# A Conic Smörgåsbord

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ISM

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# Today's contents

1. Cones, convexity and optimization
2. Duality and facial reduction
3. Bonus content

**Software**: CVXPY (there are also versions for Julia, R and others):
`https://www.cvxpy.org/`

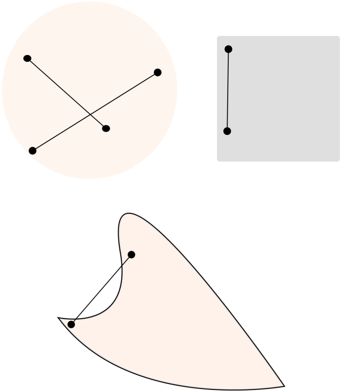
# Part 1 - Cones, convexity and optimization

# Convex sets

**Definition (Convex set)**

Let $C \subseteq \mathbb{R}^n$. $C$ is convex iff

$$x, y \in C \Rightarrow \alpha x + (1-\alpha) y \in C, \forall \alpha \in [0,1]$$



# Basic types of convex sets - affine sets

Affine set $\stackrel{\text{def}}{\Longleftrightarrow}$ the solution set of finitely many **equations**

- $\mathcal{C} \subseteq \mathbb{R}^n$ is affine $\Leftrightarrow$ exists $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$ such that $\mathcal{C} = \{x \in \mathbb{R}^n \mid Ax = b\}$

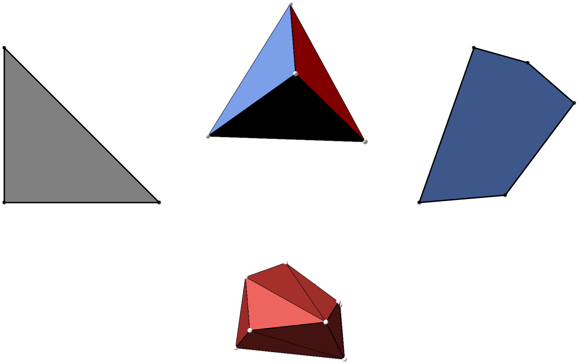
Examples

- A hyperplane $\{x \in \mathbb{R}^n \mid \langle x, v \rangle = \alpha\}$
- A vector subspace in $\mathbb{R}^n$
- Affine space = "translated subspace".

# Basic types of convex sets - polyhedral sets

Polyhedral sets $\overset{\text{def}}{\iff}$ the solution set of finitely many **equalities** and **inequalities**

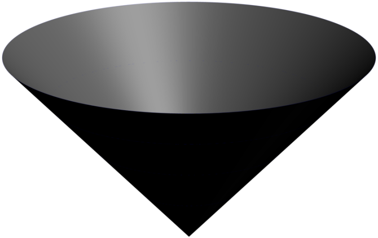
- $\mathcal{C} \subseteq \mathbb{R}^n$ is polyhedral $\Leftrightarrow$ exists $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ such that $\mathcal{C} = \{x \in \mathbb{R}^n \mid Ax \leq b\}$



# Basic types of convex sets - convex cones

$\mathcal{K}$ is a convex cone $\stackrel{\text{def}}{\iff}$ $\alpha x + \beta y \in \mathcal{K}$, whenever $x, y \in \mathcal{K}$ and $\alpha, \beta \geq 0$.

- $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_i \geq 0, \forall i\}$
- $n \times n$ symmetric positive semidefinite matrices $\mathcal{S}^n_+$.



# Convex functions

$f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$

- $f$ is convex $\stackrel{\text{def}}{\iff}$ $f(\alpha x + (1-\alpha)y) \le \alpha f(x) + (1-\alpha) f(y)$, $\forall x, y \in \mathbb{R}^n$, $\forall \alpha \in [0, 1]$.
- $f$ is convex $\Leftrightarrow$ the epigraph of $f$ given by $\operatorname{epi} f := \{(x, \mu) \mid f(x) \le \mu\}$ is a convex set.

**Examples**:

- $f(x) = x^2$
- $f(x) = ax$
- $f(x) = -\ln(x)$.

**Non-examples**:

- $f(x) = \ln(x)$
- $f(x) = x^3$

# Conic linear programming

$$
\begin{aligned}
\min_{x} \quad & \langle c, x\rangle \\
\text{subject to} \quad & \mathcal{A}x = b \\
& x \in \mathcal{K}
\end{aligned}
$$

- $\mathcal{K} \subseteq \mathcal{E}$: closed convex cone,
- $\mathcal{A} : \mathcal{E} \to \mathbb{R}^m$: linear map, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$,
- $\mathcal{E}$ is an Euclidean space equipped with an inner product $\langle \cdot, \cdot \rangle$ and a norm $\|\cdot\|$ induced by $\langle \cdot, \cdot \rangle$.

Feasible region $\{x \in \mathcal{K} \mid Ax = b\}$ = "a cone intersected by an affine set".

# Conic Linear Programming - Alternative forms

- "Minimize/Maximize a linear function, subject to equalities, inequalities and cone constraints"
- These are all CLPs:

$$\begin{aligned} \max_{x \in \mathbb{R}^n} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b, \\ & Ex - d \in \mathcal{K} \end{aligned}$$

$$\begin{aligned} \min_{x,y} \quad & c_1^T x + c_2^T y \\ \text{subject to} \quad & A_1 y \leq b_1, \\ & A_2 x = b_2 \\ & (x_1, x_2) \in \mathcal{K}_1 \times \mathcal{K}_2 \end{aligned}$$

# Linear Programming (LP)

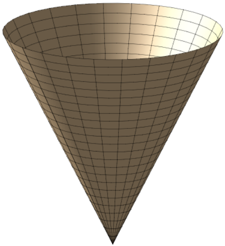
$$\begin{aligned} \min_{x} \quad & c^\top x \\ \text{subject to} \quad & \mathcal{A}x = b \\ & x \in \mathbb{R}^n_+ \end{aligned}$$

- $\mathcal{A}$ is a $m \times n$ matrix, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$.

# The second-order cone (a.k.a ice-cream cone)

$$\mathcal{Q}^{n+1} := \{(x_0, \bar{x}) \in \mathbb{R} \times \mathbb{R}^n \mid x_0 \geq \|\bar{x}\|_2\},$$

where $\|\bar{x}\|_2 = \sqrt{\bar{x}_1^2 + \cdots + \bar{x}_n^2}$



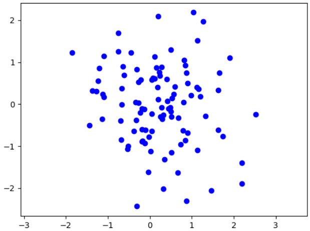
# Second-order cone programming (SOCP)

$$\begin{aligned} \min_{x} \quad & c^\top x \\ \text{subject to} \quad & \mathcal{A}x = b \\ & x \in \mathcal{Q}^{n_1} \times \cdots \times \mathcal{Q}^{n_r} \end{aligned} \tag{P}$$

- $\mathcal{A}$ is a $m \times n$ matrix, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$.

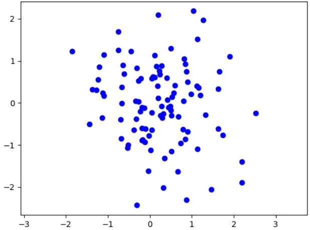
# Antenna placing problem

We want place an antenna that sends a signal that covers the whole region below.




Where should the antenna be placed and what is the minimum radius of the signal capable of covering all the points?

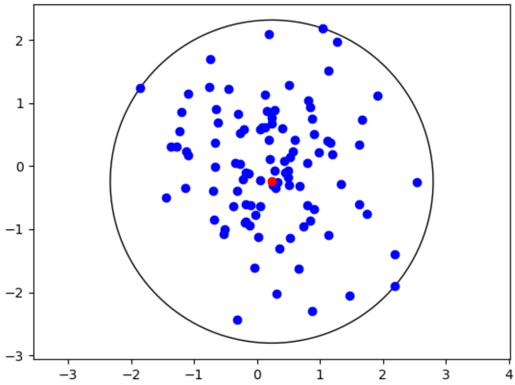
# Antenna placing problem- Formulation




$$\min_{x \in \mathbb{R}^3} \quad x_0 \qquad \text{(P)}$$
$$\text{subject to} \quad \|\bar{x} - p_i\|_2 \leq x_0, \forall i = 1, \ldots, m$$

$$\min_{x \in \mathbb{R}^3} \quad x_0 \qquad \text{(P)}$$
$$\text{subject to} \quad (x_0, \bar{x} - p_i) \in \mathcal{Q}^3, \qquad \forall i = 1, \ldots, m$$

# Antenna placing problem - solution



# Semidefinite Programming (SDP)

$$\begin{aligned} \min_{X \in \mathcal{S}^n} \quad & \langle C, X \rangle \\ \text{subject to} \quad & \langle A_i, X \rangle = b_i, \quad i = 1, \dots, m \\ & X \succeq 0 \end{aligned} \tag{P}$$

- $\mathcal{S}^n$: $n \times n$ symmetric matrices.
- $X \succeq 0 \iff X \in \mathcal{S}^n_+ \stackrel{\text{def}}{\iff} v^T X v \geq 0, \forall v \in \mathbb{R}^n$.
- $\langle X, Y \rangle := \mathrm{trace}(X^\top Y) = \sum_{i,j} X_{ij} Y_{ij}$
- $\|X\|_F := \sqrt{\mathrm{trace}(X^\top X)} = \sqrt{\sum_{i,j} X_{ij}^2}$

"Linear programming for the 21st century"

# Linear Algebra Review

Let $X \in \mathcal{S}^n$ and $v \in \mathbb{R}^n$.

- $X \succeq 0 \iff$ all the eigenvalues of $X$ are nonnegative
- $X \succeq 0 \iff$ there exists a $n \times n$ symmetric matrix $V$ such that $X = V^2$.
- $\langle X, vv^T \rangle = v^T X v$
- If $X \succeq 0$, then $\langle X, vv^T \rangle = 0 \iff Xv = 0$.

# SDP Example: Nearest correlation matrix problem

- Suppose we are given a $H \in \mathcal{S}^n$ with diagonal entries equal to 1.
- **Problem:** We want to find the correlation matrix that is the nearest possible to $H$.

$$\begin{aligned} \min_{X \in \mathcal{S}^n} \quad & \|X - H\|_F \\ \text{subject to} \quad & X_{ii} = 1, \quad i = 1, \ldots, n \\ & X \succeq 0 \end{aligned} \tag{Cor}$$

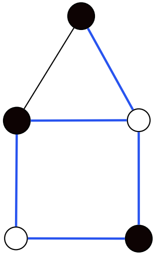
$\|\cdot\|_F$ is the Frobenius norm: $\|A\|_F = \sqrt{\mathrm{trace}(AA^\top)}$.

# SDP Example: Nearest correlation matrix problem (continued)

$$\begin{aligned} \min_{X\in\mathcal{S}^n, t\in\mathbb{R}} \quad & t \\ \text{subject to} \quad & \|X - H\|_F \leq t \\ & X_{ii} = 1, \quad i = 1, \ldots, n \\ & X \succeq 0 \end{aligned} \tag{Cor}$$

The constraint "$\|X - H\| \leq t$" can be written as a second order cone.

