Research Memorandum No.154 March 1979
The Institute of Statistical Mathematics

RANDOM COLLISION PROCESS ON ORIENTED GRAPH

YOSHIAKI ITOH

THE INSTITUTE OF STATISTICAL MATHEMATICS
4-6-7 MINAMI-AZABU,MINATO-KU
TOKYO, JAPAN

Abstract

In graph theory, an oriented graph which represents dominance relations is called a tournament. We introduce a random collision model on a tournament [V] for competition of species. Our collision rule is that an individual of species i and an individual of species j collide with each other and become two individuals of i if i dominates j. If, for 2r+1 species, each species dominates r of the other species r d is dominated by the remained r of the other species, we say that the tournament of the 2r+1 species is isomorphic to tournament $[T_r]$. We start from a population which contains m species with dominance relations of [V]. If $[V] \subset [T_s]$, the probability that it contains 2r+1 species with a tournament isomorphic to $[T_r]$ is given asymptotically in the main theorem for r = 1, 2, ··· , M, where M is defined by $M = \mathrm{Max}\{r \mid [T_r] \subset [V]\}$. Taking an appropriate time scale t for an application, we find the asymptotic probability is proportional to $e^{-{}_{2r+1}C_2 \frac{t}{n}}$ where n is the population size. Stochastic properties used in our proof are based on properties of tournaments. For example, a a martingale which represents so to speak a measure of $[T_r]$ness of the system plays a major role in this paper.

RANDOM COLLISION; FINITE NUMBER OF PARTICLES; ORIENTED GRAPHS; COMPETING SPECIES; MARTINGALES; SUPERMARTINGALES; EIGEN VALUE; ASYMPTOTIC PROBABILITY OF COEXISTENCE

1. Introduction

Problems of species competitions have been studied by many authors since Lotka[9] and Volterra [14]. Ehrenfest's urn model was discussed in detail by Kac [5], and Moran [13] treated another urn model in relation to a genetic problem. The present author treated random collision models for competing species, [2], [3], [4]. A stochastic model with finite number of individuals for each species has been introduced in [3], and several qualitative conditions for species coexistence has been studied with aid of computer simulation. Kimura [7] has calculated the asymptotic probabilities of coexistence for Wright's genetic model (which is lacking competition). The present study can be applied to the case with competition.

We use oriented graphs to represent the dominance relation in competition and introduce random collisions on the oriented graphs. According to that model, an individual of species i and an individual of species j collide (come into contact) and become two individuals of species i, if species i dominates j (see Fig. 1), or become two individuals of species j if the dominance relation is reversed. The word "dominance" is here used as a graph theoretic term, and this is not be confused with the dominance-recessive relationship in genetics. A class of oriented graphs representing dominance relations is called "tournament", (see Moon [12]).

The object of this paper is to give an asymptotic result for coexistence of species in relation to tournaments. Our theorem is stated in section 4. Here we introduce our study by a special case of it, that is to say, we illustrate a random collision model whose tournament [V] are given in Fig. 2, in which the species, denoted by the nodes, v_2 dominates v_1, v_4 dominates v_2, and so on. The number of individuals of species v_i at time u is the random variable $N_{v_i}(u)$ for $i = 1, 2, \cdots, 5$, where $\sum_{i=1}^{5} N_{v_i}(u) = n$ is a time independent constant. Denoting the vector variable $(N_{v_1}(u), N_{v_2}(u), \cdots, N_{v_5}(u))$ by $\vec{N}(u)$, we define two functions $H_{V,2}(\vec{N}(u))$ and $H_{V,1}(\vec{N}(u))$. Suppose that there are at least one individual for each species. Replacing species v_i in Fig. 2 by one individual of species v_i, we call this oriented graph as "the tournament [V] for individuals". So the number of all possible tournaments for individuals of the five competing species is now defined as function $H_{V,2}$. As is easily seen $H_{V,2}(\vec{N}(u)) = \prod_{i=1}^{5} N_{v_i}(u)$. Next, we consider five subtournaments isomorphic to $[T_1]$ in Fig. 4, with nodes $\{v_1, v_2, v_4\}$, $\{v_2, v_3, v_5\}$, $\{v_3, v_4, v_1\}$, $\{v_4, v_5, v_2\}$ and $\{v_5, v_1, v_3\}$. Applying the above replacement by individuals, the number of all possible tournaments isomorphic to $[T_1]$ for individuals of the five species of [V] is denoted by $H_{V,1}(\vec{N}(u))$, which is so to speak a measure of $[T_1]$ness of the system at time u. We have

$$\begin{aligned} &H_{V,1}(\vec{N}(u)) \\ &= N_{v_1}(u) N_{v_2}(u) N_{v_4}(u) + N_{v_2}(u) N_{v_3}(u) N_{v_5}(u) \\ &+ N_{v_3}(u) N_{v_4}(u) N_{v_1}(u) + N_{v_4}(u) N_{v_5}(u) N_{v_2}(u) \\ &+ N_{v_5}(u) N_{v_1}(u) N_{v_3}(u). \end{aligned}$$

Denoting F_u by the σ-algebra generated by $\vec{N}(0), \vec{N}(1), \cdots, \vec{N}(u)$,

$$\{(1 - 2\frac{{}_5C_2}{n(n-1)})^{-u} H_{V,2}(\vec{N}(u)),\ \ F_u\ u = 0, 1, 2, \cdots\}$$

and $\{(1 - 2\frac{{}_3C_2}{n(n-1)})^{-u} H_{V,1}(\vec{N}(u)),\ F_u\ \ u = 0, 1, 2, \cdots\}$ are martingales.

