Max semiselfdecomposable distributions and their properties

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Correspondence of limit theorem between sum scheme and max scheme

Sum	Maximum
Infinitely Divisible Sum of random variables $\hat{\mu}(z) = \hat{\mu}_n(z)^n$	Max Infinitely Divisible Maximum of random variables $F(x) = F_n(x)^n$
(Semi) Selfdecomposable Sum of Independent r.v. s $\hat{\mu}(z) = \hat{\mu}(bz)\hat{\rho(z)}$	Max (Semi) Selfdecomposble Maximum of Independent r.v.s $F(x) = F(Tx)F_\beta(x)$
(Semi) Stable Sum of i.i.d. r.v..s $\hat{\mu}(z)^a = \hat{\mu}(bz)e^{i\langle c,z\rangle}$	Max (Semi) Stable Maximum of i.i.d. r.v.s $F^t(x) = F(T_t(x))$

where $\widehat{\mu}$ is characteristic functions, F is distribution functions and T_t is an operator: $\mathbb{R}^d \to \mathbb{R}^d$

Result: Max semi selfdecomposabiliy and its characterization

For independent random vectors $\{\mathbb{X}_i\}$, suppose normalized maximum converges weakly to non-degenerate F i.e.

$$F_n(x) := P(L_n^{-1} \bigvee_{i=1}^{k_n} \mathbb{X}_i < x) \xrightarrow{\mathrm{w}} F(x), \quad n \to \infty, L_n \in \mathrm{GMA}$$

with uniformity assumption.

Then the limit distribution F is said to be a max semiselfdecomposable distributions. We denote by MSSD the class of that distributions.

Taking some $\beta \in (0, 1]$, we have decomposition

$$F(x) = F(T_\beta x)F_\beta(x), \; F_\beta \in \mathrm{MID}$$

i.e.

$$X \stackrel{\mathrm{d}}{=} T_\beta^{-1} X \vee Y_\beta, \; X \perp\!\!\!\perp Y_\beta, \; Y_\beta \in \mathrm{MID}$$

Result: Representation of max semi selfdecomp. dist.

If F is max semiselfdecomposable

then there exists (exponential) measure μ on $\overline{\mathbb{R}}^d$ such that

$$F(x) = \exp\left(-\mu(A_x^c)\right),$$

where $x = (x_1, \ldots, x_d)$ and $A_x = [-\infty, x_1] \times \ldots \times [-\infty, x_d]$.

(Balkema & Resnick 1977 J. Appl. Probal. **14** 309-319)

Further there exists

- finite measure μ_0 on $\overline{\mathbb{R}}^d$
- Borel measurable function $g_n(x)$, $n \in \mathbb{Z}$: $S_\beta \to [0, \infty)$
  with $g_n(x) - g_{n+1}(x) \geq 0$,
  where $S_\beta = \{x^* \in \mathbb{R}^d; \max x^* = \max\{x_1^*, \ldots, x_d^*\} \leq 1,\ maxT_\beta x^* > 1\}$

such that

$$\mu(A_x^c) = \int_{S_\beta} \mu_0(dx) \sum_{n \in \mathbb{Z}} g_n(x) \mathbf{1}_{A_x^c}(T_\beta^n x).$$

with some conditions. Converse is true.

(cf. Maejima, Sato, Watanabe 1999 Tokyo J. Math Vol **22** No.2)

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Central Limit Theorem (CLT)

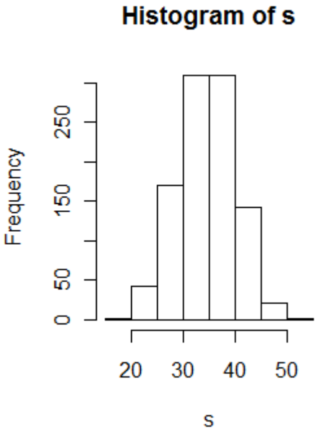
Here we show an experimental example of CLT.

Example: sum of number of ten dice

1000 trials
(Simulation)

36 40 22 36 33 19 47 37
36 31 42 35 26 34 34 34
22 38 34 32 $\cdots$

the frequency at near 10 or 60 is
low and that at near 35 is high



CLT and its extension

Suppose that i.i.d. random variables $\{X_i\}$ have mean μ and variance σ^2.

Setting $S_n = \sum_{i=1}^n X_i$ and we have

$$n^{-1/2}(S_n - n\mu) \xrightarrow{\mathrm{d}} Z \sim N(0, \sigma^2).$$

convergence to normal distribution with shift $n\mu$ and scale $n^{-1/2}$.