# MAX-CUT



**Goal**: Separate the vertices in two sets $S$, $S'$, such that the weight of the crossing edges is maximized. (**NP-Hard**)

- $a_{ij}$: weight of the edge between the $i$-th and $j$-th vertices.
- $x_i$: 1 if the $i$-th vertex is in $S$, $-1$ if in $S'$.

$$\begin{array}{lll} \max\limits_{x \in \mathbb{R}^n} & \displaystyle\sum_{i,j=1}^{n} \frac{a_{ij}}{4}(1 - x_i x_j) & \\ \text{subject to} & x_i^2 = 1, & i = 1, \ldots, n \end{array}$$

# The SDP relaxation - GW'95

- $X \in \mathcal{S}_+^n$ and $\operatorname{rank}(X) = 1 \Leftrightarrow X = xx^T$, for some $x \in \mathbb{R}^n$.
  - $X_{ij} = x_i x_j$ holds.

$$\begin{array}{lll} \max\limits_{x \in \mathcal{S}^n} & \sum\limits_{i,j=1}^{n} \frac{a_{ij}}{4}(1 - X_{ij}) & \\ \text{subject to} & X_{ii} = 1, & i = 1, \dots, n \\ & X \in \mathcal{S}_+^n, & \operatorname{rank}(X) = 1 \end{array}$$

**SDP relaxation**:

$$\begin{array}{lll} \max\limits_{x \in \mathcal{S}^n} & \sum\limits_{i,j=1}^{n} \frac{a_{ij}}{4}(1 - X_{ij}) & \\ \text{subject to} & X_{ii} = 1, & i = 1, \dots, n \\ & X \in \mathcal{S}_+^n, & \cancel{\operatorname{rank}(X) = 1} \end{array}$$

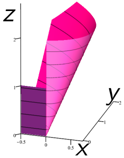
- Approximation ratio: $\frac{\text{MCUT}}{\text{SDP}} > 87\%$.
- Similar idea applies to many combinational optimization problems.

# Interlude - Some history

- **Symmetric Cone Programming**: LP + SOCP + SDP + $\alpha$.
  - SOCP and SDPs : researched intensively from the 90s on, partly because of the advent of interior point methods.
- **Non-symmetric cone optimization**: exponential cones, power cones, *p*-cones and many others.
  - More recent topic, with several new solvers developed in the past few years.

# The exponential cone

$$K_{\text{exp}} := \left\{ (x, y, z) \mid y > 0, z \geq y e^{x/y} \right\} \cup \{(x, y, z) \mid x \leq 0, z \geq 0, y = 0\}\,.$$




1. Applications to entropy optimization, logistic regression, geometric programming and etc..

V. Chandrasekaran, P. Shah
Relative entropy optimization and its applications.
*Math. Program. 161, 1–32 (2017)*

# A geometric programming example

1. B-san wants to give a box-like present to a friend.
2. However, B-san wants to wrap it using a *special wrapping paper* and B-san only has $1m^2$ of it.
3. Because B-san is pretentious, B-san wants the ratio between height of the box and its width to be in $[1.5, \phi]$, where $\phi$ is the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$
4. **Problem**: What is the biggest box (in volume) that can be wrapped with the special paper?

# A geometric programming example

$$\begin{aligned} \max_{w,h,d} \quad & whd \\ \text{subject to} \quad & 2(wh + wd + hd) \leq 1 \\ & 1.5 \leq \frac{h}{w} \leq \phi \\ & w > 0, h > 0, d > 0 \end{aligned}$$

**Not a convex problem** but if we make the substitutions $w = e^{\hat{w}}$, $d = e^{\hat{d}}$ and $h = e^{\hat{h}}$ we get

$$\begin{aligned} \max_{\hat{w},\hat{h},\hat{d}} \quad & e^{\hat{w}+\hat{h}+\hat{d}} \\ \text{subject to} \quad & (e^{\hat{w}+\hat{h}} + e^{\hat{w}+\hat{d}} + e^{\hat{h}+\hat{d}}) \leq 0.5 \\ & 1.5 \leq e^{\hat{h}-\hat{w}} \leq \phi \end{aligned}$$

# A geometric programming example

**Taking logs** linearizes the objective function and some of the constraints.

$$\begin{aligned} \max_{\hat{w},\hat{h},\hat{d}} \quad & \hat{w}+\hat{h}+\hat{d} \\ \text{subject to} \quad & e^{\hat{w}+\hat{h}} + e^{\hat{w}+\hat{d}} + e^{\hat{h}+\hat{d}} \le 0.5 \\ & \log(1.5) \le \hat{h}-\hat{w} \le \log(\phi) \end{aligned}$$

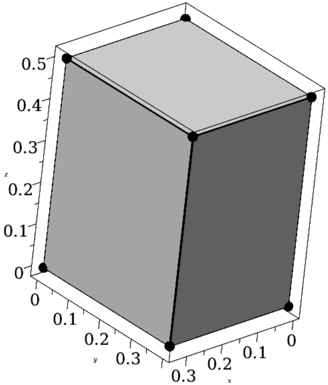
Noting that $e^x \le t$ holds if and only if $(x, 1, t) \in K_{\text{exp}}$, we have

$$\begin{aligned} \max_{\hat{w},\hat{h},\hat{d},t} \quad & \hat{w}+\hat{h}+\hat{d} \\ \text{subject to} \quad & t_1+t_2+t_3 \le 0.5 \\ & (\hat{w}+\hat{h}, 1, t_1) \in K_{\text{exp}}, (\hat{w}+\hat{d}, 1, t_2) \in K_{\text{exp}}, (\hat{h}+\hat{d}, 1, t_3) \in K_{\text{exp}} \\ & \log(1.5) \le \hat{h}-\hat{w} \le \log(\phi) \end{aligned}$$

Reminder:

$$K_{\text{exp}} := \left\{(x, y, z) \mid y > 0, z \ge ye^{x/y}\right\} \cup \{(x, y, z) \mid x \le 0, z \ge 0, y = 0\}.$$

# Solution



# Discrete distribution estimation

- We want to estimate a discrete distribution $p$ based on some prior information.
  - We might know some bounds on the moments
  - We might have some information on the $p_i$'s themselves.
- **Maximum entropy principle**: we try to find the "most random" $p$ that is consistent with the prior information $\mathcal{P}$.

$$
\begin{aligned}
\max_{p \in \mathbb{R}^n} \quad & \sum_{i=1}^{n} -p_i \ln p_i \\
\text{subject to} \quad & p \in \mathcal{P} \\
& \sum_{i=1}^{n} p_i = 1 \\
& p \in \mathbb{R}^n_+
\end{aligned}
$$

# Exponential cone formulation

Reminder:

$$K_{\exp} := \left\{(x,y,z) \mid y > 0, z \geq y e^{x/y}\right\} \cup \{(x,y,z) \mid x \leq 0, z \geq 0, y = 0\}\,.$$

$$\begin{aligned}
\max_{p,t\in\mathbb{R}^n} \quad & \sum_{i=1}^{n} t_i \\
\text{subject to} \quad & t_i \leq -p_i \ln p_i, \quad i = 1,\dots,n \\
& p \in \mathcal{P}, p \in \mathbb{R}^n_+ \\
& \sum_{i=1}^{n} p_i = 1
\end{aligned}$$

$$\begin{aligned}
\max_{p,t\in\mathbb{R}^n} \quad & \sum_{i=1}^{n} t_i \\
\text{subject to} \quad & (t_i, p_i, 1) \in K_{\exp}, \quad i = 1,\dots,n \\
& p \in \mathcal{P}, p \in \mathbb{R}^n_+ \\
& \sum_{i=1}^{n} p_i = 1
\end{aligned}$$

# Expressive power

# Convex optimization

**Convex optimization**:

$$\begin{aligned} \min_{x} \quad & f(x) \\ \text{subject to} \quad & x \in C, \end{aligned}$$

$C$ is a convex set and $f$ is a convex function.
**Conic linear programming** (CLP)

$$\begin{aligned} \min_{x} \quad & \langle c, x \rangle \\ \text{subject to} \quad & \mathcal{A}x = b \\ & x \in \mathcal{K} \end{aligned}$$

If we let $C := \{x \in \mathcal{K} \mid \mathcal{A}x = b\}$, then $C$ is convex.

- CLP is a particular case of convex optimization. However

# CLP $\cong$ Convex Optimization

$$\begin{aligned} \min_{x} \quad & f(x) \\ \text{subject to} \quad & x \in C \end{aligned} \qquad \begin{aligned} \min_{x,t} \quad & t \\ \text{subject to} \quad & x \in C \\ & f(x) \leq t \end{aligned}$$

Let $C_2 := \{(x, t) \mid x \in C, f(x) \leq t\}$ and let $\mathcal{K}$ be the convex cone in $\mathcal{E} \times \mathbb{R}^2$ generated by $C_2 \times \{1\}$. That is

$$\mathcal{K} := \{\alpha(x, t, 1) \mid \alpha \geq 0, (x, t) \in C_2\}.$$

$$\begin{aligned} \min_{x,t,\alpha} \quad & t \\ \text{subject to} \quad & \alpha = 1 \\ & (x, t, \alpha) \in \mathcal{K} \end{aligned}$$

- Every convex optimization problem has an equivalent CLP formulation!
- CLP philosophy: concentrate the hard part of the problem inside the cone.
- CVXPY works by converting a convex problem into an equivalent CLP and calling a CLP solver.

# More on expressive power

- Some researchers believe a few cones are enough to model the vast majority of convex applications.

  > *The following chapters present modeling with four types of convex cones: quadratic cones, power cones, exponential cone, semidefinite cone. It is "well-known" in the convex optimization community that this family of cones is sufficient to express almost all convex optimization problems appearing in practice.* [MOSEK Modelling cookbook, 2023]

- That said, a cone may be "too general" for a certain application ⇒ a more specific cone may be better.
- Some new solvers (alfonso, DDS, Hypatia, etc) support multiple cones
  - User can select the cone that best fit the application.

# More specific vs more general cones

$$\mathcal{Q}^{n+1} := \{(x_0, \bar{x}) \in \mathbb{R}^n \times \mid x_0 \geq \|\bar{x}\|_2\},$$

where $\|\bar{x}\|_2 = \sqrt{\bar{x}_1^2 + \cdots + \bar{x}_n^2}$

$$\mathcal{S}_+^n := \{X \in \mathcal{S}^n \mid v^T X v \geq 0, \forall v \in \mathbb{R}^n\}$$

$$(\bar{x}, x_0) \in \mathcal{Q}^{n+1} \Leftrightarrow \begin{pmatrix} x_0 & \bar{x}_1 & \cdots & \bar{x}_n \\ \bar{x}_1 & x_0 & 0 & \cdots \\ \vdots & & \ddots & \cdots \\ \bar{x}_n & 0 & \cdots & x_0 \end{pmatrix} \in \mathcal{S}_+^{n+1}$$

- Everything that can be expressed using $\mathcal{Q}^{n+1}$ can also be expressed using $\mathcal{S}_+^{n+1}$
- However, $\mathcal{S}_+^{n+1}$ requires $(n+1) \times (n+1)$ matrices, while $\mathcal{Q}^{n+1}$ is a cone in $\mathbb{R}^{n+1}$.

