The Hypergraph Network Simplex Algorithm & Railway Optimization

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joint work with Isabel Beckenbach and Markus Reuther

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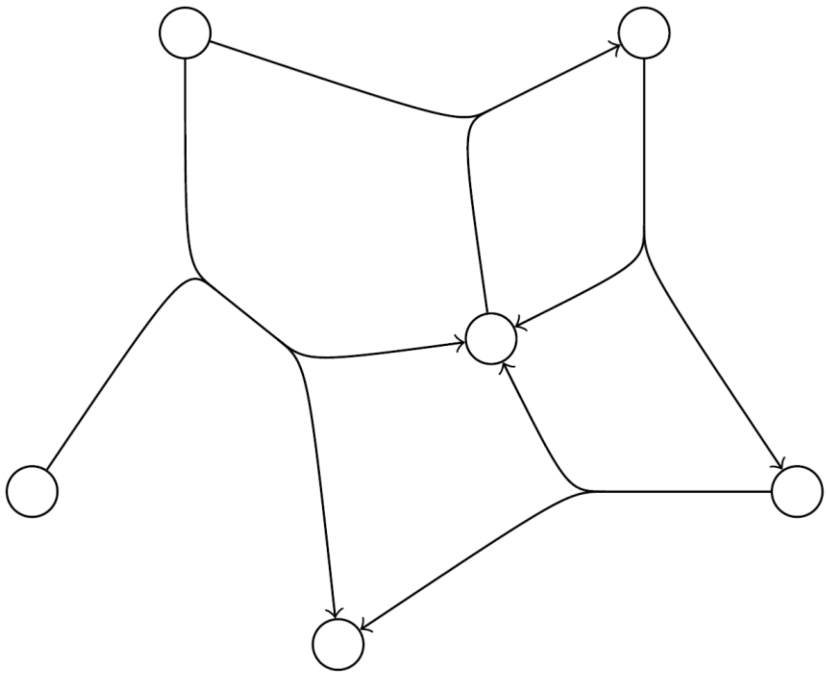
Directed Hypergraphs

Definition

A *directed hypergraph* is a pair $H = (V, \mathcal{A})$ where V is a finite set of *vertices* and $\mathcal{A}$ is a family of *hyperarcs*.

A *hyperarc* a $\in \mathcal{A}$ is a pair $a = (t, h)$ of disjoint sets $t, h \subseteq V$ sets of vertices, at least one of them non-empty; $t \subseteq V$ is called the *tail* of a, $h \subseteq V$ is the *head*.



Graph-based Hypergraphs

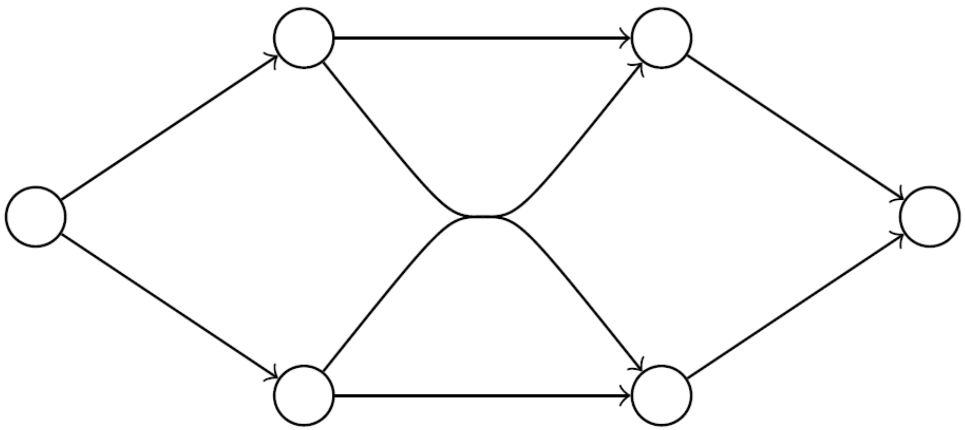
Definition

Let $D = (V, A)$ be a simple directed draph. A *directed hypergraph based on* D is a pair $H = (V, \mathcal{A})$ where $\mathcal{A} \subseteq 2^A$ is a set of non-empty subsets $a \subseteq A$ of vertex-disjoint arcs. A directed hypergraph based on some graph D is called *graph-based*.

Remark

A graph-based directed hypergraph is a directed hypergraph:

For $a \in \mathcal{A}$ let $t(a) \coloneqq \{v \in V : \exists (v, w) \in a\}$ and $h(a) \coloneqq \{w \in V : \exists (v, w) \in a\}$.



The Minimum Cost Hyperflow Problem

Definition

Let $H = (V, \mathcal{A})$ be a directed hypergraph based on a directed graph D, $c \in \mathbb{R}^{\mathcal{A}}$ a vector of *costs*, and $b \in \mathbb{R}^{V}$ of *demands* s.t. $b^T 1 = 0$.

The *minimum cost hyperflow problem* (MCH) is the linear program

$$\min \sum_{a \in \mathcal{A}} c_a x_a$$

$$\begin{aligned} \sum_{a \in \mathcal{A}: v \in h(a)} x_a - \sum_{a \in \mathcal{A}: v \in t(a)} x_a &= b_v \quad \forall v \in V \\ x &\geq 0. \end{aligned}$$

A vector $x \in \mathbb{R}^{\mathcal{A}}$ that is feasible for this LP is a *hyperflow* (in H) (actually a circulation).

The Minimum Cost Hyperflow Problem

- In contrast to the graph case, there might not exist an integral min cost hyperfow, even if all data is integral (see example later).

- Finding a minimum cost integral hyperflow is NP-hard, even if the hyperarcs consist of at most two arcs.

- If the underlying digraph D is connected and $A \subseteq \mathcal{A}$, i.e., all arcs of the underlying digraph are also hyperarcs, then

$$\begin{aligned} \sum_{a \in \mathcal{A}: v \in h(a)} x_a - \sum_{a \in \mathcal{A}: v \in t(a)} x_a &= b_v \quad \forall v \in V \\ x &\geq 0. \end{aligned}$$

has a solution if and only if $b^T 1 = 0$.

A Hypernetwork Simplex Algorithm

Earlier work

R. Cambini, G. Gallo, and M. G. Scutellà: Flows on hypergraphs. Mathematical Programming 78.2, p. 195-217 (1997).

However

- We heavily use that we work on graph-based hypergraphs.
- The algorithm for our setting is simpler and closer to the original network simplex.

Reference

I. Beckenbach: Matchings and Flows in Hypergraphs. PhD thesis, Freie Universität Berlin (2019).

Characterizing the Bases

Assumption

The underlying digraph D is connected and $A \subseteq \mathcal{A}$.

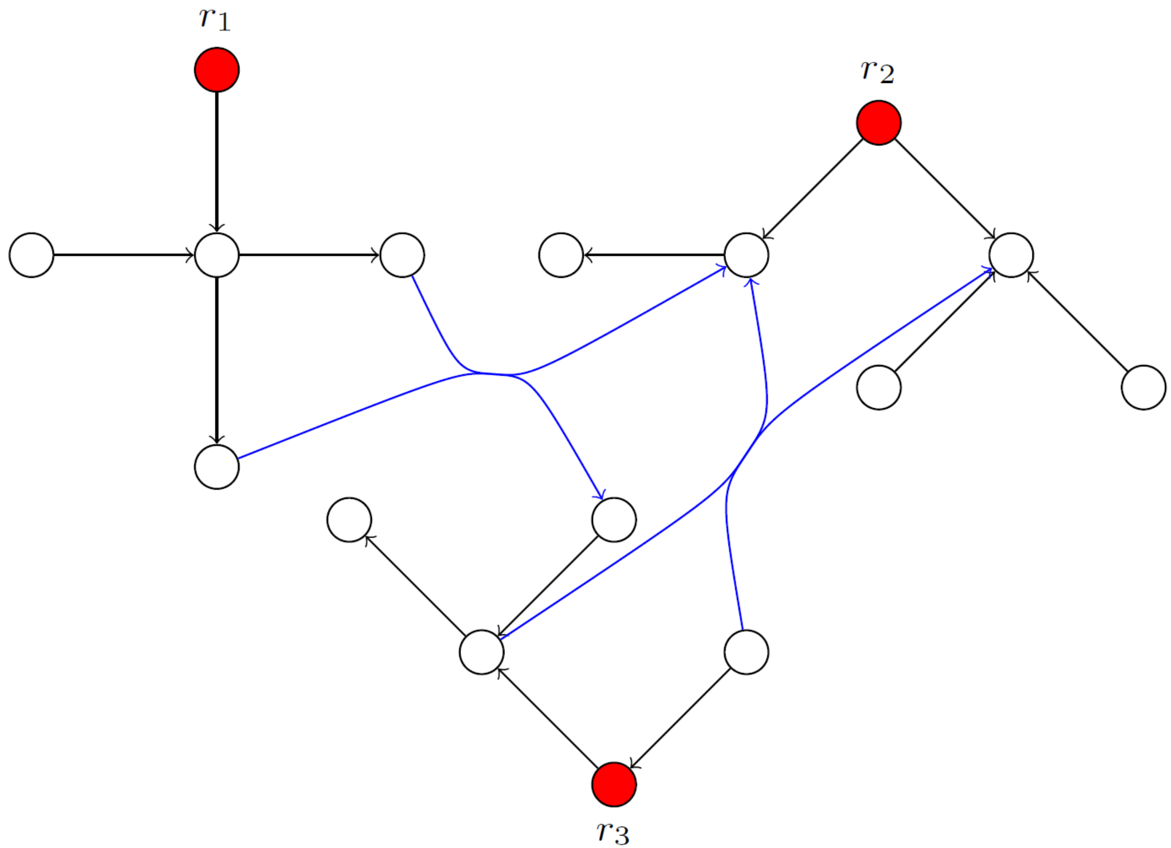
- Let $M \in \{0, \pm 1\}^{V \times \mathcal{A}}$ be the incidence matrix of H. The minimum cost hyperflow problem can then be written as

$$\min c^T x, Mx = b, x \geq 0.$$

- If $B = \{a_1, \dots, a_k\}$, then let $M_{\cdot B} \coloneqq (M_{\cdot a_1}, \dots, M_{\cdot a_k})$.
- $rk(M) = |V| - 1$
- B is a basis if and only if $rk(M_{\cdot B}) = |B| = |V| - 1$.
- If B is a basis, then $H[B \cap A] = (V, B \cap A)$ is a forest that contains $|V| - |B \cap A| = |V| - (|B| - |B \setminus A|) = |B \setminus A| + 1$ components.

Characterizing the Bases

- Let $B \subseteq \mathcal{A}$ be s.t. $|B| = |V| - 1, B_1 := B \cap A$, $B_2 := B \setminus A$.
- $H[B_1]$ is a forest with $|B_2| + 1$ components.
- For every tree of $H[B_1]$, choose a root r and denote its tree by T_r.
- Let R be the set of all such roots.

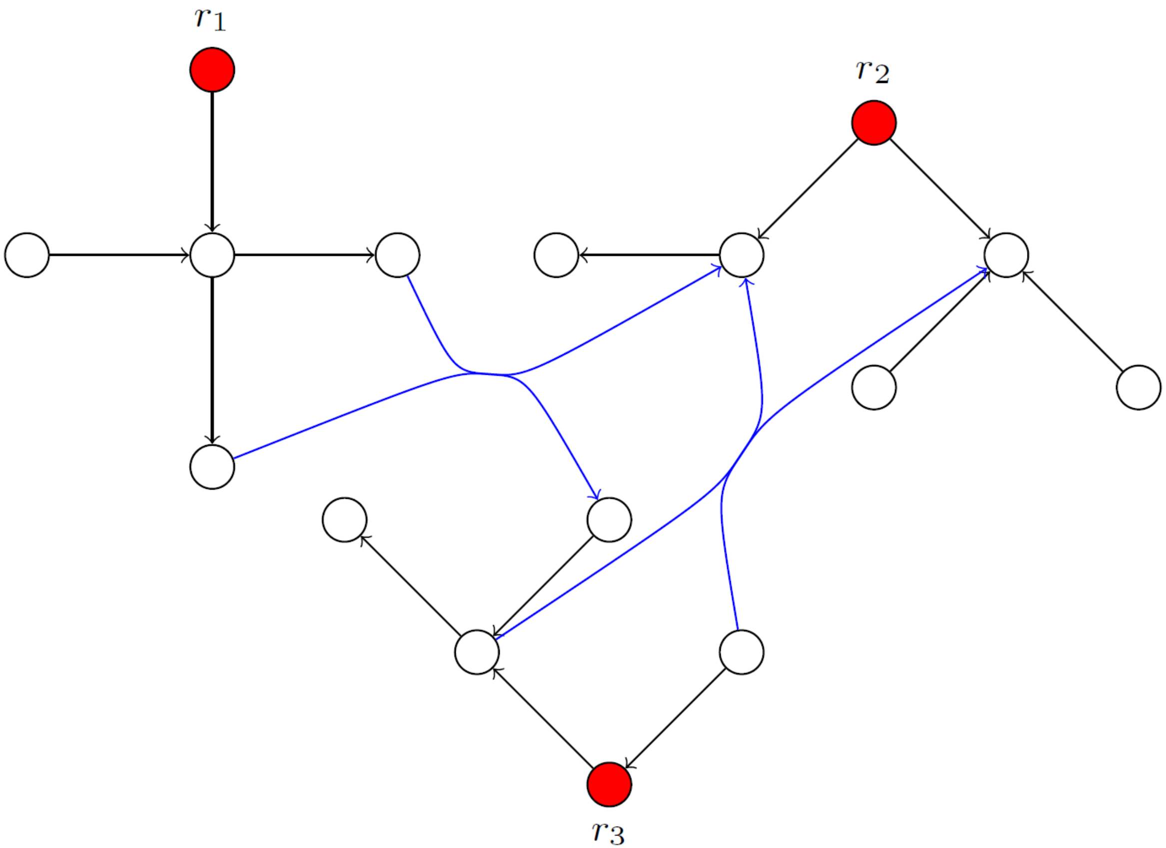


Characterizing the Bases

- If B is a basis and $r_1, r_2 \in R$ two roots, the system

$$M_{\cdot B} f = -e_{r_1} + e_{r_2}, x \geq 0$$

has a unique solution; we can send 1 unit of flow from r_1 to r_2.



Characterizing the Bases

- If B is a basis and $r_1, r_2 \in R$ two roots, the system

$$M_{\cdot B} f = -e_{r_1} + e_{r_2}, f \geq 0$$

  has a unique solution; we can (in a unique way) send 1 unit of flow from r_1 to r_2 in $H[B \cap A]$.

- The unique flow of 1 unit from an arbitrary fixed root r^* to some other other root $r \neq r^*$ is called *elementary*.

- We can send 1 unit of flow from r_1 to r_2 via an arbitrary intermediate root r^*, i.e., from r_1 to r^* to r_2.

- We can also (easily) send 1 unit of flow inside of a tree.

- Any flow in $H[B \cap A]$ is a superposition of elementary flows and flows on trees.

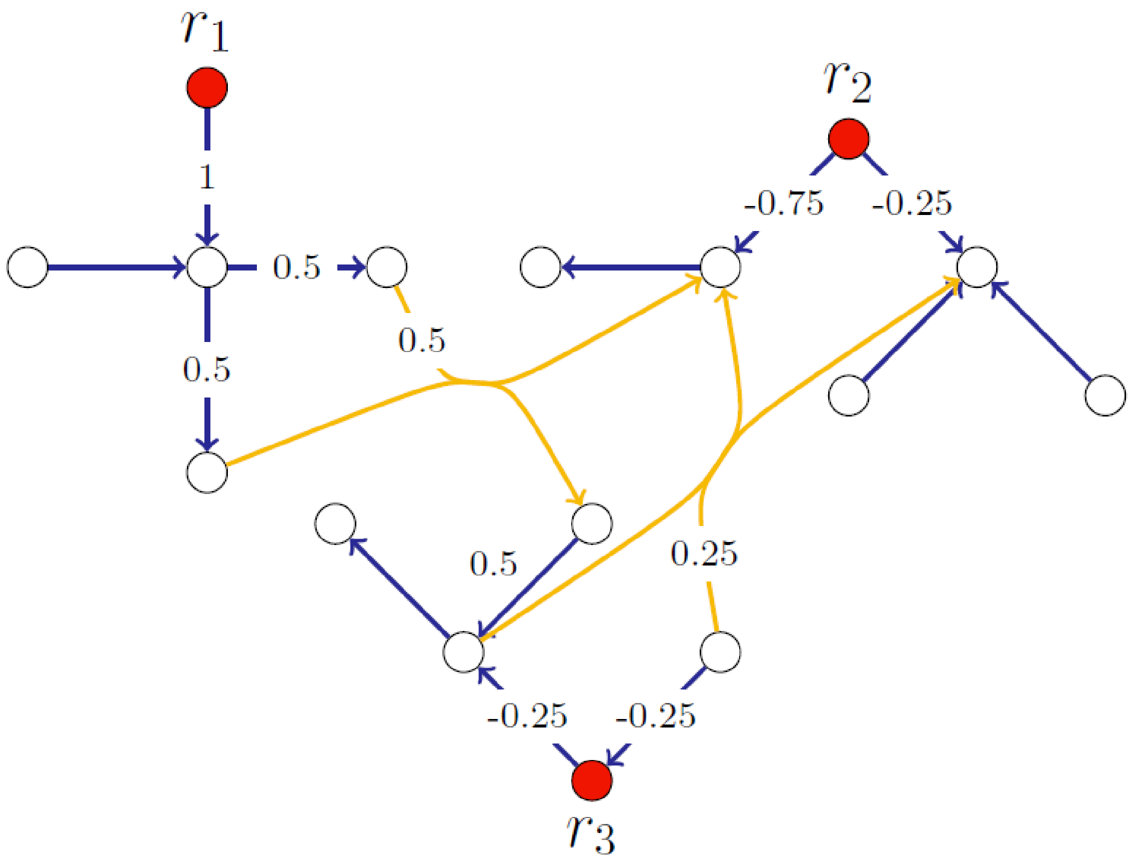
The Elementary Hyperflow Matrix

- Let B be a basis, $T_r, r \in R$, the rooted trees, $r^* \in R$ a fixed root.
- For every $r \in R \setminus \{r^*\}$ there is a unique hyperflow f_r in (V, B) that transports 1 unit from r^* to r.
- Let $F \coloneqq (f_r)_{r \in R \setminus \{r^*\}} \in \mathbb{R}^{B \times R \setminus \{r^*\}}$ be the *elementary hyperflow matrix* whose r-th column contains this flow.
- F is easily reconstructed from $F|B_2 = F_{B_2\cdot}$ by recomputing the flows on the trees (but this takes time).

The Elementary Hyperflow Matrix

Example

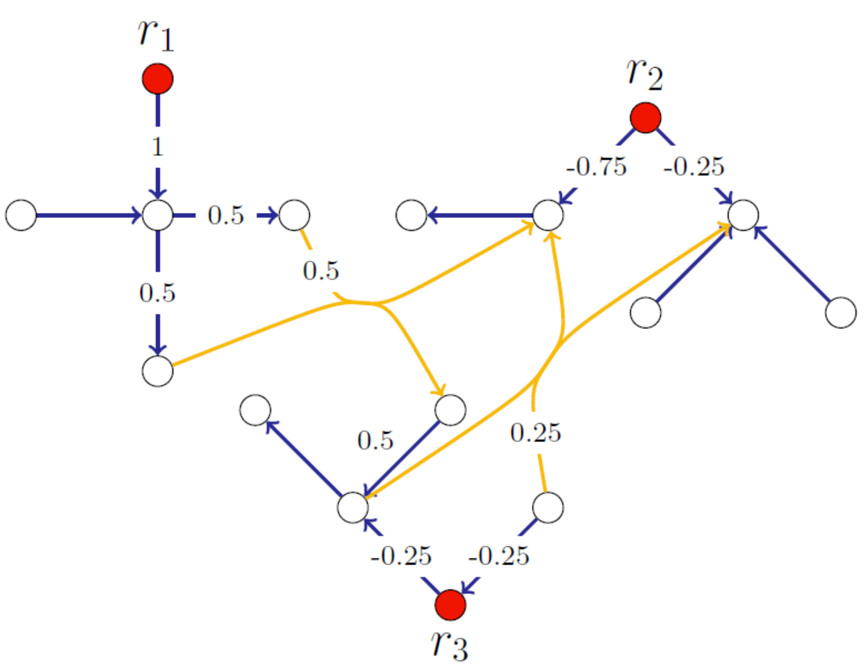
$$r^* = r_1, \qquad f_{r_2}|B_2 = \begin{pmatrix} 1/2 \\ 1/4 \end{pmatrix}$$

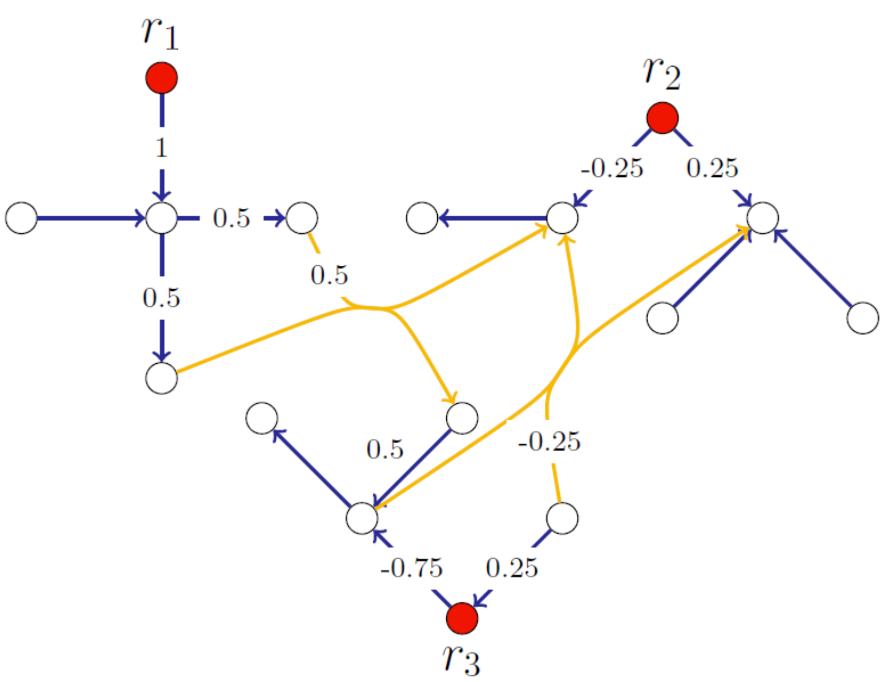


The Elementary Hyperflow Matrix

Example

$$r^* = r_1, \qquad F_{B_2\cdot} = \left(f_{r_2}, f_{r_3}\right)_{B_2\cdot} = \begin{pmatrix} 1/2 & 1/2 \\ 1/4 & -1/4 \end{pmatrix}$$

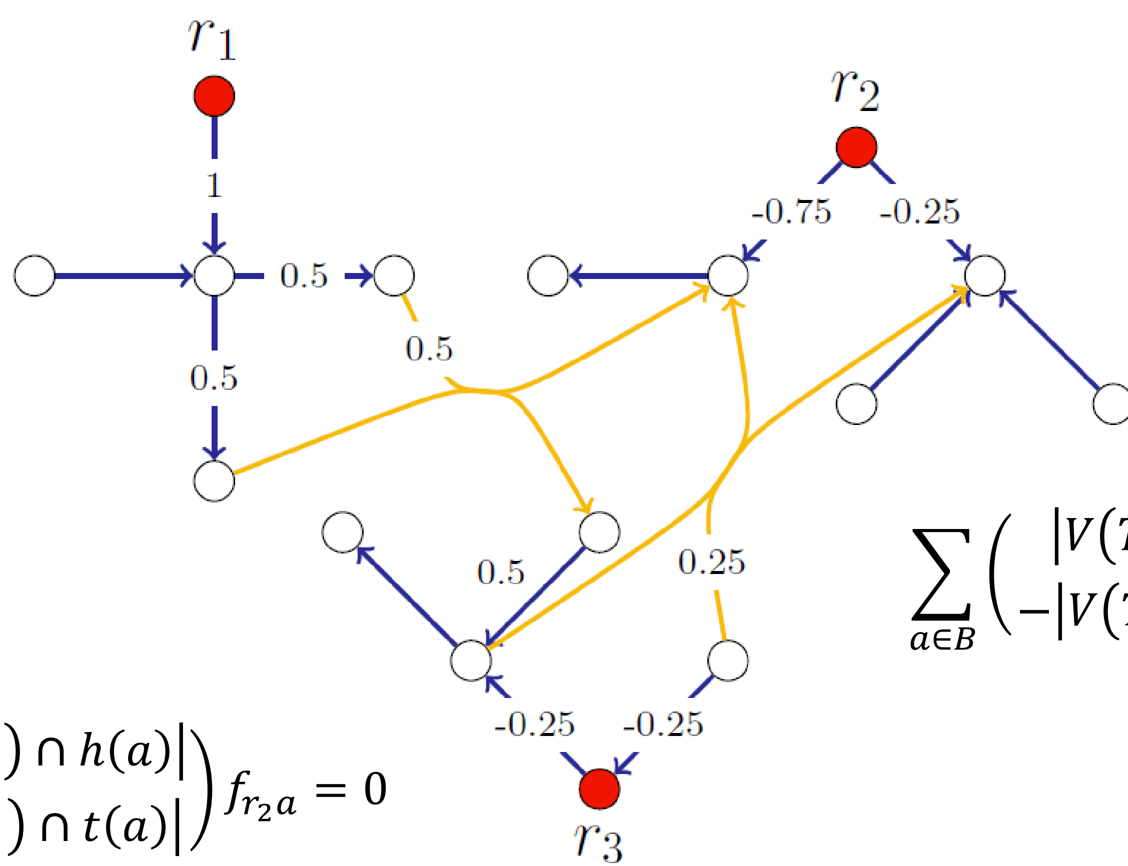




The Elementary Hyperflow Matrix

Example

$$r^* = r_1, \qquad E_{R\setminus\{r_1\}\cdot} = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}, \qquad F_{B_2\cdot} = \begin{pmatrix} 1/2 & 1/2 \\ 1/4 & -1/4 \end{pmatrix}$$



$$\sum_{a\in B} \begin{pmatrix} |V(T_{r_2}) \cap h(a)| \\ -|V(T_{r_2}) \cap t(a)| \end{pmatrix} f_{r_2 a} = 1$$

$$\sum_{a\in B} \begin{pmatrix} |V(T_{r_3}) \cap h(a)| \\ -|V(T_{r_3}) \cap t(a)| \end{pmatrix} f_{r_2 a} = 0$$

