Exercise Smörgåsbord for a Conic Summer

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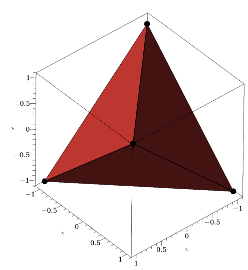
Part 1

1. Given a closed convex set $C \subseteq \mathbb{R}^n$ and point x_0, the projection of x_0 onto C is defined as the solution of the following problem
$$\inf_{x \in C} \|x - x_0\|$$
Suppose that C is a polyhedral set of the form $C \coloneqq \{x \mid Ax \leq b\}$, where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$
   (a) Show that the problem of computing the projection of x_0 onto C can be written as a second order cone program (SOCP). (You are allowed to use inequality constraints as well)
   (b) Use CVXPY to compute the projection of $(-1, 2, 3)$ onto the tetrahedron given by
$$C = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid -x_1 - x_2 - x_3 \leq 1, -x_1 + x_2 + x_3 \leq 1, x_1 - x_2 + x_3 \leq 1, x_1 + x_2 - x_3 \leq 1\}$$



2. (An example adapted from (Boyd and Vandenberghe, 2004)) Let X be a discrete random variable that takes 100 equidistant values in the interval $[-1, 1]$. We know that:
   - $EX \in [-0.1, 0.1]$
   - $EX^2 \in [0.5, 0.6]$
   - $E(3X^3 - 2X) \in [-0.3, -0.2]$
   - $P(X < 0) \in [0.3, 0.4]$

   Use CVXPY to find the the maximum entropy distribution consistent with this prior information.

Part 2

In what follows, we will always assume that $\mathcal{E}$ is a real finite dimensional space equipped with an inner product $\langle \cdot, \cdot \rangle$ and an induced norm given such that $\|x\| \coloneqq \sqrt{\langle x, x \rangle}$ for $x \in \mathcal{E}$

3. (**Warm-up**) Let $C \subseteq \mathcal{E}$ be a convex set and $\mathcal{K} \subseteq \mathcal{E}$, $\hat{\mathcal{K}} \subseteq \mathcal{E}$ be closed convex cones. Prove the following.

(a) $(\mathcal{K}+\hat{\mathcal{K}})^* = \mathcal{K}^* \cap \hat{\mathcal{K}}^*$, where $\mathcal{K}+\hat{\mathcal{K}} := \{x+y \mid x \in \mathcal{K}, y \in \hat{\mathcal{K}}\}$.

(b) $\mathcal{K} \subseteq \hat{\mathcal{K}} \Longleftrightarrow \hat{\mathcal{K}}^* \subseteq \mathcal{K}^*$.

(c) $(\operatorname{ri}\mathcal{K})^* = \mathcal{K}^*$

(d) $\mathcal{F}$ is a face of $\mathcal{K} \iff$ for all $x, y \in \mathcal{K}$, we have that $x+y \in \mathcal{F}$ implies $x, y \in \mathcal{F}$.

(e) $\mathcal{F}$ and $\hat{\mathcal{F}}$ are both faces of $C \Rightarrow \mathcal{F} \cap \hat{\mathcal{F}}$ is a face of C.

(f) Prove or give a counter-example: If $\hat{\mathcal{F}}$ is a face of $\mathcal{F}$ and $\mathcal{F}$ is a face of C then $\hat{\mathcal{F}}$ is a face of C.

(g) Prove or give a counter-example: If $\hat{\mathcal{F}}$ is an exposed face of $\mathcal{F}$ and $\mathcal{F}$ is an exposed face of C, then $\hat{\mathcal{F}}$ is an exposed face of C.

(Obs: All items admit short proofs)

4. (The dual of the dual...) Consider a conic linear program (CLP) in dual format:

$$\begin{aligned} \sup_{y\in\mathbb{R}^m} \quad & \langle b, y\rangle \\ \text{subject to} \quad & c - \mathcal{A}^* y \in \mathcal{K}^*. \end{aligned} \tag{D}$$

For $x \in \mathcal{K}$, define $\mathcal{L}(x) := \sup_{y\in\mathbb{R}^m} \left[\langle b, y\rangle + \langle c - \mathcal{A}^* y, x\rangle)\right]$.