# https://docs.mosek.com/cheatsheets/conic.pdf

mosek

## Conic Modeling Cheatsheet

### Cones

Quadratic cone $\mathcal{Q}^n$

$$x_1 \geq \sqrt{x_2^2 + \cdots + x_n^2}$$

Rotated quadratic cone $\mathcal{Q}_r^n$

$$2x_1x_2 \geq x_3^2 + \cdots + x_n^2,\ x_1, x_2 \geq 0$$

Power cone $\mathcal{P}_3^{\alpha,1-\alpha}$, $\alpha \in (0,1)$

$$x_1^\alpha x_2^{1-\alpha} \geq |x_3|,\ x_1, x_2 \geq 0$$

Exponential cone $K_{\exp}$

$$x_1 \geq x_2 e^{x_3/x_2},\ x_2 \geq 0$$

### Simple bounds

| | |
|---|---|
| $t \geq x^2$ | $(0.5, t, x) \in \mathcal{Q}_r^3$ |
| $\lvert t\rvert \leq \sqrt{x}$ | $(0.5, x, t) \in \mathcal{Q}_r^3$ |
| $t \geq \lvert x\rvert$ | $(t, x) \in \mathcal{Q}^2$ |
| $t \geq 1/x,\ x > 0$ | $(x, t, \sqrt{2}) \in \mathcal{Q}_r^3$ |
| $t \geq \lvert x\rvert^p,\ p > 1$ | $(t, 1, x) \in \mathcal{P}_3^{1/p,1-1/p}$ |
| $t \geq 1/x^p,\ x > 0,\ p > 0$ | $(t, x, 1) \in \mathcal{P}_3^{1/(1+p),p/(1+p)}$ |
| $\lvert t\rvert \leq x^p,\ x > 0,\ p \in (0,1)$ | $(x, 1, t) \in \mathcal{P}_3^{p,1-p}$ |
| $t \geq \lvert x\rvert^p/y^{p-1},\ y \geq 0$<br>$p > 1$ | $(t, y, x) \in \mathcal{P}_3^{1/p,1-1/p}$ |
| $t \geq x^T x/y,\ y \geq 0$ | $(0.5t, y, x) \in \mathcal{Q}_r^{n+2}$ |
| $t \geq e^x$ | $(t, 1, x) \in K_{\exp}$ |
| $t \leq \log x$ | $(x, 1, t) \in K_{\exp}$ |
| $t \geq 1/\log x,\ x > 1$ | $(u, t, \sqrt{2}) \in \mathcal{Q}_r^3$<br>$(x, 1, u) \in K_{\exp}$ |
| $t \geq a_1^{x_1} \cdots a_n^{x_n},\ a_i > 0$ | $(t, 1, \sum x_i \log a_i) \in K_{\exp}$ |
| $t \geq xe^x,\ x \geq 0$ | $(t, x, u) \in K_{\exp}$<br>$(0.5, u, x) \in \mathcal{Q}_r^3$ |
| $t \geq \log(1 + e^x)$ | $u + v \leq 1$<br>$(u, 1, x - t) \in K_{\exp}$<br>$(v, 1, -t) \in K_{\exp}$ |
| $t \geq \lvert x\rvert^{3/2}$ | $(t, 1, x) \in \mathcal{P}_3^{2/3,1/3}$ |
| $t \geq x^{3/2},\ x \geq 0$ | $(s, t, x), (x, 1/8, s) \in \mathcal{Q}_r^3$ |
| $t \geq 1/x^3,\ x > 0$ | $(t, x, 1) \in \mathcal{P}_3^{3/4,1/4}$ |
| $0 \leq t \leq x^{2/5},\ x \geq 0$ | $(x, 1, t) \in \mathcal{P}_3^{2/5,3/5},\ t \geq 0$ |

### Means and averaging

| | |
|---|---|
| Log-sum-exp<br>$t \geq \log(\sum e^{x_i})$ | $(z_i, 1, x_i - t) \in K_{\exp}$<br>$i = 1, \ldots, n$<br>$\sum z_i \leq 1$ |
| Harmonic mean<br>$0 \leq t \leq n(\sum x_i^{-1})^{-1}$<br>$x_i > 0$ | $(z_i, x_i, t) \in \mathcal{Q}_r^3$<br>$i = 1, \ldots, n$<br>$\sum z_i = nt/2$ |
| Geometric mean<br>$\lvert t\rvert \leq (x_1 \cdots x_n)^{1/n}$<br>$x_i > 0$ | $(z_i, x_i, z_{i+1}) \in \mathcal{P}_3^{1-1/i,1/i}$<br>$i = 2, \ldots, n$<br>$z_2 = x_1,\ z_{n+1} = t$ |
| $\lvert t\rvert \leq \sqrt{xy},\ x, y > 0$ | $(x, y, \sqrt{2}t) \in \mathcal{Q}_r^3$ |
| Weighted geom. mean<br>$\lvert t\rvert \leq x_1^{\alpha_1} \cdots x_n^{\alpha_n},\ x_i > 0$<br>$\alpha_i > 0, \sum \alpha_i = 1$ | $(z_i, x_i, z_{i+1}) \in \mathcal{P}_3^{1-\beta_i,\beta_i}$<br>$\beta_i = \alpha_i/(\alpha_1 + \cdots + \alpha_i)$<br>$i = 2, \ldots, n$<br>$z_2 = x_1,\ z_{n+1} = t$ |
| $\lvert t\rvert \leq x^{1/4}y^{5/12}z^{1/3}$<br>$x, y, z \geq 0$ | $(s, z, t) \in \mathcal{P}_3^{2/3,1/3}$<br>$(x, y, s) \in \mathcal{P}_3^{3/8,5/8}$ |

### Entropy

| | |
|---|---|
| $t \leq -x \log x$ | $(1, x, t) \in K_{\exp}$ |
| $t \geq x \log(x/y)$ | $(y, x, -t) \in K_{\exp}$ |
| $t \geq \log(1 + 1/x)$<br>$x > 0$ | $(x + 1, u, \sqrt{2}) \in \mathcal{Q}_r^3$<br>$(1 - u, 1, -t) \in K_{\exp}$ |
| $t \leq \log(1 - 1/x)$<br>$x > 1$ | $(x, u, \sqrt{2}) \in \mathcal{Q}_r^3$<br>$(1 - u, 1, t) \in K_{\exp}$ |
| $t \geq x \log(1 + x/y)$<br>$x, y > 0$ | $(y, x + y, u) \in K_{\exp}$<br>$(x + y, y, v) \in K_{\exp}$<br>$t + u + v = 0$ |

### Convex quadratic problems

Let $\Sigma \in \mathbb{R}^{n \times n}$, symmetric, p.s.d.
Find $\Sigma = LL^T$, $L \in \mathbb{R}^{n \times k}$ (Cholesky factor).
Then $x^T \Sigma x = \|L^T x\|_2^2$.

| | |
|---|---|
| $t \geq \frac{1}{2} x^T \Sigma x$ | $(1, t, L^T x) \in \mathcal{Q}_r^{k+2}$ |
| $t \geq \sqrt{x^T \Sigma x}$ | $(t, L^T x) \in \mathcal{Q}^{k+1}$ |
| $\frac{1}{2} x^T \Sigma x + p^T x + q \leq 0$ | $(1, -p^T x - q, L^T x) \in \mathcal{Q}_r^{k+2}$ |
| $\max_x c^T x - \frac{1}{2} x^T \Sigma x$ | $\max c^T x - r$<br>$(1, r, L^T x) \in \mathcal{Q}_r^{k+2}$ |
| $c^T x + d \geq \|Ax + b\|_2$ | $(c^T x + d, Ax + b) \in \mathcal{Q}^{m+1}$ |

### Norms, $x \in \mathbb{R}^n$

| | |
|---|---|
| $\|\cdot\|_1,\ t \geq \sum \lvert x_i\rvert$ | $(z_i, x_i) \in \mathcal{Q}^2,\ t = \sum z_i$ |
| $\|\cdot\|_2,\ t \geq (\sum x_i^2)^{1/2}$ | $(t, x) \in \mathcal{Q}^{n+1}$ |
| $\|\cdot\|_p,\ p > 1$<br>$t \geq (\sum \lvert x_i\rvert^p)^{1/p}$ | $(z_i, t, x_i) \in \mathcal{P}_3^{1/p,1-1/p}$<br>$i = 1, \ldots, n$<br>$\sum z_i = t$ |

### Geometry

| | |
|---|---|
| Bounding ball<br>$\min_x \max_i \|x - x_i\|_2$ | $\min r$<br>$(r, x - x_i) \in \mathcal{Q}^{n+1}$ |
| Geometric median<br>$\min_x \sum \|x - x_i\|_2$ | $\min \sum t_i$<br>$(t_i, x - x_i) \in \mathcal{Q}^{n+1}$ |
| Analytic center<br>$\max_x \sum \log(b_i - a_i^T x)$ | $\max \sum t_i$<br>$(b_i - a_i^T x, 1, t_i) \in K_{\exp}$ |

### Regression and fitting

| | |
|---|---|
| Regularized least squares<br>$\min_w \|Xw - y\|_2^2 + \lambda \|w\|_2^2$ | $\min t + \lambda r$<br>$(0.5, t, Xw - y) \in \mathcal{Q}_r^{m+2}$<br>$(0.5, r, w) \in \mathcal{Q}_r^{n+2}$ |
| Max likelihood<br>$\max_p p_1^{a_1} \cdots p_n^{a_n}$ | $\max \sum a_i t_i$<br>$(p_i, 1, t_i) \in K_{\exp}$ |
| Logistic cost function<br>$t \geq -\log(1/(1 + e^{-\theta^T x}))$ | $u + v \leq 1$<br>$(u, 1, -\theta^T x - t) \in K_{\exp}$<br>$(v, 1, -t) \in K_{\exp}$ |

### Risk-return

$\Sigma \in \mathbb{R}^{n \times n}$ – covariance, $\Sigma = LL^T,\ L \in \mathbb{R}^{n \times k}$

| | |
|---|---|
| $\max_x \alpha^T x$<br>s.t. $x^T \Sigma x \leq \gamma$ | $\max_x \alpha^T x$<br>$(\sqrt{\gamma}, L^T x) \in \mathcal{Q}^{k+1}$ |
| $\max_x \alpha^T x - \delta x^T \Sigma x$ | $\max_x \alpha^T x - \delta r$<br>$(0.5, r, L^T x) \in \mathcal{Q}_r^{k+2}$ |
| Risk plus $x^{1.5}$ impact cost<br>$t \geq \delta x^T \Sigma x + \beta \sum \lvert x_i\rvert^{3/2}$ | $t \geq \delta r + \beta \sum u_i$<br>$(0.5, r, L^T x) \in \mathcal{Q}_r^{k+2}$<br>$(u_i, 1, x_i) \in \mathcal{P}_3^{2/3,1/3}$ |
| Risk in factor model<br>$\gamma \geq x^T(D + FSF^T)x$<br>$D$ – specific risk (diag.)<br>$F \in \mathbb{R}^{n \times k}$ – factor loads<br>$S = UU^T$ – factor cov. | $\gamma \geq t + s$<br>$(0.5, t, \sqrt{D}x) \in \mathcal{Q}_r^{n+2}$<br>$(0.5, s, U^T F^T x) \in \mathcal{Q}_r^{k+2}$ |

More: https://docs.mosek.com/modeling-cookbook/index.html

# Part 2 - Duality and facial reduction

# More convex analysis

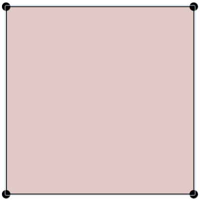
# Topological Interior

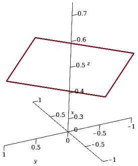
- $\mathcal{E}$: Euclidean space (i.e., $\mathbb{R}^n$) with inner product $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$
- $B(x,r)$ is the open ball centered in $x$ with radius $r$, i.e., $B(x,r) = \{y \mid \|y-x\| < r\}$.

