RETAS: a restricted ETAS model inspired by Bath's law

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Abstract

A version of the restricted trigger model is used to analyse the temporal behaviour of some aftershock sequences. The conditional intensity function of the model is similar to that of the Epidemic Type Aftershock-Sequence (ETAS) model with the restriction that only the aftershocks of magnitude bigger than or equal to some threshold M_{tr} can trigger secondary events. For this reason we have named the model restricted epidemic type aftershock-sequence (RETAS) model. Varying the triggering threshold we examine the variants of the RETAS model which range from the Modified Omori Formula (MOF) to the ETAS model, including such models as limit cases.

1. Introduction

THE most widely used models to describe clusters of events are: the modified Omori law [5] and the ETAS model [2]. The first assumes that all the events in an aftershock sequence are triggered by the stress field change due to the main shock, are conditionally independent and follow a nonstationary Poisson process; the second extends the capability of generating secondary events to every aftershock of the sequence. Between these two extreme cases there is a range of different versions of trigger models which differ in the nature of the primary events.

According to Bath's law the difference between the main shock magnitude and the magnitude of the strongest aftershock is constant, on average 1.2 for Richter and 1.4 for Utsu [4]. By extending this principle to the subsequences generated by primary events as well, we get that the difference between the weakest primary event and the weakest event in the aftershock sequence must be at least 1.2. This suggests choosing the primary events among those of magnitude larger than a suitable threshold M_{tr}.

The aim of our work is to examine the class of trigger models obtained by varying M_{tr} between the cut-off M_0 and the maximum magnitude M_{max}. At first we have simulated two data sets according to the ETAS and RETAS model respectively, and we checked that the true model was identified using the Akaike criterion. Then we have applied the RETAS model to the analysis of two aftershock sequences with different characteristics: the first is formed by the events which followed the strong earthquake of $M = 7.8$ occurred in Kresna, SW Bulgaria, in 1904, the second includes three main shocks and a large swarm of minor shocks following the quake of 26 September 1997 in Umbria-Marche region, central Italy.

2. Trigger models

THE conditional intensity function of a trigger model is given by

$$\lambda(t|\mathcal{H}_t) = \mu \; + \sum_{\{m\in\mathcal{P}:t_m<t\}} \frac{K_m}{(t-t_m+c)^p} \qquad ,$$

being $\mathcal{P} = \{m = i_k, k = 1,\ldots,L\}$ the subset of primary events triggering offspring. This set is the discriminant element among the models of the class. In particular, when $\mathcal{P}$ is constituted only by the main shock we have the inverse power law of the MOF [5]

$$\lambda(t) = \frac{K}{(t+c)^p} \qquad ,$$

where t is the time elapsed from the occurrence of the main shock; whereas $\mathcal{P}$ is formed by the entire sequence in the ETAS model [2]

$$\lambda(t|\mathcal{H}_t) = \mu \; + \sum_{t_i<t} \frac{K_i}{(t-t_i+c)^p} \qquad ,$$

where the term K_i may have the explicit formulation

$$K_i = K_0 \; e^{\alpha(M_i - M_0)} \qquad . \tag{1}$$

In intermediate situations $\mathcal{P}$ may be composed by events randomly distributed [6], or identified before estimation [2] and possibly, in their turn, triggered by events belonging to the same subset $\mathcal{P}$, occurred previously [4].

In this work we have examined a trigger model in which the primary events are those having magnitude larger than or equal to a threshold M_{tr}, but, contrary to [4], the parameters K_m obey the restriction (1); hence the conditional intensity function of the model turns out to be

$$\lambda(t|\mathcal{H}_t) = \mu \; + \sum_{\substack{t_i<t \\ M_i\geq M_{tr}}} \frac{K_0 \; e^{\alpha(M_i-M_0)}}{(t-t_i+c)^p} \qquad . \tag{2}$$

The idea of a gap between the magnitude M_{tr} of the triggering event and that of the largest event generated is borrowed by Bath's law, but we have not fixed the size of this gap, even better by varying M_{tr} we have examined all the models between the MOF and the ETAS model on the basis of the Akaike criterion.