Characterizing the Bases

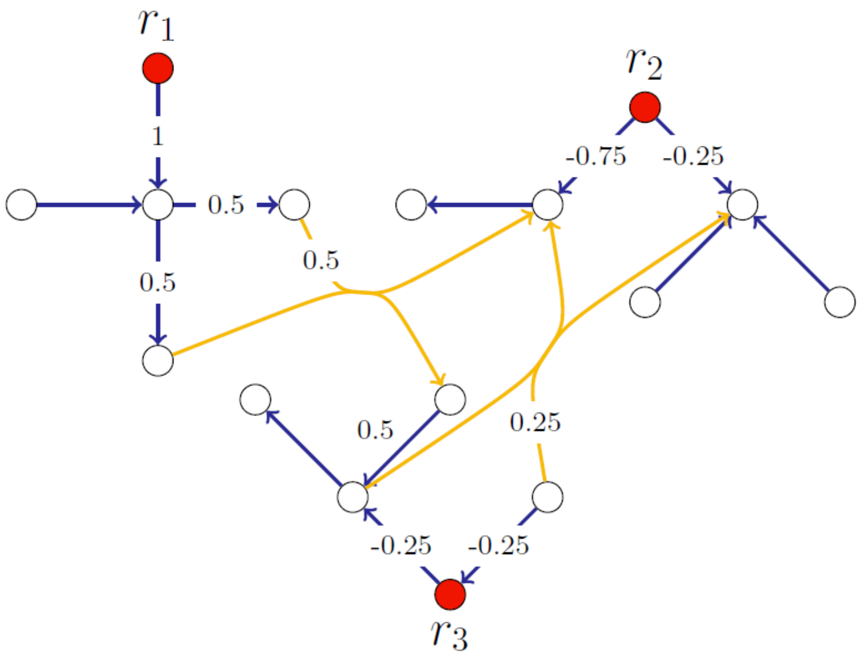
- Define $E \in \mathbb{Z}^{R \times B_2}$ as

$$E_{ra} := |V(T_r) \cap h(a)| - |V(T_r) \cap t(a)|.$$

Lemma

- $E_{R\setminus\{r^*\}\cdot} = F_{B_2\cdot}^{-1}$
- B basis $\Leftrightarrow H[B_1]$ forest of $|B_2| + 1$ components $\wedge\ rk\left(F_{B_2\cdot}\right) = |B_2|$

  $\Leftrightarrow H[B_1]$ forest of $|B_2| + 1$ components $\wedge\ rk(E) = |B_2|$.

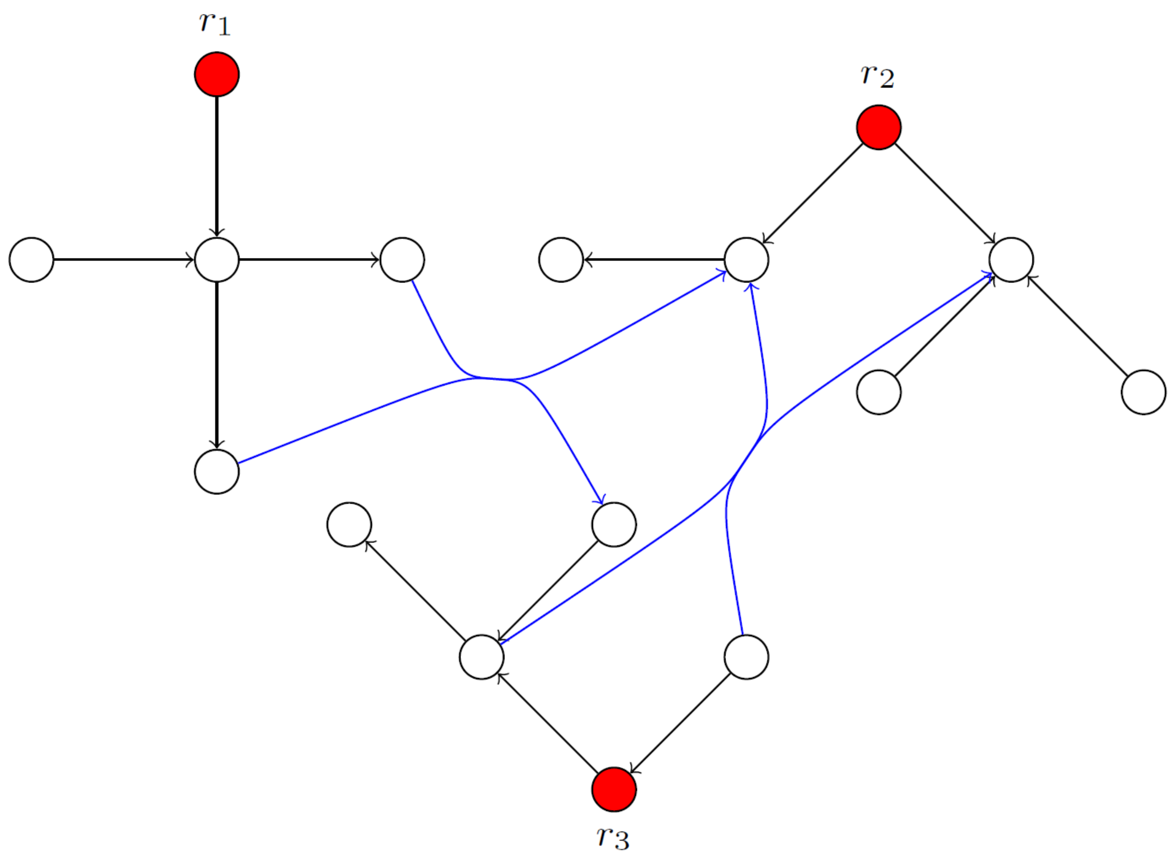


MODAL

Characterizing the Bases

Example

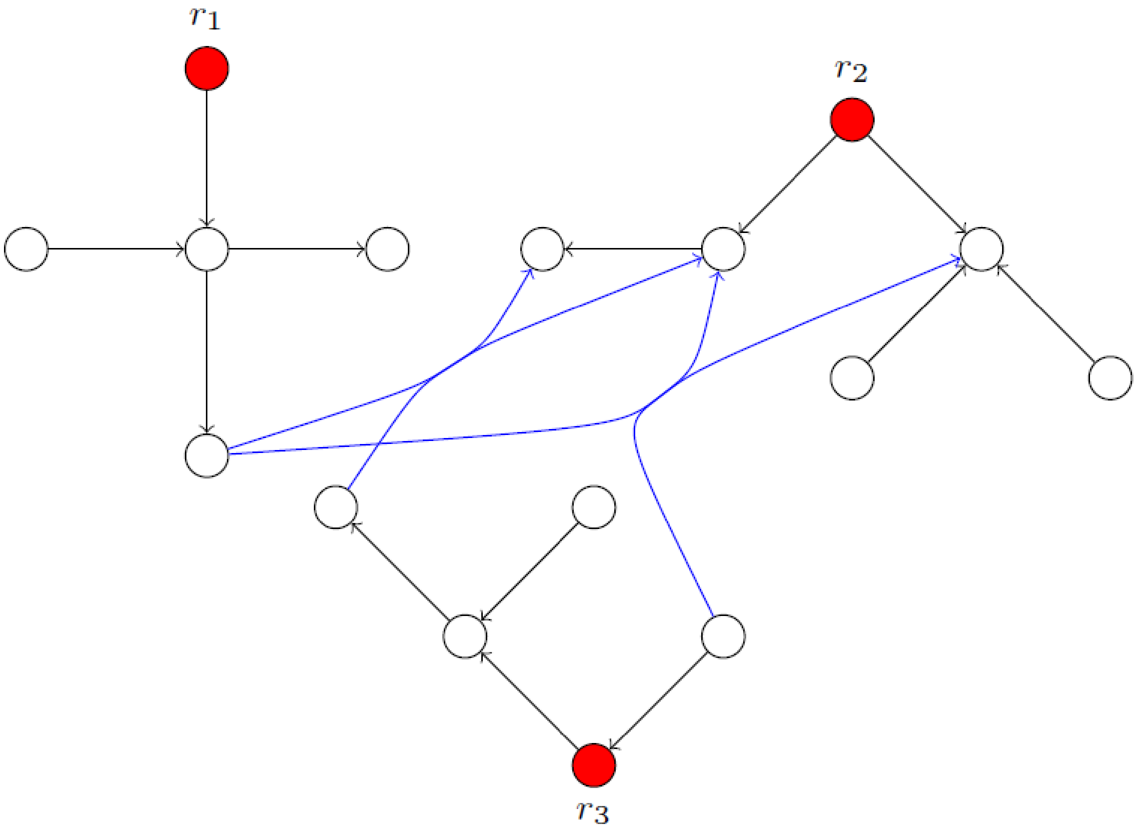
$$E = \begin{pmatrix} -2 & 0 \\ 1 & 2 \\ 1 & -2 \end{pmatrix}$$ has rank 2 $\Rightarrow B$ is a basis



Characterizing the Bases

Example

$$E = \begin{pmatrix} -1 & -1 \\ 2 & 2 \\ -1 & -1 \end{pmatrix} \text{ has rank 1} \Rightarrow B \text{ is not a basis}$$



Elementary Hyperflow and Intersection Count Matrix

- Let B be a basis, $T_r, r \in R$, the rooted trees, $r^* \in R$ a fixed root.
- For every $r \in R \setminus \{r^*\}$ there is a unique hyperflow f_r in (V, B) that transports 1 unit from r^* to r.
- Let $F := (f_r)_{r \in R \setminus \{r^*\}} \in \mathbb{R}^{B \times R \setminus \{r^*\}}$ be the *elementary hyperflow matrix* whose r-th column contains this flow.
- F is easily reconstructed from $F|B_2 = F_{B_2 \cdot}$ by recomputing the flows on the trees.
- Define an *intersection count matrix* $E \in \mathbb{Z}^{R \times B_2}$ as

$$E_{ra} := |V(T_r) \cap h(a)| - |V(T_r) \cap t(a)|.$$

- $E_{R \setminus \{r^*\} \cdot} = F_{B_2 \cdot}^{-1}$

The Hyperflow Network Simplex Algorithm

Input: Hypergraph $H = (V, \mathcal{A})$ based on digraph $D = (V, A)$, cost $c \geq 0$, demand b s.t. $b^T 1 = 0$, feasible basic (hyper)flow x s.t. supp(x) $\subseteq A$, associated basis B and tree T_r.

Output: (Fractional) Minimum cost hyperflow x.

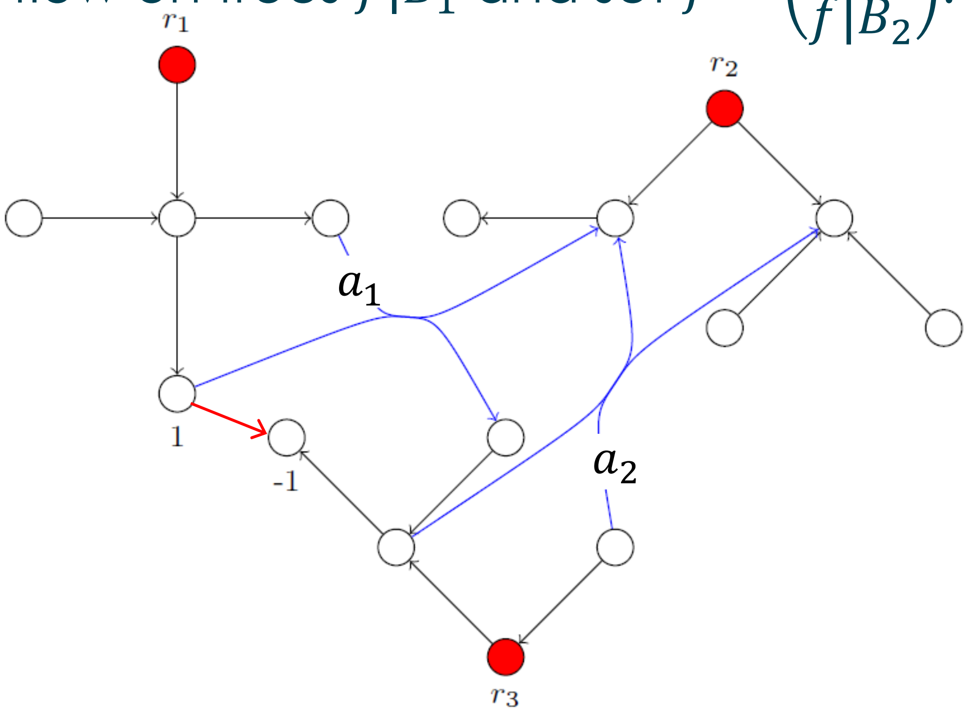
1. (BTRAN) Solve $\pi^T M_{.B} = c_B^T$ and compute reduced costs
$$\bar{c}_a = c_a - \pi(V\big(h(a)\big) + \pi(V(t(a)).$$
2. (PRICE) If $\bar{c} \geq 0$ then output x, stop; else choose a^{in} s.t. $\bar{c}_{a^{in}} < 0$.
3. (FTRAN) Solve $M_{.B} f = -M_{a^{\text{in}}}$.
4. (CHUZR) Choose $a^{\text{out}} \in \operatorname{argmin}\left\{-\frac{x_a}{f_a} : f_a < 0, a \in B\right\}$.
5. (UPDATE)
$$x_a \leftarrow \begin{cases} -x_{a^{\text{out}}}/f_{a^{\text{out}}}, & a = a^{\text{in}} \\ x_a - x_{a^{\text{out}}}/f_{a^{\text{out}}}, & a \in B \\ x_a, & a \notin B \cup \{a^{\text{in}}\} \end{cases}$$
, update $B, R, \{T_r\}, M_B, F$.
6. Go to 1.

(FTRAN) $M_{\cdot B} f = -M_{\cdot a^{\text{in}}}$

- 1 unit of flow on entering arc a^{in} creates at tree T_r demand

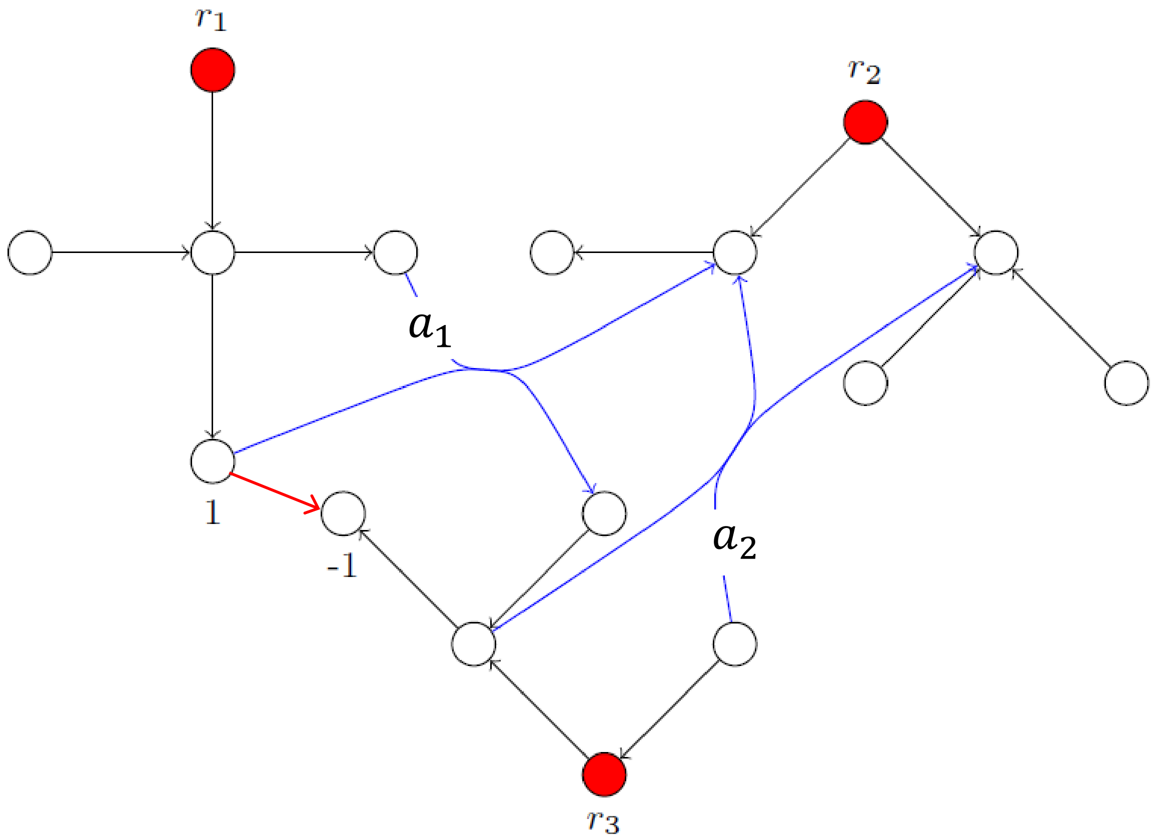
$$E_{ra^{\text{in}}} \coloneqq -(|V(T_r) \cap h(a)| - |V(T_r) \cap t(a)|)$$

- Compute hyperflow on basic hyperarcs as superposition of elementary flows $f|B_2 = F_{B_2 \cdot} E_{R \setminus \{r^*\} a^{\text{in}}}$.
- Compute associated flow on trees $f|B_1$ and set $f = \begin{pmatrix} f|B_1 \\ f|B_2 \end{pmatrix}$.
- $M_{\cdot B} f = -M_{\cdot a^{in}}$



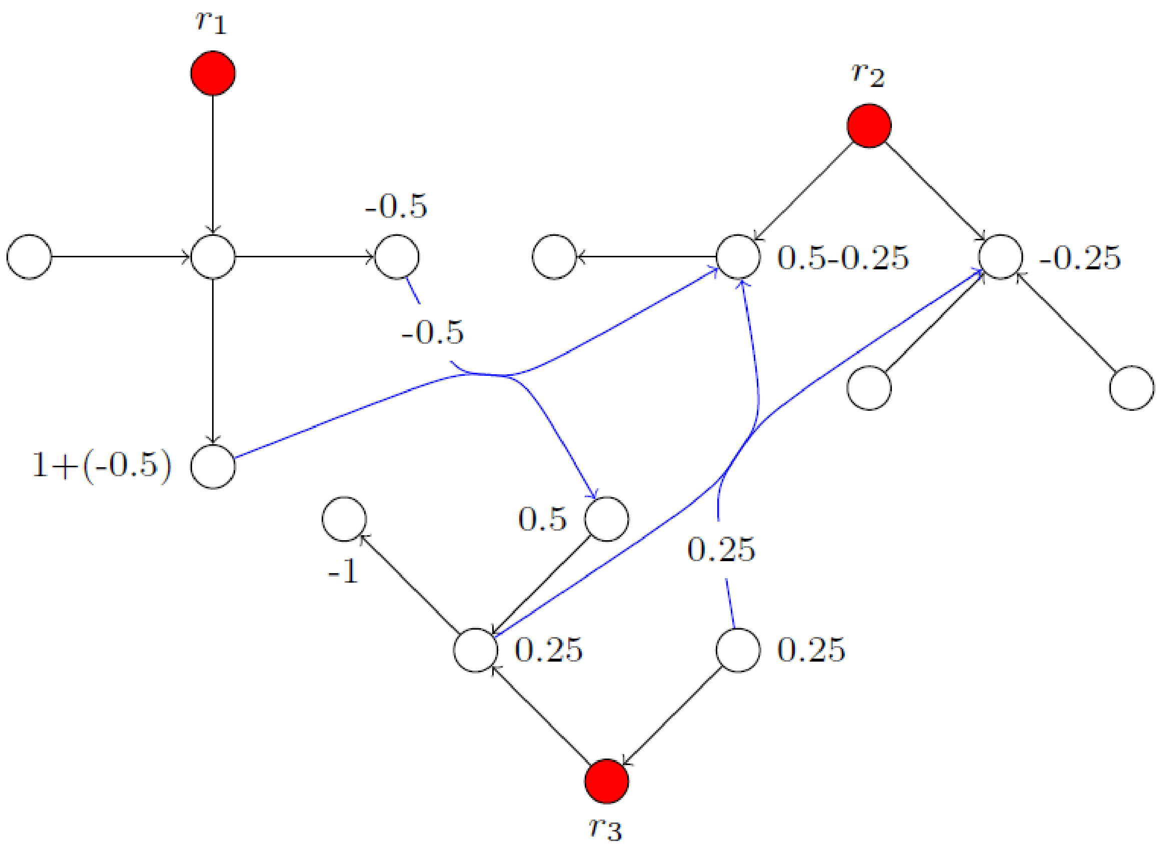
Example: (FTRAN) $M_{\cdot B} f = -M_{\cdot a^{\text{in}}}$

- a^{in} is the red arc
- $E_{R a^{\text{in}}} = (1,0,-1)^T$
- $\begin{pmatrix} f_{a_1} \\ f_{a_2} \end{pmatrix} = F \cdot E_{R\setminus\{r^*\} a^{\text{in}}} = \begin{pmatrix} 1/2 & 1/2 \\ 1/4 & -1/4 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/4 \end{pmatrix}$



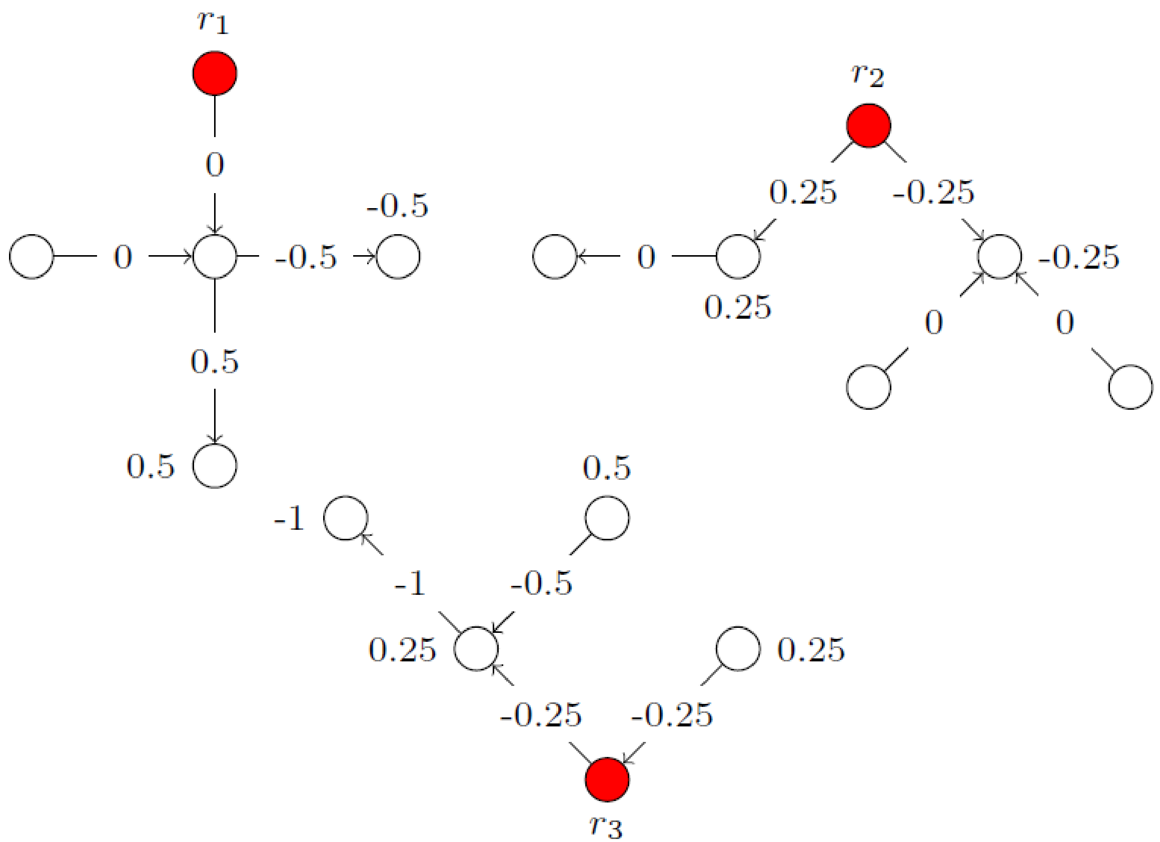
Example: (FTRAN) $M_{\cdot B} f = -M_{\cdot a^{\text{in}}}$

- Compute demand on tree vertices induced by flow on basic hyerparcs.
- Total demands (including demands on entering hyperarc) sum up to 0 on every tree.