If one of N_{v_i}, say N_{v_5}, is zero, the corresponding tournament are represented by Fig. 3. In this case, evidently $H_{V,2}(\vec{N}(u)) \equiv 0$.

$\{(1 - 2\frac{{}_3C_2}{n(n-1)})^{-u} H_{V,1}(\vec{N}(u)),\ F_u\ \ u = 0, 1, 2, \cdots\}$ is also a martingale.

where

$$H_{V,1}(\vec{N}(u)) = N_{v_1}(u)\,N_{v_2}(u)\,N_{v_4}(u) + N_{v_3}(u)\,N_{v_4}(u)\,N_{v_1}(u).$$

Besides, we have

$$E(N_{v_1}(u+1)\,N_{v_3}(u+1)\,N_{v_4}(u+1) \mid F_u) = (1 - \frac{2\,{}_3C_2 + N_{v_2}(u)}{n(n-1)})\,N_{v_1}(u)\,N_{v_3}(u)\,N_{v_4}(u)\,.$$

Using these relations and calculating eigen values and their eigen vectors of the transition matrix for the model, we have an asymptotic result on species coexistence in relation to $[T_1]$ and $[T_2]$.

A computer simulation given in Fig. 5 will help the reader to understand the model. At time 0 there are five species with the tournament [V] in Fig. 2, each with 20 individuals. From time 0 to time 1392 the five species coexist. From 1393 to 1921 the four species with the tournament in Fig. 3 exist. From 1922 to 4786 three species with a tournament isomorphic to $[T_1]$ in Fig. 4 coexist.

For the general case we need the formulations and lemmas which are stated in the following sections.

2. Tournament

A class of oriented graphs, such as given in Fig.2, 3 and 4, is named "tournament". In this section we use a term "node" instead of the term "species". A tournament [T] consists of a set of m nodes $T = \{1, 2, \cdots, m\}$ in which each pair of distinct nodes i and j is joined by one and only one of the oriented arcs $\overrightarrow{ij}$ or $\overrightarrow{ji}$. If the arc $\overrightarrow{ji}$ is in [T], then we say i dominates j and write as $i \succ j$. Two tournaments are isomorphic if there exists a one-to-one dominance preserving correspondence between their nodes. We write $[T] \cong [T']$ if [T] is isomorphic to [T']. A tournament [V] is a subtournament of a tournament [T] if there exists a one-to-one mapping Π between the nodes of [V] and a subset of the nodes of [T] such that if $p \succ q$, then $\Pi(p) \succ \Pi(q)$. We write $[V] \subset [T]$ if [V] is a subtournament of [T]. The subtournament of [T] generated by $X \subset T$ is the tournament with X as its nodes set and with all the dominance relations in [T] that have both their endpoints in X. We define a_{ij} for a tournament [V] as

$$a_{ij} = \begin{cases} 1 & \text{if } i \succ j \\ 0 & \text{if } i = j \\ -1 & \text{if } i \prec j. \end{cases}$$

For a node α of the tournament [V] and the subset W of V, where $\alpha \cap W = \phi$, a dominance relation is defined as

$$\alpha \succ W \quad \text{if} \quad \sum_{i \in W} a_{i\alpha} < 0$$

$$\alpha \prec W \quad \text{if} \quad \sum_{i \in W} a_{i\alpha} > 0.$$

$[T_r]$ is a tournament which consists of a set of nodes $T_r = \{0, 1, 2,$

$\cdots$, 2r} in which each pair of distinct nodes i and j is joined by the dominance relations

$$i \succ j \text{ if } i - j \equiv 1, 2, \cdots, r \pmod{2r+1}.$$

$f_r(\cdot)$ is a periodic function of integers defined by the followings:

i) $f_r(i) \equiv i$ for $0 \le i \le 2r$

ii) $f_r(i+(2r+1)m) = f_r(i)$ for any integer m.

3. Model and its property

We consider a system which satisfies the following.

i) In a population there are individuals of m species v_1, v_2, $\cdots$, v_m whose numbers are $n_{v_1}(0)$, $n_{v_2}(0)$, $\cdots$, $n_{v_m}(0)$ respectively at time zero.

ii) We assume that in a unit time one collision occurs and assume the uniform distribution of colliding pairs.

iii) An individual of species v_i and an individual of species v_j collide with each other and become two individuals of species v_i if $v_i \succ v_j$ or $v_i = v_j$. v's are nodes of the tournament $[V] \subset [T_s]$.

To clarify the above i), ii), iii), we pose iv).

iv) The numbers of individuals of the m species at time u are expressed by the random vector $\vec{N}(u) = (N_{v_1}(u), N_{v_2}(u), \cdots, N_{v_m}(u))$ which are defined by the following markov chain

$$P(\vec{N}(u+1) = \vec{n}_{ij} \mid \vec{N}(u) = \vec{n}) = 2\,\frac{n_{v_i} n_{v_j}}{n(n-1)}$$

$$P(\vec{N}(u+1) = \vec{n} \mid \vec{N}(u) = \vec{n}) = \sum_{i=1}^{m} \frac{n_{v_i}(n_{v_i}-1)}{n(n-1)}\ ;$$

where $\vec{n} \equiv (n_{v_1}, n_{v_2}, \cdots, n_{v_m})$, $n = \sum_{i=1}^{m} n_{v_i}$, and $\vec{n}_{ij}$ is a m dimensional vector defined for $i \neq j$ in which i's component n_{v_i} and j's component n_{v_j} of $\vec{n}$ are replaced by $n_{v_i} + a_{v_i v_j}$ and $n_{v_j} + a_{v_j v_i}$ respectively.