3. Case study: Kresna region, SW Bulgaria

ON April 4, 1904 an earthquake of $M = 7.8$ struck the region and triggered an aftershock sequence which continued for a long period. The sequence considered is constituted by 60 earthquakes of magnitude $M \geq 4.0$ covering a time period of 1230 days (Fig. 1(a)). Fig. 1(b) shows the AIC values obtained by fitting the class of the RETAS models to the sequence. The best model corresponds to $M_{tr} = 7.8$, that is, to the MOF. The AIC value is not sensitive to the M_{tr} variations because of the very strong dependence of the aftershock frequency on the magnitude of the triggering event as indicated by the values of the parameter estimates, high for α and small for K.

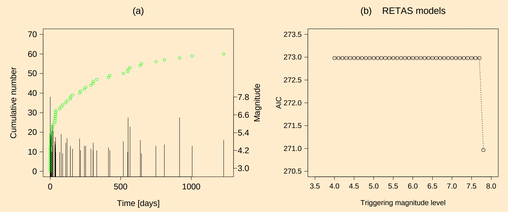

Figure 1: *(a) Cumulative number and magnitude of the events in the sequence; (b) AIC value of the RETAS model for different M_{tr} triggering magnitudes. The minimum AIC value indicates that the best model is the RETAS model with $M_{tr} = 7.8$, equivalent to the MOF in which only the main shock can trigger descendants.*

The residual analysis (Fig. 2(b)) reveals that the observations exceed the expected number of events at the beginning of the sequence. We think that this is due to the series of foreshocks so strong as to influence the start of the aftershock process.

The analysis performed on the relaxation process indicates that we have a typical aftershock sequence in which the process is just controlled by the stress field change caused by the main shock.

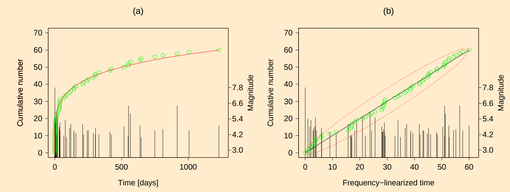

Figure 2: *Observed (circles) and estimated (continuous line) cumulative number of events versus (a) the ordinary and (b) transformed time. The plug-in estimate corresponds to the RETAS = MOF model ($M_{tr} = 7.8$). The dotted lines in (b) are the twofold standard deviations of the residual point process.*

4. Case study: Umbria-Marche region, central Italy

THE sequence is characterized by the occurrence of six earthquakes of magnitude larger than 5 in a period of 20 days; it contains 508 events with $M \geq 2.9$, covering the period from September 26 up to November 3, 1997.

First we have fitted the class of RETAS models to this data set. The AIC values obtained (Fig. 3(a)) show that the ETAS model is the best; according to Ogata [3] the small value of the α parameter estimate, $\hat{\alpha} = 0.89$, classifies the sequence as a swarm.

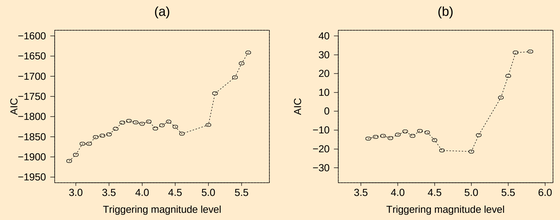

Figure 3: *AIC value of the RETAS model for different M_{tr} triggering magnitudes with cut-off magnitude (a) $M_0 = 2.9$ and (b) $M_0 = 3.6$. The best model is (a) the ETAS and (b) the RETAS model with $M_{tr} = 5.0$.*

The residual analysis reveals that the number of the observed events exceeds that of the expected ones nearly in the whole time interval (Fig. 4(a)).

