymbol{\psi}_1(t) \\ \lim\limits_{t\to\infty} \boldsymbol{\psi}_2(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}. \tag{5.4}$$

Once $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ are calculated, $\boldsymbol{u}^o$ can be obtained by (5.1). For this, however, we must solve the system of differential equations (5.4), which is difficult to solve because of the mixed existence of initial and final conditions. We shall represent $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ in terms of $\boldsymbol{x}^*$ and $\boldsymbol{x}^{**}$.

For (5.4), let a transition matrix between arbitrary τ and t, $0 \leqq \tau < t \leqq t_1$, be

$$\Phi(t,\tau) = \Phi(t-\tau) = \begin{bmatrix} \Phi_{11}(t-\tau) & \cdots & \Phi_{14}(t-\tau) \\ \vdots & & \vdots \\ \Phi_{41}(t-\tau) & \cdots & \Phi_{44}(t-\tau) \end{bmatrix}.$$

With this transition matrix, the solutions to (5.2b) and (5.3b) can be given as follows:

$$\boldsymbol{\psi}_1(t_1) = \Phi_{31}(t_1-t)\,\boldsymbol{x}^*(t) + \Phi_{32}(t_1-t)\,\boldsymbol{x}^{**}(t) + \Phi_{33}(t_1-t)\,\boldsymbol{\psi}_1(t) + \Phi_{34}(t_1-t)\,\boldsymbol{\psi}_2(t),$$
$$\boldsymbol{\psi}_2(t_1) = \Phi_{41}(t_1-t)\,\boldsymbol{x}^*(t) + \Phi_{42}(t_1-t)\,\boldsymbol{x}^{**}(t) + \Phi_{43}(t_1-t)\,\boldsymbol{\psi}_1(t) + \Phi_{44}(t_1-t)\,\boldsymbol{\psi}_2(t).$$

Substituting these into

$$\lim_{t_1\to\infty} \begin{bmatrix} \boldsymbol{\psi}_1(t_1) \\ \boldsymbol{\psi}_2(t_1) \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}$$

leads to

$$\lim_{t_1\to\infty} \begin{bmatrix} \boldsymbol{\psi}_1(t_1) \\ \boldsymbol{\psi}_2(t_1) \end{bmatrix} = \begin{bmatrix} \Phi_{31}(\infty) & \Phi_{32}(\infty) \\ \Phi_{41}(\infty) & \Phi_{42}(\infty) \end{bmatrix} \begin{bmatrix} \boldsymbol{x}^*(t) \\ \boldsymbol{x}^{**}(t) \end{bmatrix} + \begin{bmatrix} \Phi_{33}(\infty) & \Phi_{34}(\infty) \\ \Phi_{43}(\infty) & \Phi_{44}(\infty) \end{bmatrix} \begin{bmatrix} \boldsymbol{\psi}_1(t) \\ \boldsymbol{\psi}_2(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}.$$

Hence

$$\begin{bmatrix} \boldsymbol{\psi}_1(t) \\ \boldsymbol{\psi}_2(t) \end{bmatrix} = -\begin{bmatrix} \Phi_{33}(\infty) & \Phi_{34}(\infty) \\ \Phi_{43}(\infty) & \Phi_{44}(\infty) \end{bmatrix}^{-1} \begin{bmatrix} \Phi_{31}(\infty) & \Phi_{32}(\infty) \\ \Phi_{41}(\infty) & \Phi_{42}(\infty) \end{bmatrix} \begin{bmatrix} \boldsymbol{x}^*(t) \\ \boldsymbol{x}^{**}(t) \end{bmatrix}.$$

For simplicity, let

$$\begin{bmatrix} \boldsymbol{\psi}_1(t) \\ \boldsymbol{\psi}_2(t) \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}^*(t) \\ \boldsymbol{x}^{**}(t) \end{bmatrix}. \tag{5.5}$$

This is to say, $\begin{bmatrix} \boldsymbol{\psi}_1(t) \\ \boldsymbol{\psi}_2(t) \end{bmatrix}$ and $\begin{bmatrix} \boldsymbol{x}^*(t) \\ \boldsymbol{x}^{**}(t) \end{bmatrix}$ are in a linear relation.