Let $C \subseteq \mathcal{E}$

**Interior**

$\operatorname{int} C := \{x \in C \mid \exists r > 0, s.t., B(x,r) \subseteq C\}$.





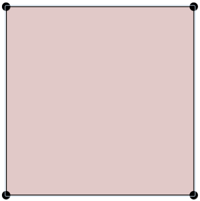
# Relative interior

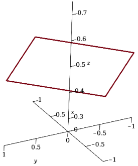
**Definition (Relative interior)**

$x$ is a relative interior point of $C$ (i.e., $x \in \operatorname{ri} C$) if for every $y \in C$, the line segment connecting $x$ and $y$ can be extended past $x$ while staying inside $C$.

$$x \in \operatorname{ri} C \overset{\text{def}}{\iff} \forall y \in C, \exists \mu > 1, \text{ s.t. } \mu x + (1-\mu) y \in C$$

- $\operatorname{ri} C = C \overset{\text{def}}{\iff} C$ is **relatively open**.





# Closure

$\mathcal{E}$: finite dimensional Euclidean space
$C \subseteq \mathcal{E}$: convex set

**Definition (Closure)**

The closure $\operatorname{cl} C$ of $C$ is the set of limit points of $C$ $\Leftrightarrow$ smallest closed set containing $C$.

- $\operatorname{cl} C = C \overset{\text{def}}{\iff} C$ is **closed**

# Properties of closures and relative interiors

$\mathcal{E}$: finite dimensional Euclidean space
$C \subseteq \mathcal{E}$: convex set

- $\mathrm{ri}\, C$ and $\mathrm{cl}\, C$ are convex.
- $\mathrm{ri}\, C \neq \emptyset$ if $C \neq \emptyset$.
- $\mathrm{ri}\,(\mathrm{cl}\, C) = \mathrm{ri}\, C$
- $\mathrm{ri}\,\mathrm{ri}\,(C) = \mathrm{ri}\, C$ "relative interiors are relatively open"
- $\mathrm{cl}\,(\mathrm{cl}\, C) = \mathrm{cl}\, C$ "closures are closed"

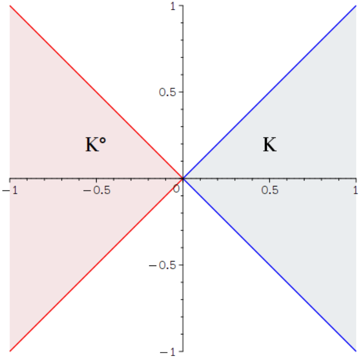
**Examples**

- $\mathrm{ri}\, \mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_i > 0, \forall i\}$
- $\mathrm{ri}\, \mathcal{S}^n_+ =$ symmetric positive definite matrices.

# Polars and duals of cones

$\mathcal{K} \subseteq \mathcal{E}$: convex cone.

$$\mathcal{K}^\circ = \{y \in \mathcal{E} \mid \langle x, y \rangle \leq 0, \forall x \in \mathcal{K}\}.$$



# Polars of cones - Examples and Properties

$\mathcal{K} \subseteq \mathcal{E}$: convex cone.

- **Bipolar Theorem**: $\mathcal{K}^{\circ\circ} = \mathrm{cl}\,(\mathcal{K})$.
- $(\mathbb{R}^n_+)^\circ = -\mathbb{R}^n_+$
- $(\mathcal{S}^n_+)^\circ = -\mathcal{S}^n_+$.
- $(\mathcal{Q}^n_p)^\circ = -\mathcal{Q}^n_q$, where $1/p + 1/q = 1$, $p \in (1, \infty)$,
  $\mathcal{Q}^n_p := \{(x_0, \bar{x}) \mid \|\bar{x}\|_p \leq x_0\}$.

**Dual cone**

$$\mathcal{K}^* := -\mathcal{K}^\circ = \{y \in \mathcal{E} \mid \langle x, y \rangle \geq 0, \forall x \in \mathcal{K}\}$$

- **Bipolar Theorem**: $\mathcal{K}^{**} = \mathrm{cl}\,(\mathcal{K})$.
- $(\mathbb{R}^n_+)^* = \mathbb{R}^n_+$.
- $(\mathcal{S}^n_+)^* = \mathcal{S}^n_+$.
- $(\mathcal{Q}^n_p)^* = \mathcal{Q}^n_q$, where $1/p + 1/q = 1$, $p \in (1, \infty)$,
  $\mathcal{Q}^n_p := \{(x_0, \bar{x}) \mid \|\bar{x}\|_p \leq x_0\}$.

# Recall our basic conic linear program

$$\begin{aligned} \cancel{\min_x}\ \inf \quad & \langle c, x \rangle \\ \text{subject to} \quad & \mathcal{A}x = b \\ & x \in \mathcal{K} \end{aligned} \tag{P}$$

Suppose we wish to **relax the linear constraints**:

$$\begin{aligned} \mathcal{L}(y) &:= \inf_{x \in \mathcal{K}} \left[ \langle c, x \rangle + \langle y, b - \mathcal{A}x \rangle) \right] \\ &= \inf_{x \in \mathcal{K}} \left[ \langle c - \mathcal{A}^* y, x \rangle + \langle b, y \rangle \right] \\ &= \langle b, y \rangle + \inf_{x \in \mathcal{K}} \langle c - \mathcal{A}^* y, x \rangle \\ &= \begin{cases} \langle b, y \rangle & \text{if } c - \mathcal{A}^* y \in \mathcal{K}^* \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

Denote the optimal value of (P) by $\theta_P$. Then:

$$\theta_P \geq \mathcal{L}(y), \qquad \forall y$$

# Relaxing the CLP

$$\begin{aligned} \inf_x \quad & \langle c, x \rangle \\ \text{subject to} \quad & \mathcal{A}x = b \\ & x \in \mathcal{K} \end{aligned} \tag{P}$$

We have

$$\mathcal{L}(y) = \begin{cases} \langle b, y \rangle & \text{if } c - \mathcal{A}^* y \in \mathcal{K}^* \\ -\infty & \text{otherwise} \end{cases}$$

and

$$\theta_P \geq \mathcal{L}(y), \qquad \forall y$$

Which leads to

$$\theta_P \geq \sup_y \mathcal{L}(y)$$

- The dual problem is the task of finding the $y$ that provides the **tightest (Lagrangian) relaxation** to (P)

# Primal dual conic linear program (CLP)

$$\begin{aligned} \inf_x \quad & \langle c, x \rangle \qquad \text{(P)} \\ \text{subject to} \quad & \mathcal{A}x = b \\ & x \in \mathcal{K} \end{aligned}$$

$$\begin{aligned} \sup_y \quad & \langle b, y \rangle \qquad \text{(D)} \\ \text{subject to} \quad & c - \mathcal{A}^* y \in \mathcal{K}^*. \end{aligned}$$

- $\mathcal{K}^* := \{s \in \mathcal{E} \mid \langle s, x \rangle \geq 0, \forall x \in \mathcal{K}\}$. (dual cone)
- We denote the primal and dual optimal values by $\theta_P$ and $\theta_D$.

**Proposition (Weak duality)**

$$\theta_P \geq \theta_D$$

# Example - Eigenvalues via SDP duality

Suppose that $C \in \mathcal{S}^n$ is a fixed matrix and consider the SDP:

$$\begin{aligned} \sup_{y \in \mathbb{R}} \quad & y \\ \text{s.t.} \quad & C - yI_n \succeq 0, \end{aligned} \tag{D}$$

where $I_n$ is the $n \times n$ identity matrix. Then $\theta_D = \lambda_{\min}(C)$, where $\lambda_{\min}(C)$ is the minimum eigenvalue of $C$. The primal is:

$$\begin{aligned} \inf_{X \in \mathcal{S}^n} \quad & \langle C, X \rangle \\ \text{s.t.} \quad & \langle I_n, X \rangle = \mathrm{trace}(X) = 1 \\ & X \succeq 0 \end{aligned} \tag{P}$$

If $v \in \mathbb{R}^n$ is an eigenvector of $C$ associated to $\lambda_{\min}(C)$ with $\|v\| = 1$, then $X^* := vv^\top$ is optimal to (P).

$$\theta_P = \theta_D = \lambda_{\min}(C).$$

# Strong duality in CLP

**Theorem (Strong duality Theorem - Primal version)**

*Suppose that*

- (P) *has a relative interior feasible solution, i.e., there exists $x$ such that $\mathcal{A}x = b$ and $x \in \operatorname{ri} \mathcal{K}$ (Primal Slater Condition)*

*Then:*

- $\theta_P = \theta_D$.
- (D) *has optimal solutions if $\theta_P$ is finite.*

**Theorem (Strong duality Theorem - Dual version)**

*Suppose that*

- (D) *has a relative interior feasible solution, i.e., there exists $y$ such that $c - \mathcal{A}^* y \in \operatorname{ri} \mathcal{K}^*$. (Dual Slater Condition)*

*Then:*

- $\theta_P = \theta_D$.
- (P) *has optimal solutions if $\theta_D$ is finite.*

# Optimality conditions

$$\begin{array}{rll} \inf_x & \langle c, x\rangle & \text{(P)} \\ \text{subject to} & \mathcal{A}x = b & \\ & x \in \mathcal{K} & \end{array} \qquad \begin{array}{rll} \sup_y & \langle b, y\rangle & \text{(D)} \\ \text{subject to} & c - \mathcal{A}^* y \in \mathcal{K}^*. & \end{array}$$

A **sufficient condition** for $(x^*, y^*)$ to be optimal is that the following are satisfied:

- Primal feasibility: $\mathcal{A}x^* = b$, $x^* \in \mathcal{K}$
- Dual feasibility: $s^* \in \mathcal{K}^*$, where $s^* := c - \mathcal{A}^* y^*$
- **Complementary slackness** (i.e., zero duality gap[1]): $\langle s^*, x^*\rangle = 0$.

If the primal and dual Slater conditions hold, the conditions above are **necessary** too.