Writing $P(\vec{n}_{ij} \mid \vec{n})$ instead of $P(\vec{N}(u+1) = \vec{n}_{ij} \mid \vec{N}(u) = \vec{n})$ and $P(\vec{n};u)$ instead of $P(\vec{N}(u)=\vec{n})$, we have

(1) $$P(\vec{n};u+1) = P(\vec{n} \mid \vec{n})P(\vec{n};u) + \frac{1}{2} \sum_{\substack{\vec{n}=\vec{n}'_{ij} \\ i \neq j}} P(\vec{n}'_{ij} \mid \vec{n}')P(\vec{n}';u).$$

Putting $P_{v_i}(u) = \frac{n_{v_i}(u)}{n}$, and taking an appropriate time scale, we have the following deterministic approximation for the process, which is used in the proof of Lemma 2.

(2) $$\frac{d}{dt} P_{v_i}(t) = P_{v_i}(t)\left(\sum_{v_j \varepsilon V}^{m} a_{v_i v_j} P_{v_j}(t)\right)$$

for $v_i \varepsilon V$, with $P_{v_i}(t) > 0$ and $\sum_{v_i \varepsilon V} P_{v_i}(t) = 1$.

Notations and assumptions for Lemmas and Theorem. [X], [W], [W'] and [V] are the subtournaments of $[T_s]$ generated by nodes sets X, W, W' and V respectively where X, W, W', $V \subset T_s$ and W, $W' \subset V$.

$$g_X(\vec{n}) \equiv \prod_{v_i \varepsilon X} n_{v_i}.$$

$$H_{V,r}(\vec{n}) \equiv \sum_{\substack{X \subset V \\ [X] \cong [T_r]}} g_X(\vec{n}).$$

t's, v's, and w's are nodes of the tournaments $[T_s]$, [V] and [W] respectively. a's are defined for the tournament $[T_s]$ (See section 2).

Lemma 1. Let $[W] \cong [T_r]$,

$$\sum_{v_i \varepsilon W} a_{v_i v_j} = \begin{cases} 1 & \text{if } v_j \prec W \\ 0 & \text{if } v_j \varepsilon W \\ -1 & \text{if } v_j \succ W. \end{cases}$$

Lemma 2. Let $[W] \cong [T_r]$ and $[W'] \cong [T_r]$.

If $W \succ v_i$ ($W \prec v_i$) for every $v_i \in V \backslash W$ and $W' \succ v_i$ ($W' \prec v_i$) for every $v_i \in V \backslash W'$, then $W = W'$.

Proof. If $[W] \cong [T_r]$, and $W \succ v_i$ for every $v_i \in V \backslash W$,

$$(2) \qquad \frac{d}{dt} \prod_{v_i \in W} P_{v_i}(t) = \prod_{v_i \in W} P_{v_i}(t) \Big(\sum_{v_i \in V \backslash W} P_{v_i}(t) \Big) > 0.$$

Since $\sum_{v_i \in V} P_{v_i}(t) = 1$, the following two must hold for Eq. (2).

i) $\prod_{v_i \in W} P_{v_i}(t)$ increases and $\sum_{v_i \in V \backslash W} P_{v_i}(t)$ approaches to zero.

ii) $\prod_{v_i \in W'} P_{v_i}(t)$ increases and $\sum_{v_i \in V \backslash W'} P_{v_i}(t)$ approaches to zero.

If $W' \neq W$ the above two is not mutually consistent from the uniqueness of the solution of Eq. (2). Q.E.D.

Lemma 3. Let $[W] \cong [T_r]$, $W = \{w_0, w_1, \cdots, w_{2r}\}$ where $0 \leq w_0 < \cdots < W_{2r} \leq 2s$, and $w_i \prec v_\alpha \prec w_{f_r(i+1)}$. If $W \prec v_\alpha$ ($W \succ v_\alpha$), W' and $v_{\alpha'}$ which satisfy $W' \cup v_{\alpha'} = W \cup v_\alpha$, $[W'] \cong [T_r]$, and $v_{\alpha'} \prec W'$ ($v_{\alpha'} \succ W'$), are uniquely determined as $W' = (W \backslash w_i) \cup v_\alpha$ ($W' = (W \backslash w_{f_r(i+1)}) \cup v_\alpha$), and $v_{\alpha'} = w_i$ ($v_{\alpha'} = w_{f_r(i+1)}$).

We write the above W' as $W(v_\alpha)$ and $v_{\alpha'}$ as $v_\alpha(W)$.

Proof. From the assumptions,

$$v_\alpha \prec w_{f_r(i+j)} \quad \text{for} \quad j = 1, 2, \cdots, r,$$

and

$$v_\alpha \succ w_{f_r(i+j)} \quad \text{for} \quad j = 0, -1, \cdots, -r .$$

Thus we see

$$\{x \mid v_\alpha \succ x,\ x \in (W \setminus w_i)\} = \{x \mid w_i \succ x,\ x \in W\}$$

$$\{x \mid v_\alpha \prec x,\ x \in (W \setminus w_i)\} = \{x \mid w_i \prec x,\ x \in W\}.$$

From this we have

$$W' = (W \setminus w_i) \cup v_\alpha,\ v_{\alpha'} = w_i .$$

We see the uniqueness from Lemma 2. Q.E.D.

Lemma 4. Let $[W] \equiv [T_M]$ for $M = \text{Max}\,\{r \mid [T_r] \subset [V]\}$, $W \prec v_i$ for every $v_i \in V \setminus W$ and let $t_\alpha \prec W$ for t_α, which satisfies $t_\alpha \cap V = \phi$, then one of the following two holds (See Notations and assumptions).

i) $\text{Max}\,\{r \mid [T_r] \subset [V \cup t_\alpha]\} = M + 1$

ii) $W(t_\alpha) \prec t_i$ for every $t_i \in (V \cup t_\alpha) \setminus W(t_\alpha)$.