By substituting (5.5) into (5.1), $\boldsymbol{u}^o$ takes the form

$$\boldsymbol{u}^o(t) = -\frac{1}{2} R_1^{-1} B_1^T \{(P_{11} + \lambda P_{21})\boldsymbol{x}^*(t) + (P_{12} + \lambda P_{22})\boldsymbol{x}^{**}(t)\}. \tag{5.6}$$

Besides, by substituting (5.5) into (4.20) and (4.21), the maximizing disturbances $\boldsymbol{v}^*$ for the upper-level problem and $\boldsymbol{v}^{**}$ for the lower-level problem can be represented as

$$\boldsymbol{v}^*(t) = \frac{1}{2} R_2^{-1} B_2^T \left(P_{11}\boldsymbol{x}^*(t) + P_{12}\boldsymbol{x}^{**}(t)\right), \quad \boldsymbol{v}^{**}(t) = \frac{1}{2\gamma^2} B_2^T \left(P_{21}\boldsymbol{x}^*(t) + P_{22}\boldsymbol{x}^{**}(t)\right). \tag{5.7}$$

Meanwhile, substituting (5.5) into (5.2a), we get

$$\dot{\boldsymbol{x}}^*(t) = \Big\{A + \frac{1}{2}(-B_1R_1^{-1}B_1^T + B_2R_2^{-1}B_2^T)P_{11} - \frac{1}{2}\lambda B_1R_1^{-1}B_1^TP_{21}\Big\}\boldsymbol{x}^*(t)$$
$$+ \Big\{\frac{1}{2}(-B_1R_1^{-1}B_1^T + B_2R_2^{-1}B_2^T)P_{12} - \frac{1}{2}\lambda B_1R_1^{-1}B_1^TP_{22}\Big\}\boldsymbol{x}^{**}(t), \quad \boldsymbol{x}^*(0) = \boldsymbol{x}_0, \tag{5.8a}$$

and substituting the upper half of (5.5) into (5.2b) yields

$$P_{11}\dot{\boldsymbol{x}}^*(t) + P_{12}\dot{\boldsymbol{x}}^{**}(t) = (-2Q - A^TP_{11})\boldsymbol{x}^*(t) - A^TP_{12}\boldsymbol{x}^{**}(t). \tag{5.8b}$$

Similarly, substituting (5.5) into (5.3a), we have

$$\dot{\boldsymbol{x}}^{**}(t) = \Big\{-\frac{1}{2}B_1R_1^{-1}B_1^TP_{11} + \Big(-\frac{1}{2}\lambda B_1R_1^{-1}B_1^T + \frac{1}{2\gamma^2}B_2B_2^T\Big)P_{21}\Big\}\boldsymbol{x}^*(t)$$
$$+ \Big\{A - \frac{1}{2}B_1R_1^{-1}B_1^TP_{12} + \Big(-\frac{1}{2}\lambda B_1R_1^{-1}B_1^T + \frac{1}{2\gamma^2}B_2B_2^T\Big)P_{22}\Big\}\boldsymbol{x}^{**}(t), \quad \boldsymbol{x}^{**}(0) = \boldsymbol{0}, \tag{5.9a}$$

and substituting the lower half of (5.5) into (5.3b) yields

$$P_{21}\dot{\boldsymbol{x}}^*(t) + P_{22}\dot{\boldsymbol{x}}^{**}(t) = -A^TP_{21}\boldsymbol{x}^*(t) - (2C^TC + A^TP_{22})\boldsymbol{x}^{**}(t). \tag{5.9b}$$