Example: (FTRAN) $M_{\cdot B} f = -M_{\cdot a^{\text{in}}}$

- Compute flow on each tree by reverse BFS.



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$$\bar{c}_a = c_a - \pi(V\big(h(a)\big) + \pi(V(t(a)).$$
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5. (UPDATE)
$$x_a \leftarrow \begin{cases} -x_{a^{\text{out}}}/f_{a^{\text{out}}}, & a = a^{\text{in}} \\ x_a - x_{a^{\text{out}}}/f_{a^{\text{out}}}, & a \in B \\ x_a, & a \notin B \cup \{a^{\text{in}}\} \end{cases}$$
, update $B, R, \{T_r\}, M_B, F$.
6. Go to 1.

(BTRAN) $\pi^T M_{.B} = c_B^T$

- Set $\pi'_R := 0$ and extend to potential π' s.t. all tree arcs have reduced cost zero.
- The basic hyperarcs have (preliminary) reduced costs

$$\bar{c}'_a = c_a - \pi'(V(h(a)) + \pi'(V(t(a))_{.a}.$$

- Adjust potentials at trees (except r^*, i.e., $\pi_{r^*} = 0$) such that basic hyperarcs get zero reduced costs by setting

$$\pi_r := \bar{c}'^T_{B_2} F_{B_2 r} \quad \text{and} \quad \pi_{V(T_r)} \leftarrow \pi'_{V(T_r)} + \pi_r \cdot 1.$$

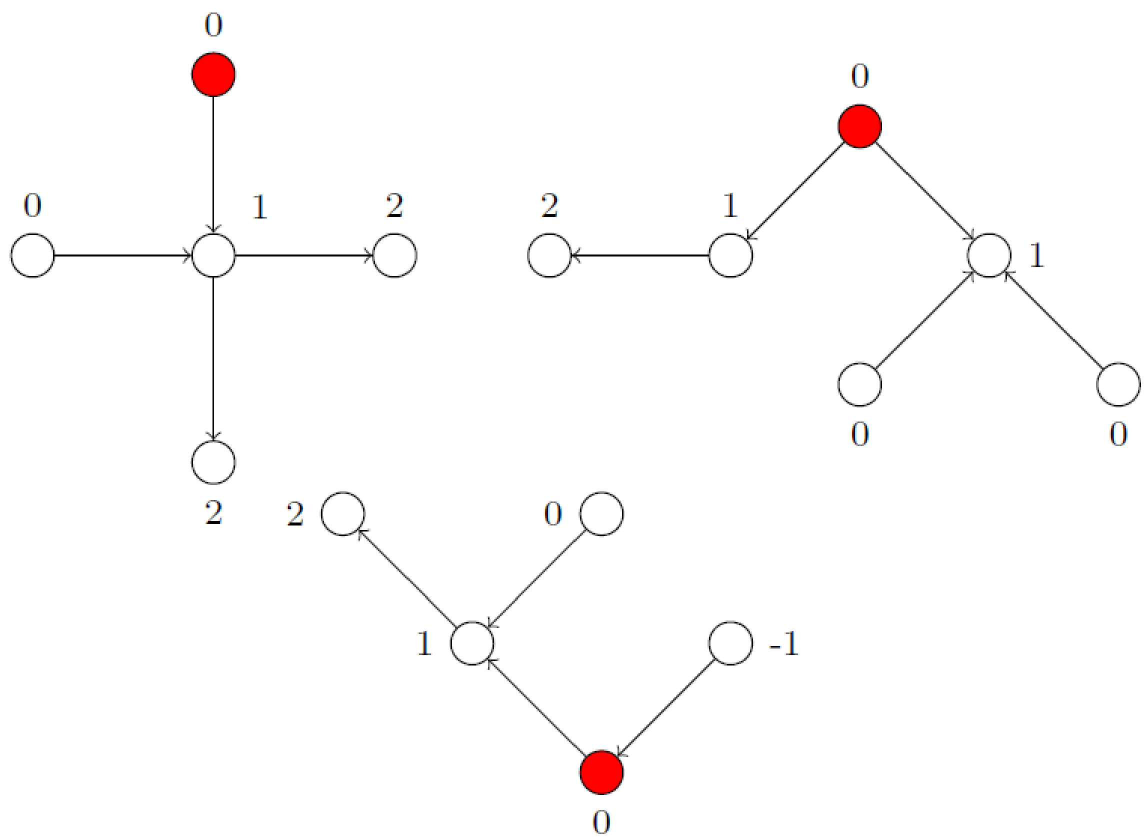
- $\bar{c}_a = \bar{c}'_a = 0, \quad a \in B_1$
- $\bar{c}_a = \bar{c}'_a - \sum_{r \in R \setminus \{r*\}} \pi_r(|V(T_r) \cap h(a)| - |V(T_r) \cap t(a)|)$

$$= \bar{c}'_a - \sum_{r \in R \setminus \{r*\}} \bar{c}'^T_{B_2} F_{B_2 r} E_{ar}$$
$$= \bar{c}'_a - \bar{c}'_a = 0, \quad a \in B_2$$

- $\pi^T M_{.B} = c_B^T.$

Example: (BTRAN) $\pi^T M_{.B} = c_B^T$

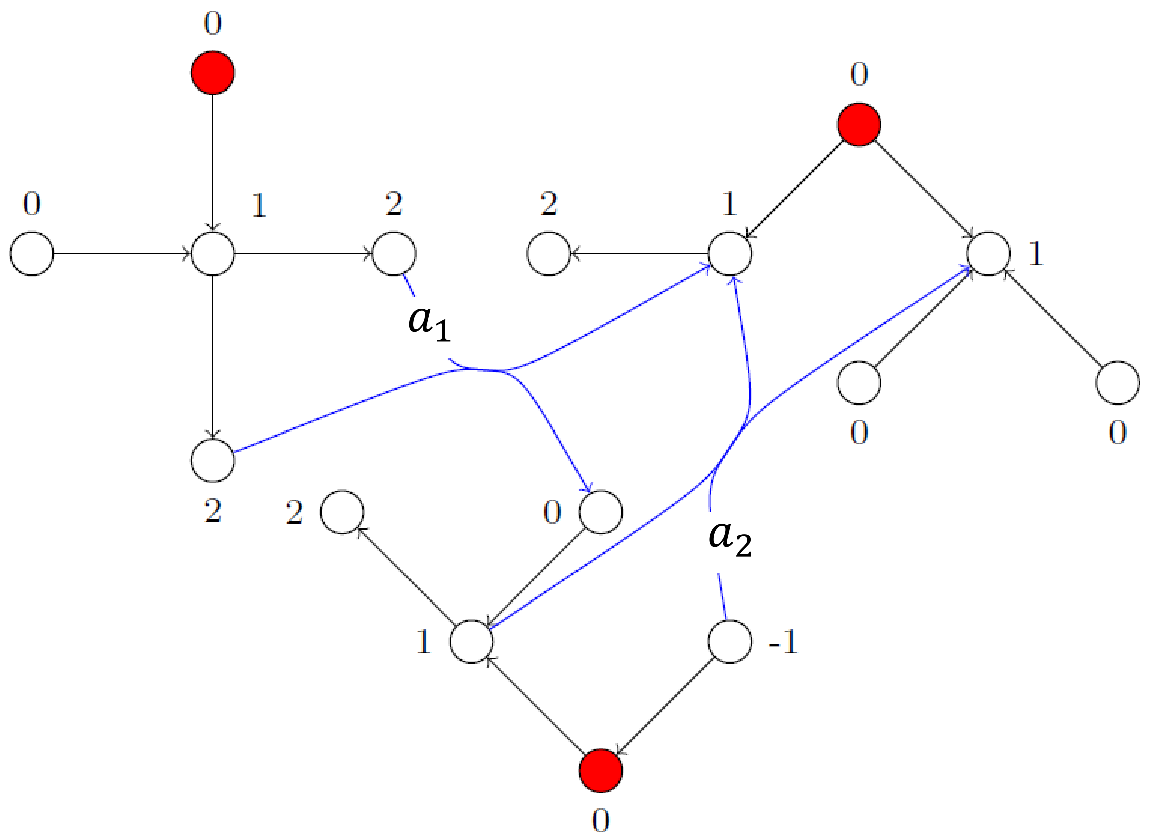
- Set $\pi'_R := 0$ and extend to potentials π' such that all tree arcs have reduced cost zero.



Example: (BTRAN) $\pi^T M_{.B} = c_B^T$

- The basic hyperarcs have preliminary reduced costs

$$\bar{c}'_{a_1} = 1 \; - \; (1 + 0) + (2 + 2) = 4$$
$$\bar{c}'_{a_2} = 1 \; - \; (1 + 1) + (1 \; - 1) = -1$$

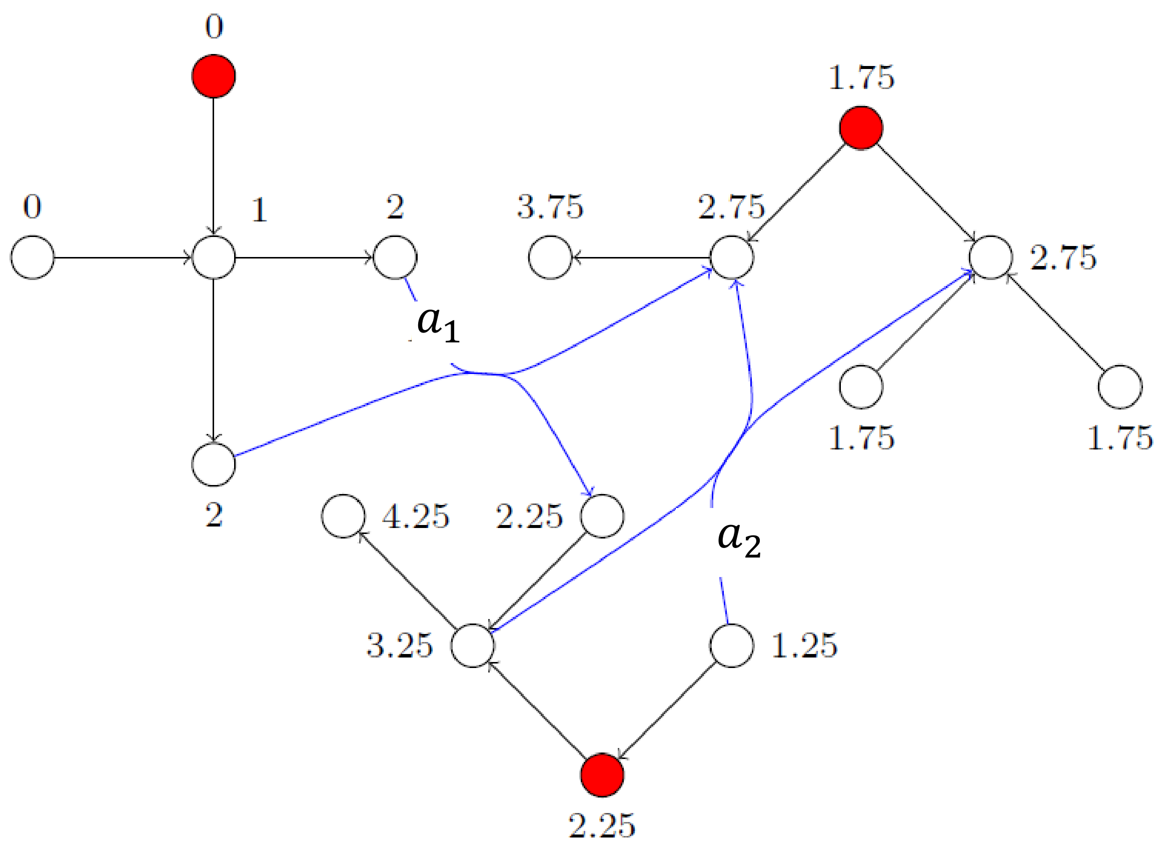


Example: (BTRAN) $\pi^T M_{.B} = c_B^T$

- Adjust potentials at tree roots (except r^*) such that basic hyperarcs get zero reduced costs by setting

$$\pi^T_{R\setminus\{r_1\}} := \bar{c}'^T_{B_2} F_{B_2\cdot} = (4,-1)\begin{pmatrix} 1/2 & 1/2 \\ 1/4 & -1/4 \end{pmatrix} = (1.75, 2.25).$$

and raising all tree potentials according to the roots.



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, update $B, R, \{T_r\}, M_B, F$.
6. Go to 1.

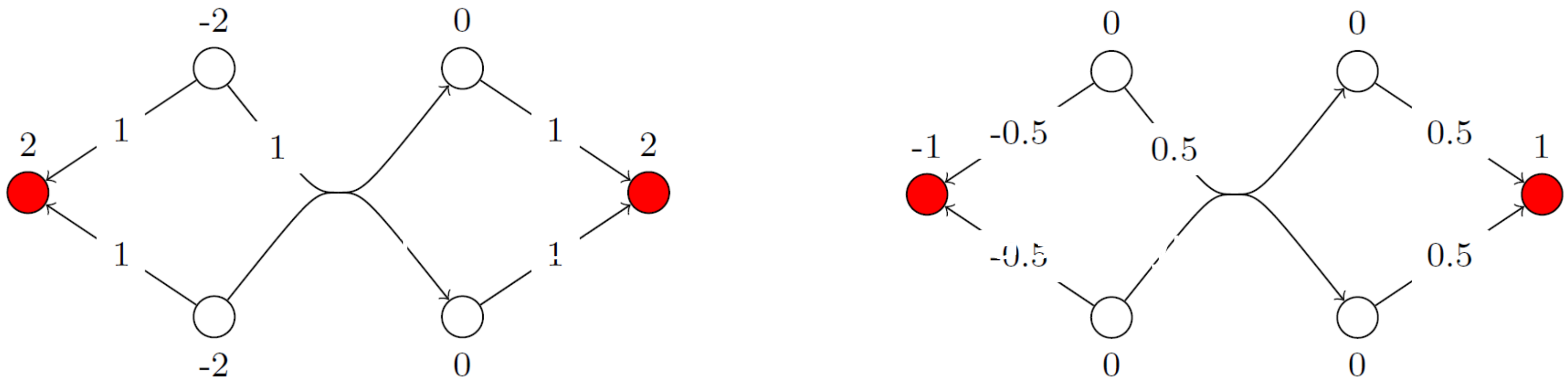
UPDATE: Root Set & Elementary Hyperflow Matrix

- New basis $B' \leftarrow B \cup \{a^{\text{in}}\} \setminus \{a^{\text{out}}\}$, $B'' \leftarrow B \cup \{a^{\text{in}}\}$
- Find new root set R' s.t. $R' \ni r^*$ and $|R \Delta R'| = 1$
- $M_{.B} f = -M_{a^{\text{in}}} \Leftrightarrow M_{.B''} \begin{pmatrix} f \\ 1 \end{pmatrix} =: M_{.B''} f'' = 0$
- $f_r'' \leftarrow \begin{pmatrix} f_r \\ 0 \end{pmatrix}$, $r \in R \cup R'$, where $M_B f_r = e_r - e_{r^*}$ if there is $r \in R' \setminus R$
- $F'_{ar} \leftarrow f_r''(a) - \frac{f_r''(a^{\text{out}})}{f(a^{\text{out}})} f''(a), \quad$ a $\in B'', r \in R'$
- $F'_{a^{\text{out}}r} = 0$
- $M_{.B'} F'_{B'r} = M_{.B''} F'_{B''r}$

$$
\begin{aligned}
&= M_{.B''} f_r'' - \frac{f_r''(a^{\text{out}})}{f(a^{\text{out}})} M_{.B''} f'' \\
&= M_{.B''} f_r'' \\
&= M_{.B} f_r \\
&= e_r - e_{r^*}
\end{aligned}
$$

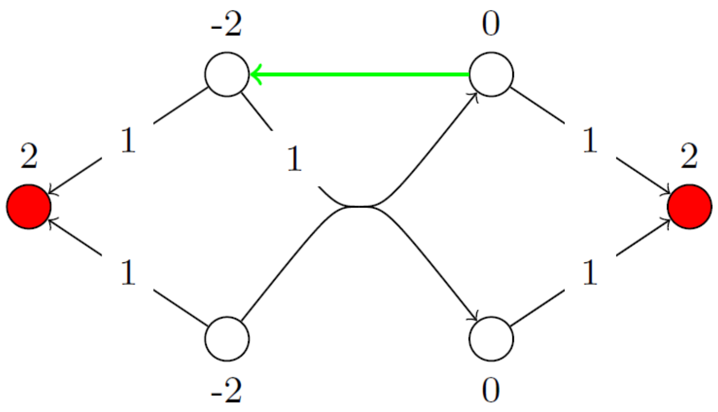
Example: UPDATE

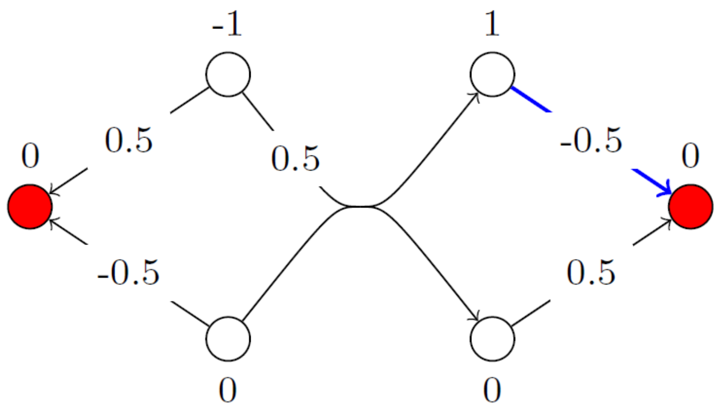
$$F = (-0.5, -0.5, 0.5, 0.5, 0.5)^T$$



Example: UPDATE

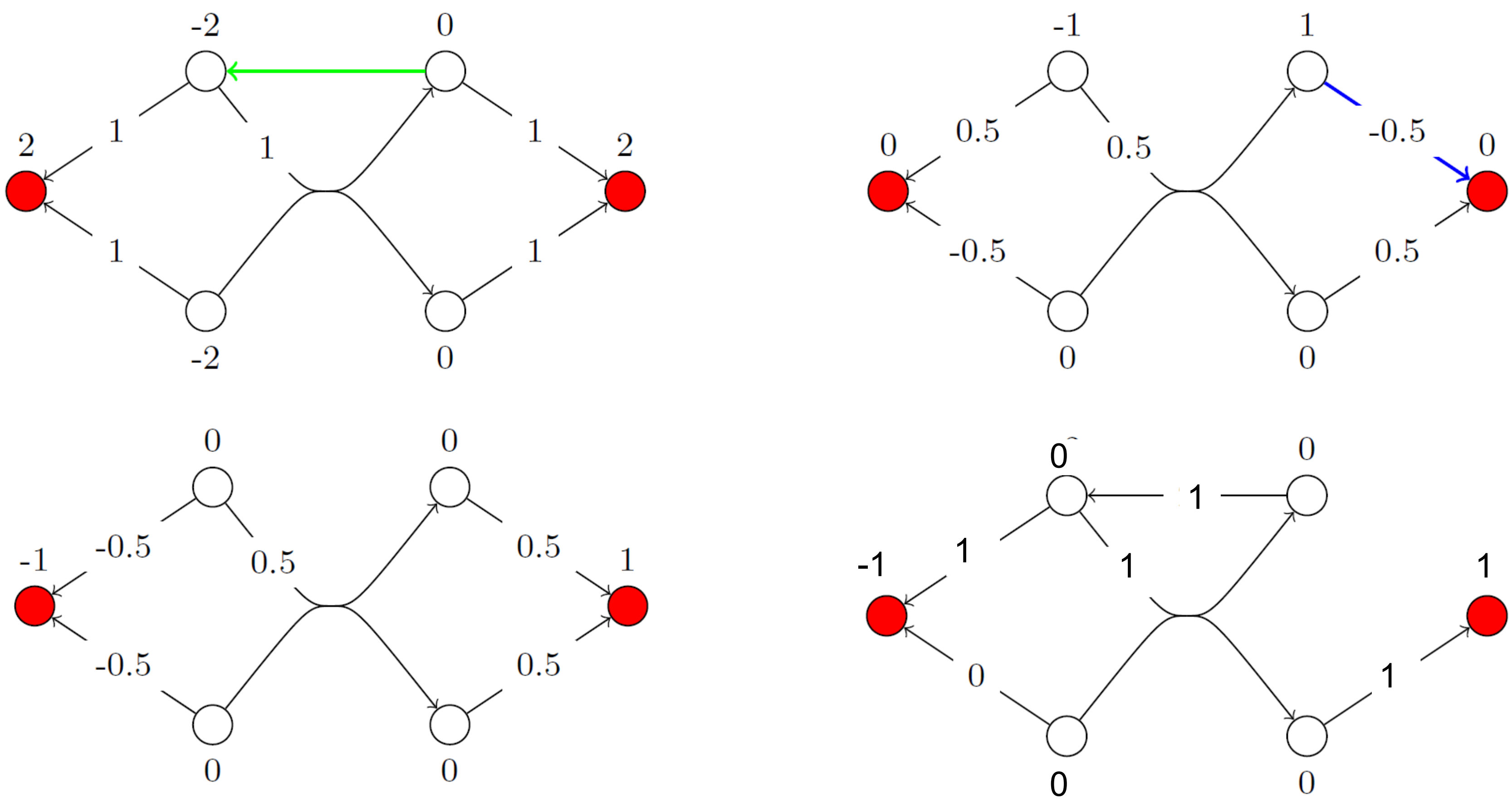
$$f = (0.5, -0.5, 0.5, 0.5, -0.5)^T$$





Example: UPDATE

- $F = (-0.5, -0.5, 0.5, 0.5, 0.5)^T, \quad f = (0.5, -0.5, 0.5, 0.5, -0.5)^T$
- Ratio is $-\frac{0.5}{-0.5} = 1 \Rightarrow F'_{a^{\text{in}}} = 1$
- $F'_{B\cap B'} = (-0.5, -0,5, 0.5, 0.5)^T - (-1) \cdot (0.5, -0.5, 0.5, 0.5)^T = (0, -1, 1, 1)^T$



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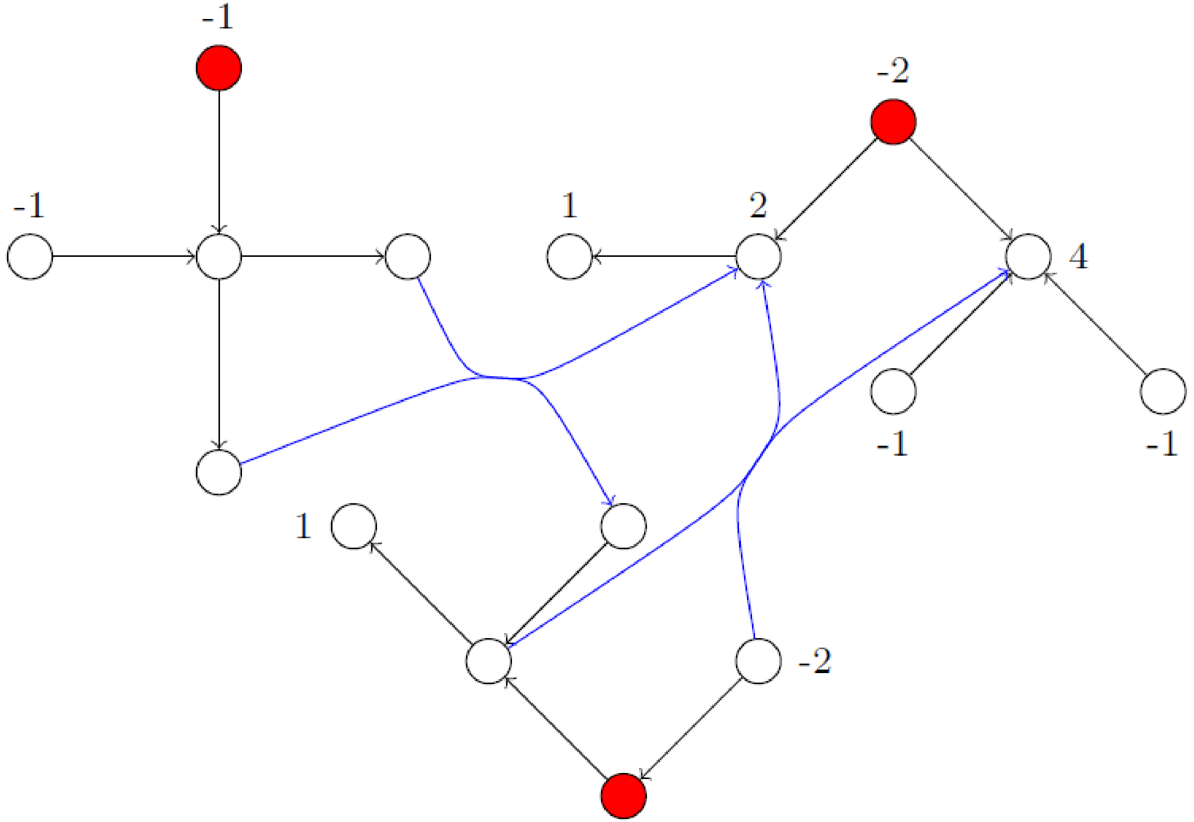
Output: (Fractional) Minimum cost hyperflow x.