Proof. Based on Lemma 3, we can prove the statement;

Let $(W \cup v_{\alpha_1} \cup v_{\alpha_2}) \subset T_s$, $[W] \cong [T_r]$, $v_{\alpha_1} \succ W$, $v_{\alpha_2} \prec W$, and $W(v_{\alpha_2}) \succ v_{\alpha_1}$, then $[W \cup v_{\alpha_1} \cup v_{\alpha_2}] \cong [T_{r+1}]$. From this, if ii) does not hold, there exists v_α which satisfies $v_\alpha \in V \setminus W$ and $W(t_\alpha) \succ v_\alpha$. So i) must hold. Q.E.D.

Lemma 5. Let

$$\text{Max}\,\{r \mid [T_r] \subset [V]\} = M,$$

i) then W which satisfies the followings is uniquely determined.

$$[W] \cong [T_M]$$

and

$$v_i \succ W$$

for every $v_i^* \in V \setminus W$.

ii) for the W, let F_u be the σ-algebra generated by $\vec{N}(0), \vec{N}(1), \cdots, \vec{N}(u)$,

$$E(g_W(\vec{N}(u+1)) \mid F_u) = \left(1 - \frac{{}_{2M+1}C_2 + \sum_{i \in V \setminus W} N_i(u)}{n(n-1)}\right) g_W(\vec{N}(u)). \tag{3}$$

Proof.

i) We construct W from an arbitrary X which satisfies

$$[X] \equiv [T_M].$$

From Lemma 3 one of the following two holds, for X and v_α with $X \cap v_\alpha = \phi$.

a) $X \prec v_\alpha$,

b) $X(v_\alpha) \prec v_\alpha(X)$.

Since $\mathrm{Max}\{r \mid [T_r] \subset [V]\} = M$, Lemma 4 assures us that W can be constructed recursively from the X.

ii) Using Lemma 1, we have

$$\frac{1}{2} \sum_{\substack{i,j \in W \\ i \neq j}} (g_W(\vec{n}_{ij}) - g_W(\vec{n}))\, P(\vec{n}_{ij} \mid \vec{n}) = - \frac{{}_{2M+1}C_2}{n(n-1)} g_W(\vec{n}),$$

and

$$\sum_{\substack{i \in W \\ j \in V \setminus W}} (g_W(\vec{n}_{ij}) - g_W(\vec{n}))\, P(\vec{n}_{ij} \mid \vec{n}) = \frac{- \sum_{i \in V \setminus W} n_i}{n(n-1)} g_W(\vec{n}).$$

Using these relations, we have

$$E(g_W(\vec{N}(u+1)) \mid \vec{N}(u) = \vec{n}) = \left(1 - \frac{{}_{2M+1}C_2 + \sum_{i \in V \setminus W} n_i}{n(n-1)}\right) g_W(\vec{n}). \qquad \text{Q.E.D.}$$

Lemma 6. Let F_u be the σ-algebra generated by $\vec{N}(0)$, $\vec{N}(1)$, $\cdots$, $\vec{N}(u)$, and let $\mathrm{Max}\{r \mid [T_r] \subset [V]\} = M$, then the sequence $\{W_{V,r}(u), F_u\ u = 0, 1, 2, \cdots\}$ is a martingale for $r = 1, 2, \cdots, M$, in which

$$W_{V,r}(u) = \left(1 - 2\frac{{}_{2r+1}C_2}{n(n-1)}\right)^{-u} H_{V,r}(\vec{N}(u)). \tag{4}$$

Proof. From Lemma 3 and Lemma 1, if $[X] \equiv [T_r]$, for $j \in V\setminus X$

$$\sum_{i\in X}(g_X(\vec{n}_{ij}) - g_X(\vec{n}))P(\vec{n}_{ij} \mid \vec{n}) + \sum_{i\in X(j)}(g_{X(j)}(\vec{n}_{ij(X)}) - g_{X(j)}(\vec{n}))P(\vec{n}_{ij(X)} \mid \vec{n}) = 0.$$

($X(j)$ and $j(X)$ are defined in Lemma 3.) So we have

$$\begin{aligned}
&\sum_{\substack{V\supset X\\ [T_r]\equiv[X]}} \sum_{\substack{i\in X\\ j\in V\setminus X}} (g_X(\vec{n}_{ij}) - g_X(\vec{n}))P(\vec{n}_{ij} \mid \vec{n})\\
&= \frac{1}{2}\sum_{\substack{V\supset X\\ [T_r]\equiv[X]}} \sum_{j\in V\setminus X} \Big(\sum_{i\in X}(g_X(\vec{n}_{ij}) - g_X(\vec{n}))P(\vec{n}_{ij} \mid \vec{n})\\
&\quad + \sum_{i\in X(j)}(g_{X(j)}(\vec{n}_{ij(X)}) - g_{X(j)}(\vec{n}))P(\vec{n}_{ij(X)} \mid \vec{n})\Big)\\
&= 0.
\end{aligned}$$

Since

$$g_X(\vec{n}_{ij}) - g_X(\vec{n}) = 0 \quad \text{for } i, j \in V\setminus X,$$

$$\sum_{\substack{V\supset X\\ [T_r]\equiv[X]}} \sum_{\substack{i\neq j\\ i,j\in V\setminus X}} (g_X(\vec{n}_{ij}) - g_X(\vec{n}))P(\vec{n}_{ij} \mid \vec{n}) = 0.$$