Substituting (5.8a) and (5.9a) into (5.8b), we can delete $\dot{\boldsymbol{x}}^*$ and $\dot{\boldsymbol{x}}^{**}$ to obtain

$$\Big[P_{11}A + \frac{1}{2}P_{11}(-B_1R_1^{-1}B_1^T + B_2R_2^{-1}B_2^T)P_{11}$$
$$- \frac{1}{2}P_{11}\lambda B_1R_1^{-1}B_1^TP_{21} - \frac{1}{2}P_{12}B_1R_1^{-1}B_1^TP_{11}$$
$$+ P_{12}\Big(-\frac{1}{2}\lambda B_1R_1^{-1}B_1^T + \frac{1}{2\gamma^2}B_2B_2^T\Big)P_{21} + 2Q + A^TP_{11}\Big]\boldsymbol{x}^*(t)$$
$$+ \Big[\frac{1}{2}P_{11}(-B_1R_1^{-1}B_1^T + B_2R_2^{-1}B_2^T)P_{12}$$
$$- \frac{1}{2}P_{11}\lambda B_1R_1^{-1}B_1^TP_{22} + P_{12}A - \frac{1}{2}P_{12}B_1R_1^{-1}B_1^TP_{12}$$
$$+ P_{12}\Big(-\frac{1}{2}\lambda B_1R_1^{-1}B_1^T + \frac{1}{2\gamma^2}B_2B_2^T\Big)P_{22} + A^TP_{12}\Big]\boldsymbol{x}^{**}(t) = \boldsymbol{0}. \tag{5.10}$$

Similarly, from (5.8a), (5.9a) and (5.9b), it follows that

$$\Big[P_{21}A + \frac{1}{2}P_{21}(-B_1R_1^{-1}B_1^T + B_2R_2^{-1}B_2^T)P_{11}$$
$$- \frac{1}{2}P_{21}\lambda B_1R_1^{-1}B_1^TP_{21} - \frac{1}{2}P_{22}B_1R_1^{-1}B_1^TP_{11}$$
$$+ P_{22}\Big(-\frac{1}{2}\lambda B_1R_1^{-1}B_1^T + \frac{1}{2\gamma^2}B_2B_2^T\Big)P_{21} + A^TP_{21}\Big]\boldsymbol{x}^*(t)$$
$$+ \Big[\frac{1}{2}P_{21}(-B_1R_1^{-1}B_1^T + B_2R_2^{-1}B_2^T)P_{12}$$
$$- \frac{1}{2}P_{21}\lambda B_1R_1^{-1}B_1^TP_{22} + P_{22}A - \frac{1}{2}P_{22}B_1R_1^{-1}B_1^TP_{12}$$
$$+ P_{22}\Big(-\frac{1}{2}\lambda B_1R_1^{-1}B_1^T + \frac{1}{2\gamma^2}B_2B_2^T\Big)P_{22}$$
$$+ 2C^TC + A^TP_{22}\Big]\boldsymbol{x}^{**}(t) = \boldsymbol{0}. \tag{5.11}$$

Relations (5.10) and (5.11) must hold for any $\boldsymbol{x}^*$ and $\boldsymbol{x}^{**}$. From their coefficients, we have the following four Riccati equations:

$$\begin{aligned}
&P_{11}A + A^T P_{11} + \frac{1}{2}P_{11}(-B_1 R_1^{-1} B_1^T + B_2 R_2^{-1} B_2^T)P_{11} \\
&\quad - \frac{1}{2}\lambda P_{11} B_1 R_1^{-1} B_1^T P_{21} - \frac{1}{2}P_{12} B_1 R_1^{-1} B_1^T P_{11} \\
&\quad + P_{12}\Big(-\frac{1}{2}\lambda B_1 R_1^{-1} B_1^T + \frac{1}{2\gamma^2} B_2 B_2^T\Big) P_{21} + 2Q = O, \\
&P_{12}A + A^T P_{12} + \frac{1}{2}P_{11}(-B_1 R_1^{-1} B_1^T + B_2 R_2^{-1} B_2^T)P_{12} \\
&\quad - \frac{1}{2}\lambda P_{11} B_1 R_1^{-1} B_1^T P_{22} - \frac{1}{2}P_{12} B_1 R_1^{-1} B_1^T P_{12} \\
&\quad + P_{12}\Big(-\frac{1}{2}\lambda B_1 R_1^{-1} B_1^T + \frac{1}{2\gamma^2} B_2 B_2^T\Big) P_{22} = O, \\
&P_{21}A + A^T P_{21} + \frac{1}{2}P_{21}(-B_1 R_1^{-1} B_1^T + B_2 R_2^{-1} B_2^T)P_{11} \\
&\quad - \frac{1}{2}\lambda P_{21} B_1 R_1^{-1} B_1^T P_{21} - \frac{1}{2}P_{22} B_1 R_1^{-1} B_1^T P_{11} \\
&\quad + P_{22}\Big(-\frac{1}{2}\lambda B_1 R_1^{-1} B_1^T + \frac{1}{2\gamma^2} B_2 B_2^T\Big) P_{21} = O, \\
&P_{22}A + A^T P_{22} + \frac{1}{2}P_{21}(-B_1 R_1^{-1} B_1^T + B_2 R_2^{-1} B_2^T)P_{12} \\
&\quad - \frac{1}{2}\lambda P_{21} B_1 R_1^{-1} B_1^T P_{22} - \frac{1}{2}P_{22} B_1 R_1^{-1} B_1^T P_{12} \\
&\quad + P_{22}\Big(-\frac{1}{2}\lambda B_1 R_1^{-1} B_1^T + \frac{1}{2\gamma^2} B_2 B_2^T\Big) P_{22} + 2C^T C = O.
\end{aligned}$$