[1] Note that $\langle s^*, x^*\rangle = \langle c, x^*\rangle - \langle b, y^*\rangle$

# Ex1 - MAXCUT-SDP

$$\begin{array}{lll} \inf\limits_{x \in \mathcal{S}^n} & \langle A, X \rangle & \text{(P)} \\ \text{s.t} & X_{ii} = 1, \quad i = 1, \ldots, n & \\ & X \in \mathcal{S}^n_+ & \end{array} \qquad \begin{array}{lll} \sup\limits_{y \in \mathbb{R}^n} & y_1 + \cdots + y_n & \text{(D)} \\ \text{s.t.} & A - \sum_{i=1}^n E_i y_i \in \mathcal{S}^n_+, & \end{array}$$

where $E_i$ is the matrix that has 1 in the $(i, i)$-entry and zero elsewhere.

- Both primal and dual Slater conditions are satisfied $\Rightarrow \quad \theta_P = \theta_D$ and both problems are attained.

# Ex2 - Dual Slater Condition holds, but no dual optimal solution

$$\begin{array}{rl} \sup_{t,s} & -s \\ \text{s.t.} & \begin{pmatrix} t & 1 \\ 1 & s \end{pmatrix} \succeq 0 \end{array} \qquad \text{(D)}$$

$$\begin{array}{rl} \inf_{X \in \mathcal{S}^2} & 2X_{12} \\ \text{s.t.} & -X_{11} = 0 \\ & -X_{22} = -1 \\ & X \succeq 0. \end{array} \qquad \text{(P)}$$

- The dual satisfies Slater condition, $\theta_D$ is finite but no dual optimal solutions exists. $\theta_D$ **is unattained**.
- The primal does not satisfy Slater conditions, but has an optimal solution.
- $\theta_P = \theta_D$ holds.

# Some clarification

Keep in mind the following:

$\inf(0,1) = 0$, but $0 \notin (0,1)$. "The infimum is finite but an optimal solution does not exist".

**Primal side**

- $\theta_P$ is **finite** $\Leftrightarrow$ $\theta_P$ is a **real number**.
- $\theta_P$ is **attained** $\Leftrightarrow$ there is a feasible $x^*$ such that $\theta_P = \langle c, x^* \rangle$.
- $\theta_P = -\infty$ ((P) is **unbounded**) $\Leftrightarrow$ there is a sequence $\{x^k\}$ of feasible solutions such that $\lim_{k\to\infty} \langle c, x^k \rangle \to -\infty$
- By convention $\theta_P = +\infty$ iff (P) is infeasible

**Dual side**

- $\theta_D$ is **finite** $\Leftrightarrow$ $\theta_D$ is a **real number**.
- $\theta_D$ is **attained** $\Leftrightarrow$ there is a feasible $y^*$ such that $\theta_D = \langle b, y^* \rangle$.
- $\theta_D = -\infty$ ((D) is **unbounded**) $\Leftrightarrow$ there is a sequence $\{y^k\}$ of feasible solutions such that $\lim_{k\to\infty} \langle b, y^k \rangle \to -\infty$
- By convention $\theta_D = +\infty$ iff (D) is infeasible

# Ex3 - Positive gap SDP

$$\sup_{t,s} \quad -s \qquad \text{(D)}$$
$$\text{s.t.} \quad \begin{pmatrix} t & 1 & s-1 \\ 1 & s & 0 \\ s-1 & 0 & 0 \end{pmatrix} \succeq 0$$

$$\inf_{X \in \mathcal{S}^3} \quad 2X_{12} - 2X_{13} \qquad \text{(P)}$$
$$\text{s.t.} \quad X_{11} = 0$$
$$-X_{22} - 2X_{13} = -1$$
$$X \succeq 0.$$

# Positive gap SDP

$$\sup_{t,s} \; -s \to -1 \qquad \text{(D)}$$
$$\text{s.t.} \quad \begin{pmatrix} t & 1 & s-1 \\ 1 & s & 0 \\ s-1 & 0 & 0 \end{pmatrix} \succeq 0$$

$$\inf_{X \in \mathcal{S}^3} \; 2X_{12} \to 0 \; - 2X_{13} \to 0 \qquad \text{(P)}$$
$$\text{s.t.} \quad X_{11} = 0$$
$$-X_{22} - 2X_{13} = -1$$
$$X \succeq 0.$$

$\theta_D = -1$ and $\theta_P = 0$. **Neither the primal nor the dual satisfy Slater**

- Ok, so what? How bad can this be?

To correct this we substitute $\mathcal{S}^3_+$ for

$$\mathcal{S}^2_+ \oplus 0 = \left\{ \begin{pmatrix} a & b & 0 \\ b & c & 0 \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in \mathcal{S}^2_+ \right\}.$$

# Example

$$\sup_{t,s} \quad \cancel{-s} \to -1 \qquad \text{(D')}$$

$$\text{s.t.} \quad \begin{pmatrix} t & 1 & s-1 \\ 1 & s & 0 \\ s-1 & 0 & 0 \end{pmatrix} \in \mathcal{S}^2_+ \oplus 0$$

Still, $\theta_{D'} = -1$. Let's take a look at the primal problem...

$$(\mathcal{S}^2_+ \oplus 0)^* = \left\{ \begin{pmatrix} a & b & * \\ b & c & * \\ * & * & * \end{pmatrix} \middle| \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in \mathcal{S}^2_+ \right\}.$$

$$\inf_{x} \quad \cancel{2x_{12}} \to 0 \; \cancel{-2x_{13}} \to -1 \qquad \text{(P')}$$

$$\text{s.t.} \quad x_{11} = 0$$

$$-x_{22} - 2x_{13} = -1$$

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in \mathcal{S}^2_+.$$

# What happened? 

- $\mathcal{S}^3_+ \Rightarrow \mathcal{S}^2_+ \oplus 0$: The feasible region of (D) stays the same $\Rightarrow$ $\theta_D = \theta_{D'} = -1$.
- $\mathcal{S}^3_+ \Rightarrow (\mathcal{S}^2_+ \oplus 0)^*$: The feasible region of (P) expands $\Rightarrow$ $-1 = \theta_{P'} \leq \theta_P = 0$.

$\mathcal{S}^2_+ \oplus 0$ a *face* of $\mathcal{S}^3_+$ with two key properties:

- it contains the feasible region of (D)
- Slater's condition is satisfied at (D').

This is an example of *Facial Reduction*

# Even more convex analysis

# Separating hyperplanes

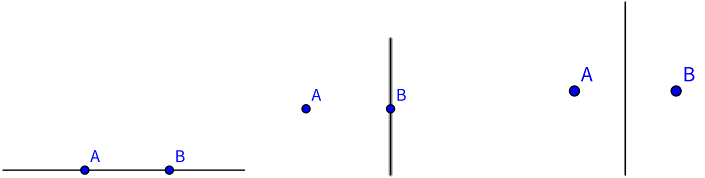
$H = \{x \in \mathcal{E} \mid \langle x, y \rangle = \theta\}$: hyperplane ($x \neq 0$)
$C_1, C_2$: convex sets
Define the closed half-spaces

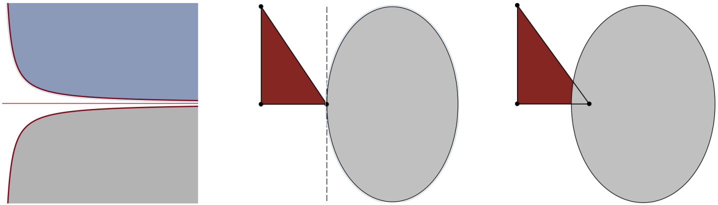
$$H^+ := \{x \in \mathcal{E} \mid \langle x, y \rangle \geq \theta\}, \qquad H^- := \{x \in \mathcal{E} \mid \langle x, y \rangle \leq \theta\}$$

$B_\epsilon$: unit ball of radius $\epsilon$

- $C_1$ and $C_2$ are **separated** by $H \overset{\text{def}}{\Longleftrightarrow}$ $C_1$ and $C_2$ belong to different closed half-spaces defined by $H$.
- $C_1$ and $C_2$ are **properly separated** by $H \overset{\text{def}}{\Longleftrightarrow}$ $C_1$ and $C_2$ belong to different closed half-spaces and **at least one of them is not contained in** $H$.
- $C_1$ and $C_2$ are **strongly separated** by $H \overset{\text{def}}{\Longleftrightarrow}$ $\exists \epsilon > 0$ such that $C_1 + B_\epsilon$ and $C_2 + B_\epsilon$ belong to different **open** half-spaces defined by $H$.

# Separating hyperplanes - Examples



# Some results

$C_1, C_2 \subseteq \mathcal{E}$: nonempty closed convex sets.

1. $C_1$ and $C_2$ can be **strongly separated** $\Leftrightarrow$ $\mathrm{dist}\,(C_1, C_2) = \inf_{x,y} \|x - y\| > 0 \Leftrightarrow 0 \notin \mathrm{cl}\,(C_1 - C_2)$
2. $C_1$ and $C_2$ can be **properly separated** $\Leftrightarrow (\mathrm{ri}\, C_1) \cap (\mathrm{ri}\, C_2) = \emptyset$.

# Faces of convex sets

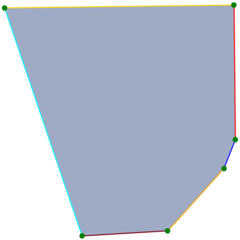
**Definition (Face)**

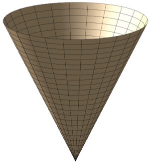
Let $C, \mathcal{F}$ be convex sets such that $\mathcal{F} \subseteq C$. $\mathcal{F}$ is a face of $C$ $\overset{\text{def}}{\Longleftrightarrow}$ for every $\alpha \in (0,1)$ and every $x, y \in C$

$$\alpha x + (1-\alpha) y \in \mathcal{F} \Rightarrow x, y \in \mathcal{F}$$

We write $\mathcal{F} \trianglelefteq C$.

- A face that is a singleton $\{x\}$ is called an **extreme point**
- A face $\mathcal{F}$ of dimension 1 of a cone $\mathcal{K}$ is called an **extreme ray**.



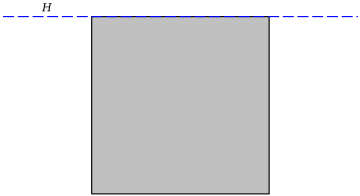


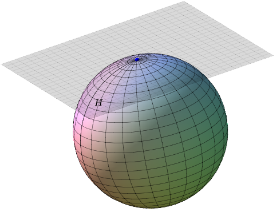
# Supporting hyperplanes

$H = \{x \in \mathcal{E} \mid \langle c, x \rangle = \theta\}$: hyperplane
$C \subseteq \mathcal{E}$: convex set
$H^+ := \{x \in \mathcal{E} \mid \langle c, x \rangle \geq \theta\}, \qquad H^- := \{x \in \mathcal{E} \mid \langle c, x \rangle \leq \theta\}$

$H$ is a supporting hyperplane of $C \overset{\text{def}}{\iff} H \cap C \neq \emptyset$ and $C$ is contained in one of the closed half-spaces defined by $H$.