We have

$$\begin{aligned}
&\frac{1}{2}\sum_{\substack{V\supset X\\ [T_r]\equiv[X]}} \sum_{\substack{i\neq j\\ i,j\in X}} (g_X(\vec{n}_{ij}) - g_X(\vec{n}))P(\vec{n}_{ij} \mid \vec{n})\\
&= -\frac{2\,{}_{2r+1}C_2}{n(n-1)} \sum_{\substack{V\supset X\\ [T_r]\equiv[X]}} g_X(\vec{n}).
\end{aligned}$$

From these relations, we have

$$E(H_{V,r}(\vec{N}(u+1)) \mid \vec{N}(u) = \vec{n}) - H_{V,r}(\vec{n}) = -\frac{2\,{}_{2r+1}C_2}{n(n-1)} H_{V,r}(\vec{n}).$$

Q.E.D.

4. Main theorem

We use the following notations.

$$E_{V,r} = \bigcup_{\substack{X \subset V \\ [T_r] \cong [X]}} E_V(X)$$

where

$$E_V(X) = \{(n_{v_1}, n_{v_2}, \cdots, n_{v_m}) \mid \sum_{i=1}^{m} n_{v_i} = n, \text{ n's are nonnegative integers, } n_{v_i} > 0 \text{ for every } v_i \in X, n_{v_i} = 0 \text{ for every } v_i \in V \setminus X, X \subset V\}.$$

L_r is the number of elements of the set $E_V(V)$ for $m = 2r+1$.

Theorem. Let $[V] \subset [T_s]$ and $M = \text{Max}\,\{r \mid [T_r] \subset [V]\}$,

then

$$(5) \lim_{u\to\infty} \frac{P(\vec{N}(u) \in E_{V,r} \mid \vec{N}(0) = \vec{y})}{H_{V,r}(\vec{y})\left(\frac{1}{L_r} \sum_{\substack{n_o+n_1+\cdots+n_{2r}=n \\ n_o,n_1,\cdots,n_{2r}>0}} n_o n_1 \cdots n_{2r}\right)^{-1}\left(1 - 2\,\frac{{}_{2r+1}C_2}{n(n-1)}\right)^u} = 1$$

for $r = 1, 2, \cdots, M$.

Proof of theorem.

i) the case of $[V] \cong [T_r]$

Without loss of generality we can assume $v_{i+1} = i$ for $i = 0, 1, 2, \cdots, 2r$. We call the set of vectors,

$$\{(n_o, n_1, \cdots, n_i, \cdots, n_{2r}) \mid \sum_{i=o}^{2r} n_i = n, \text{ n's are nonnegative integers, } n_i = 0 \text{ for at least one of n's}\},$$

state 0. By an appropriate numbering for the distinct elements of $E_V(V)$ from 1 to L_r, we can consider the state space $S = (0, 1, \cdots, L_r)$.

The transition matrix is written as

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ P_{1o} & & & & \\ P_{2o} & & & & \\ \vdots & & Q & & \\ P_{Lo} & & & & \end{bmatrix} .$$

The matrix A has maximum eigenvalue 1 corresponding to the left eigenvector $\vec{x}_o = (1, 0, 0, \cdots, 0)$ since the state 0 is absorbing. The second eigen value λ_1 of A is the maximum eigenvalue of Q, and hence is real and satisfies

$$\lambda_o = 1 \geq \lambda_1 \geq |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_{L_r}| .$$

We can find $\lambda_1 = 1 - 2 \frac{{}_{2r+1}C_2}{n(n-1)} > |\lambda_2|$ and the left eigen vector of λ_1 is $\vec{x}_1 = (d, \frac{1}{L_r}, \frac{1}{L_r}, \cdots, \frac{1}{L_r})$ for an appropriate d.

Let $\vec{x}_i$ be the left eigen vector corresponding to λ_i. Any $1 + L_r$ dimensional vector is expressed as

$$\vec{x} = C_o(\vec{x})\vec{x}_o + C_1(\vec{x})\vec{x}_1 + \cdots + C_{L_r}(\vec{x})\vec{x}_{L_r} .$$

Thus we have

$$\vec{x}A^u = C_o(\vec{x})\vec{x}_o + C_1(\vec{x})(1 - 2 \frac{{}_{2r+1}C_2}{n(n-1)})^u \vec{x}_1 + O(1 - 2 \frac{{}_{2r+1}C_2}{n(n-1)})^u ,$$

where

$$\vec{x}_o = (1, 0, 0, \cdots, 0)$$

and

$$\vec{x}_1 = (d, \frac{1}{L_r}, \frac{1}{L_r}, \cdots, \frac{1}{L_r}).$$

Since

$$[V] \cong [T_r], \quad H_{V,r}(\vec{N}(u)) = \prod_{i=0}^{2r} N_i(u),$$

we have

$$\lim_{u\to\infty} E(H_{V,r}(\vec{N}(u) \mid \vec{N}(u) \in E_{V,r}) = \sum_{\substack{n_0+n_1+\cdots+n_{2r}=n \\ n_0>0, n_1>0, \cdots, n_{2r}>0}} n_0 n_1 \cdots n_{2r} \frac{1}{L_r}.$$

From Lemma 6, we have

$$E(H_{V,r}(\vec{N}(u)) \mid \vec{N}(0) = \vec{y}) = H_{V,r}(y)(1 - 2\frac{{}_{2r+1}C_2}{n(n-1)})^u.$$