These can be equivalently expressed as

$$\widetilde{P}\widetilde{A} + \widetilde{A}^T\widetilde{P} + \widetilde{P}\widetilde{R}\widetilde{P} + \widetilde{Q} = O, \tag{5.12}$$

where

$$\begin{aligned}
\widetilde{P} &= \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \qquad \widetilde{A} = \begin{bmatrix} A & O \\ O & A \end{bmatrix}, \qquad \widetilde{Q} = \begin{bmatrix} 2Q & O \\ O & 2C^T C \end{bmatrix}, \\
\widetilde{R} &= \begin{bmatrix} \frac{1}{2}(-B_1 R_1^{-1} B_1^T + B_2 R_2^{-1} B_2^T) & -\frac{1}{2}\lambda B_1 R_1^{-1} B_1^T \\ -\frac{1}{2} B_1 R_1^{-1} B_1^T & -\frac{1}{2}\lambda B_1 R_1^{-1} B_1^T + \frac{1}{2\gamma^2} B_2 B_2^T \end{bmatrix}.
\end{aligned}$$

Consequently, the matrices P_{11}, P_{12}, P_{21} and P_{22} in the satisfactory optimal control (5.6) and the maximizing disturbance (5.7) can be given by the solution of equation (5.12). By combining (5.8a) and (5.9a), $\boldsymbol{x}^*$ and $\boldsymbol{x}^{**}$ must solve the following closed-loop system

$$\begin{aligned}
\begin{bmatrix} \dot{\boldsymbol{x}}^*(t) \\ \dot{\boldsymbol{x}}^{**}(t) \end{bmatrix} = \begin{bmatrix} A - \frac{1}{2}B_1 R_1^{-1} B_1^T (P_{11} + \lambda P_{21}) + \frac{1}{2} B_2 R_2^{-1} B_2^T P_{11} \\ \quad -\frac{1}{2} B_1 R_1^{-1} B_1^T (P_{11} + \lambda P_{21}) + \frac{1}{2\gamma^2} B_2 B_2^T P_{21} \end{bmatrix} & \\
\begin{matrix} -\frac{1}{2} B_1 R_1^{-1} B_1^T (P_{12} + \lambda P_{22}) + \frac{1}{2} B_2 R_2^{-1} B_2^T P_{12} \\ A - \frac{1}{2} B_1 R_1^{-1} B_1^T (P_{12} + \lambda P_{22}) + \frac{1}{2\gamma^2} B_2 B_2^T P_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}^*(t) \\ \boldsymbol{x}^{**}(t) \end{bmatrix}, & \\
\begin{bmatrix} \boldsymbol{x}^*(0) \\ \boldsymbol{x}^{**}(0) \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{0} \end{bmatrix}. &
\end{aligned} \tag{5.13}$$

6. Conclusion

We have studied a satisfactory optimal control problem for linear-quadratic systems, which is a min-max control problem subject to the H_∞ norm condition. The satisfactory optimal control is given in a state-feedback-like form, using the state of an imaginary closed-loop system driven by the maximizing disturbances, with a $2n \times 2n$ sized Riccati equation. In this paper we have not argued the complementary slackness condition (4.17c) and the value of λ. Since λ is a scalar, however, by changing λ continuously, we can check whether the complementary slackness condition is satisfied or not. In fact, by searching λ in this way, we succeeded in finding an optimal λ for several numerical examples.