# Examples of supporting hyperplanes



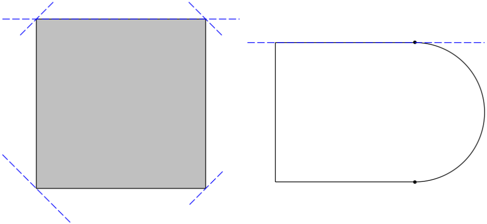



# Exposed faces

$\mathcal{F} \trianglelefteq C$ is **exposed** $\overset{\text{def}}{\iff}$ $\mathcal{F} = C \cap H$ holds for some supporting hyperplane $H$ of $C$

If all nonempty faces of $C$ are exposed we say that $C$ is facially exposed.

1. If $\mathcal{K}$ is a cone, $\mathcal{F} \trianglelefteq \mathcal{K}$ is exposed iff $\mathcal{F} = \mathcal{K} \cap \{s\}^\perp$ for some some $s \in \mathcal{K}^*$.



# Faces of $\mathcal{S}^n_+$

**Theorem**

*Let $\emptyset \neq \mathcal{F} \trianglelefteq \mathcal{S}^n_+$. Then, there exists a $n \times n$ orthogonal matrix $Q$ such that*

$$Q^\top \mathcal{F} Q = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \mid A \in \mathcal{S}^r_+ \right\}$$

Every nonempty face of $\mathcal{S}^n_+$ is exposed and is linearly isomorphic to a $\mathcal{S}^s_+$ for $s \leq n$.

# Minimal Face

$C, K \subseteq \mathcal{E}$: convex sets

**Definition (Minimal Face)**

Suppose $C \subseteq K$. The minimal face of $C$ with respect to $K$, is the *smallest* face of $K$ containing $C$. We write $\mathcal{F}_{\min}(C, K)$.

$$\mathcal{F}_{\min}(C, K) = \bigcap_{\substack{\mathcal{F} \trianglelefteq K \\ C \subseteq \mathcal{F}}} \mathcal{F}$$

**Key property**

Let $\emptyset \neq C \subseteq K$.

$$\mathcal{F}_{\min}(C, K) = \mathcal{F} \iff C \subseteq \mathcal{F} \text{ and } C \cap \operatorname{ri} \mathcal{F} \neq \emptyset.$$

# Facial Reduction - The basic idea

$$\sup_{y\in\mathbb{R}^m} \quad \langle b, y\rangle \qquad \text{(D)}$$
$$\text{s.t.} \quad c - \mathcal{A}^* y \in \mathcal{K}.$$

- Let $\mathcal{F}_{\text{D}} = \{c - \mathcal{A}^* y \mid c - \mathcal{A}^* y \in \mathcal{K}\} = (c + \operatorname{range} \mathcal{A}^*) \cap \mathcal{K}$, this are the feasible slacks of (D).
- We define the minimal face of (D) as $\mathcal{F}^D_{\text{min}} = \mathcal{F}_{\text{min}}(\mathcal{F}_{\text{D}}, \mathcal{K})$.
- Note: $\mathcal{F}^D_{\text{min}} = \mathcal{K} \iff$ (D) satisfies Slater's condition.

$$\inf_x \quad \langle c, x\rangle \qquad (\hat{\text{P}})$$
$$\text{subject to} \quad \mathcal{A}x = b$$
$$x \in (\mathcal{F}^D_{\text{min}})^*$$

$$\sup_y \quad \langle b, y\rangle \qquad (\hat{\text{D}})$$
$$\text{subject to} \quad c - \mathcal{A}^* y \in \mathcal{F}^D_{\text{min}}.$$

- Now, $(\hat{\text{D}})$ satisfies Slater's condition.

# Facial Reduction - Example

$$\begin{array}{ll} \sup\limits_{t,s} & -s \\ \text{s.t.} & \begin{pmatrix} t & 1 & s-1 \\ 1 & s & 0 \\ s-1 & 0 & 0 \end{pmatrix} \succeq 0 \end{array} \qquad \text{(D)}$$

$$\mathcal{F}_{\mathrm{D}} = \left\{ \begin{pmatrix} t & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid \begin{pmatrix} t & 1 \\ 1 & 1 \end{pmatrix} \succeq 0 \right\}$$

$$\mathcal{F}^D_{\min} = \left\{ \begin{pmatrix} a & b & 0 \\ b & c & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid \begin{pmatrix} a & b \\ b & c \end{pmatrix} \succeq 0 \right\} = \mathcal{S}^2_+ \oplus 0.$$

# Facial Reduction - Continued

$$\begin{array}{ll} \sup\limits_{t,s} & -s \\ \text{s.t.} & \begin{pmatrix} t & 1 & s-1 \\ 1 & s & 0 \\ s-1 & 0 & 0 \end{pmatrix} \succeq 0 \end{array} \qquad \text{(D)}$$

$$\mathcal{F}^D_{\min} = \left\{ \begin{pmatrix} a & b & 0 \\ b & c & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid \begin{pmatrix} a & b \\ b & c \end{pmatrix} \succeq 0 \right\} = \mathcal{S}^2_+ \oplus 0.$$

$$(\mathcal{F}^D_{\min})^* = \left\{ \begin{pmatrix} a & b & * \\ b & c & * \\ * & * & * \end{pmatrix} \mid \begin{pmatrix} a & b \\ b & c \end{pmatrix} \succeq 0 \right\} = (\mathcal{S}^2_+ \oplus 0)^*.$$

# The Facial Reduction Algorithm

- So... How do we compute $\mathcal{F}^D_{\min}$ in practice?

**Answer**: separating hyperplane theorem.

$\mathcal{V}$: Polyhedral set and $K$ a convex set

- $\mathcal{V} \cap (\operatorname{ri} K) = \emptyset \Leftrightarrow \mathcal{V}$ and $K$ can be properly separated by $H$ in such a way that $H$ does not contain $K$.

# The Facial Reduction Algorithm

$$\sup_{y \in \mathbb{R}^m} \quad \sum_{i=1}^m b_i y_i \qquad \text{(D)}$$
$$\text{s.t.} \quad C - \sum_{i=1}^m y_i A_i \succeq 0.$$

- Let $\mathcal{V} = \{C - \sum_{i=1}^m y_i A_i \mid y \in \mathbb{R}^m\}$ and $\mathcal{K} = \mathcal{S}_+^n$.
- Slater's condition is **not** satisfied $\iff \mathcal{V} \cap (\operatorname{ri} \mathcal{K}) = \emptyset$.

There exists $0 \neq X \in \mathcal{S}^n$ and $\alpha \in \mathbb{R}$ such that

$$\langle X, C - \sum_{i=1}^m y_i A_i \rangle \leq \alpha \leq \langle X, Z \rangle, \quad \forall y \in \mathbb{R}^m, \forall Z \in \mathcal{S}_+^n.$$

Therefore

- $\alpha \leq 0$, $X \in \mathcal{S}_+^n$
- $\langle X, C \rangle \leq 0$ and $\langle X, A_i \rangle = 0$, for every $i$.

# The Facial Reduction Algorithm

$$\sup_{y\in\mathbb{R}^m} \quad \sum_{i=1}^m b_i y_i \qquad \text{(D)}$$
$$\text{s.t.} \quad C - \sum_{i=1}^m y_i A_i \in \mathcal{S}_+^n.$$

Suppose Slater's condition is not satisfied, then there exists $0 \neq X_1 \in \mathcal{S}^n$ such that

- $X_1 \in \mathcal{F}_1 := \mathcal{S}_+^n$
- $\langle X_1, C\rangle \leq 0$ and $\langle X_1, A_i\rangle = 0$, for every $i$.

Two cases:

1. $\langle X_1, C\rangle < 0 \Rightarrow$ (D) is **infeasible**.
2. $\langle X_1, C\rangle = 0 \Rightarrow X_1 \notin (\mathcal{S}_+^n)^\perp$, so $\mathcal{F}_\mathrm{D} \subseteq \mathcal{S}_+^n \cap \{X_1\}^\perp \subsetneq \mathcal{S}_+^n$. $\mathcal{F}_2 := \mathcal{S}_+^n \cap \{X_1\}^\perp$ is a *face* of $\mathcal{S}_+^n$ that is smaller than $\mathcal{S}_+^n$.

# The Facial Reduction Algorithm

$$\sup_{y\in\mathbb{R}^m} \quad \sum_{i=1}^m b_i y_i \qquad (D_2)$$
$$\text{s.t.} \quad C - \sum_{i=1}^m y_i A_i \in \mathcal{F}_2.$$

If Slater's condition is not satisfied for $(D_2)$, then there exists $0 \neq X_2 \in \mathcal{S}^n$ such that

- $X_2 \in (\mathcal{F}_2)^*$.
- $\langle X_2, C\rangle \leq 0$ and $\langle X_2, A_i\rangle = 0$, for every $i$.

Two cases:

1. $\langle X_2, C\rangle < 0 \Rightarrow$ (D) is infeasible.
2. $\langle X_2, C\rangle = 0 \Rightarrow X_2 \notin (\mathcal{F}_2)^\perp$, so $\mathcal{F}_D \subseteq \mathcal{F}_2 \cap \{X_2\}^\perp \subsetneq \mathcal{F}_2$.
   $\mathcal{F}_3 = \mathcal{F}_2 \cap \{X_2\}^\perp$ is a *face* of $\mathcal{S}^n_+$ that is smaller than $\mathcal{F}_2$.

# The Facial Reduction Algorithm

$$\begin{aligned} \sup_{y \in \mathbb{R}^m} \quad & \sum_{i=1}^m b_i y_i \\ \text{s.t.} \quad & C - \sum_{i=1}^m y_i A_i \in \mathcal{F}_3. \end{aligned} \tag{$D_3$}$$

If Slater's condition is not satisfied for $(D_3)$, then there exists $0 \neq X_3 \in \mathcal{S}^n$ such that

- $X_3 \in (\mathcal{F}_3)^*$.
- $\langle X_3, C \rangle \leq 0$ and $\langle X_3, A_i \rangle = 0$, for every $i$.

Two cases:

1. $\langle X_3, C \rangle < 0 \Rightarrow$ (D) is infeasible.
2. $\langle X_3, C \rangle = 0 \Rightarrow X_3 \notin (\mathcal{F}_3)^\perp$, so $\mathcal{F}_D \subseteq \mathcal{F}_3 \cap \{X_3\}^\perp \subsetneq \mathcal{F}_3$.
   $\mathcal{F}_4 := \mathcal{F}_3 \cap \{X_3\}^\perp$ is a *face* of $\mathcal{S}_+^n$ that is smaller than $\mathcal{F}_3$.

# The Facial Reduction Algorithm - General form

**Assumptions**:

$$(c + \operatorname{range} \mathcal{A}^*) \cap \mathcal{K} \neq \emptyset.$$

1. Let $\mathcal{F}_1 = \mathcal{K}$ and $i \leftarrow 1$.
2. If $(c + \operatorname{range} \mathcal{A}^*) \cap \operatorname{ri} \mathcal{F}_i \neq \emptyset$, we are done.
3. If $(c + \operatorname{range} \mathcal{A}^*) \cap \operatorname{ri} \mathcal{F}_i = \emptyset$, then we invoke a separation theorem.
    - There exists $x_i \in \mathcal{F}_i^* \setminus \mathcal{F}_i^\perp$ and $x_i \in \ker \mathcal{A} \cap \{c\}^\perp$.
    - Let $\mathcal{F}_{i+1} \leftarrow \mathcal{F}_i \cap \{x_i\}^\perp$ and $i \leftarrow i + 1$. Go to Step 2.