Replacing shift with b_n and scale with a_n, we consider $\dfrac{S_n - b_n}{a_n}$.

in case without variance? in case without mean?
in case of non-identical distributions? in case of subsequent convergence?
$\rightarrow$ Infinitely divisible distributions and its subclasses

Infinitely divisible distributions and its subclasses

Take random sequence $\{X_i\}$. Mean and variance are unnecessary.
Set sum $S_n = \sum_{i=1}^{n} X_i$ and scaled and shifted sum $a_n^{-1}(S_n - b_n)$. The latter converges to the followings:

Distributions	Identicalness	Convergence	Independence
Infinitely divisible	×	Subsequent	×
Stable	◯	Full sequent	◯
Semi Stable	◯	Subsequent	◯
Selfdecomposable	×	Full sequent	◯
Semi Selfdecomposable	×	Subsequent	◯

◯: usually supposed, ×: usually not supposed

Uniformity condition (few r.v.s are not unnaturally big) is necessary.

For extremes (max scheme), we consider maximum $M_n = \max_{1 \le i \le n} X_i$ and scaled and shifted maximum $a_n^{-1}(M_n - b_n)$.

In multidimensional cases, we have similar distributions.

Characterization by ch.f. and distribution function

Characterizing:
Infinitely divisible distributions (sum scheme) by ch.f. $\widehat{\mu}(z)$,
Extreme distributions (max scheme) by distribution function$F(x)$.

In case of sum of random variables:
Put ch.f. (Fourier Transform) for r.v. X as $\widehat{\mu}_X(z) = E(e^{izX})$.
For independent random variables X_1, X_2, we have

$$\widehat{\mu}_{X_1+X_2}(z) = E(e^{iz(X_1+X_2)}) = E(e^{izX_1})E(e^{izX_2}) = \widehat{\mu}_{X_1}(z)\widehat{\mu}_{X_2}(z).$$

It is suitable for sum scheme.

In case of maximum of random variables:
Put distirbution function for r.v. X as $F(x) = P(X \le x)$.
For independent random variables X_1, X_2, we have

$$\begin{aligned} F_{X_1 \vee X_2}(x) &= P(X_1 \vee X_2 \le x) = P(X_1 \le x)P(X_2 \le x) \\ &= F_{X_1}(x)F_{X_2}(x) \end{aligned}$$

It is suitable for maximum scheme.

Correspondence between sum scheme and max scheme

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where $\widehat{\mu}$ is characteristic functions, F is distribution functions and T_t is an operator: $\mathbb{R}^d \to \mathbb{R}^d$

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History of limit theorems for max scheme

Year	Researcher	Limit	paper
1928	Fisher & Tippet	Stable	Limiting forms of the frequency dist. largest or smallest member of a sample *Proc. Camb. Philos. Soc.*, **24.** 180-190
1977	Balkema & Resnick	Infinitely Divisible	Max-infinite divisibility. *J. Appl. Prob.* **14.** 309-319
1986	Gerritse & (Vervaat)	Self decomp.	Supremum selfdecomp. random vectors *Prob. Theo. Relat. Fields* **72** 17–33.
1990	Pancheva	Self decomp.	Selfdecomposable Dist. for Maxima of Independent Random Vectors *Prob. Theo. Relat. Fields* **84** 267–278
1993	Grinevich	Semi Stable	Dom. of att. for max-semistable laws under linear and power normalizations. Theory Probab. Appl. **38** no. 4 640–650

Limit theorem for maximum of i.i.d. r.v.s

Let X_i be i.i.d. r.v.s with mutual distribution F and set maximum

$$M_n = \max_{1 \le i \le n} X_i.$$

The distribution function of M_n is

$$P(M_n < x) = (F(x))^n.$$

Here we take limit as $n \to \infty$, we only have

$$\lim_{n \to \infty} M_n = \sup\{x : F(x) < 1\} \quad \text{a.s.}$$

which is degenerate. But for some distributions F, taking appropriate scale and shift, we have

$$P((M_n - b_n)/a_n \le x) = (F(a_n x + b_n))^n \xrightarrow{\text{w}} G(x)$$

which is non-degenerate distribution.

Distribution function of max stable distributions

It is known that by taking this limit we have three limit distributions except type equivalence:

$$V(x) = U(ax + b).$$

The class of limit distributions is coincide with the class satisfying

$$G^t(x) = G(\alpha(t)x + \beta(t)).$$

These discussion is very similar to that of sum scheme: for i.i.d. r.v.s with mutual distributions μ, taking sum $S_n = \sum_{i=1}^n X_i$ and we have

$$E\left(i\langle z, (S_n - b_n)/a_n\rangle\right) = (\widehat{\mu}(a_n z + b_n))^n \to \widehat{\nu}(z)$$

and $\widehat{\nu}(z)^t = \widehat{\nu}(a(t)z)e^{i\langle b(t), z\rangle}$. The limit distribution ν is called stable distributions. So the former limit distribution G is called max stable.