1. (BTRAN) Solve $\pi^T M_{.B} = c_B^T$ and compute reduced costs
$$\bar{c}_a = c_a - \pi(V\big(h(a)\big) + \pi(V(t(a)).$$
2. (PRICE) If $\bar{c} \geq 0$ then output x, stop; else choose a^{in} s.t. $\bar{c}_{a^{in}} < 0$.
3. (FTRAN) Solve $M_{.B} f = -M_{a^{\text{in}}}$.
4. (CHUZR) Choose $a^{\text{out}} \in \operatorname{argmin}\left\{-\frac{x_a}{f_a} : f_a < 0, a \in B\right\}$.
5. (UPDATE)
$$x_a \leftarrow \begin{cases} -x_{a^{\text{out}}}/f_{a^{\text{out}}}, & a = a^{\text{in}} \\ x_a - x_{a^{\text{out}}}/f_{a^{\text{out}}}, & a \in B \\ x_a, & a \notin B \cup \{a^{\text{in}}\} \end{cases}$$, update $B, R, \{T_r\}, M_B, F$.
6. Go to 1.

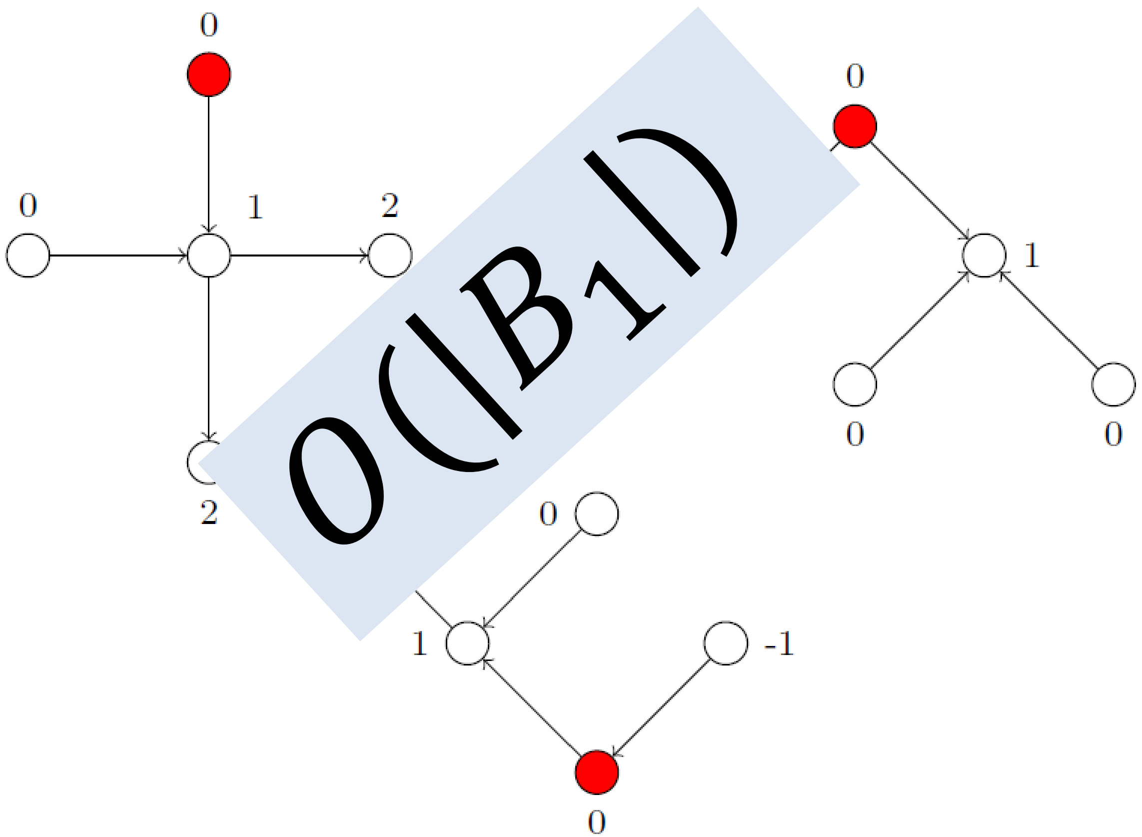
Example: Start with a Feasible Hyperflow

- Only arcs/hyperarcs of the current basis B are shown.
- Flow $x_a = \begin{cases} 1, & a \in B \\ 0, & a \notin B \end{cases}$
- Costs $c = 1$, demands b as labeled.



Example: (BTRAN) $\pi^T M_{.B} = c_B^T$

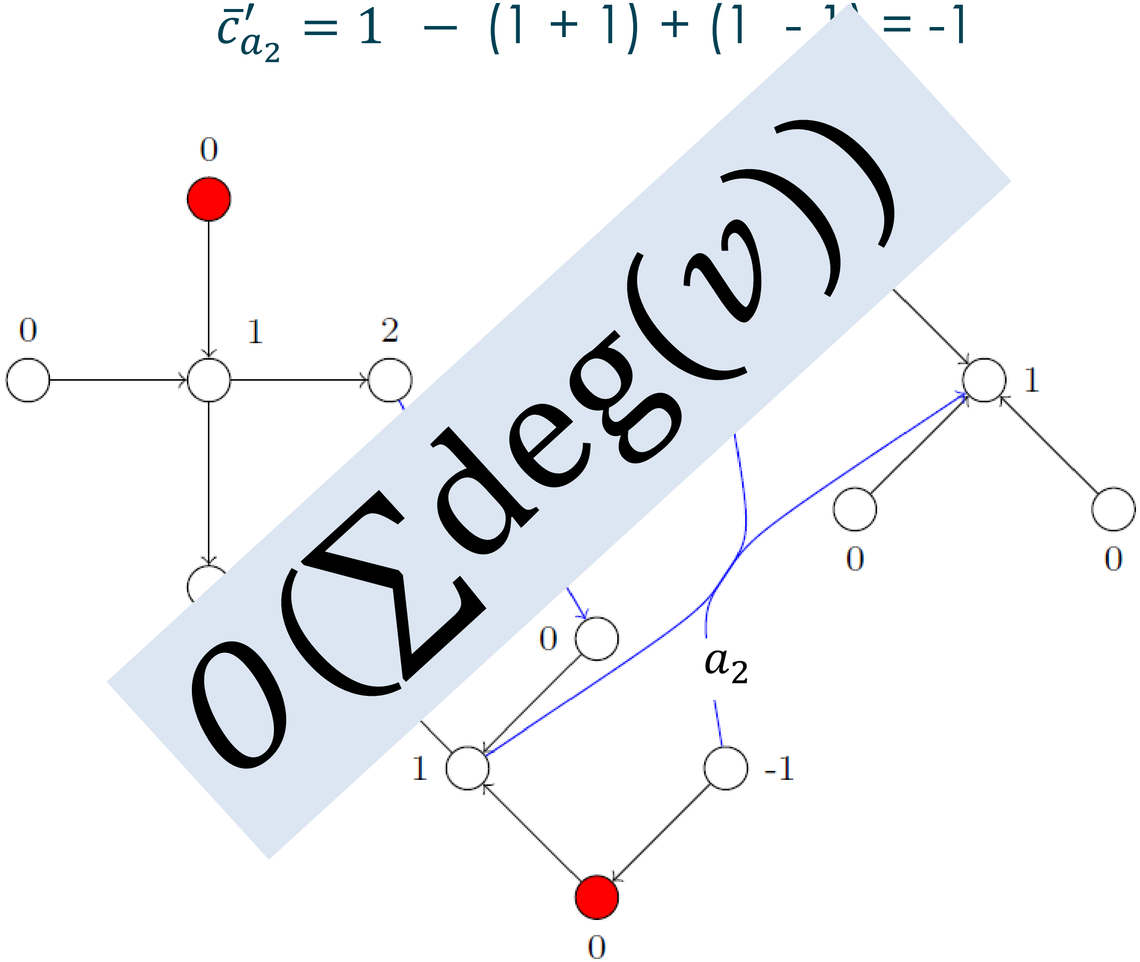
- Set $\pi'_R := 0$ and extend to potentials π' such that all tree arcs have reduced cost zero.



Example: (BTRAN) $\pi^T M_{.B} = c_B^T$

- The basic hyperarcs have preliminary reduced costs

$$\bar{c}'_{a_1} = 1 \;-\; (1 + 0) + (2 + 2) = 4$$
$$\bar{c}'_{a_2} = 1 \;-\; (1 + 1) + (1 - 1) = -1$$

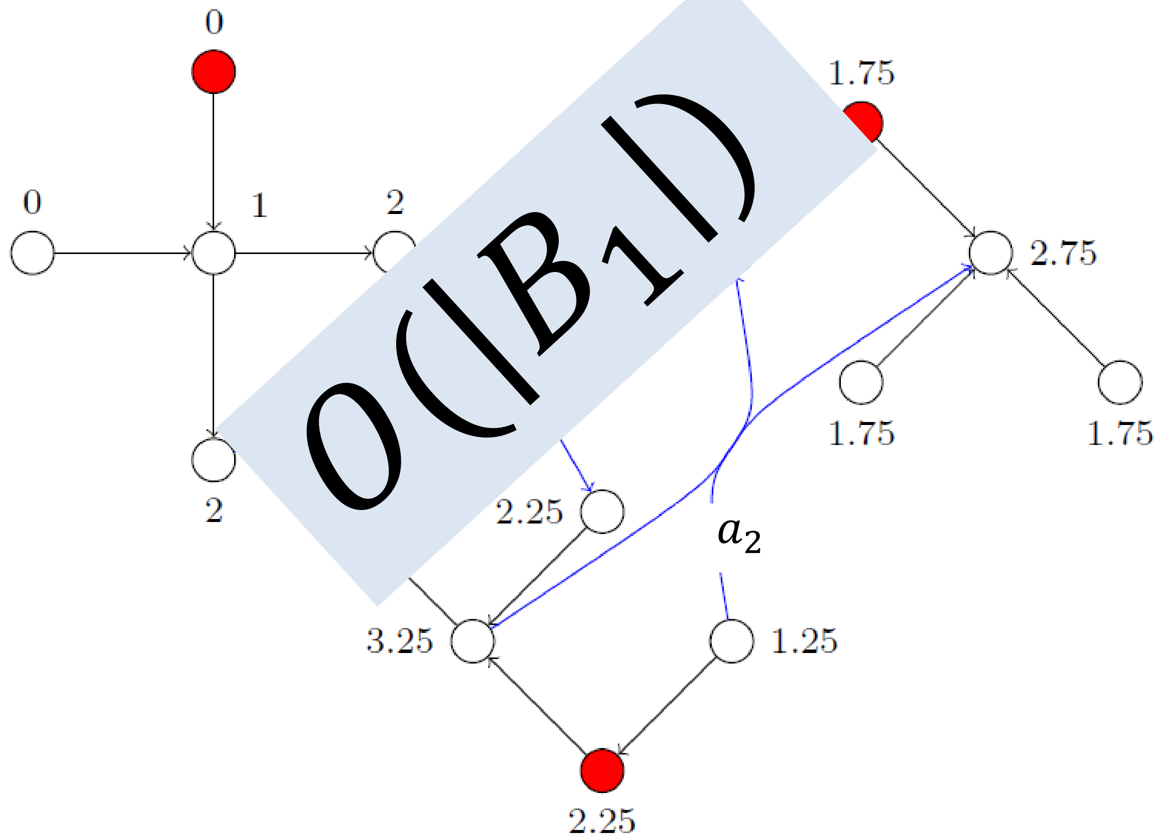


Example: (BTRAN) $\pi^T M_{.B} = c_B^T$

- Adjust potentials at tree roots (except r^*) such that basic hyperarcs get zero reduced costs by setting

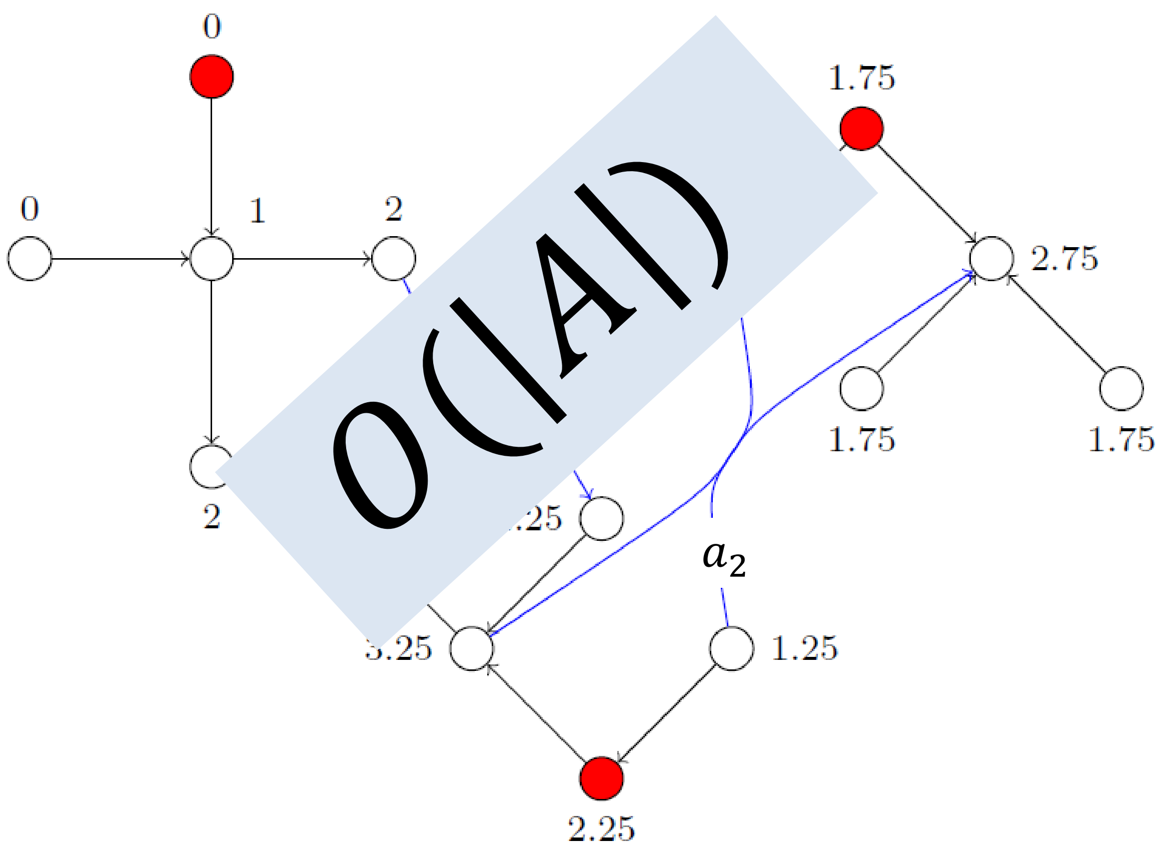
$$\pi^T_{R\setminus\{r_1\}} := \bar{c}'^T_{B_2} F_{B_2\cdot} = (4,-1)\begin{pmatrix} 1/2 & 1/2 \\ 1/4 & -1/4 \end{pmatrix} = (1.75, 2.25).$$

and raising all tree potentials according to the roots.



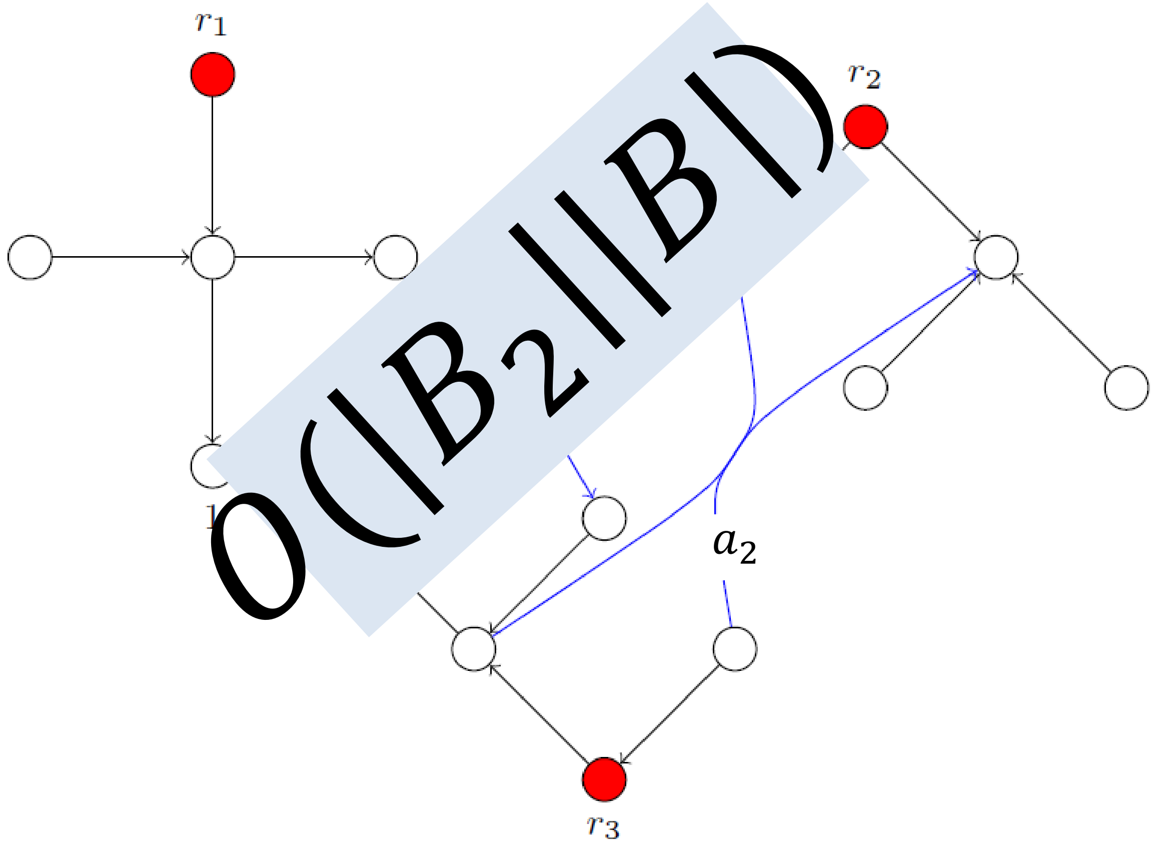
Example: (PRICE) Choose $\bar{c}_{a^{in}} < 0$

- Green arc has reduced cost 1 - 4:25 + 2 = -1.25 < 0.
- Add this arc to the basis.



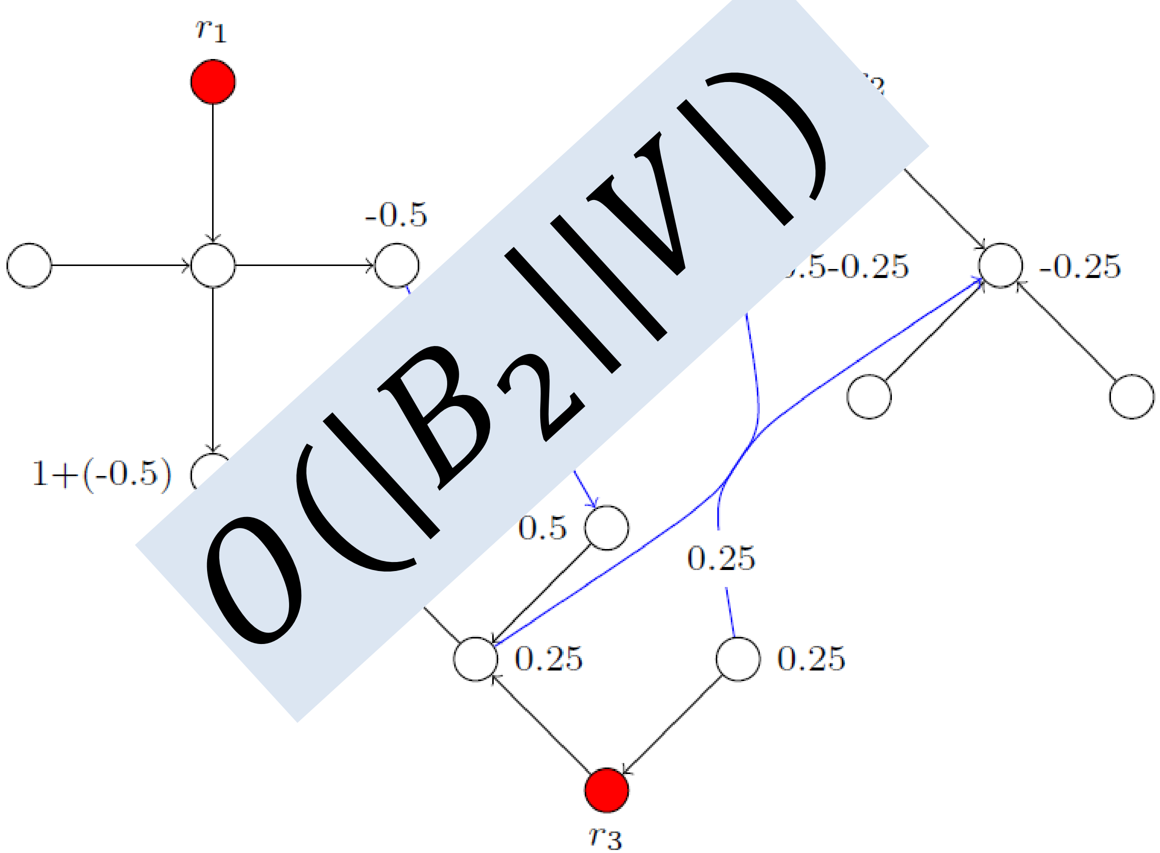
Example: (FTRAN) $M_{\cdot B} f = -M_{\cdot a^{\text{in}}}$

- a^{in} is the red arc
- $E_{R a^{\text{in}}} = (1,0,-1)^T$
- $\begin{pmatrix} f_{a_1} \\ f_{a_2} \end{pmatrix} = F \cdot E_{R\setminus\{r^*\} a^{\text{in}}} = \begin{pmatrix} 1/2 & 1/2 \\ 1/4 & -1/4 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/4 \end{pmatrix}$



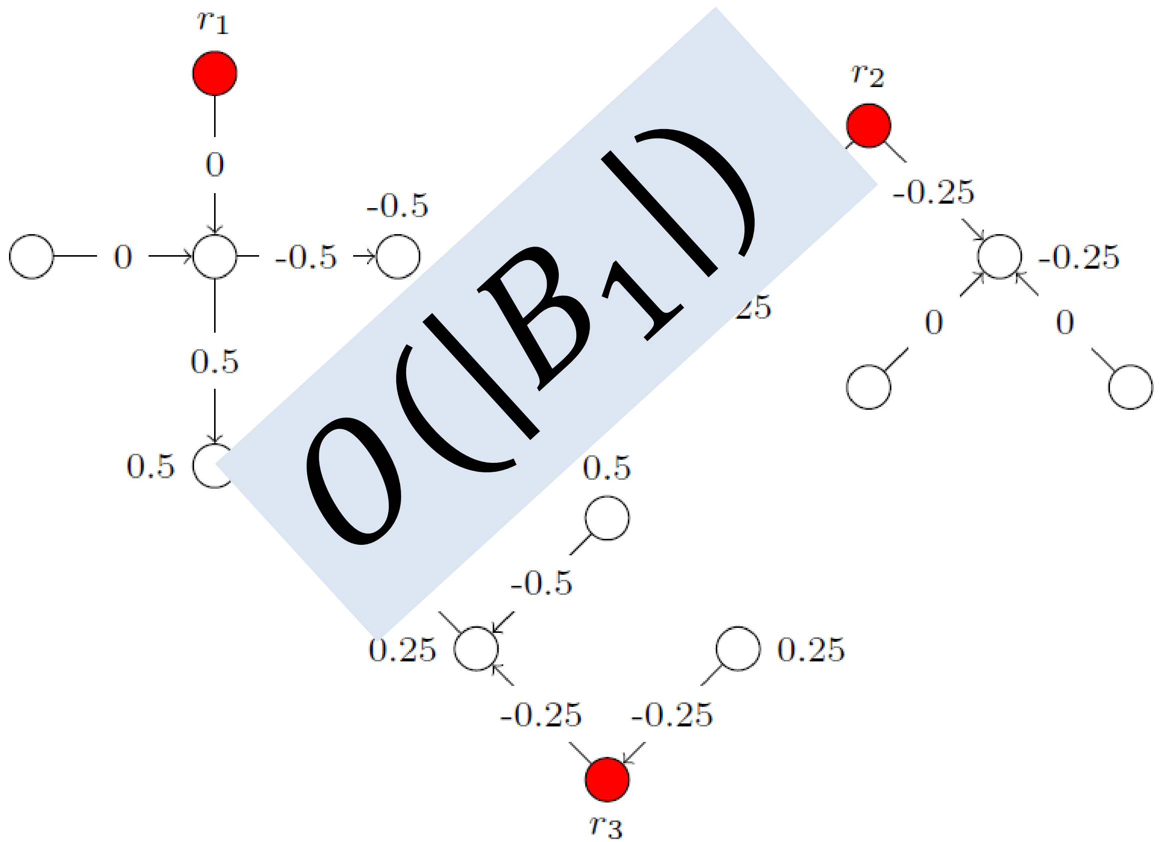
Example: (FTRAN) $M_{\cdot B} f = -M_{\cdot a^{\text{in}}}$

- Compute demand on tree vertices induced by flow on basic hyerparcs.
- Total demands (including demands on entering hyperarc) sum up to 0 on every tree.



