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NON-ASSOCIATIVE ALGEBRA AND LOTKA–VOLTERRA EQUATION WITH TERNARY INTERACTION

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IN THIS paper we consider a Lotka–Volterra system taking into account the interactions of neighbouring two or three individuals. The system with binary interaction treated by Kimura [6] and Mather [8] is represented by

$$\frac{\mathrm{d}}{\mathrm{d}t} p_i(t) = p_i(t) \sum_{j=1}^{m} a_{ij} p_j(t) \quad \text{for } t \geqslant t_0, \quad \text{where } a_{ij} + a_{ji} = 0, \tag{1}$$

$$P_i(t_0) > 0, \qquad \sum_{i=1}^{m} p_i(t_0) = 1, \quad \text{for } i, j = 1, 2, \ldots, m.$$

We cannot neglect ternary interactions for higher density of individuals. As stated in Mather [8], a plant may feel the effects of competition from a number of other individuals growing at various distances from it and interacting with one another in their effects on it. Non-associative algebras are applied to discuss quadratic differential equations. (See Markus [7] and Kaplan and Yorke [5].) We define a non-associative algebra to give a systematic description for binary or higher order interactions of competition. The notion of our non-associative algebra is naturally obtained from a random collision model for Lotka–Volterra system which is introduced by an analogy of kinetic theory of gasses. (Itoh [1–4]). We introduce a system of differential equations using the algebra. We prove, for a system in which ternary interaction is not neglected, a Lyapunov function increases until the system attains equilibrium, while for a system only with binary interaction the function is time invariant. Hence the term usually neglected can produce the qualitative difference. This gives a reason of being for our ternary interaction model. If a stability raised by an increase of density of individuals is recognized in some experiments or observations, our result may give one of the possible explanations. For four or more than four order interaction, the asymptotic stability is expected from numerical studies. But the proof is our future problem. A special case of the present result was proved in the previous paper [3], assuming a symmetry for interaction. Now we do not assume the symmetry. So the condition treated here is sufficiently general for actual situations.

Definition. The non-associative algebra A^m is defined as follows:

(I) $A^m = \left\{ \sum_{i=1}^{m} x_i E_i \middle| x_i \in R, i = 1, 2, \ldots, m \right\}$

is a m-dimensional linear space over a field R which is generated by linearly independent elements E_i, $i = 1, 2, \ldots, m$.

(II) The products of the bases are defined as

$$E_i \circ E_j = (\tfrac{1}{2} + a_{ij})E_i + (\tfrac{1}{2} + a_{ji})E_j$$

where

$$a_{ij} = -a_{ji} \quad \text{and} \quad -\tfrac{1}{2} \leqslant a_{ij} \leqslant \tfrac{1}{2}.$$

(III) The product of two elements

$$x = \sum_{i=1}^{m} x_i E_i, \qquad y = \sum_{j=1}^{m} y_j E_j \in A^m,$$

is defined as

$$\sum_{i=1}^{m} x_i E_i \circ \sum_{j=1}^{m} y_j E_j = \sum_{i,j=1}^{m} x_i y_j E_i \circ E_j.$$

A^m has the following properties.

Property 1. We see from the above definition,

$$E_i \circ E_j = E_j \circ E_i, \qquad E_i \circ E_i = E_i.$$

Thus the algebra is commutative.

Hereafter we write i's component of $x \in A^m$ as x_i.

Property 2. For $x, y \in A^m$, we have

$$\sum_{i=1}^{m} (x \circ y)_i = \sum_{i,j=1}^{m} x_i y_j = \left(\sum_{i=1}^{m} x_i\right)\left(\sum_{j=1}^{m} y_j\right).$$

Using the non-associative algebra A^m (1) is expressed by

$$\frac{\mathrm{d}}{\mathrm{d}t} p(t) = p(t) \circ p(t) - p(t).$$

The motivation of such representation is given in Itoh [3].

The system with ternary interactions is represented by

$$\frac{\mathrm{d}}{\mathrm{d}t} p(t) = k_1(p(t) \circ p(t) - p(t)) + k_2((p(t) \circ p(t)) \circ p(t) - p(t)) \quad \text{for } p(t) \in A^m. \tag{2}$$

The biological meaning of this equation will be published elsewhere.

Consider an initial value problem for the system of differential equations associated with (2). We assume

$$(p_1(\tau), p_2(\tau), \ldots, p_m(\tau)) \in B = \left\{ p \,\middle|\, \sum_{i=1}^{m} p_i = 1, p_i > 0 \quad \text{for} \quad i = 1, 2, \ldots, m \right\}.$$

We can prove that there exists a unique solution such that $(p_1(t), p_2(t), \ldots, p_m(t)) \in \bar{B}$ on $0 \leqslant t - \tau \leqslant \infty$. We have the following theorem.