Notations

For $\overline{\mathbb{R}}^d (= [-\infty, \infty)^d)$–valued random vector sequence $\{\mathbb{X}_k\} = \left\{ \left(X_k^{(i)},\ 1 \le i \le d \right) \right\},\ 1 \le k \le n$, we set the maximum vector

$$\mathbb{M}_n = \mathbb{X}_1 \vee \ldots \vee \mathbb{X}_n = \left(\max_{1 \le j \le n} X_j^{(i)},\ 1 \le j \le d \right).$$

Denote by GMA the set of all continuous $\vee$–automorphisms L of $\overline{\mathbb{R}}^d$.
i.e. $L(x \vee y) = L(x) \vee L(y)$ and there exists inverse mapping L^{-1}.

Max Infinitely Divisible Distributions

Reset $\mathbb{M}_n = \mathbb{X}_{n1} \vee \ldots \vee \mathbb{X}_{nk(n)}$ and
Suppose $P(\mathbb{M}_n < x) \stackrel{\mathrm{w}}{\to} F(x)$ non-degenerate.
Assume uniformity assumption

$$\max_{1 \le j \le k_n} P(\mathbb{X}_{nj} > x) \to 0, \quad n \to \infty$$

at any continuity point of F.

Then we call the limit distribution F Max infinitely divisible and denote the class by MID.

If $F \in$ MID then for any integer n, there exists F_n such that

$$F(x) = (F_n(x))^n.$$

Balkema & Resnick (1977), Max-Infinite Divisibility, *Journal of Applied Probability* **14**, No. 2.

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Max Selfdecomposable Distributions

For independent random vectors $\{\mathbb{X}_k\}$, suppose normalized maximum converges weakly to non-degenerate F i.e.

$$F_n(x) := P(L_n^{-1}\mathbb{M}_n < x) \xrightarrow{\mathrm{w}} F(x), \quad n \to \infty, L_n \in \mathrm{GMA}$$

And assume uniformity assumption.
Then the limit distribution F is said to be a max selfdecomposable distributions. We denote by MSD the class of that distributions.

Here, for any $\beta \in (0,1]$ there exists

$$T_\beta(x) = \lim L_{[n\beta]}^{-1} \cdot L_n(x),\ L_n \in \mathrm{GMA}\left(\overline{\mathbb{R}}^d\right)$$

and that $\mathcal{T} = \left\{T_\beta : \beta \in (0,1]\right\}$ is 1-parameter semi-group with

$$T_\alpha(T_\beta x) = T_{\alpha\cdot\beta}(x),\ \alpha,\beta \in (0,1].$$

Pancheva, E. I.(1990) Selfdecomposable Distributions for Maxima of Independent Random Vectors *Prob. Theo. Relat. Fields* **84** 267–278

MSD & MID characterization by distribution function

Setting $\mathbb{X}_{nj} = L_n^{-1}\mathbb{X}_j$, we have MSD $\subset$ MID.

Suppose F $\in$MSD and $F_n(x) = P(L_n^{-1}\mathbb{M}_n < x) \xrightarrow{\mathrm{w}} F(x)$.
For a $\beta \in (0,1]$, set

$$F_n^{(1)}(x) = P(\mathbb{M}_{n\beta} < L_n x),\ F_n^{(2)}(x) = P\left(\max_{[n\beta]+1 \le k \le n} \mathbb{X}_k < L_n x\right)$$

and we have decomposition $F_n(x) = F_n^{(1)}(x)F_n^{(2)}(x)$.

Since $P(\mathbb{M}_{[n\beta]} < L_{[n\beta]}x) = F_{[n\beta]}(x) \xrightarrow{\mathrm{w}} F(x)$, we have

$$\begin{aligned} F_n^{(1)}(x) &= P(\mathbb{M}_{n\beta} < L_{[n\beta]} \cdot (L_{[n\beta]}^{-1} \cdot L_n x)) \\ &= F_{[n\beta]}(L_{[n\beta]}^{-1} \cdot L_n x) \xrightarrow{\mathrm{w}} F(T_\beta x), \end{aligned}$$

where, $T_\beta(x) = \lim L_{[n\beta]}^{-1} \cdot L_n(x)$.We also have $F_n^{(2)} \xrightarrow{\mathrm{w}} \exists F_\beta \in$ MID and limit decomposition $F(x) = F(T_\beta x)F_\beta(x)$ for any $\beta \in (0,1]$.

