A Study on the Limited Power Law Model

T. Wang\textsuperscript{1}, L. Ma\textsuperscript{2}, and Y. Li\textsuperscript{1}

\textsuperscript{1}School of Mathematical Sciences, Statistical Data Analysis Laboratory, Beijing Normal University, Beijing 100875, China
\textsuperscript{2}Institute of Earthquake Science, China Earthquake Administration, Beijing 100036, China

Corresponding Author: T. Wang, athena_tingwang@yahoo.com.cn; athena_tingwang@hotmail.com

\textbf{Abstract:} This paper modifies the ETAS model for the aftershock rate by using the Limited Power Law (LPL) formula in place of the modified Omori law. The parameters are estimated by the maximum likelihood method. The model is then applied to the Chi-Chi sequence and the resulting AIC values show that the new model performs better than the original ETAS model for the earliest aftershocks.

1. Introduction

From the Omori law (Omori, 1894) to the modified Omori law (Utsu, 1961), the aftershock decay rate has been well described as a power function (Zhuang and Ma, 2000). This provided the basis of Epidemic Type Aftershock Sequence (ETAS) model (Ogata, 1988). The modified Omori law suggests that the aftershock decay rate is

\[ \Lambda(t) = \frac{K}{(t + c)^p} \]

where \( \Lambda(t) \) is the aftershock rate at time \( t \), \( K \) depends on the lower bound of the magnitude of aftershocks considered, but \( c \) and \( p \) are usually regarded independent of this bound (Ogata, 1983).

However, the earliest aftershocks are always less frequent than predicted by both the Omori law and the modified Omori law. Narteau, et al. (2002) investigated a great number of aftershock sequences in a seismic region and proposed the Limited Power Law (LPL) to relate this phenomenon to stress characteristics in the aftershock zone, resulting in an aftershock rate of

\[ \lambda_0 = \lambda(s) \text{ describes the stress relaxation decay rate at } s, \text{ while it is assumed that the overload (stress minus strength) is uniformly distributed on } [0, s] \text{ and } \lambda_0 \text{ is usually described as a characteristic rate associated with the threshold of crack growth (Narteau, et al., 2002; Narteau, et al., 2003).} \]

The present paper utilizes the LPL formula (2) in place of the modified Omori law (1) in the ETAS model to model the rate of aftershocks. The maximum likelihood method is used to estimate the parameters, and the model is applied to the Chi-Chi sequence to illustrate its performance.

2. ETAS model with LPL formula

The conditional intensity function of the original ETAS model (Ogata, 1983) is given by

\[ \lambda(t | H_t) = \mu + K \sum_{i \in c} \frac{e^{\alpha(M_i - M_c)}}{(t - t_i + c)^p}, \quad M_i \geq M_c \]
where $\mu$ is the background rate; $K$, $c$, and $p$ are the parameters corresponding to (1), $\alpha$ is the parameter “measuring the effect of magnitude in the production of aftershocks”, and shocks with magnitude $M_i$ occur at times $t_i$ while $M_c$ is the magnitude threshold.

Each earthquake produces an inhomogeneous field of increased stress in individual parts of a special region in and around the earthquake source. The stress will then be gradually released in the form of aftershocks. Under the same assumptions in (Shebalin, 2004), we can use a stationary decay rate of $\lambda(\sigma_i)$ to describe such relaxation, where $\sigma_i$ is the overload. The present paper is concerned with an exponential expression of the transition rates (Shebalin, 2004),

$$\lambda(\sigma_0) = \lambda_0 e^{\sigma_0/\sigma_a},$$

where $\sigma_a > 0$ is a scaling parameter. Thus (2) becomes (according to Shebalin, 2004, $q = 1$)

$$\Lambda(t) = A e^{-\lambda_i t} - e^{-\lambda_0 t}.$$ (5)

Therefore, the conditional intensity function in ETAS model will be modified to

$$\lambda(t \mid H_t) = \mu + \sum_{i = 1}^n \frac{e^{\alpha(M_i - M_c)} e^{-\lambda_i (t - t_i)} - e^{-\lambda_0 (t - t_i)}}{t - t_i}, \quad M_i \geq M_c$$ (6)

where $\mu$, $\alpha$, $M_i$, and $M_c$ have the same meaning as in (3).