THEOREM. Let there exist $q \in A^m$ which satisfies

$$q \circ q - q = 0, \quad \sum_{i=1}^{m} q_i = 1 \quad \text{and} \quad q_i > 0 \quad \text{for} \quad i = 1, 2, \ldots, m,$$

then:

(a) $\displaystyle \frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{m} q_i \log p_i(t) = 2k_2 \sum_{i=1}^{m} q_i \left(\sum_{j=1}^{m} a_{ij} p_j(t) \right)^2 \geqslant 0,$ if $(p_1(\tau), p_2(\tau), \ldots, p_m(\tau)) \in B$.

(b) Every trajectory in the set B converges to $(q_1, q_2, \ldots, q_m)$.

Proof. From Property 2 we see

$$\sum_{i=1}^{m} p_i(t) = 1 \quad \text{for } t \geqslant \tau.$$

Since

$$(p \circ p) \circ p - p = (p \circ p - p) \circ p + p \circ p - p,$$

we have

$$\frac{1}{k_2} \frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{m} q_i \log p_i(t) = \frac{1}{k_2} \sum_{i=1}^{m} q_i \frac{(\mathrm{d}/\mathrm{d}t)\, p_i}{p_i}$$

$$= \sum_{i=1}^{m} q_i \frac{((p \circ p - p) \circ p)_i}{p_i} + \frac{k_1 + k_2}{k_2} \sum_{i=1}^{m} q_i \frac{(p \circ p - p)_i}{p_i}$$

Since

$$p \circ p - p = \sum_{i=1}^{m} p_i \sum_{j=1}^{m} a_{ij} p_j E_i,$$

we have

$$\sum_{i=1}^{m} q_i a_{ij} = \sum_{i=1}^{m} q_i a_{ji} = 0.$$

Hence we have,

$$\sum_{i=1}^{m} \left(q_i \frac{(p \circ p - p)_i}{p_i} \right) = 2 \sum_{i=1}^{m} q_i \sum_{j=1}^{m} a_{ij} p_j = 2 \sum_{j=1}^{m} \left(\sum_{i=1}^{m} q_i a_{ij} \right) p_j = 0.$$

We have

$$(p \circ p - p) \circ p = 2 \left(\sum_{i,j=1}^{m} a_{ij} p_i p_j E_i \right) \circ p$$

$$= 2 \left(\sum_{i,j=1}^{m} a_{ij} p_i p_j E_i \right) \circ \left(\sum_{k=1}^{m} p_k E_k \right)$$

$$= \sum_{i,j,k=1}^{m} a_{ij} p_i p_j p_k (E_i + 2a_{ik} E_i + E_k + 2a_{ki} E_k).$$

Thus

$$\sum_\rho q_\rho \frac{((p \circ p - p) \circ p)_\rho}{p_\rho} = \sum_{\rho, j, k} q_\rho a_{\rho j} p_j p_k + 2 \sum_{\rho, j, k} q_\rho a_{\rho j} a_{\rho k} p_j p_k$$

$$+ \sum_{i, j, \rho} q_\rho a_{ij} p_i p_j + 2 \sum_{i, j, \rho} q_\rho a_{ij} a_{\rho i} p_i p_j$$

$$= 2 \sum_{\rho, j, k} q_\rho a_{\rho j} p_j a_{\rho k} p_k$$

$$= 2 \sum_\rho q_\rho \left(\sum_j a_{\rho j} p_j \right)^2,$$

since

$$\sum_{\rho, j, k} q_\rho a_{\rho j} p_j p_k = \sum_{j, k} \left(\sum_\rho q_\rho a_{\rho j} \right) p_j p_k = 0,$$

$$\sum_{i, j, \rho} q_\rho a_{ij} p_i p_j = \sum_\rho q_\rho \sum_{i, j} a_{ij} p_i p_j = \sum_\rho q_\rho \sum_i (p \circ p - p)_i = 0$$

(from Property 2), and

$$\sum_{i, j, \rho} q_\rho a_{ij} a_{\rho i} p_i p_j = \sum_{i, j} \left(\sum_\rho q_\rho a_{\rho i} \right) a_{ij} p_i p_j = 0.$$

Hence we see $\sum_{i=1}^{m} q_i \log p_i(t)$ is a Lyapunov function on the invariant set B whose maximum is given by q.

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