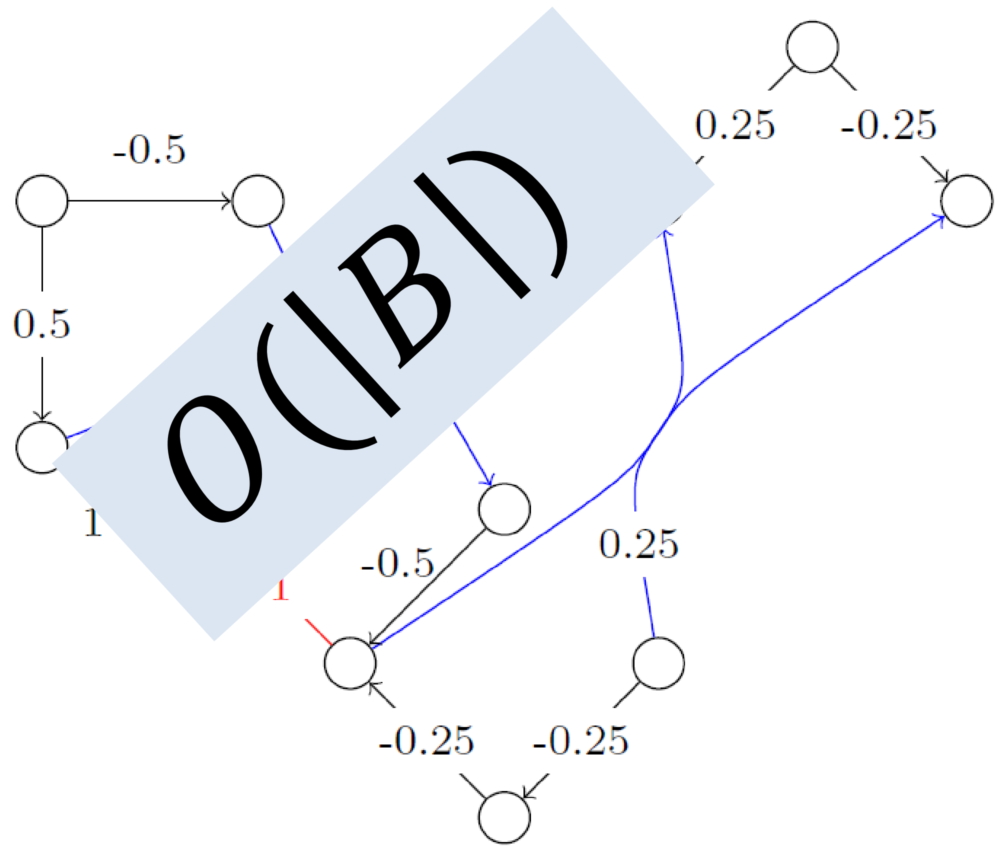
Example: (FTRAN) $M_{\cdot B} f = -M_{\cdot a^{\text{in}}}$

- Compute flow on each tree by reverse BFS.



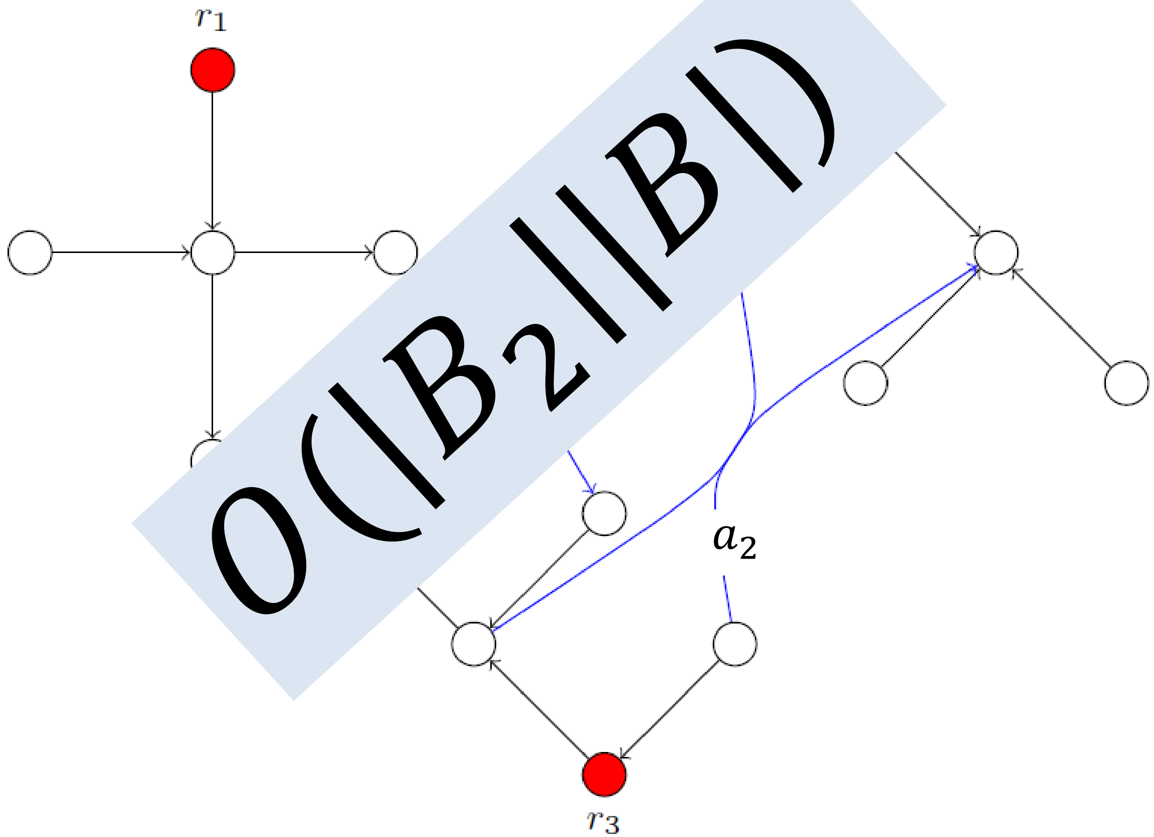
Example: (CHUZR) $a^{\text{out}} \in \operatorname{argmin}\left\{-\frac{x_a}{f_a} : f_a < 0, a \in B\right\}$

- $\min\left\{-\frac{x_a}{f_a} : f_a < 0, a \in B\right\} = 1.$
- a^{out} = red arc.



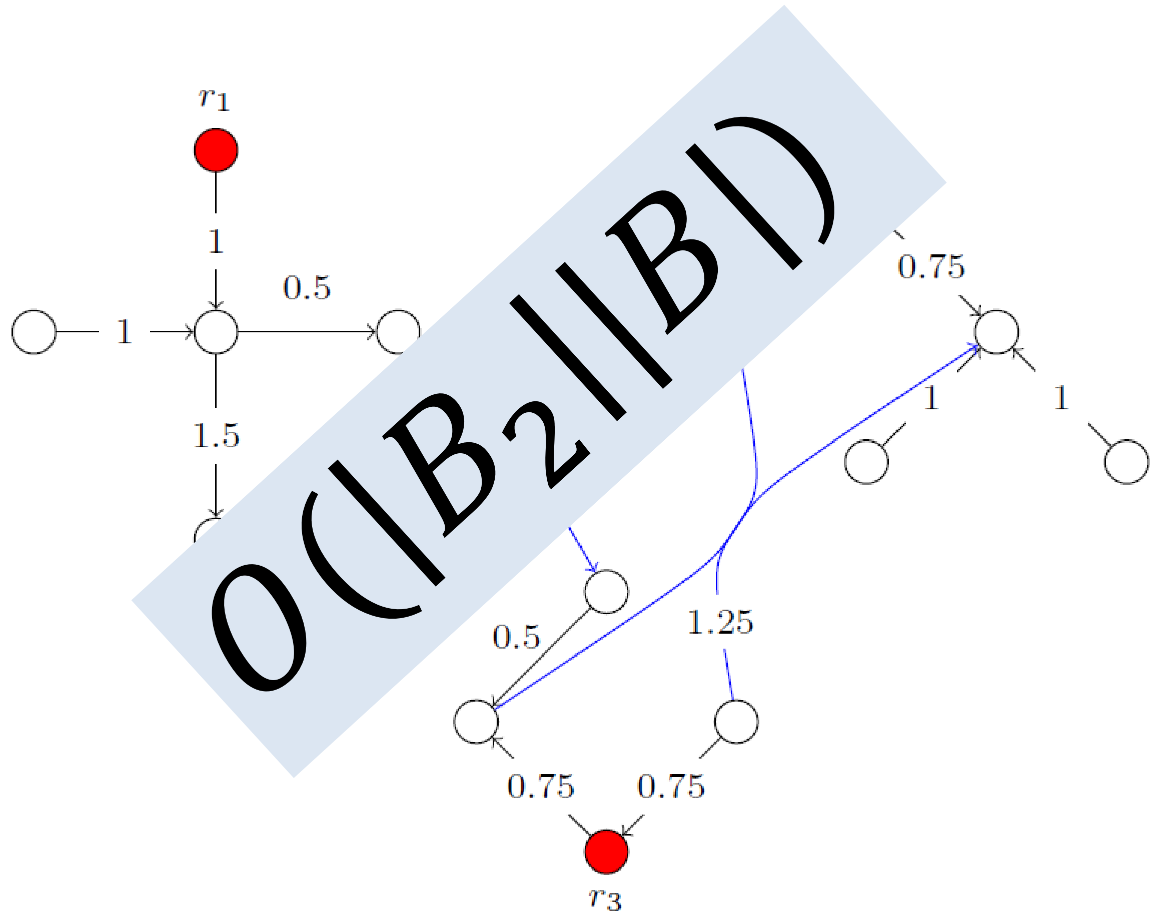
Example: (FTRAN) $M_{\cdot B} f = -M_{\cdot a^{\text{in}}}$

- a^{in} is the red arc
- $E_{R a^{\text{in}}} = (1,0,-1)^T$
- $\begin{pmatrix} f_{a_1} \\ f_{a_2} \end{pmatrix} = F \cdot E_{R\setminus\{r^*\} a^{\text{in}}} = \begin{pmatrix} 1/2 & 1/2 \\ 1/4 & -1/4 \end{pmatrix} \begin{pmatrix} 0 \\ - \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/4 \end{pmatrix}$



Example: UPDATE

- $x \leftarrow x + 1 \cdot f$.
- Trees T_{r_1} and T_{r_3} change.
- R, B_2, F do not change.

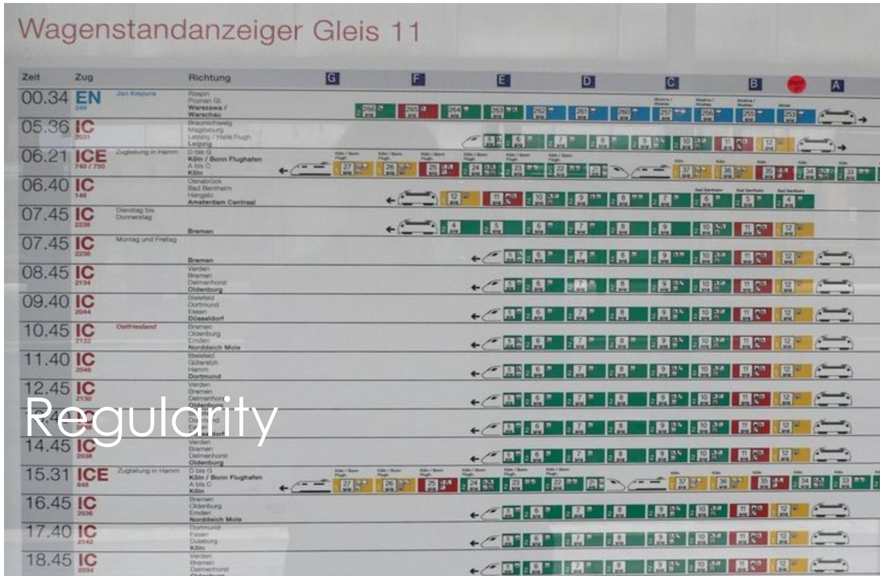


Summary

- One iteration of the Hyperflow Network Simplex Algorithm can be implemented in $O(\sum \deg(v) + |V|^2)$.
- If Bland's rule is used, the algorithm is finite. Also polynomial?
- Generalization of Network Simplex Method to Graph Based Hypergraphs.
- Combinatorial: Each pivot consists of (fast) graph theoretical computations.
- Can handle upper bounds on the variables.