If $\vec{N}(u) \notin E_V(V)$, $H_{V,r}(\vec{N}(u)) = 0$. So

$$E(H_{V,r}(\vec{N}(u)) \mid \vec{N}(0) = \vec{y}) = E(H_{V,r}(\vec{N}(u)) \mid \vec{N}(u) \in E_{V,r}) P(\vec{N}(u) \in E_{V,r} \mid \vec{N}(0) = \vec{y})$$

Thus we have

$$\lim_{u\to\infty} \frac{P(\vec{N}(u) \in E_{V,r} \mid \vec{N}(0) = \vec{y})}{H_{V,r}(\vec{y})(\frac{1}{L_r} \sum_{\substack{n_0+n_1+\cdots+n_{2r}=n \\ n_0, n_1, \cdots, n_{2r}>0}} n_0 n_1 \cdots n_{2r})^{-1} (1 - 2\frac{{}_{2r+1}C_2}{n(n-1)})^u}$$

$= 1$, for the case $[V] \cong [T_r]$.

ii) the general case.

If $[T_r] \subset [X]$ and $[T_r] \not\cong [X]$,

$$P(\vec{N}(u) \in E_V(X) \mid \vec{N}(0) = \vec{y})$$

$$= O(1 - 2\frac{{}_{2M(X)+1}C_2}{n(n-1)})^u = O(1 - 2\frac{{}_{2r+1}C_2}{n(n-1)})^u$$

for $M(X) = \mathrm{Max}\{r \mid [T_r] \subset [X]\}$, from Lemma 5. Using Lemma 6 and the above i), we have Eq. (5) for $r = 1, 2, \cdots, M$. Q.E.D.

5. Applications

1) Kimura [8] gave an example for his fundamental theorem for natural selections, as the following. Consider three alleles A_1, A_2 and A_3 with frequencies p_1, p_2 and p_3. For simplicity assume a haplont situation. Suppose an individual with gene A_1 has its fitness decreased by the amount c when it is surrounded by individuals with A_2, while A_2 if surrounded by the former A_1 gains fitness by the same amount c. Similarly A_2 when surrounded by A_3 individuals loses fitness by c, while the reverse situation whould add fitness c to an A_3 individual, and so on cyclically. Then the law of change of gene frequencies will be the deterministic approximation of our process in the case $[V] \equiv [T_1]$, as

$$\begin{aligned} \frac{d}{dt} p_1 &= cp_1(p_3 - p_2) \\ \frac{d}{dt} p_2 &= cp_2(p_1 - p_3) \qquad (6) \\ \frac{d}{dt} p_3 &= cp_3(p_2 - p_1) \end{aligned}$$

In this case

$$\frac{d}{dt} p_1 p_2 p_3 = 0.$$

Mather [11] considered selections through competition of plants in various situations. He dicussed a difference equation corresponding to the following differential equation

$$\frac{d}{dt} p_i = p_i\left(\sum_{j=1}^{m} a_{ij} p_j\right), \quad a_{ij} = -a_{ji} \text{ for } i = 1, 2, \cdots, m.$$

Our model represents the stochastic case of the situations treated by Kimura and Mather.

For applications the following assumption ii)' is more reasonable than ii) of our model.

ii)' each individual collide with another individual one time on the average in a unit time interval.

So u of the denominator of Eq. (5) must be replaced by $\frac{n}{2}u$. Since

$$H_{V,r}(\vec{y})\Big(\frac{1}{L_r}\sum_{\substack{n_o+n_1+\cdots+n_{2r}=n\\ n_o,n_1,\cdots,n_{2r}>0}} n_o n_1\cdots n_{2r}\Big)^{-1}$$

$$= H_{V,r}(\frac{1}{n}\vec{y})\Big(\frac{1}{L_r}\sum_{\substack{n_o+n_1+\cdots+n_{2r}=n\\ n_o,n_1,\cdots,n_{2r}>0}} \frac{n_o}{n}\frac{n_1}{n}\cdots\frac{n_{2r}}{n}\Big)^{-1};$$

and since

$$\frac{1}{L_r}\sum_{\substack{n_o+n_1+\cdots+n_{2r}=n\\ n_o,n_1,\cdots,n_{2r}>0}} \frac{n_o}{n}\frac{n_1}{n}\cdots\frac{n_{2r}}{n}$$

is approximated by $k = \int\int\cdots\int_{\substack{p_o+p_1+\cdots+p_{2r}=1\\ p_o,p_1,\cdots,p_{2r}>0}} p_o p_1\cdots p_{2r}\, dp_o dp_1\cdots dp_{2r}$

for sufficiently large n, we have an asymptotic formulla

$$P(\vec{N}(u)\in E_{V,r}\mid \vec{N}(0)=\vec{y}) \approx k' H_{V,r}(\frac{1}{n}\vec{y})\, e^{-{}_{2r+1}C_2 \frac{u}{n}}$$

for a constant $k' \doteqdot k$, where $\frac{1}{n}\vec{y}$ represents frequencies of species.

Consider a random collision model with the collision rule that one i and one j become two i's with probability $\frac{1}{2}+a$ and two j's with probability $\frac{1}{2}-a$ for $0 \le a \le \frac{1}{2}$ if $i \succ j$. Wright's model is the case $a = 0$ for which the asymptotic probabilities of species coexistence are obtained for an arbitrary combination of species. (See Kimura [7]). The case $a = \frac{1}{2}$ is treated in our theorem. Analogous results can be obtained for the other cases.