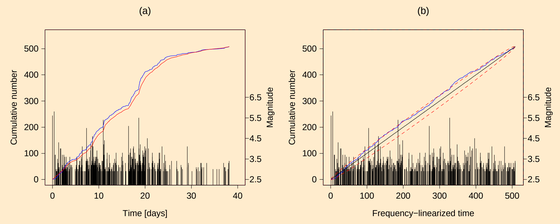

Figure 4: *Observed (blue) and estimated (red) cumulative number of events versus (a) the ordinary and (b) transformed time. The plug-in estimate corresponds to the RETAS = ETAS model ($M_{tr} = 2.9$). The dotted lines in (b) are the twofold standard deviations of the residual point process.*

As the non-monotonic trend of the curve (Fig. 3(a)) of the AIC values is similar to that obtained by analysing a sequence simulated by a RETAS model, and as that curve has a local minimum in correspondence of $M_{tr} = 4.6$, we repeat the analysis on the subset of the aftershocks sequence with cut-off magnitude $M_0 = 3.6$. The minimum AIC value (Fig. 3(b)) is achieved for $M_{tr} = 5.0$; this means that, by raising the threshold magnitude, secondary clusters appear in the so reduced Umbria-Marche sequence, generated by the six strongest events.

The RETAS model we have proposed can be applied for modelling compound seismic sequences in regions of complex tectonic structure with highly fractured earth crust.

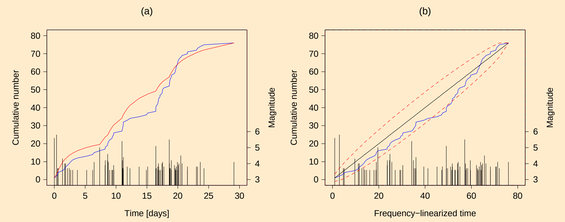

Figure 5: *Cumulative number and magnitude of the 1997 aftershocks in Umbria-Marche region, central Italy, for cut-off magnitude $M_0 = 3.6$; observed (blue) and estimated (red) cumulative number of events versus (a) the ordinary and (b) transformed time. The plug-in estimate corresponds to the RETAS model with $M_{tr} = 5.0$. The dotted lines in (b) are the twofold standard deviations of the residual point process.*

The algorithm

The algorithm we have applied is composed of two stages. The first consists in the following steps:

i. order the set $\mathcal{D} = \{M_i\}_{i=1}^{N}$ in increasing way and build the set of different magnitude levels $\{Mag_k\}_{k=1}^{K}$ obtained by removing from $\mathcal{D}$ the repeated values, so that $Mag_k < Mag_{k+1}$, $k = 1,\ldots,K-1$, $\forall k \; Mag_k \in \mathcal{D}$, and $\forall i \; \exists k$: $M_i = Mag_k$;

ii. set $k = 0$;

iii. $k \leftarrow k+1$, $M_{tr} = Mag_k$;

iv. given the conditional intensity function (2), maximize $\log\lambda(\cdot)$ and evaluate AIC_k;

v. if $k = 1$, then set $k_{best} = 1$ and $AIC_{best} = AIC_1$; otherwise, if $AIC_k < AIC_{best}$, then set $k_{best} = k$ and $AIC_{best} = AIC_k$;

vi. if $k < K$ then go to (iii), otherwise stop.

The final output, expressed by the value k_{best}, identifies the best model: ETAS if $k_{best} = 1$, MOF if $k_{best} = K$ and RETAS with triggering magnitude $Mag_{k_{best}}$ if $1 < k_{best} < K$.

If the ETAS model comes out to be the best model, it is interesting to examine whether particular correlations between the strongest aftershocks and the remaining seismicity emerge by raising the threshold M_0. In this second stage the analysis can be carried out throught the following steps:

i. plot the set of pairs $\{(Mag_k, AIC_k)\}_{k=1}^{K}$;

ii. if this set shows a monotonic trend, then stop; otherwise find the index of the local minimum, let's say j, set $M_{tr} = Mag_j$ and $M_0 \approx M_{tr} - 1$, and go back to the first stage considering the subset formed by the events of size exceeding M_0.

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