InterCity Express (ICE) High Speed Train

Railway Constraints



Regularity

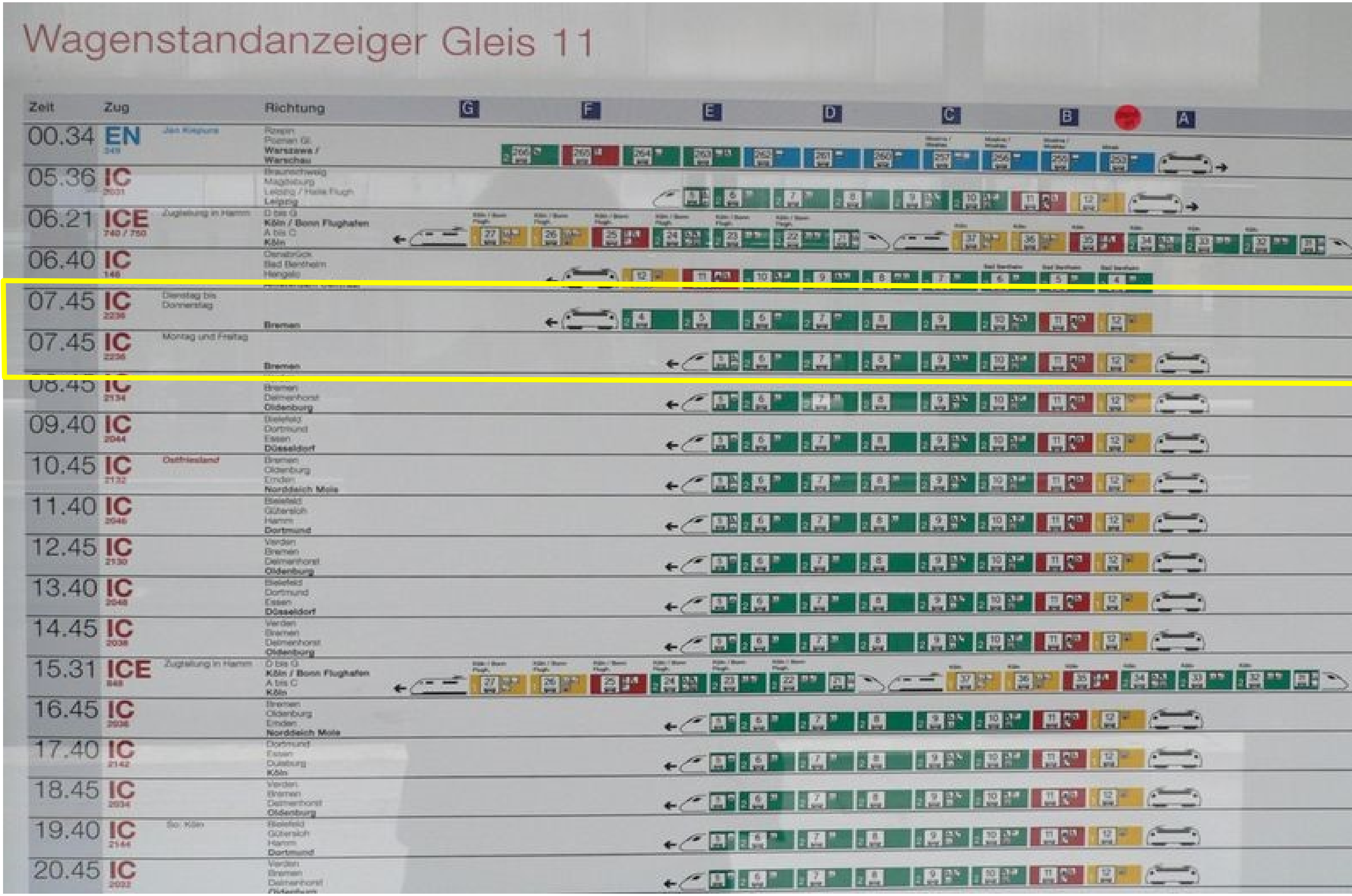
Maintenance

Parking

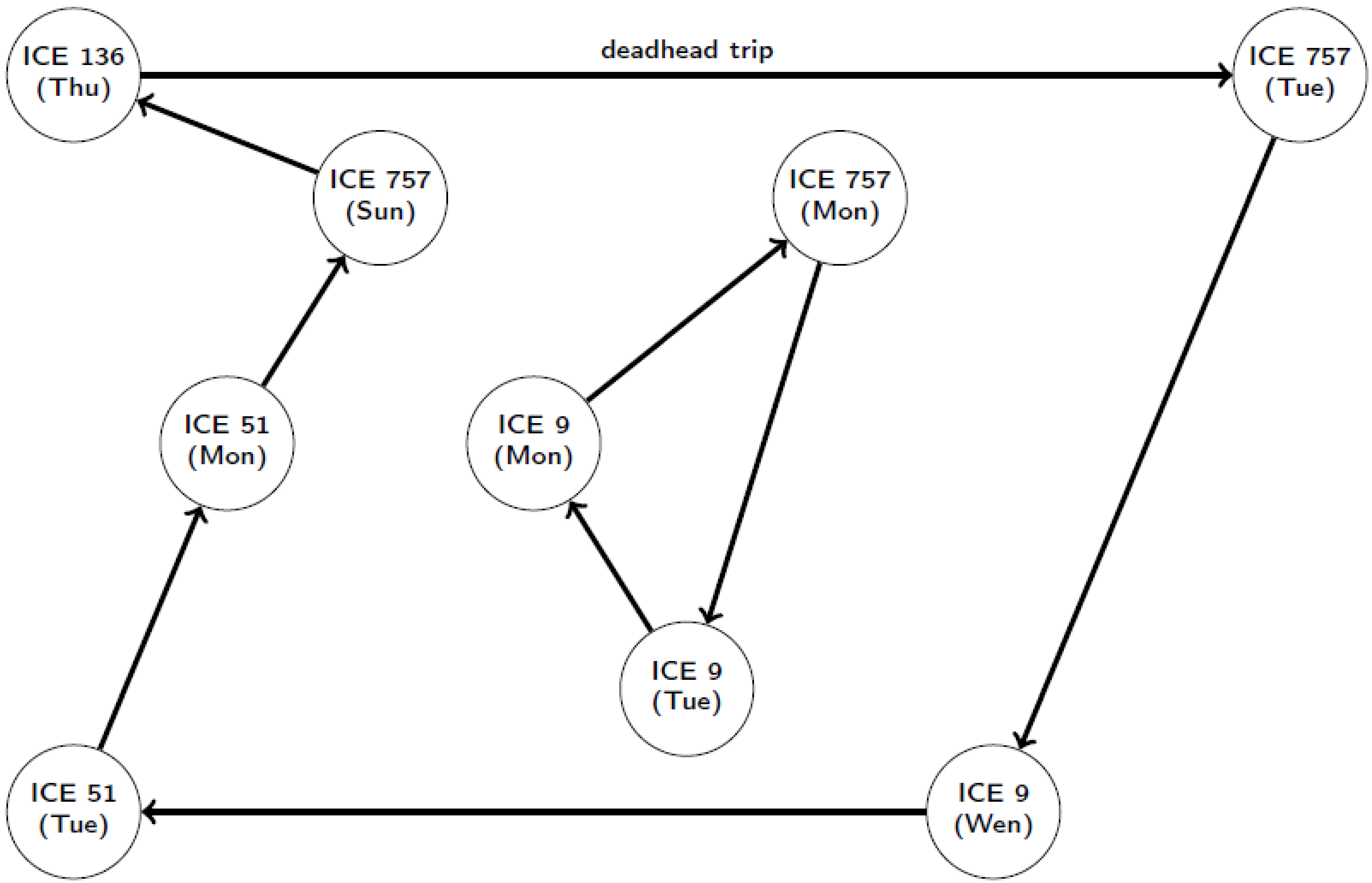
Train Composition

Photos courtesy of DB Mobility Logistics AG

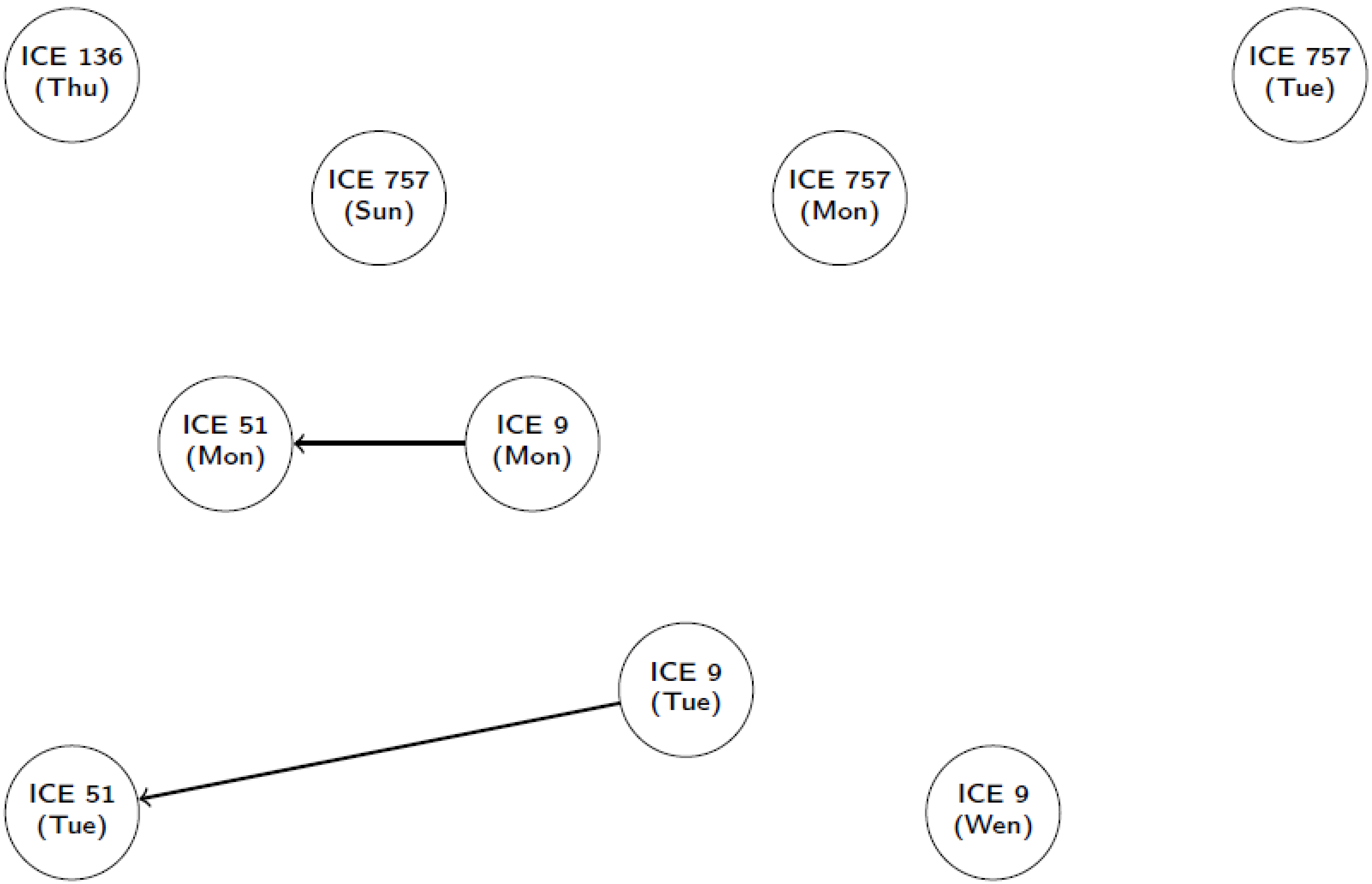
Timetable Regularity



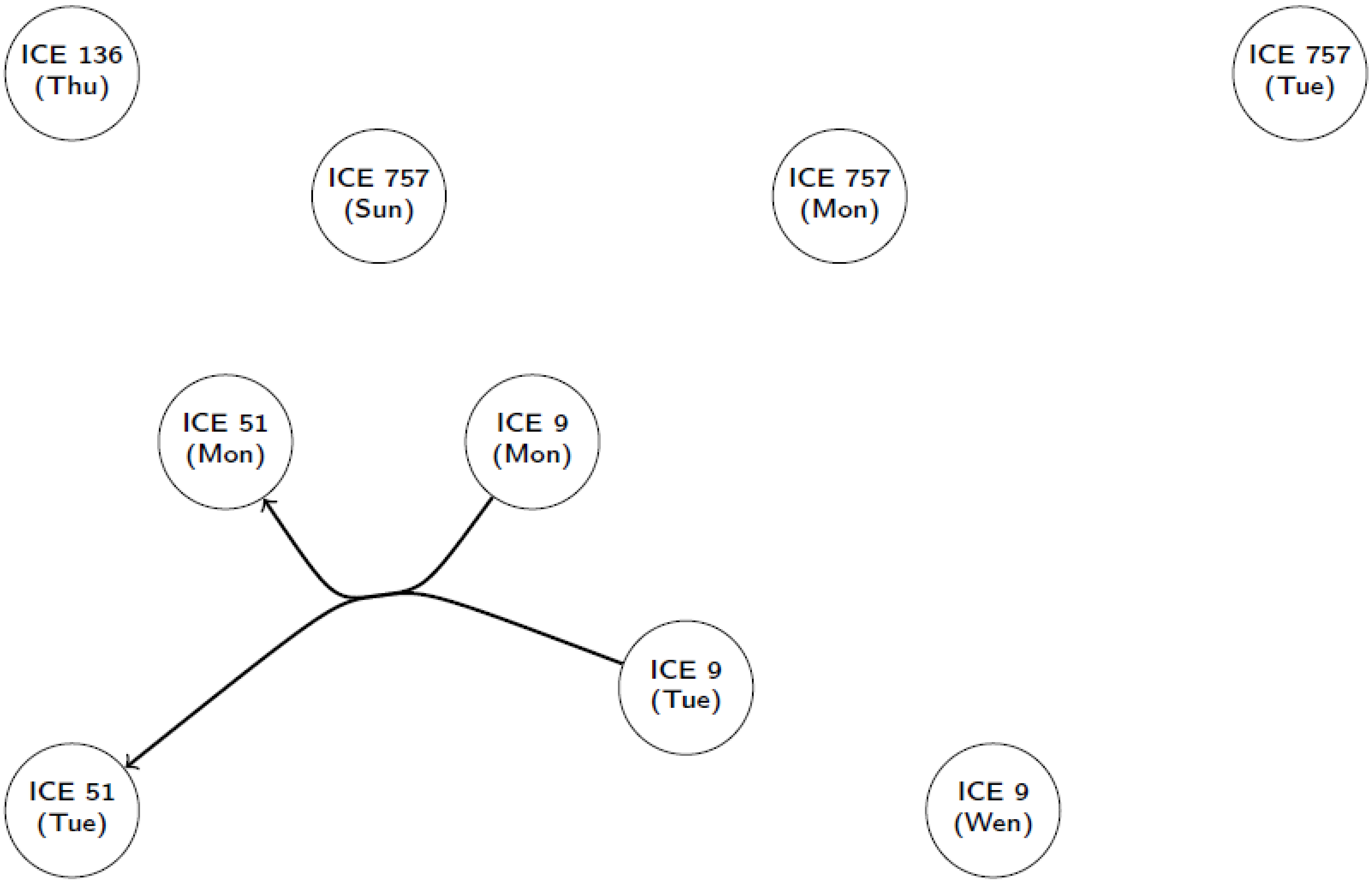
Assignment Solution



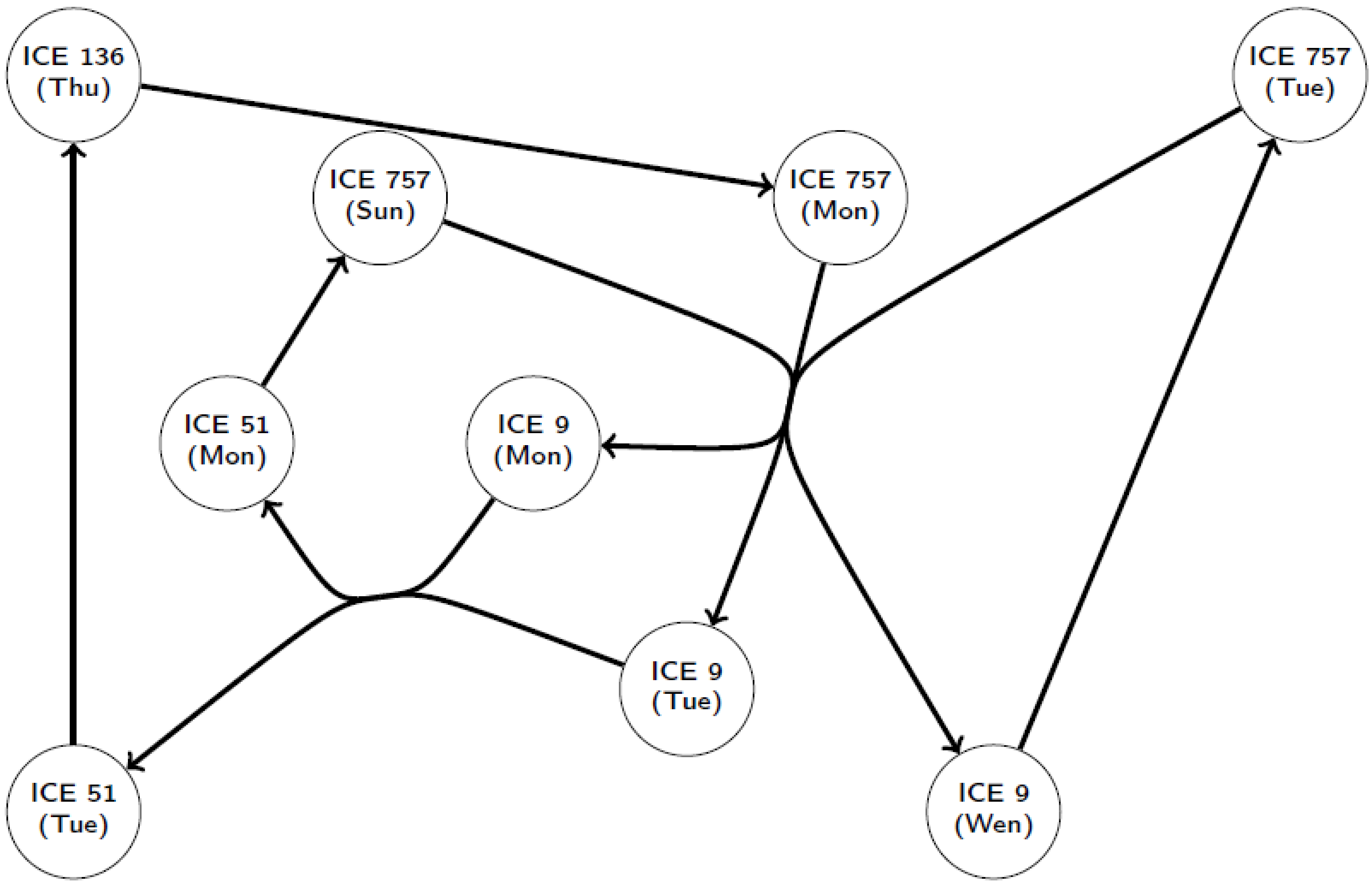
Rotation Regularity



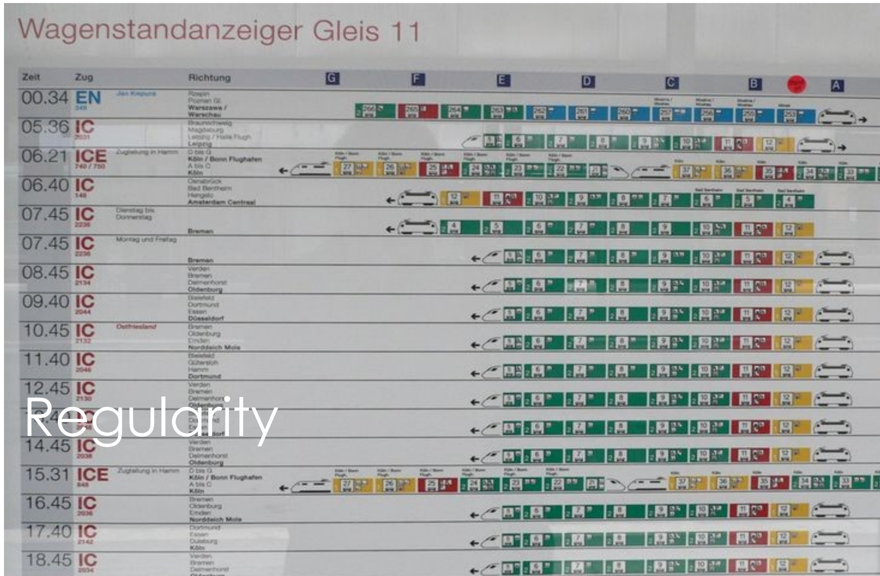
Modeling Regularity via Hyperedges



Hyperflow Model



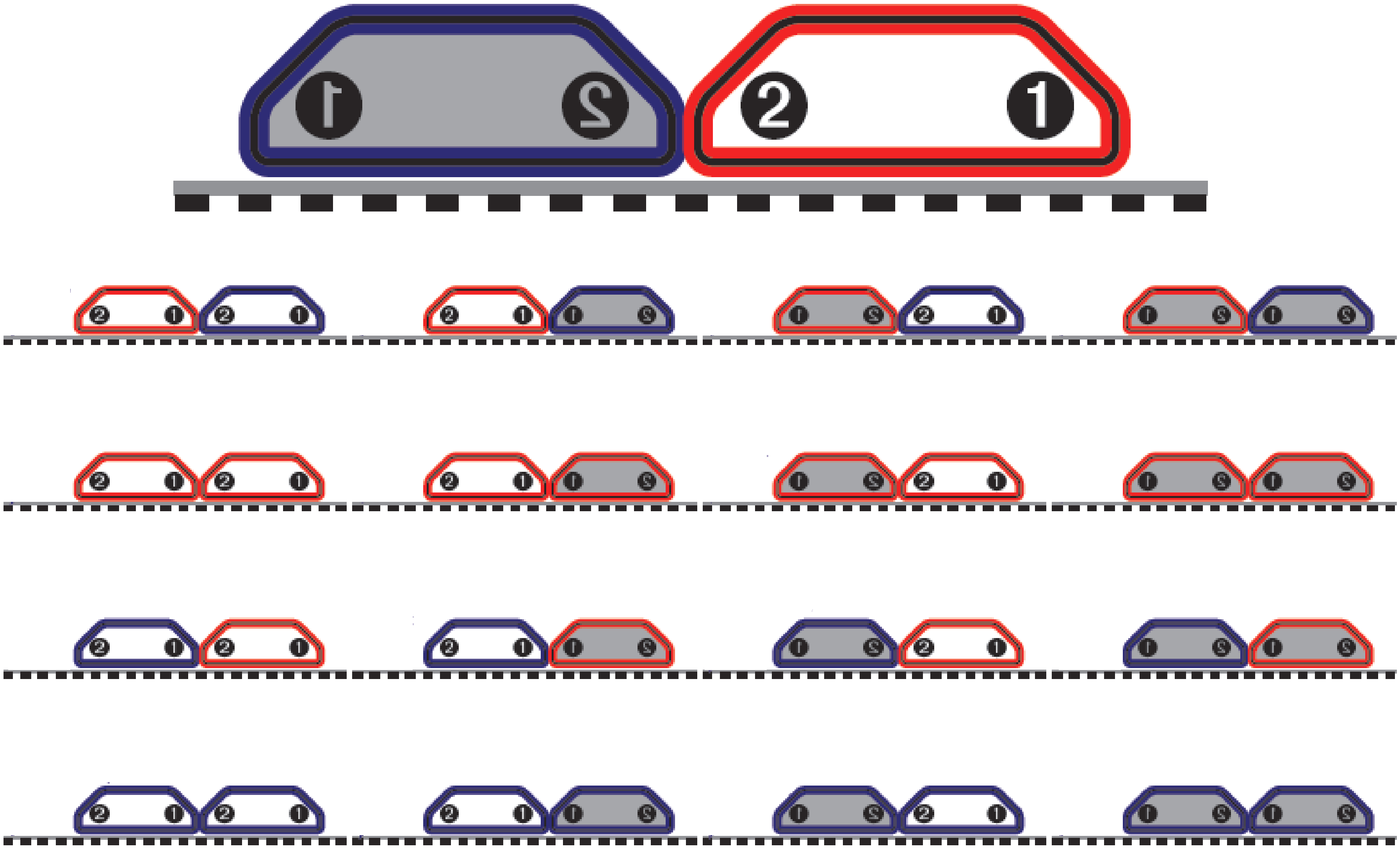
Railway Constraints



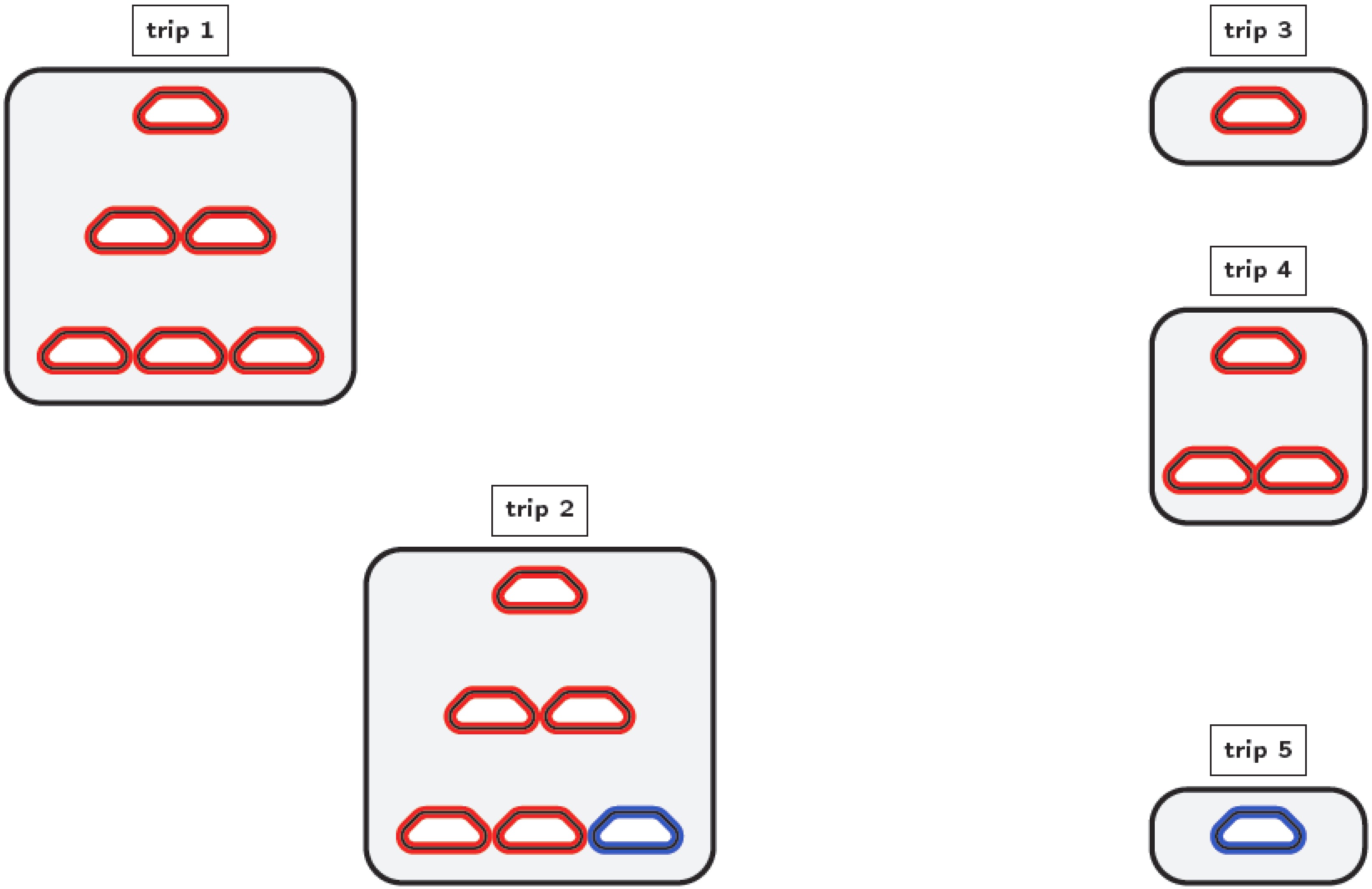
Photos courtesy of DB Mobility Logistics AG

Rare Train Composition Example

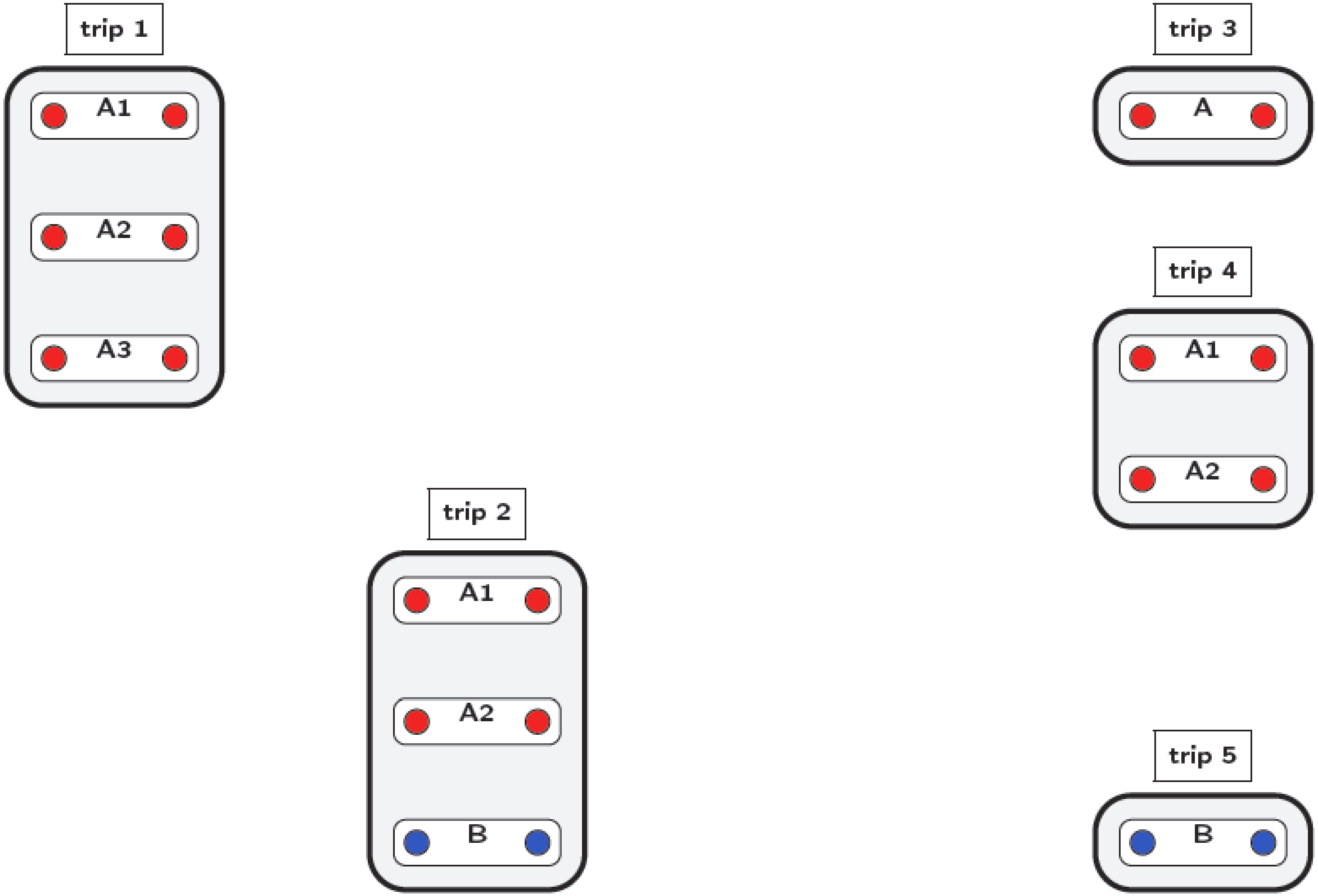
Train Composition: Type, Order, Orientation



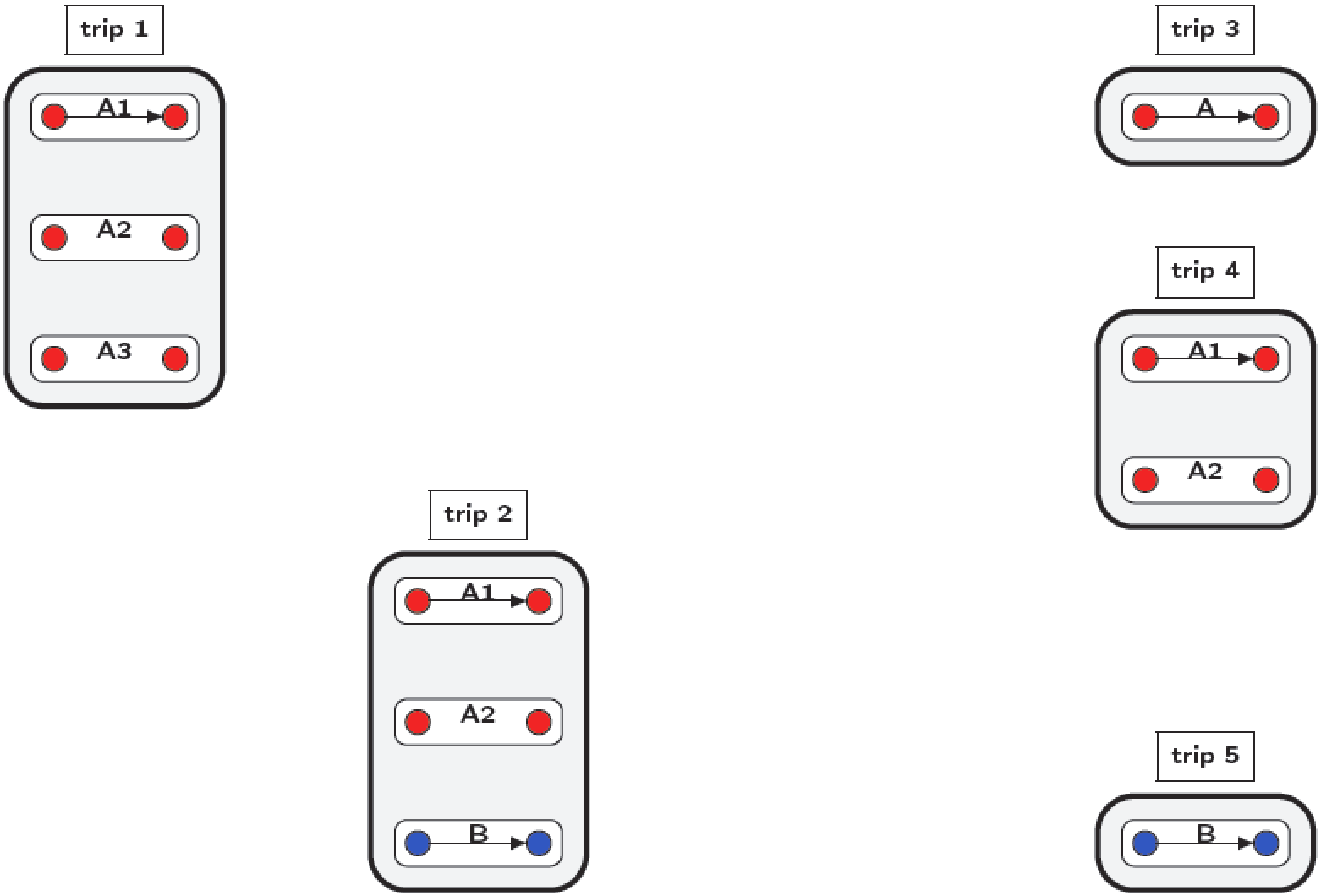
Hypergraph Model: Possible Train Compositions