2) Our random collision process is approximated by the system of stochastic differential equations,

$$
\begin{aligned}
dX_1(t) &= c_1X_1(t)(X_3(t) - X_2(t))dt \\
&\quad + c_2 \sqrt{X_1(t)X_3(t)}\, db_{31}(t) - c_2 \sqrt{X_1(t)X_2(t)}\, db_{12}(t), \\
dX_2(t) &= c_1X_2(t)(X_1(t) - X_3(t))dt \\
&\quad + c_2 \sqrt{X_2(t)X_1(t)}\, db_{12}(t) - c_2 \sqrt{X_2(t)X_3(t)}\, db_{23}(t), \\
dX_3(t) &= c_1X_3(t)(X_2(t) - X_1(t))dt \\
&\quad + c_2 \sqrt{X_3(t)X_2(t)}\, db_{23}(t) - c_2 \sqrt{X_3(t)X_1(t)}\, db_{31}(t),
\end{aligned} \tag{7}
$$

where $b_{12}(t)$, $b_{23}(t)$ and $b_{31}(t)$ are mutually independent Brownien motion. Using Itô's formulla (See [1]), we have

$$d(X_1(t) + X_2(t) + X_3(t)) = 0,$$

and for the σ-algebra F_t generated by $X(s) \equiv (X_1(s),X_2(s),X_3(s))$ for $0 \leq s \leq t$,

$$E(d(X_1(t)X_2(t)X_3(t))| F_t) = -\frac{1}{2} c_2^2 \; {}_3C_2 X_1(t)X_2(t)X_3(t)dt.$$

If $c_1 = 0$, Eq. (7) represents Wright's model for three alleles. If $c_2 = 0$, Eq. (7) coincides to Eq. (6). Eq. (7) seems to have interesting properties. But we could not prove the existence and uniqueness of the solution.

3) MacArthur [10] considered from a census of birds that, if m-1 points are thrown at random on a stick and if the stick is broken at these points, the length of the m resulting segments represent the relative abundances of the m species. As we see in the proof of our theorem, each element of the set $E_V(V)$ with $[V] \equiv [T_r]$ tends to be equally probable. So our results give a justification for MacArthur's consideration.

4) We consider a system of competing brands in a market. For example, one who usually takes cigarettes of brand j has opportunities of comparing to the other brands. The collision rule of the model treated here corresponds to that he compares to his j to a cigarette of i and decides to change his favorite brand to i. So the model treated here represents an idealized situation of competing brands in a market. The speed of approach to the monopoly is estimated by our results.

Acknowledgement.

The author is indebted to Profs. K. Kunisawa, T. Maruyama and R. L. Tweedie for advices and discussions. He is indebted to the referee for helpful suggestions. He thanks Miss S. Ueda for the computer program of the simulation.

References

[1] Itô, K. (1951) On a formula concerning stochastic differentials, Nagoya Math. J. 3, 55-65.

[2] Itoh, Y. (1971) The Boltzmann equation on some algebraic structure concerning struggle for existence, Proc. Japan Acad. 47, 854-858.

[3] Itoh, Y. (1973) On a ruin problem with interaction, Ann. Inst. Statist. Math. 25, 635-641.

[4] Itoh, Y. (1975) An H-theorem for a system of competing species, Proc. Japan Acad. 51, 374-379.

[5] Kac, M. (1959) Probability and related topics in physical sciences, Interscience Publisher, London.

[6] Karlin, S. (1966) A first course in stochastic processes, New York and London.

[7] Kimura, M. (1955) Random genetic drift in multi-allelic Locus, Evolution 9, 419-435.

[8] Kimura, M. (1958) On the change of population fitness by natural selection, Heredity 12, 145-167.

[9] Lotka, A. J. (1925) Elements of physical biology, Williams & Wilkins.

[10] MacArthur, R. H. (1957) On the relative abundance of bird species, Proc. Nat. Acad. Sci. U. S. A. 43, 293-295.

[11] Mather, K. (1969) Selection through competition, Heredity 24, 529-540.

[12] Moon, J. W. (1968) Topics on tournaments, New York.

[13] Moran, P. A. P. (1958) Random processes in genetics, Proc. Camb. Phil. Soc. 54, 60-71.

[14] Volterra, V. (1931) Lecons sur la theorie mathematique de la lutte pour la vie, Cahiers Scientifiques VII, Gauthier-Villars, Paris.

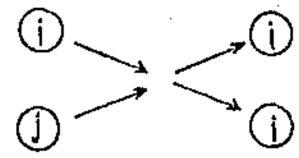

Fig.1

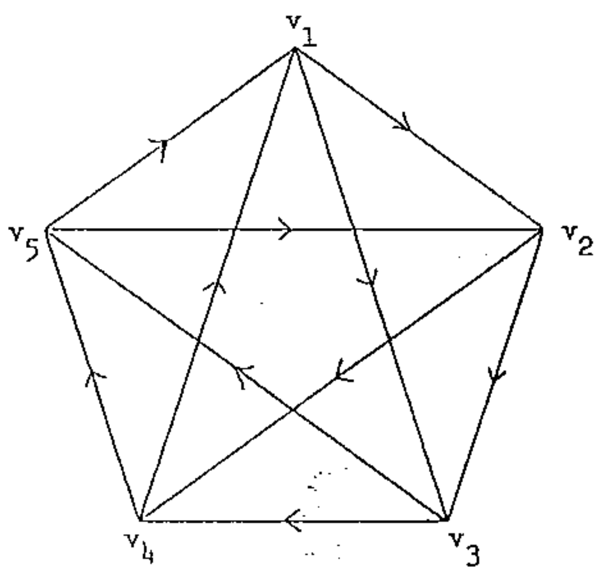

Fig.2 (isomorphic to $[T_2]$)