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Appendix

This appendix gives the gradient of an integral-type performance index:

$$\phi[\boldsymbol{u},\boldsymbol{v}] = \int_0^\infty f_0\big(\boldsymbol{x}(t),\boldsymbol{u}(t),\boldsymbol{v}(t)\big)\,dt \tag{A.1a}$$

$$\text{subj. to} \quad \dot{\boldsymbol{x}}(t) = \boldsymbol{f}\big(\boldsymbol{x}(t),\boldsymbol{u}(t),\boldsymbol{v}(t)\big), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{A.1b}$$

where the vectors $\boldsymbol{x}$, $\boldsymbol{u}$, $\boldsymbol{v}$ and the functions f_0 and $\boldsymbol{f}$ are the same as those defined in Section 3.

Let U and V be subsets of the Hilbert spaces $L_2([0,\infty),R^r)$ and $L_2([0,\infty),R^q)$, respectively. We assume that, for any pair $(\boldsymbol{u},\boldsymbol{v}) \in U \times V$, equation (A.1b) has a unique solution $\boldsymbol{x}(\boldsymbol{u},\boldsymbol{v})$ and integral (A.1a) has a finite value. Assuming the partial differentiability of the functional

$$\phi[\boldsymbol{u},\boldsymbol{v}] = \int_0^\infty f_0\big(\boldsymbol{x}(t;\boldsymbol{u},\boldsymbol{v}),\boldsymbol{u}(t),\boldsymbol{v}(t)\big)\,dt, \tag{A.2}$$

we have the following gradient representation, which is a generalization of that for a standard performance index without $\boldsymbol{v}$. (For additional assumptions, see [14, 15].)

Theorem A. *The partial gradients of the functional $\phi : U \times V \to R$, defined by (A.2), at $(\boldsymbol{u},\boldsymbol{v}) \in U \times V$ are given by*

$$\nabla_{\boldsymbol{u}}\phi[\boldsymbol{u},\boldsymbol{v}](t) = \nabla_{\boldsymbol{u}} H\big(\boldsymbol{x}(t;\boldsymbol{u},\boldsymbol{v}),\boldsymbol{u}(t),\boldsymbol{v}(t),\boldsymbol{\psi}(t)\big), \tag{A.3}$$

$$\nabla_{\boldsymbol{v}}\phi[\boldsymbol{u},\boldsymbol{v}](t) = \nabla_{\boldsymbol{v}} H\big(\boldsymbol{x}(t;\boldsymbol{u},\boldsymbol{v}),\boldsymbol{u}(t),\boldsymbol{v}(t),\boldsymbol{\psi}(t)\big) \tag{A.4}$$

for each $t \in [0,\infty)$, where

$$H(\boldsymbol{x},\boldsymbol{u},\boldsymbol{v},\boldsymbol{\psi}) = f_0(\boldsymbol{x},\boldsymbol{u},\boldsymbol{v}) + \boldsymbol{\psi}^T\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{v}), \tag{A.5}$$

and $\boldsymbol{\psi}$ is a continuous R^n-valued function defined on $[0,\infty)$ by

$$\dot{\boldsymbol{\psi}}(t) = -H_{\boldsymbol{x}}\big(\boldsymbol{x}(t;\boldsymbol{u},\boldsymbol{v}),\boldsymbol{u}(t),\boldsymbol{v}(t),\boldsymbol{\psi}(t)\big)^T, \quad \lim_{t\to\infty}\boldsymbol{\psi}(t) = \boldsymbol{0}. \tag{A.6}$$