(a) Show that $\theta_D \leq \mathcal{L}(x)$ holds for all $x \in \mathcal{K}$.

(b) The previous item implies that $\theta_D \leq \inf_{x\in\mathcal{K}} \mathcal{L}(x)$. What is the relation between $\inf_{x\in\mathcal{K}} \mathcal{L}(x)$ and the primal counterpart of (D)?

5. (Slightly harder stuff) Let $C \subseteq \mathcal{E}$ be a convex set.

(a) Let $\mathcal{F} \trianglelefteq C$. Show that if $\operatorname{cl}\mathcal{F} \subseteq C$, then $\mathcal{F}$ is closed. Conclude that faces of closed convex sets are closed. (A hint that may make the problem too easy: For a convex set S, $x \in \operatorname{ri} S$ and $y \in \operatorname{cl} S$, we have $\alpha x + (1-\alpha) y \in \operatorname{ri} S$ for all $\alpha \in (0,1)$.)

(b) Let $\mathcal{F}_1 \trianglelefteq C$, $\mathcal{F}_2 \trianglelefteq C$ be nonempty. Show that $(\operatorname{ri}\mathcal{F}_1) \cap (\mathcal{F}_2) \neq \emptyset \iff \mathcal{F}_1 \subseteq \mathcal{F}_2$. Conclude that the relative interiors of different faces never intersect.

(c) Let $\mathcal{F} \trianglelefteq C$ be nonempty with $\mathcal{F} \neq C$. Show that there exists an *exposed* face $\mathcal{F}' \trianglelefteq C$ such that $\mathcal{F}' \neq C$ and $\mathcal{F} \subseteq \mathcal{F}'$ (In particular, $\mathcal{F} \trianglelefteq \mathcal{F}'$). Hint: use a separation theorem.

6. (Faces of the intersection) Let $\mathcal{K}_1 \subseteq \mathcal{E}$, $\mathcal{K}_2 \subseteq \mathcal{E}$ be closed convex cones.

(a) (Easy) Prove that if $\mathcal{F}_1 \trianglelefteq \mathcal{K}_1$, $\mathcal{F}_2 \trianglelefteq \mathcal{K}_2$ then $\mathcal{F}_1 \cap \mathcal{F}_2 \trianglelefteq \mathcal{K}_1 \cap \mathcal{K}_2$

(b) (A bit harder) Prove that if $\mathcal{F} \trianglelefteq \mathcal{K}_1 \cap \mathcal{K}_2$ is nonempty, then there exists $\mathcal{F}_1 \trianglelefteq \mathcal{K}_1$ and $\mathcal{F}_2 \trianglelefteq \mathcal{K}_2$ such that $\mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2$ and $(\operatorname{ri}\mathcal{F}_1) \cap (\operatorname{ri}\mathcal{F}_2) \neq \emptyset$. Hint: (Use $\mathcal{F}_{\min}(\mathcal{F}, \mathcal{K}_1)$, $\mathcal{F}_{\min}(\mathcal{F}, \mathcal{K}_2)$.)

7. (Faces of the positive semidefinite cone)

(a) Let $\mathcal{F} \trianglelefteq \mathcal{S}^n_+$, $X \in \operatorname{ri}\mathcal{F}$ and $Y \in \mathcal{F}$. Show that $\ker X \subseteq \ker Y$. Conclude that the subspace $\ker X$ is constant over $X \in \operatorname{ri}\mathcal{F}$.

(b) Let $\mathcal{F} \trianglelefteq \mathcal{S}^n_+$, $X \in \operatorname{ri}\mathcal{F}$ and $Y \in \mathcal{S}^n_+$. Show that $\ker X \subseteq \ker Y$ implies $Y \in \mathcal{F}$. (Hint: first show that there exists $\alpha > 0$ such that $X - \alpha Y \in \mathcal{S}^n_+$)

(c) From the previous item conclude that $\mathcal{F} = \{X \in \mathcal{S}^n_+ \mid V \subseteq \ker X\}$, where V is the common kernel of the elements in $\operatorname{ri}\mathcal{F}$.