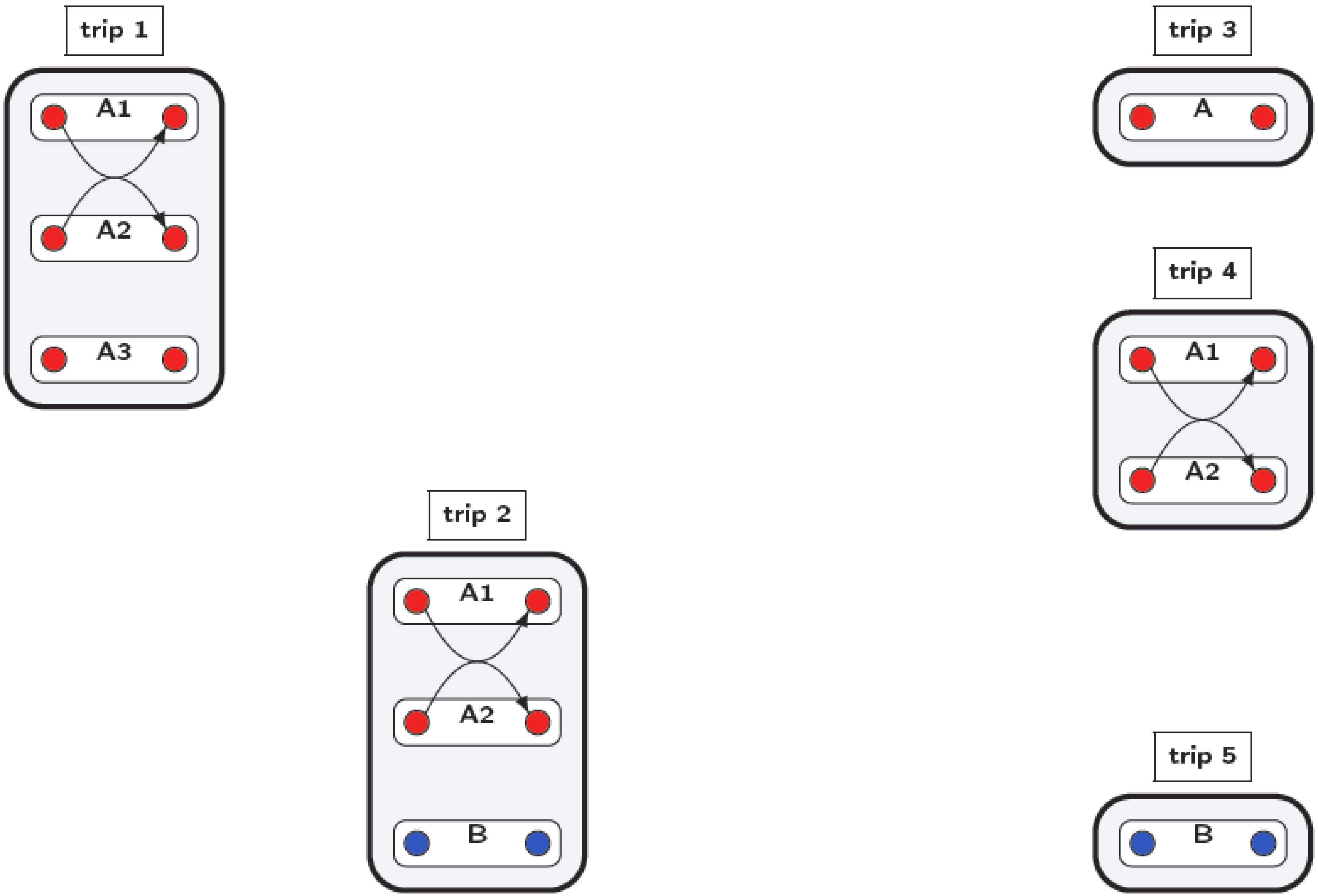
Hypergraph Model: Arrival and Departure Nodes



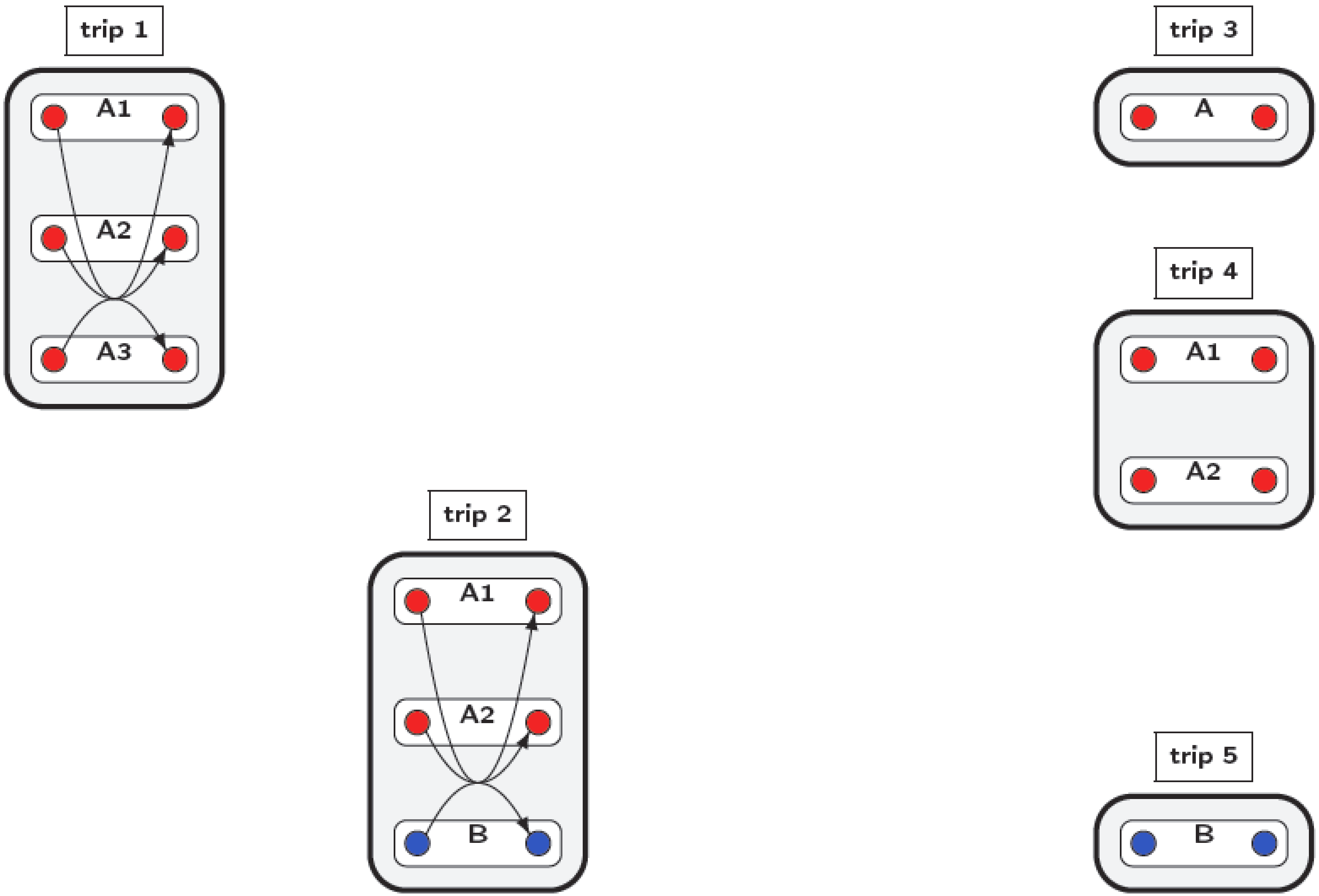
Hypergraph Model: Single Traction



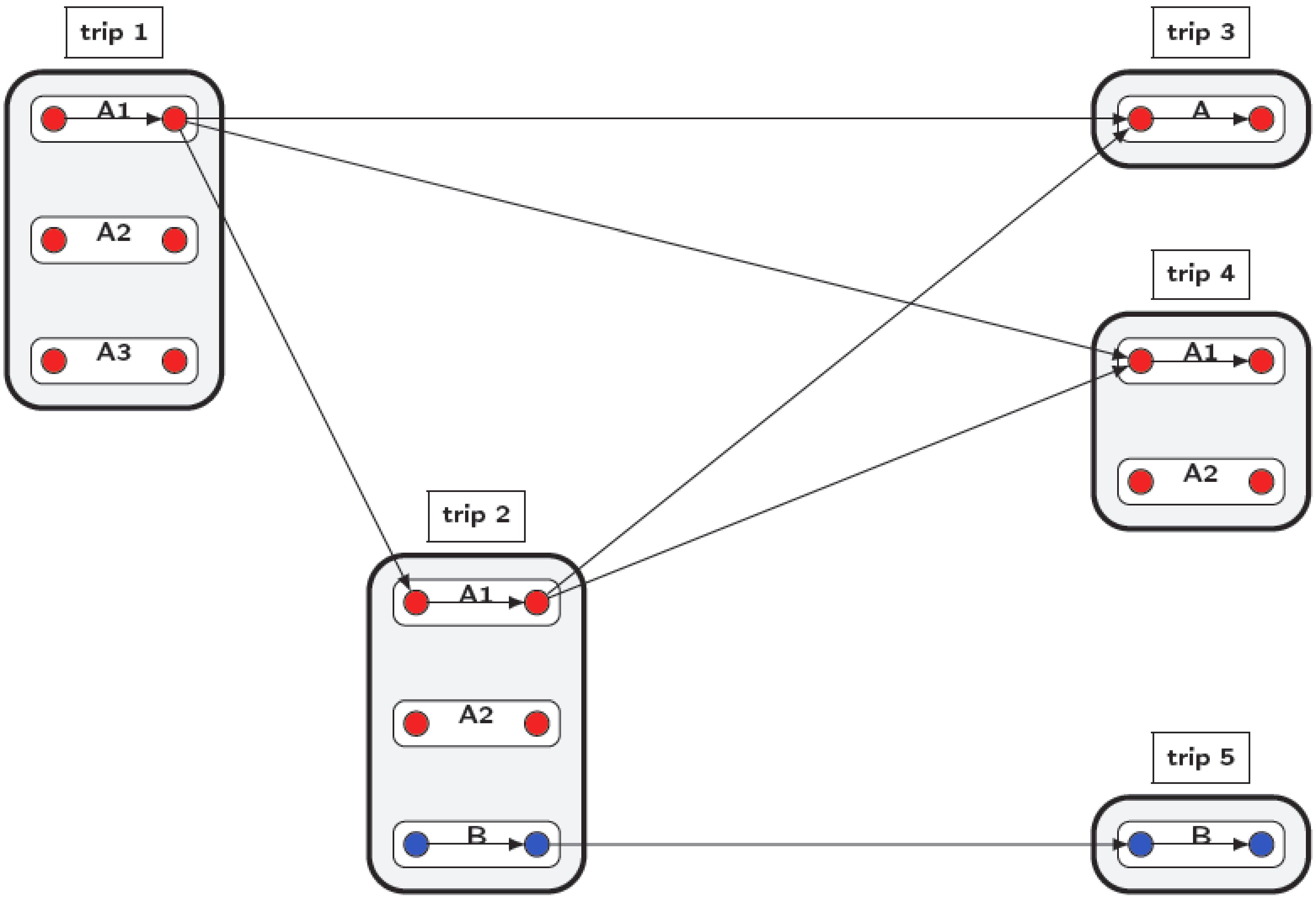
Hypergraph Model: Double Traction



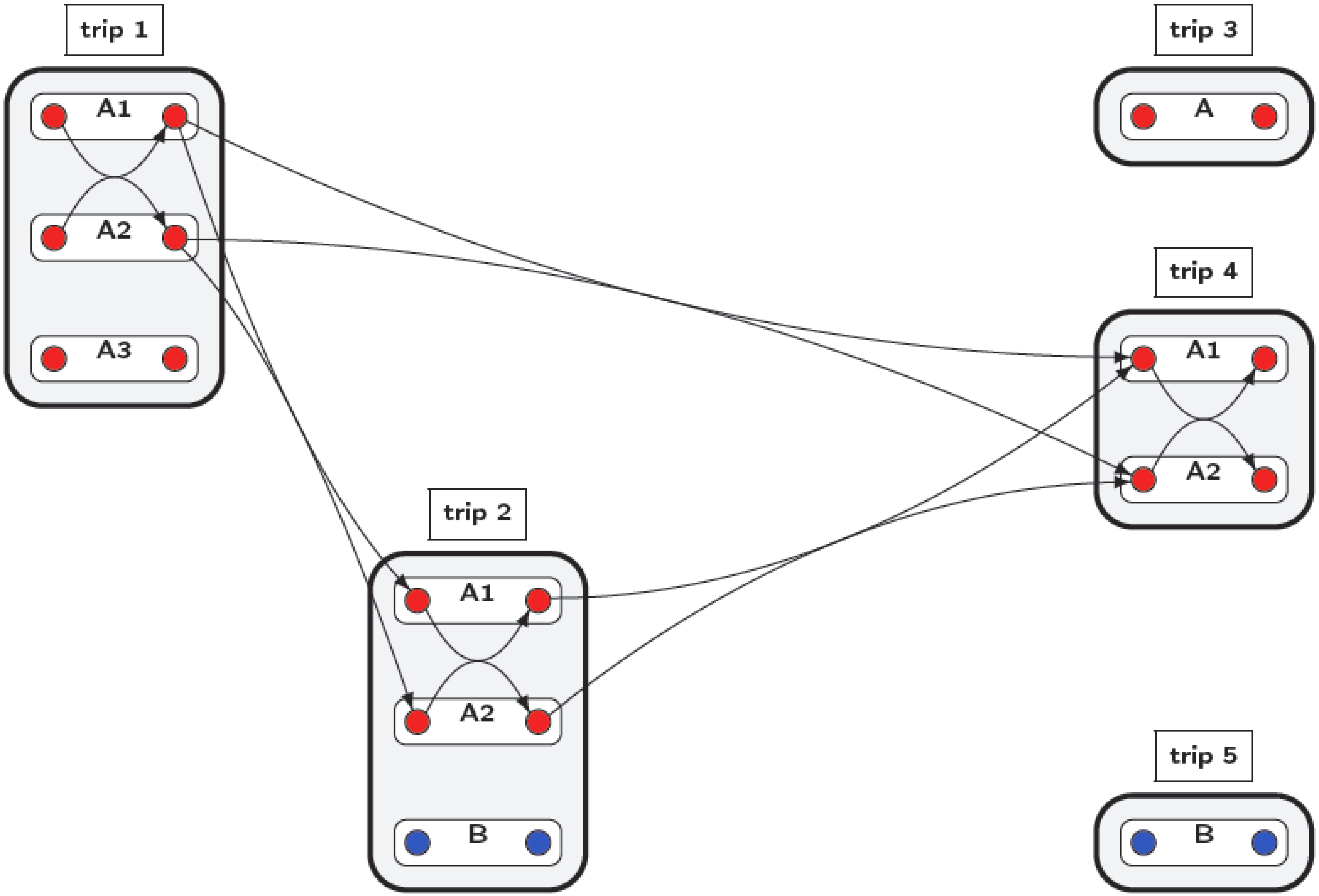
Hypergraph Model: Triple Traction



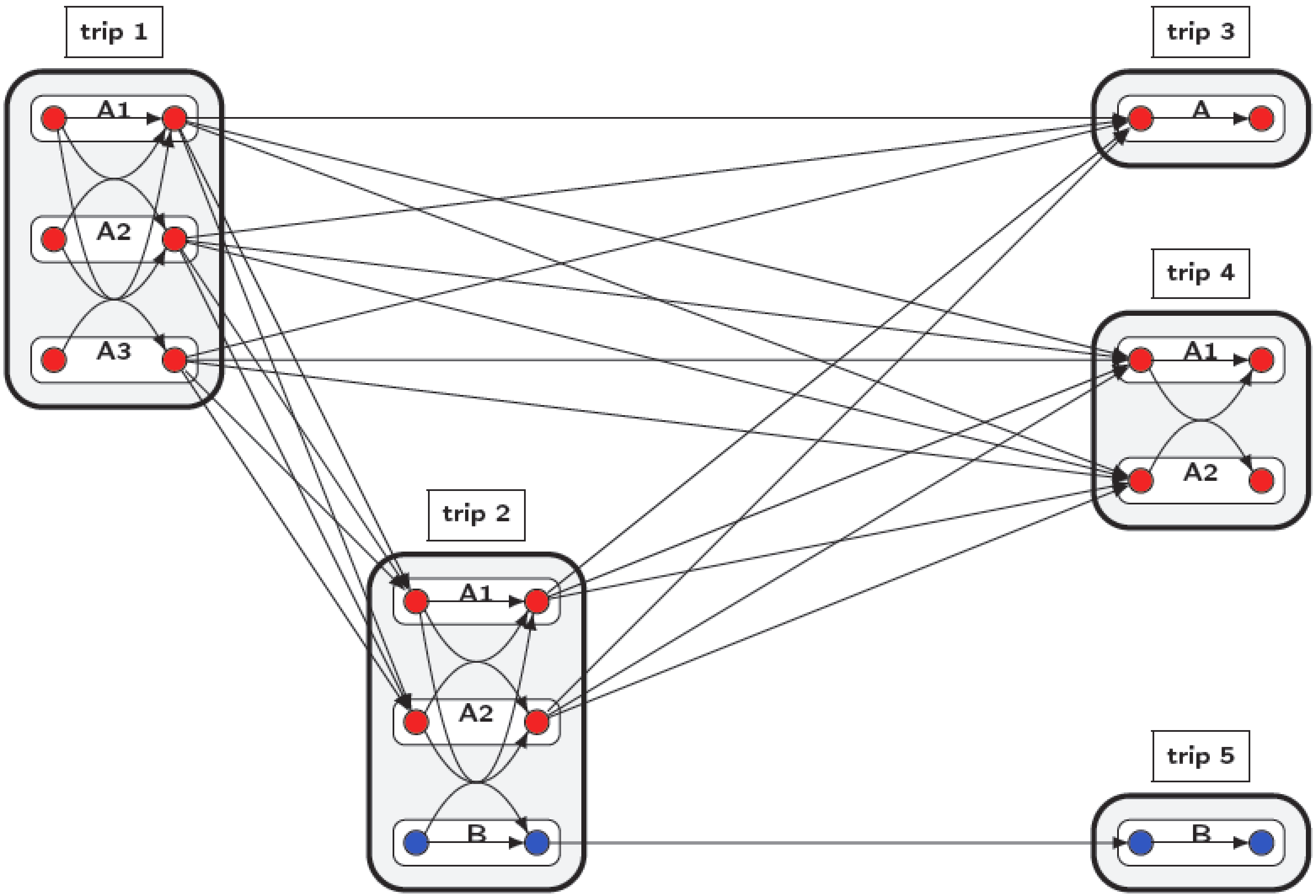
Hypergraph Model: Pass-Through Connections



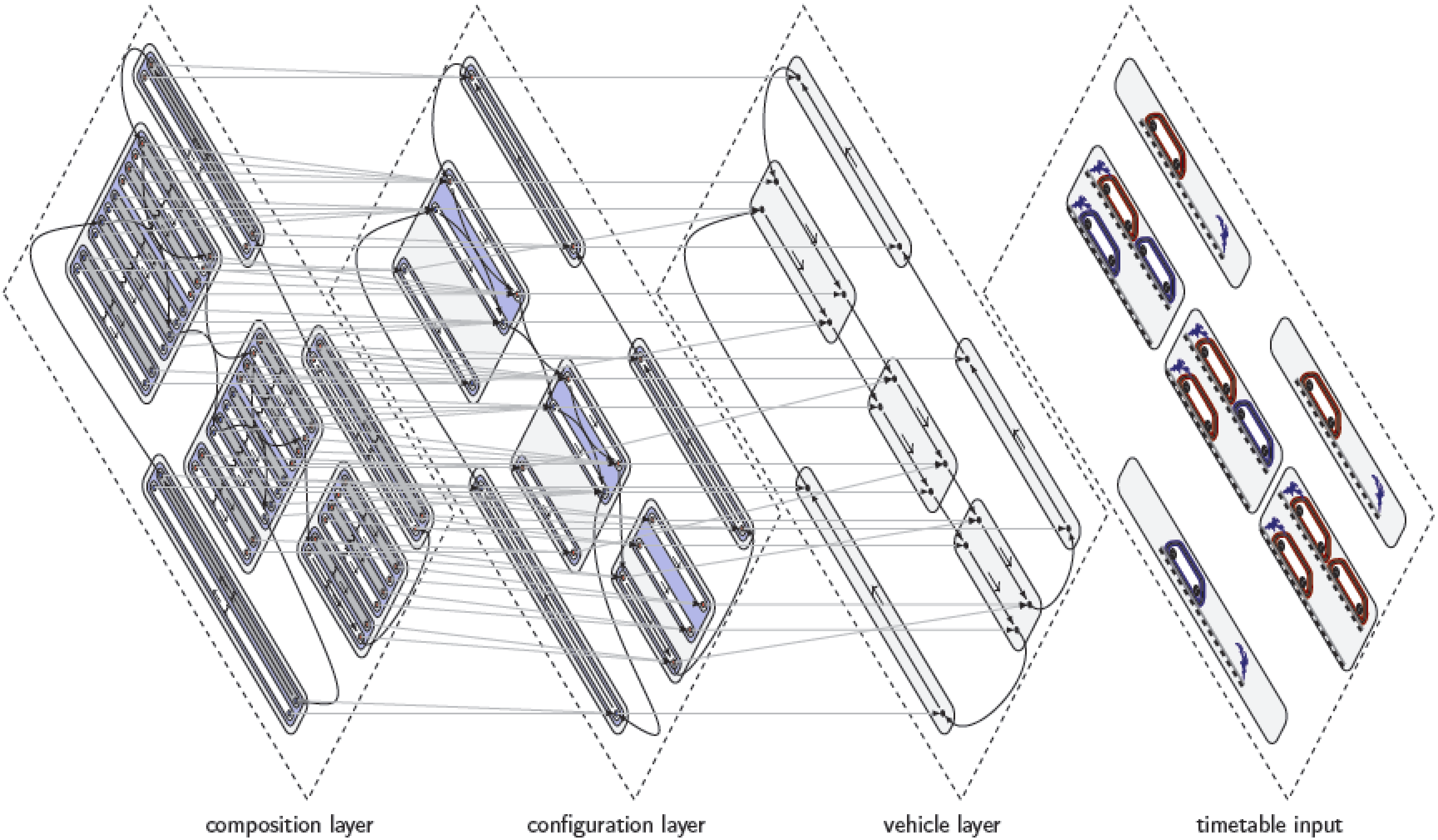
Hypergraph Model: Pass-Through Connections