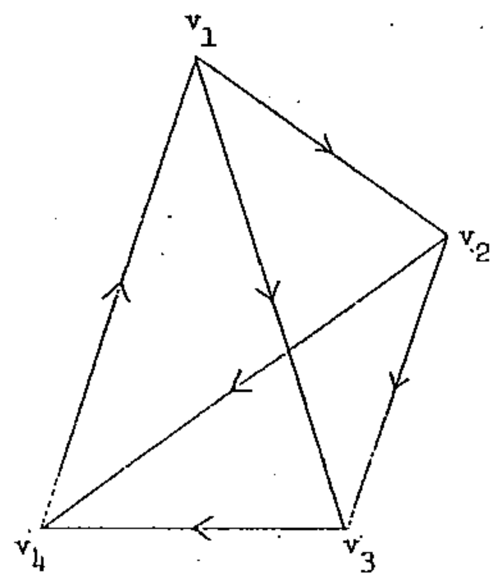

Fig.3

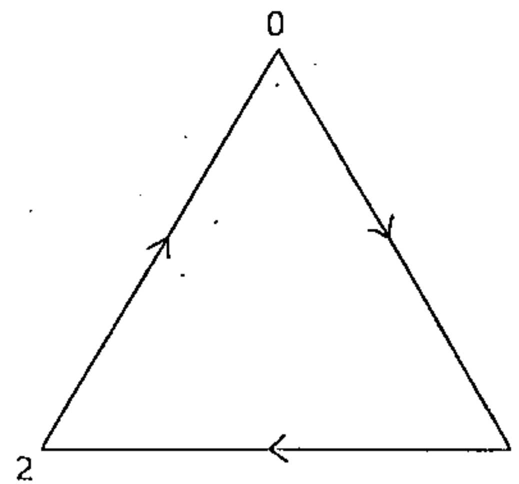

Fig.4 $[T_1]$

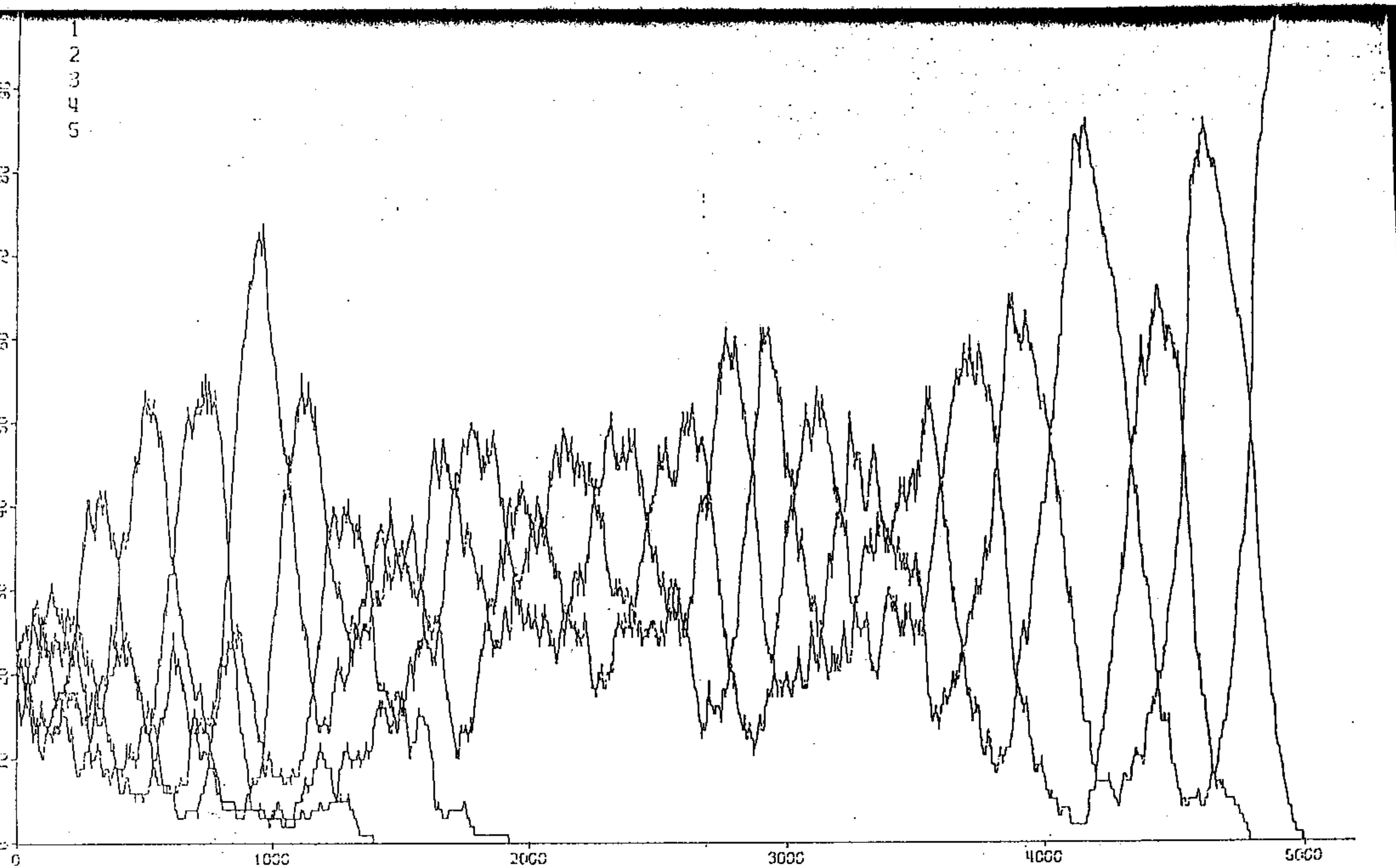

Fig.5