Max SemiSelfdecomposable Distributions

For independent random vectors $\{\mathbb{X}_k\}$, suppose normalized maximum converges weakly to non-degenerate F i.e.

$$F_n(x) := P(L_n^{-1}\mathbb{M}_{k_n} < x) \xrightarrow{\mathrm{w}} F(x), \quad n \to \infty, L_n \in \mathrm{GMA}$$

with uniformity assumption.
Then the limit distribution F is said to be a max semiselfdecomposable distributions. We denote by MSSD the class of that distributions.
MSD is subset of MSSD.

we have decomposition for some $\beta \in (0, 1]$

$$F(x) = F(T_\beta x)F_\beta(x),\ F_\beta \in \mathrm{MID}$$

where $\mathcal{T} = \left\{T_\beta\right\}$ is 1-parameter semi-group.

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Representation of max semi selfdecomp. dist.

If F is max infinitely divisible

then there exists measure μ on $\overline{\mathbb{R}}^d$ such that

$$F(x) = \exp\left(-\mu(A_x^c)\right),$$

where $x = (x_1, \ldots, x_d)$ and $A_x = [-\infty, x_1] \times \ldots \times [-\infty, x_d]$.
(cf. Balkema & Resnick)

If F is max semiselfdecomposable there exists

- finite measure μ_0 on $\overline{\mathbb{R}}^d$
- Borel measurable function $g_n(x),\ n \in \mathbb{Z}$: $S_\beta \to [0, \infty)$
  with $g_n(x) - g_{n+1}(x) \geq 0$,
  where $S_\beta = \{x^* \in \mathbb{R}^d; \max x^* = \max\{x_1^*, \ldots, x_d^*\} \leq 1,\ maxT_\beta x^* > 1\}$

such that

$$\mu(A_x^c) = \int_{S_\beta} \mu_0(dx) \sum_{n \in \mathbb{Z}} g_n(x) \mathbf{1}_{A_x^c}(T_\beta^n x).$$

with some conditions.

Converse is true.

Sketch of Proof 1

First, we establish representation for max infinitely divisible. Define

$$\mu_0(E) = \sum_{n\in\mathbb{Z}} 2^{-|n|} \frac{\mu(T_\beta^{-n}E)}{\mu(T_\beta^{-n}S_\beta)}, \; E \in \mathcal{B}(S_\beta).$$

Let $[\mu]_{T_\beta^{-n}S_\beta}$ be the restriction of μ to $T_\beta^{-n}S_\beta$.
Then $[\mu]_{T_\beta^{-n}S_\beta}$ is absolutely continuous with respect to $\mu_0 \cdot T_\beta^n$.
So there exists Randon Nikodym derivative $h_n(x)$ such that

$$\mu(dx) = h_n(x)(\mu_0 \cdot T_\beta^n)(dx) \text{ on } T_\beta^{-n}S_\beta.$$

So

$$\begin{aligned}\mu(E) &= \sum_{n\in\mathbb{Z}} \mu(E \cap T_\beta^{-n}S_\beta) \\ &= \sum_{n\in\mathbb{Z}} \int_{S_\beta} \mathbf{1}_E(T_\beta^{-n}x) h_n(T_\beta^{-n}x)\mu_0(dx).\end{aligned}$$

Here we take $g_{-n}(x) = h_n(T_\beta^{-n}x)$.

Sketch of Proof 2

Next from the decompostion $F(x) = F(T_\beta x)F_\beta(x),\ F_\beta$, we have that $F \in$ MSSD if and only if $\mu(A) - \mu(T_\beta A) = \nu_\beta(A) \geq 0$ for any $A \in \mathcal{B}(\mathbb{R}^d)$.
So

$$\begin{aligned}\mu(A) - \mu(T_\beta A) &= \int_{S_\beta} \mu_0(dx)\left(\sum_{n\in\mathbb{Z}} g_n(x)\mathbf{1}_A(T_\beta^n x) - \sum_{n\in\mathbb{Z}} g_n(x)\mathbf{1}_{T_\beta A}(T_\beta^n x)\right) \\ &= \int_{S_\beta} \mu_0(dx)\sum_{n\in\mathbb{Z}}(g_n(x) - g_{n+1}(x))\mathbf{1}_A(T_\beta^n x) \geq 0\end{aligned}$$

Thus $g_n(x) - g_{n+1}(x) \geq 0$.