(d) Conversely, show that if V is a subspace of $\mathbb{R}^n$, then $\mathcal{F}_V := \{X \in \mathcal{S}^n_+ \mid V \subseteq \ker X\}$ is a face of $\mathcal{S}^n_+$.

(e) Conclude that the map $V \mapsto \mathcal{F}_V$ is a bijection between the subspaces of $\mathbb{R}^n$ and the nonempty faces of $\mathcal{S}^n_+$.

(f) Show that all nonempty faces of $\mathcal{S}^n_+$ are exposed. (Hint: For a face $\mathcal{F}_V$ construct an exposing matrix from a basis of $V^\perp$.)

8. (An exercise in facial reduction) Consider the following SDP in dual format.

$$\begin{aligned} \sup_{t,s} \quad & -s \\ \text{s.t.} \quad & \begin{pmatrix} 0 & s-1 & 0 \\ s-1 & t & 0 \\ 0 & 0 & s \end{pmatrix} \succeq 0 \end{aligned} \tag{D}$$

(a) Write down the corresponding primal problem.

(b) Compute the primal and dual optimal values (i.e., θ_P and θ_D). Also, compute a primal optimal solution and a dual optimal solution.

(c) Compute $\mathcal{F}^D_{\min}$ and $(\mathcal{F}^D_{\min})^*$.

(d) Consider the following problem

$$\begin{aligned} \sup_{t,s} \quad & -s \\ \text{s.t.} \quad & \begin{pmatrix} 0 & s-1 & 0 \\ s-1 & t & 0 \\ 0 & 0 & s \end{pmatrix} \in \mathcal{F}^D_{\min} \end{aligned} \tag{$\hat{\text{D}}$}$$

Write down the corresponding primal problem ($\hat{\text{P}}$) and compute the optimal value of both ($\hat{\text{D}}$) and ($\hat{\text{P}}$).

(e) With respect the previous item, compute an optimal solution X^* to ($\hat{\text{P}}$). Is X^* feasible for (P) (the primal problem of (D))?

Part 3

9. (A FR Farkas' Lemma exercise) Use facial reduction to prove that the following semidefinite feasibility problem is infeasible

$$\text{find } t, s \in \mathbb{R} \text{ such that } \begin{pmatrix} t & 1 & s \\ 1 & s & 1 \\ s & 1 & 0 \end{pmatrix} \succeq 0$$

10. (A closedness criterion via duality) Let $\mathcal{K}_1 \subseteq \mathcal{E}$, $\mathcal{K}_2 \subseteq \mathcal{E}$ be closed convex cones. Let $c \in \operatorname{cl}(\mathcal{K}_1 + \mathcal{K}_2)$ and consider the following problem:

$$\begin{aligned} \inf_{(x,z)\in\mathcal{E}\times\mathcal{E}} \quad & \langle c, x \rangle \\ \text{subject to} \quad & x = z \\ & (x,z) \in (\mathcal{K}_1^*) \times (\mathcal{K}_2^*) \end{aligned} \tag{P}$$

(a) Prove that $\theta_P = 0$.

(b) Write down the dual of (P).

(c) Conclude that if $\operatorname{ri}(\mathcal{K}_1^*) \cap \operatorname{ri}(\mathcal{K}_2^*) \neq \emptyset$ then $\mathcal{K}_1 + \mathcal{K}_2$ is closed.

11. (Conjugate faces) Let $\mathcal{F} \trianglelefteq \mathcal{K}$ be nonempty, where $\mathcal{K} \subseteq \mathcal{E}$ is a closed convex cone.

(a) Show that $\mathcal{F}^\Delta = \mathcal{K}^* \cap \{x\}^\perp$ holds for $x \in \operatorname{ri} \mathcal{F}$. In particular $\mathcal{F}^\Delta$ is an exposed face of $\mathcal{K}^*$

(b) (Harder but doable) Show that $\mathcal{F}$ is exposed if and only if $\mathcal{F} = \mathcal{F}^{\Delta\Delta}$.