# Facial Reduction - Continued

$$\begin{aligned} \sup_{t,s} \quad & -s \qquad \text{(D)} \\ \text{s.t.} \quad & \begin{pmatrix} t & 1 & s-1 \\ 1 & s & 0 \\ s-1 & 0 & 0 \end{pmatrix} \succeq 0 \end{aligned}$$

- We can take $X_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

# The Facial Reduction Algorithm

- If (D) is feasible, the algorithm construct a chain of faces:

$$\mathcal{F}^D_{\min} = \mathcal{F}_\ell \subsetneq \cdots \subseteq \mathcal{F}_1 = \mathcal{K}.$$

  Therefore, the Facial Reduction Algorithm always finds the minimal face $\mathcal{F}^D_{\min}$.

# Part 3 - Bonus contents

# Farkas Lemma' in LP

$$\nexists y \text{ s.t., } \; c - \mathcal{A}^* y \geq 0 \iff \exists x \geq 0, \text{s.t.} \quad \langle c, x \rangle = -1, \mathcal{A}x = 0$$

Let $e := (1, 1, \ldots, 1)$.

## Proof.

$$\begin{aligned} \inf_{x} \quad & \langle c, x \rangle & \text{(P)} \\ \text{subject to} \quad & \mathcal{A}x = 0 \\ & x_1 + \cdots + x_n = 1 \\ & x \geq 0 \end{aligned}$$

$$\begin{aligned} \sup_{t,y} \quad & t & \text{(D)} \\ \text{subject to} \quad & c - te - \mathcal{A}^* y \geq 0. \end{aligned}$$

First, (D) is always feasible.
$\theta_D < 0 \iff \nexists y \text{ s.t.}, c - \mathcal{A}^* y \geq 0$
By LP strong duality,
$\theta_D < 0 \iff \exists x^* \geq 0, \langle c, x^* \rangle = \theta_D, \mathcal{A}x^* = 0, x_1^* + \cdots + x_n^* = 1.$ (Divide $x^*$ by $-\theta_D$) □

# Find the problem in the "proof" below

$$\nexists y \text{ s.t., } C - \mathcal{A}^* y \succeq 0 \iff \exists X \succeq 0, \text{s.t. } \langle C, X \rangle = -1, \mathcal{A}X = 0$$

Let $I$ be the identity matrix.

**"Proof."**

$$\begin{aligned} \inf_{X} \quad & \langle C, X \rangle && \text{(P)} \\ \text{subject to} \quad & \mathcal{A}X = 0 \\ & \mathrm{trace}(X) = 1 \\ & X \succeq 0 \end{aligned}$$

$$\begin{aligned} \sup_{t,y} \quad & t && \text{(D)} \\ \text{subject to} \quad & C - tI - \mathcal{A}^* y \succeq 0. \end{aligned}$$

First, (D) is always feasible and satisfies Slater.
$\theta_D < 0 \iff \nexists y \text{ s.t.}, C - \mathcal{A}^* y \succeq 0$
By CLP strong duality under Slater,
$\theta_D < 0 \iff \exists X^* \geq 0, \langle C, X^* \rangle = \theta_D, \mathcal{A}X^* = 0, \mathrm{trace}(X) = 1.$ (Divide $X^*$ by $-\theta_D$)

# Farkas' Lemma in SDP?

$$\nexists y \text{ s.t. }, \ C - \mathcal{A}^* y \succeq 0 \overset{?}{\Longleftrightarrow} \exists X \succeq 0, \text{ s.t. } \langle C, X \rangle = -1, \mathcal{A}X = 0$$

$$\begin{aligned} \sup_t \quad & 0 \\ \text{s.t.} \quad & \begin{pmatrix} t & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - t \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \succeq 0 \end{aligned} \qquad \text{(D)}$$

However, $\langle C, X \rangle = -1, \mathcal{A}X = 0 \Rightarrow X_{12} = -0.5$, $X_{11} = 0$, $X$ cannot be positive semidefinite.

- (D) is infeasible but there is no $X \succeq 0$ with $\langle C, X \rangle = -1, \mathcal{A}X = 0$
- (D) is **weakly infeasible**, i.e., $(C + \operatorname{range} \mathcal{A}) \cap \mathcal{K} = \emptyset$ but $\operatorname{dist}(C + \operatorname{range} \mathcal{A}, \mathcal{K}) = 0$.

# The Facial Reduction Algorithm Again

**Assumptions**: ~~$(c + \operatorname{range} \mathcal{A}^*) \cap \mathcal{K} \neq \emptyset$~~.

1. Let $\mathcal{F}_1 = \mathcal{K}$ and $i \leftarrow 1$.
2. If $(c + \operatorname{range} \mathcal{A}^*) \cap \operatorname{ri} \mathcal{F}_i \neq \emptyset$, we are done, $\mathcal{F}_i$ is the minimal face.
3. If $(c + \operatorname{range} \mathcal{A}^*) \cap \operatorname{ri} \mathcal{F}_i = \emptyset$, then we invoke the (partial polyhedral) proper separation theorem.

   There exists $x_i \in \mathcal{E}$ and $\alpha \in \mathbb{R}$ such that

   $$\langle x_i, c - \mathcal{A}^* y \rangle \leq \alpha \leq \langle x_i, z \rangle, \quad \forall y \in \mathbb{R}^m, \forall z \in \mathcal{F}_i$$

   Therefore

   - $\alpha \leq 0$, $x_i \in \mathcal{F}_i^*$
   - $\langle x_i, c \rangle \leq 0$ and $\mathcal{A} x_i = 0$.

   Two cases:

   a. If $\alpha < 0$, then $(c + \operatorname{range} \mathcal{A}^*) \cap \mathcal{K} = \emptyset$ (Infeasibility detected)
   b. If $\alpha = 0$, then $x_i \notin \mathcal{F}_i^\perp$ holds and we let $\mathcal{F}_{i+1} \leftarrow \mathcal{F}_i \cap \{x_i\}^\perp$ and $i \leftarrow i + 1$. Go to Step 2. ($c + \operatorname{range} \mathcal{A}^* \subseteq \{x_i\}^\perp$ holds)

# The FR Farkas' Lemma

"$c - \mathcal{A}^* y \in \mathcal{K}$" is infeasible **if and only if** there are $x_1, \ldots, x_\ell$ such that

- $x_i \in \mathcal{F}_i^* \cap \ker \mathcal{A} \cap \{c\}^\perp$, for $i = 1, \ldots, \ell - 1$, where
  - $\mathcal{F}_1 = \mathcal{K}$
  - $\mathcal{F}_i = \mathcal{F}_{i-1} \cap \{x_{i-1}\}^\perp$, for $i \geq 2$.
- $x_\ell \in \mathcal{F}_\ell^* \cap \ker \mathcal{A}$ and $\langle c, x_\ell \rangle = -1$.

**Theorem**

*An infeasible CLP has a finite certificate of infeasibility.*

# Example

$$\begin{aligned} \sup_t \quad & 0 \\ \text{s.t.} \quad & \begin{pmatrix} t & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - t \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \succeq 0 \end{aligned} \tag{D}$$

- $X_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{S}^2_+ \cap \ker \mathcal{A} \cap \{C\}^\perp$
- $X_2 = \begin{pmatrix} 0 & -0.5 \\ -0.5 & 0 \end{pmatrix} \in (\mathcal{S}^2_+ \cap \{X_1\}^\perp)^* = \left\{ \begin{pmatrix} a & * \\ * & * \end{pmatrix} \mid a \geq 0 \right\}$ and $\langle C, X_2 \rangle = -1$.

$X_1$ and $X_2$ form a certificate that (D) is infeasible.

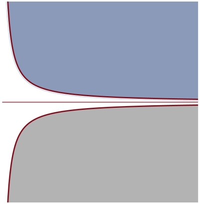
# Roots of bad behavior in CLP

$$\mathrm{dist}\,(U, V) := \inf_{x \in U, y \in V} \|x - y\|$$

Note that $\mathrm{dist}\,(U, V) = \mathrm{dist}\,(0, U - V)$.

- $\mathrm{dist}\,(U, V) = 0 \Leftrightarrow 0 \in \mathrm{cl}\,((U - V))$
- $U \cap V = \emptyset$ and $\mathrm{dist}\,(U, V) = 0 \Rightarrow U - V$ **is not closed.**

Many strange phenomena in CLP can be traced to the lack of closedness of certain maps or sums



# Example 1 - Failure of Farkas' Lemma

$$C - \mathcal{A}t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - t \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} t & 1 \\ 1 & 0 \end{pmatrix} \succeq 0$$

- $(C + \operatorname{range} \mathcal{A}) \cap \mathcal{S}^2_+ = \emptyset$ but $\operatorname{dist}(C + \operatorname{range} \mathcal{A}, \mathcal{S}^2_+) = 0$, so $\mathcal{S}^2_+ - \operatorname{range} \mathcal{A} - C$ **is not closed.**
  - In particular, $\mathcal{S}^2_+ + \operatorname{range} \mathcal{A}$ **is not closed.**

# Example 2 - Unattained optima

$$\sup_{t,s} \quad -s \tag{D}$$
$$\text{s.t.} \quad \begin{pmatrix} t & 1 \\ 1 & s \end{pmatrix} \succeq 0$$

$\theta_D = 0$ but there is no feasible solution with $s = 0$.
Define

$$\hat{\mathcal{A}}(t,s) := \left(-s, \begin{pmatrix} t & 0 \\ 0 & s \end{pmatrix}\right)$$

and

$$\hat{C} := \left(-\theta_D, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$$

Then, $\operatorname{dist}(\hat{C} + \operatorname{range}\hat{\mathcal{A}}, 0 \times \mathcal{S}^2_+) = \operatorname{dist}(\hat{C}, 0 \times \mathcal{S}^2_+ + \operatorname{range}\hat{\mathcal{A}}) = 0$

- $(\hat{C} + \operatorname{range}\hat{\mathcal{A}}) \cap (0 \times \mathcal{S}^2_+) = \emptyset$, so $(0 \times \mathcal{S}^2_+) + \operatorname{range}\hat{\mathcal{A}}$ **is not closed.**

# Fundamental questions

**A common pattern**:

- Strange thing happens $\Rightarrow \mathcal{K} + \mathcal{L}$ fails to be closed, for a certain closed convex cone $\mathcal{K}$ and subspace $\mathcal{L}$.

**Fundamental question**

Given convex cones $\mathcal{K}_1, \mathcal{K}_2$ when is $\mathcal{K}_1 + \mathcal{K}_2$ closed?

Let $S(x, y) := x + y$.

- $\mathcal{K}_1 + \mathcal{K}_2$ is closed $\iff$ $S(\mathcal{K}_1 \times \mathcal{K}_2)$ is closed.