3. Estimation of the parameters and the goodness-of-fit test

We use the maximum likelihood method (Ogata, 1983) to estimate the parameters $\mu$, $A$, $\alpha$, $\lambda_a$, and $\lambda_b$ from the observed aftershocks. As for the modified ETAS model, the likelihood function is

$$L(\mu, A, \alpha, \lambda_a, \lambda_b) = \left\{ \prod_{j=1}^{n} \lambda(t_j \mid H_{t_j}) \right\} \exp \left\{ -\int_{T_1}^{T_2} \lambda(t \mid H_t) dt \right\}$$

where $t_1, t_2, \cdots, t_n$ are the times at which the aftershocks occurs within a time interval $[T_1, T_2]$. Thus the log-likelihood function is

$$\ln L(\mu, A, \alpha, \lambda_a, \lambda_b)$$

$$= \sum_{j=1}^{n} \ln \left( \mu + \sum_{i \neq j} \frac{e^{\alpha(M_i - M_c)} e^{-\lambda_i (t_j - t_i)} - e^{-\lambda_0 (t_j - t_i)}}{t_j - t_i} \right) - \mu(T_2 - T_1)$$

$$- A \int_{T_1}^{T_2} \sum_{i \neq j} \frac{e^{\alpha(M_i - M_c)} e^{-\lambda_i (t - t_i)} - e^{-\lambda_0 (t - t_i)}}{t - t_i} dt.$$ (8)

The maximum likelihood estimates of these parameters are those which maximize the above log-likelihood function.

The Akaike Information Criterion (Akaike, 1974) is used here to test the goodness of fit of this model, which is defined by

$$\text{AIC} = -2 \ln L + 2k,$$ (9)
where $k$ is the number of the parameters in the model. A smaller AIC shows a better fit.

4. Application to Chi-Chi sequence

The data are chosen from SSLib (Harte, 1998) in the region with latitude between 23.3N and 24.24N, longitude between 120.2E and 121.2E, during the period 1999.09.20 to 2001.03.31. This includes the well-known Chi-Chi sequence. For the Chi-Chi sequence, the main shock on September 20, 1999 with magnitude 7.3. The Gutenberg-Richter law is used here to determine the cut-off magnitudes, which is M1.3 (Liu and Ma, 2004). We select the first 150 aftershocks following the main shock to examine the performance of the ETAS model using the LPL formula and compare it with the original ETAS model. The estimates of the parameters (Healy, et al., 1997; Ma and Vere-Jones, 1997) and the AIC values are shown in Table 1, from which we see that the ETAS model using the LPL formula is better than the original ETAS model for the earliest aftershocks.

Table 1. The parameter estimates and AIC values for the original ETAS model, and the ETAS model using LPL formula.

<table>
<thead>
<tr>
<th>$M_c \geq 1.3$</th>
<th>$\mu$</th>
<th>$K (A)$</th>
<th>$\alpha$</th>
<th>$c(\lambda_c)$</th>
<th>$p(\lambda_c)$</th>
<th>AIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original</td>
<td>0.0150</td>
<td>0.0638</td>
<td>1.4855</td>
<td>0.0107</td>
<td>1.0841</td>
<td>-1055.232</td>
</tr>
<tr>
<td>LPL</td>
<td>0.0150</td>
<td>0.0257</td>
<td>0.4739</td>
<td>0.3579(\lambda_c)</td>
<td>2820.6899(\lambda_c)</td>
<td>-1067.752</td>
</tr>
</tbody>
</table>

5. Conclusion

Since the earliest aftershocks are always less frequent than predicted by the modified Omori law due to the purely physical causes of aftershock shortage, and since the LPL formula performs better than the Omori law for the earliest aftershocks, this paper replaces the Omori law in the original ETAS model by the LPL formula. From the analysis of the Chi-Chi sequence, it shows that the modified ETAS model using the LPL formula describes the decay of earliest aftershock much better than the original ETAS model. Further work using more different sequences to compare these two models should be done to enhance this conclusion. Moreover, it is helpful to investigate the relationship between the parameters and strong aftershocks in different periods.

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7. References