Hypergraph Model: All Connections

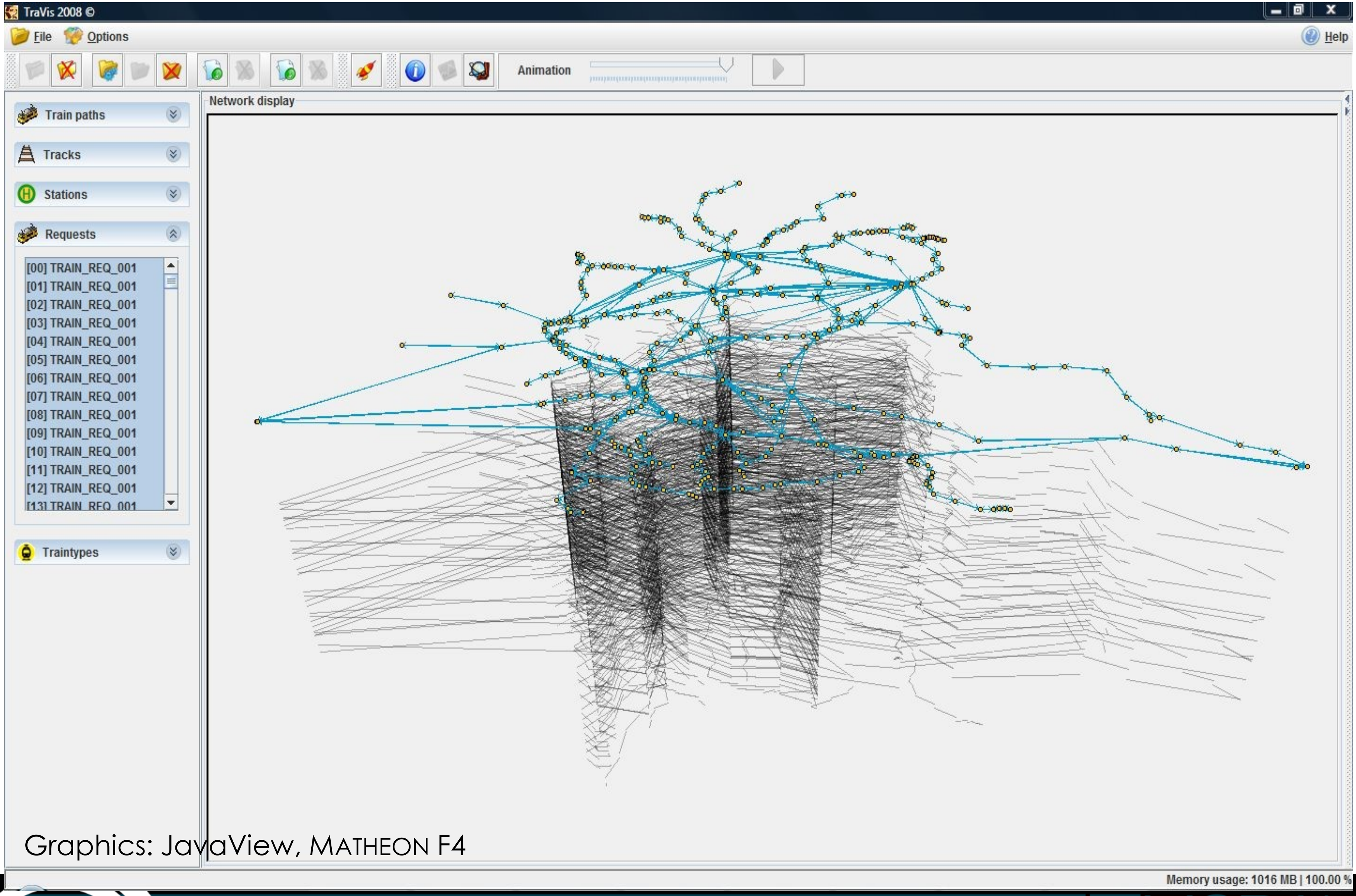


The Coarse-to-Fine Method



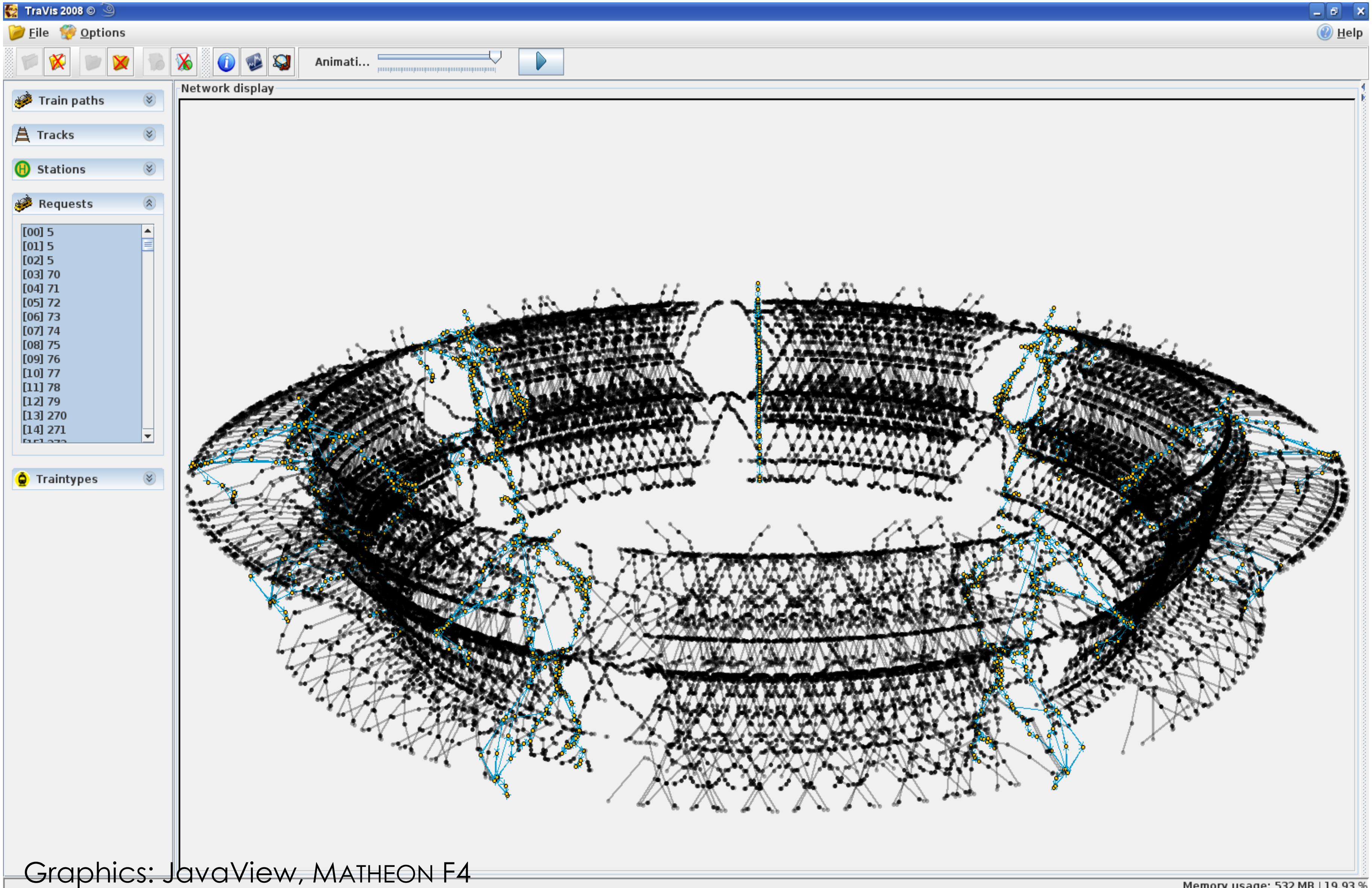
Motivation: ICE Connections

Timetabled Trips: 1 Day



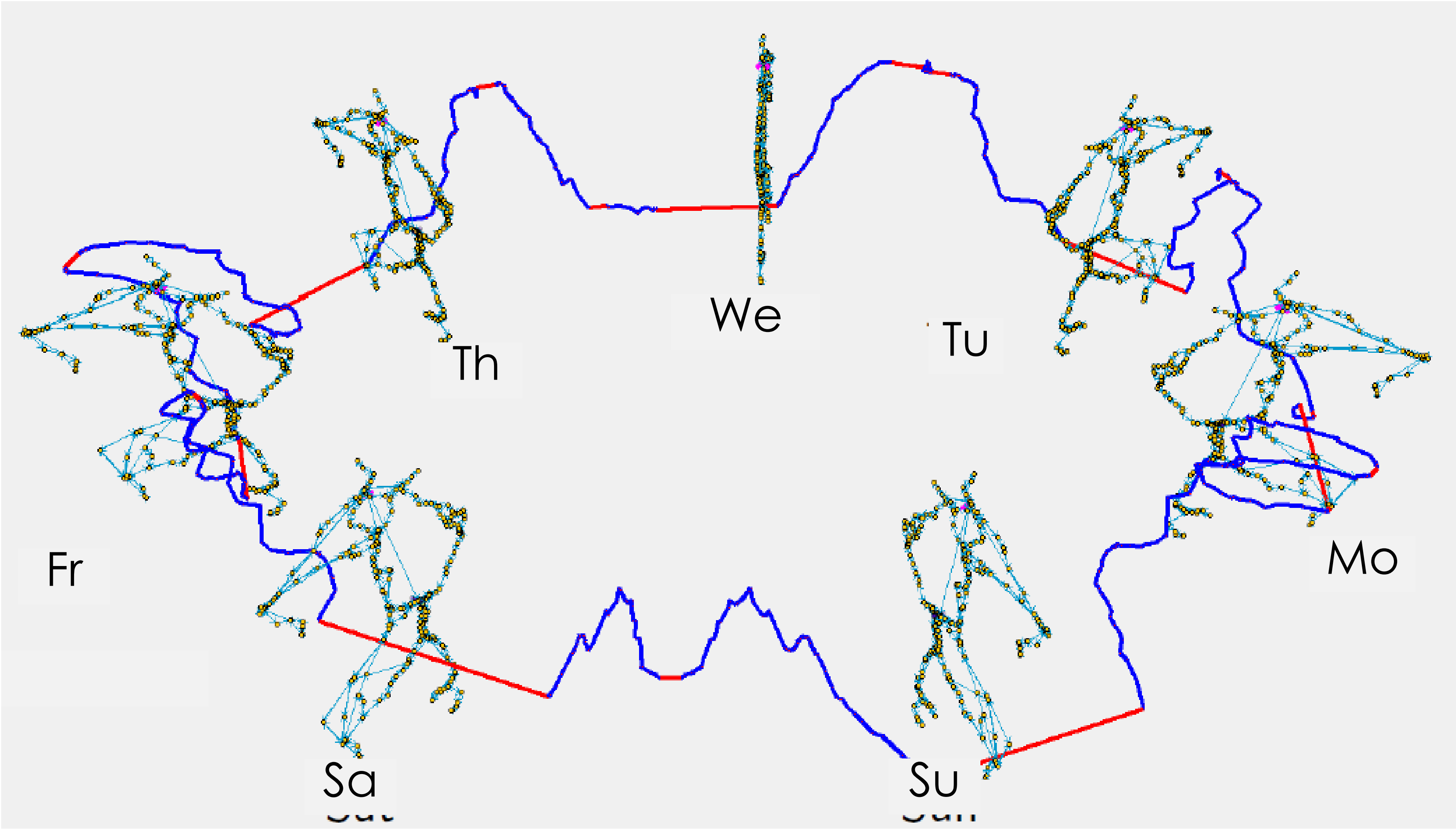
Graphics: JavaView, MATHEON F4

Timetabled Trips: Standard Week



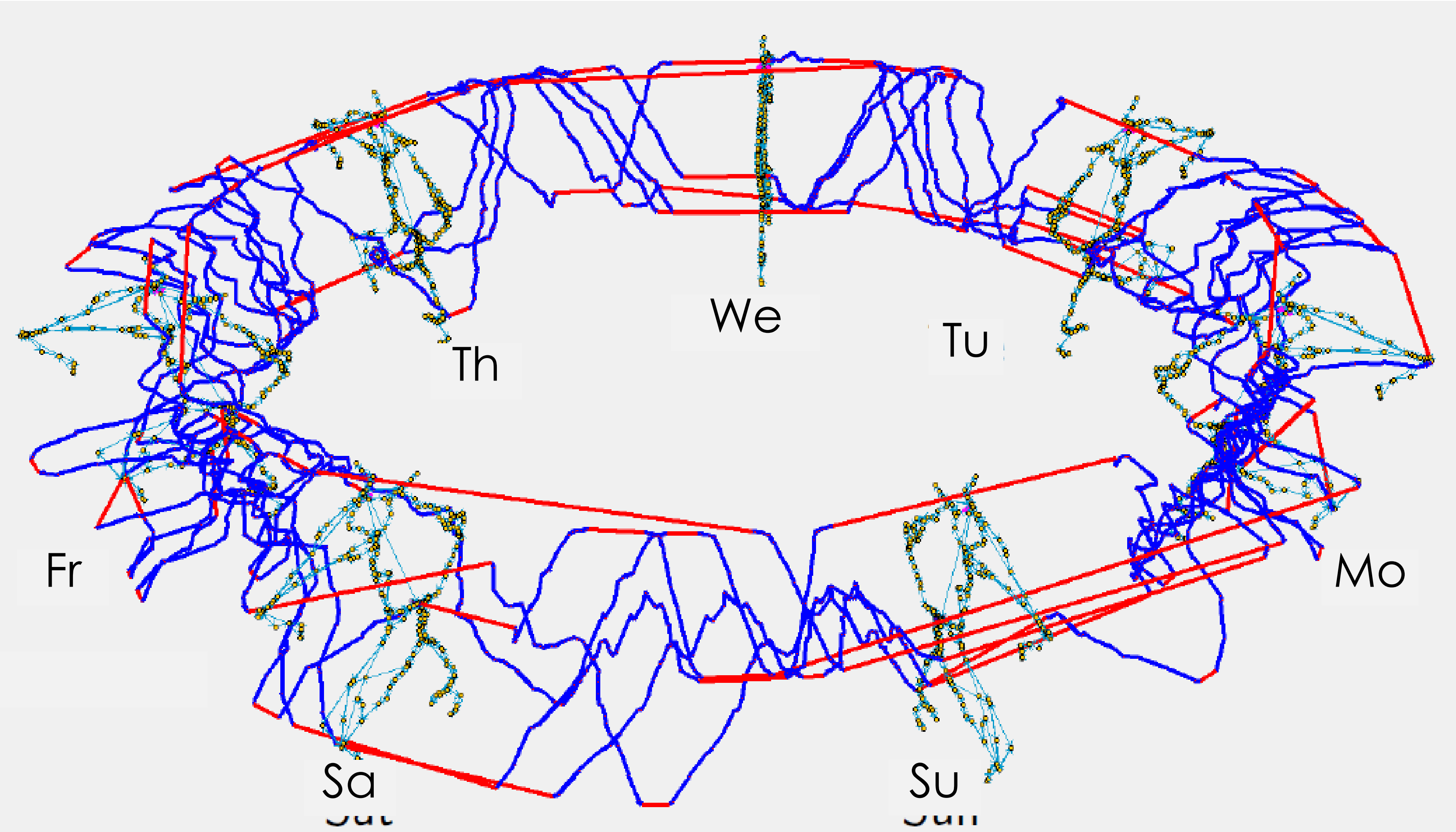
Graphics: JavaView, MATHEON F4

Vehicle Rotation (1 Week)



Graphics: JavaView, MATHEON F4

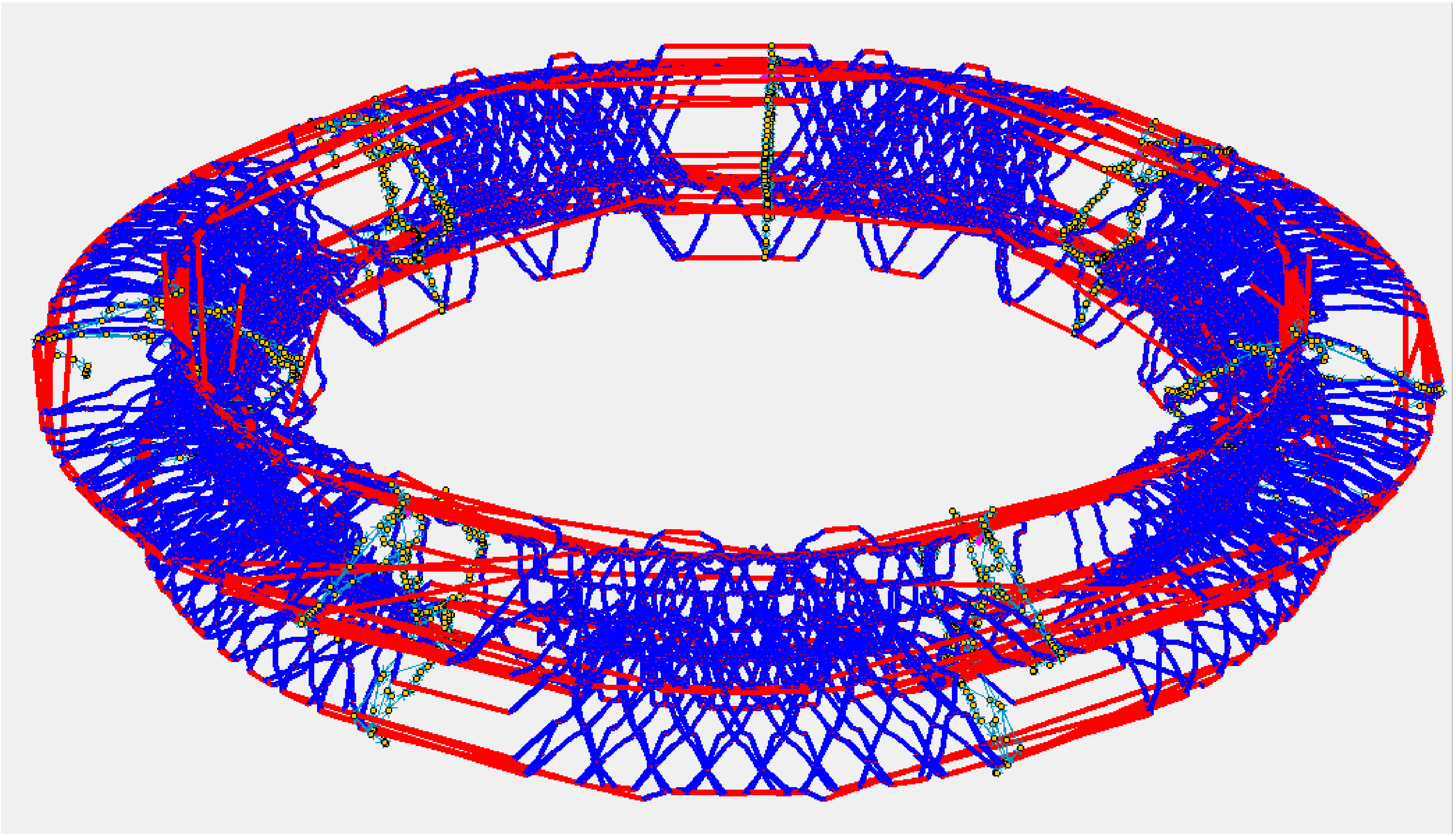
Vehicle Rotation (6 Weeks)



Graphics: JavaView, MATHEON F4

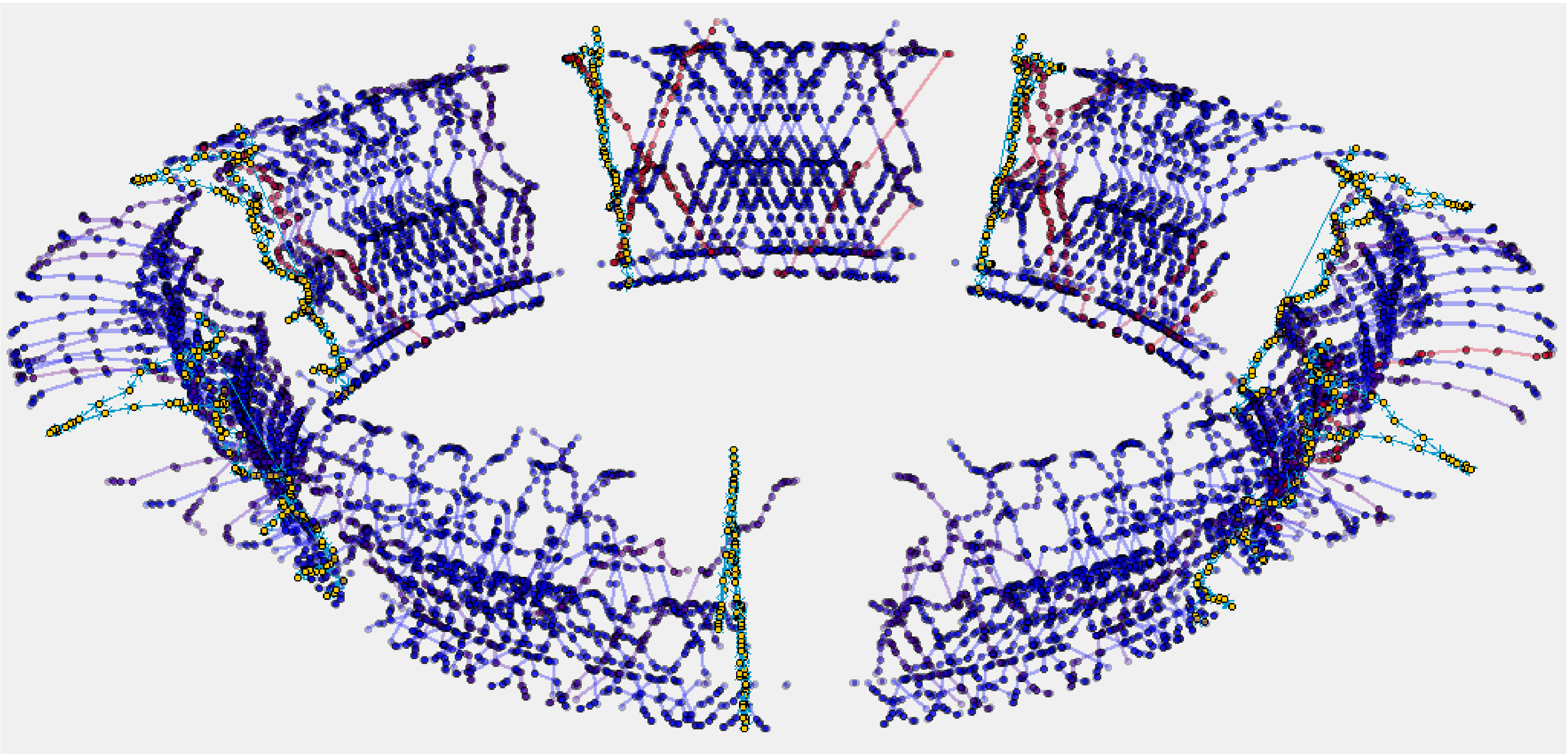
Rotation Plan: Follow-on Trip Assignment

(Blue: Timetabled Trips, Red: Deadhead Trips)



Graphics: JavaView, MATHEON F4

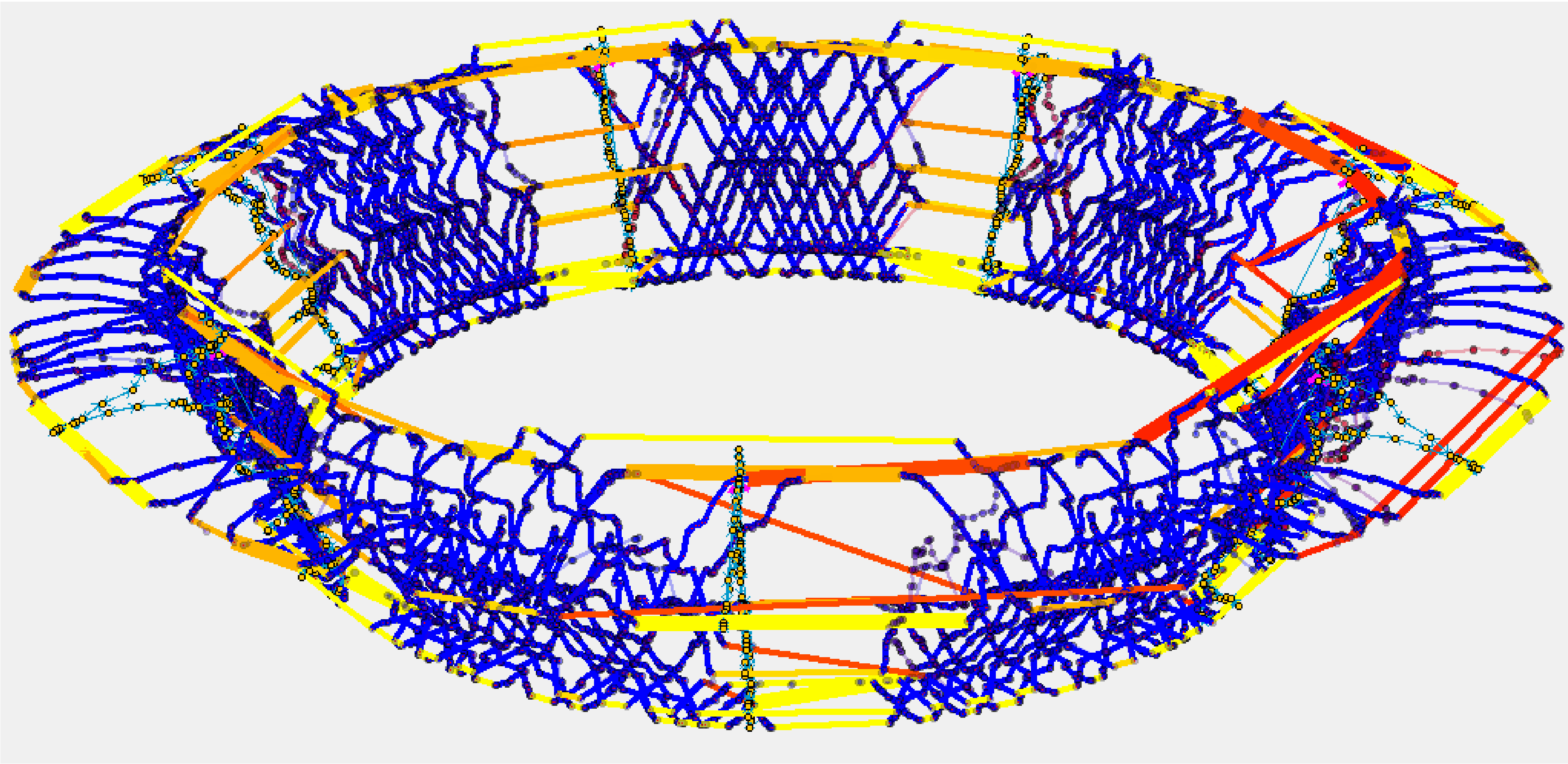
Regular Timetable



Regularity of the timetable trips:
- Red: Trip done on one day of the week
- Blue: Trip done on all days of the week

Graphics: JavaView, MATHEON F4

Regular Rotations



Regularity of the deadhead trips:

- Yellow: regular
- Red: irregular

Graphics: JavaView, MATHEON F4

Thank you for your attention

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