**Fundamental question 2**

Let $\mathcal{K}$ be a convex cone and $M$ a linear map. When is $M\mathcal{K}$ closed?

# A classical result

If $\mathrm{ri}\,(\mathcal{K}_1^*) \cap \mathrm{ri}\,(\mathcal{K}_2^*) \neq \emptyset$ then $\mathcal{K}_1 + \mathcal{K}_2$ is closed.

Proof. See exercise list.

# Nice cones

For $\mathcal{F} \trianglelefteq \mathcal{K}, \mathcal{F} \neq \emptyset$ we have

$$\mathcal{F} = \mathcal{K} \cap \operatorname{span} \mathcal{F}.$$

Therefore

$$\mathcal{F}^* = \operatorname{cl}(\mathcal{K}^* + \mathcal{F}^\perp).$$

$\mathcal{K}$ is **nice** $\iff \mathcal{F}^* = \mathcal{K}^* + \mathcal{F}^\perp, \forall \mathcal{F} \trianglelefteq \mathcal{K} \iff$
$\mathcal{K}^* + \mathcal{F}^\perp$ is closed, $\forall \mathcal{F} \trianglelefteq \mathcal{K}$.

- $\mathbb{R}^n_+, \mathcal{Q}^n, \mathcal{S}^n_+$ (and all symmetric cones) are nice.
- Many applications we will not discuss here: extended duals, lifts of convex sets...

# Preliminary - Conjugate Faces

Let $\mathcal{F} \trianglelefteq \mathcal{K}$, $\mathcal{F} \neq \emptyset$.

The conjugate face of $\mathcal{F}$ is the face $\mathcal{F}^\Delta := \mathcal{K}^* \cap \mathcal{F}^\perp$

- (Exercise) $\mathcal{F}^\Delta = \mathcal{K}^* \cap \{x\}^\perp$ holds for $x \in \operatorname{ri} \mathcal{F}$. In particular $\mathcal{F}^\Delta$ is an exposed face of $\mathcal{K}^*$.

**Example** Let $\mathcal{F} \trianglelefteq \mathcal{S}^n_+$ be such that

$$\mathcal{F} = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \mid A \in \mathcal{S}^r_+ \right\}$$

Then

$$\mathcal{F}^\Delta = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} \mid C \in \mathcal{S}^{n-r}_+ \right\}$$

# A closedness criterion by Pataki

$\mathcal{K}_1, \mathcal{K}_2$: closed convex **nice** cones.
Let $x \in \mathrm{ri}\,(\mathcal{K}_1 \cap \mathcal{K}_2)$, $\mathcal{F}_1 := \mathcal{F}_{\min}(x, \mathcal{K}_1)$ and $\mathcal{F}_2 := \mathcal{F}_{\min}(x, \mathcal{K}_2)$. Then

$\mathcal{K}_1^* + \mathcal{K}_2^*$ is closed if and only if $\mathcal{F}_1^\Delta + \mathcal{F}_2^\Delta = \mathcal{F}_1^\perp + \mathcal{F}_2^\perp$.

G. Pataki,
On the closedness of the linear image of a closed convex cone,
Math. Oper. Res., 32 (2007), pp. 395–412.

# Other exposedness properties

- Nices cones are nice, but niceness is hard to check.
- There are simpler **sufficient conditions**: *projectional exposedness*, *amenability*

J. M. Borwein and H. Wolkowicz.
Regularizing the abstract convex program.
*Journal of Mathematical Analysis and Applications*, 83(2):495 – 530, 1981.

B. F. Lourenço,
Amenable cones: error bounds without constraint qualifications,
Mathematical Programming, 186 (2021), pp. 1–48,

# Amenability

## Definition (Amenable cones)

$\mathcal{K}$ is **amenable** if for every (nonempty) face $\mathcal{F}$ of $\mathcal{K}$ there is $\kappa > 0$ such that

$$\mathrm{dist}\,(x, \mathcal{F}) \leq \kappa \mathrm{dist}\,(x, \mathcal{K}), \qquad \forall x \in \mathrm{span}\,\mathcal{F}.$$

Or, equivalently, if there is $\kappa > 0$ such that

$$\mathrm{dist}\,(x, \mathcal{F}) \leq \kappa(\mathrm{dist}\,(x, \mathcal{K}) + \mathrm{dist}\,(x, \mathrm{span}\,\mathcal{F})).$$

Amenable cones are (particularly) nice


B. F. Lourenço,
Amenable cones: error bounds without constraint qualifications,
Mathematical Programming, 186 (2021), pp. 1–48,

B. F. Lourenço, V. Roshchina, and J. Saunderson.
Amenable cones are particularly nice.
SIAM J. Optim., 32(3):2347–2375, September 2022. arXiv:2011.07745.

# A comparison table

| | | Exposed | Nice | Amenable | Projectionally |
|---|---|---|---|---|---|
| Preserved under | finite intersections | ✓ | ✓ | ✓ | ? |
| | direct product | ✓ | ✓ | ✓ | ✓ |
| | injective linear image | ✓ | ✓ | ✓ | ✓ |
| Symmetric cones | | ✓ | ✓(CT'08) | ✓ | ✓L'21 |
| Homogeneous cones | | ✓ | ✓(CT'08) | ✓LRS'22 | ? |
| Hyperbolicity cones | | ✓(R'05) | ✓ | ✓LRS'23 | ? |

- Facially exposed $\overset{\text{P'13}}{\Leftarrow}$ Nice $\overset{\text{L'21}}{\Leftarrow}$ **Amenable** $\overset{\text{EPBR}}{\Leftarrow}$ Projectionally exposed.
- There exists a 4D cone that is facially exposed but not nice (Roschina, SIOPT'14).
- There exists a 4D cone that is nice but not amenable LRS'22
- In dimension 4 or less: Amenable $\Leftrightarrow$ Projectionally exposed. LRS'22

# Feasibility vs Optimization

**Optimization problem**:

$$\begin{array}{rl} \sup\limits_{y} & \langle b, y\rangle \\ \text{subject to} & c - \mathcal{A}^* y \in \mathcal{K} \end{array}$$

**Feasibility problem**:

$$\begin{array}{rl} \text{find} & y \\ \text{subject to} & c - \mathcal{A}^* y \in \mathcal{K} \end{array}$$

Are optimization problems harder than feasibility problems?

Depends, but in a very important sense **no**.

# In Linear Programming

Consider two oracles:

- **Feasibility Oracle**: Receives LP data and returns a feasible solution if one exists or **NO** if no solution exists.
- **Optimization Oracle**: Receives LP data and returns an optimal solution if one exists or **NO** if no solution exists.

1 call to **Feasibility Oracle** is enough to simulate the **Optimization Oracle**

**Proof.**

Ask the **Feasibility Oracle** for a solution to the KKT system $\{(x, y) \mid Ax = b, c - \mathcal{A}^T y \geq 0, x \geq 0, \langle c, x \rangle - \langle b, y \rangle = 0\}$. □

# In Linear Programming

Maybe you thought that was unfair. How about this?

- **Dangerous Feasibility Oracle**: Receives LP data and returns a feasible solution if one exists or **EXPLODES** if no solution exists.
- **Optimization oracle**: Receives LP data and returns an optimal solution if one exists or **NO** if no solution exists.

Can **Dangerous Feasibility Oracle** simulate **Optimization Oracle** in finite calls (without exploding)?

The short answer is **yes**.

# In Conic Linear Programming

Consider two oracles:

- **Dangerous Feasibility Oracle**: Receives CLP data and returns a feasible solution if one exists or **EXPLODES** if no solution exists.
- **Optimization Oracle**: Receives CLP data returns an optimal solution if one exists or **NO** if no solution exists.

KKT trick no longer works because (P) or (D) may be unattained and/or there may be a duality gap.

# A FR subproblem

The directions appearing in facial reduction can be found by solving the following subproblem

$$
\begin{aligned}
\inf_{x,t,w}\quad & t && && (P_{\mathcal{K}})\\
\text{subject to}\quad & -\langle c, x - te^*\rangle + t - w && = 0 && (1)\\
& \langle e, x\rangle + w && = 1 && (2)\\
& \mathcal{A}x - t\mathcal{A}e^* && = 0 && (3)\\
& (x, t, w) \in \mathcal{K}^* \times \mathbb{R}_+ \times \mathbb{R}_+
\end{aligned}
$$

$$
\begin{aligned}
\sup_{y_1,y_2,y_3}\quad & y_2 && (D_{\mathcal{K}})\\
\text{subject to}\quad & cy_1 - ey_2 - \mathcal{A}^\top y_3 \in \mathcal{K} && (4)\\
& 1 - y_1(1 + \langle c, e^*\rangle) + \langle e^*, \mathcal{A}^\top y_3\rangle \geq 0 && (5)\\
& y_1 - y_2 \geq 0 && (6)
\end{aligned}
$$

It has the following properties:

- Slater's condition is satisfied at both sides. Common optimal value is finite.
- **KKT trick** works and **Dangerous Feasibility Oracle** never explodes.

# Dangerously doing Facial Reduction

FR applied to $c - \mathcal{A}^* y \in \mathcal{K}$

1. Let $\mathcal{F}_1 = \mathcal{K}$ and $i \leftarrow 1$.
2. We invoke the **Dangerous Feasibility oracle with the KKT trick applied to the auxiliary problems** to get either $y$ such that $c - \mathcal{A}^* y \in \operatorname{ri} \mathcal{F}_i$ (in this case, we stop) or $x_i$, $\alpha$ such that

$$\langle x_i, c - \mathcal{A}^* y \rangle \leq \alpha \leq \langle x_i, z \rangle, \quad \forall y \in \mathbb{R}^m, \forall z \in \mathcal{F}_i$$

Therefore

- $\alpha \leq 0$, $x_i \in \mathcal{F}_i^*$
- $\langle x_i, c \rangle \leq 0$ and $\mathcal{A} x_i = 0$.

Two cases:

(a) If $\alpha < 0$, then $(c + \operatorname{range} \mathcal{A}^*) \cap \mathcal{K} = \emptyset$ (Infeasibility detected)

(b) If $\alpha = 0$, then $x \notin \mathcal{F}_i^*$ holds and we let $\mathcal{F}_{i+1} \leftarrow \mathcal{F}_i \cap \{x_i\}^\perp$ and $i \leftarrow i + 1$. Go to Step 2. ($c + \operatorname{range} \mathcal{A}^* \subseteq \{x_i\}^\perp$ holds)

# Dangerous Optimization

Simulating the **Optimization Oracle**:

1. Do Facial Reduction **twice** to get a pair of problems $(\hat{D})$ $(\hat{P})$ satisfying Slater's condition. If FR declares infeasibility at some point return **NO**
2. Call the **Dangerous Feasibility Oracle** to solve the pair $(\hat{D})$ $(\hat{P})$ and obtain $\theta_D$.
3. Use FR to either compute a solution with value or $\theta_D$ or to check that none exists (return **NO** in this case).

**Dangerous Feasibility Oracle** can simulate **Optimization Oracle** with at most $O(\dim \mathcal{E})$ calls.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