12. Prove that $\mathcal{S}^n_+$ is nice.

13. (Hard, but we will split in small parts so that it is manageable) Let $\mathcal{K}_1 \subseteq \mathcal{E}$, $\mathcal{K}_2 \subseteq \mathcal{E}$ be nonempty nice closed convex cones. Here, we prove $\mathcal{K}_1 \cap \mathcal{K}_2$ is nice as well.

   (a) The span of a convex cone $\mathcal{K} \subseteq \mathcal{E}$ is the smallest subspace containing $\mathcal{K}$ and one may check that $\operatorname{span}\mathcal{K} = \mathcal{K} - \mathcal{K}$. Prove that if $x \in \operatorname{ri}\mathcal{K}$, then $\operatorname{span}\mathcal{K} = \{x + z \mid \exists t > 0, x \pm tz \in \mathcal{K}\}$.

   (b) Prove that if $\mathcal{F}_1$ and $\mathcal{F}_2$ are convex cones satisfying $(\operatorname{ri}\mathcal{F}_1) \cap (\operatorname{ri}\mathcal{F}_2) \neq \emptyset$, then $\operatorname{span}(\mathcal{F}_1 \cap \mathcal{F}_2) = \operatorname{span}\mathcal{F}_1 \cap \operatorname{span}\mathcal{F}_2$. Compute $(\mathcal{F}_1 \cap \mathcal{F}_2)^\perp$. (Hint: you may use the fact that $\operatorname{ri}(C \cap D) = \operatorname{ri}(C) \cap \operatorname{ri}(D)$ holds, if $(\operatorname{ri} C) \cap (\operatorname{ri} D) \neq \emptyset$)

   Also, give a counter-example showing that if $(\operatorname{ri}\mathcal{F}_1) \cap (\operatorname{ri}\mathcal{F}_2) = \emptyset$, then it may happen that $\operatorname{span}(\mathcal{F}_1 \cap \mathcal{F}_2) \neq \operatorname{span}\mathcal{F}_1 \cap \operatorname{span}\mathcal{F}_2$.

   (c) Suppose that $(\operatorname{ri}\mathcal{K}_1) \cap (\operatorname{ri}\mathcal{K}_2) \neq \emptyset$ and show that $\mathcal{K}_1 \cap \mathcal{K}_2$ is nice. Hint: (Use 6.(b), 10.(c) and the previous items)

   (d) Show that $\mathcal{K}_1 \cap \mathcal{K}_2$ is nice without assuming that $(\operatorname{ri}\mathcal{K}_1) \cap (\operatorname{ri}\mathcal{K}_2) \neq \emptyset$.

14. (For those who have some knowledge of convex analysis) We will show that if a nonempty closed convex cone $\mathcal{K} \subseteq \mathcal{E}$ is amenable, then $\mathcal{K}$ is nice.

   (a) For a nonempty $\mathcal{F} \trianglelefteq \mathcal{K}$, show that the amenability condition implies the existence of $\alpha > 0$ such that

$$\frac{\alpha}{2}\operatorname{dist}(x, \mathcal{F})^2 \leq \frac{\operatorname{dist}(x, \mathcal{K})^2}{2} + \frac{\operatorname{dist}(x, \operatorname{span}\mathcal{F})^2}{2}, \forall x \in \mathcal{E}$$

   (b) Let $f, g : \mathcal{E} \to \mathbb{R} \cup \{+\infty\}$ be closed proper convex functions. Their infimal convolution is defined as

$$(f \Box g)(x) := \inf\{f(x_1) + g(x_2) \mid x_1 + x_2 = x\}$$

   For $C \subseteq \mathcal{E}$ a convex set, $\frac{1}{2}\operatorname{dist}(x, C)^2$ is the infimal convolution between the indicator function δ_C and the quadratic map $\|\cdot\|^2/2$.

   We also recall that for closed proper convex functions f, g and their conjugates we have $f \leq g \Leftrightarrow g^* \leq f^*$ and $(f + g)^* = \operatorname{cl}(f^* \Box g^*)$. In the particular case where the domains of f and g are $\mathcal{E}$, we have $(f + g)^* = (f^* \Box g^*)$.

   Using item (a) and the facts mentioned above, show that if $\mathcal{K}$ is amenable then it is nice.