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ON A RUIN PROBLEM WITH INTERACTION

YOSHIAKI ITOH

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In the previous paper [1] we interpreted the Volterra's model as a Boltzmann equation on some algebraic structure. Here we will consider the discrete model which corresponds to the Volterra's model concerning struggle for existence.

Ehrenfest considered an interesting model which explains Boltzmann's H theorem. P. A. P. Moran extended the Ehrenfest's model and considered some problem of genetics.

We consider a similar model like Moran's and treat a Markov chain which is a discrete analogue of Volterra's equation.

Volterra treated the following nonlinear equation.

$$(1) \qquad \frac{\partial}{\partial t} n_i = \left(\sum_{j=1}^{r} a_{ij} n_j \right) n_i \qquad a_{ij} = -a_{ji} \ .$$

Since n_i means the number of species i, it is essentially discrete variable. So it is not meaningless to think a Markov chain corresponds to (1). We can take this problem as a ruin problem with interaction.

1. On a Markov chain

i) In a box there are three types of particles, ①, ② and ③ whose numbers are n_1, n_2 and n_3 with $\sum_{j=1}^{3} n_i = N$.

ii) We define the following collision rule,

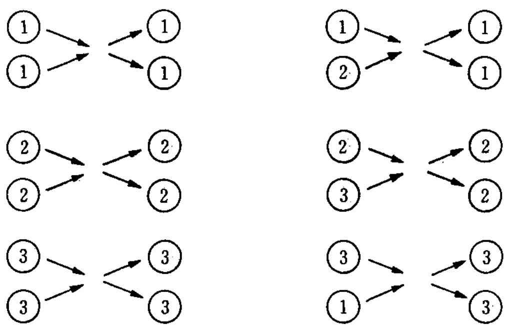


Fig. 1

iii) We assume that in a unit time one collision occurs, and assume the uniform distribution of colliding pairs. For simplifying the formulation we permit a collision with itself.

By i), ii) and iii) we can derive the following Markov chain.

The state is in (n_1, n_2, n_3) with probability $\mathrm{P}(n_1, n_2, n_3; t)$ at time t, for the appropriate initial condition.

$$\begin{aligned}\mathrm{P}(n_1, n_2, n_3; t+1)=\frac{1}{(n_1+n_2+n_3)^2}\{&(n_1^2+n_2^2+n_3^2)\mathrm{P}(n_1, n_2, n_3; t)\\&+2(n_1-1)(n_2+1)\mathrm{P}(n_1-1, n_2+1, n_3; t)\\&+2(n_2-1)(n_3+1)\mathrm{P}(n_1, n_2-1, n_3+1; t)\\&+2(n_3-1)(n_1+1)\mathrm{P}(n_1+1, n_2, n_3-1; t)\}\ .\end{aligned}$$

Consider the product of numbers of three species at time t. It is represented by a random variable $H_3(t)$.

THEOREM 1.

$$\mathrm{E}(H_3(t+1)\,|\,H_3(t))=\left(1-2\frac{{}_3C_2}{N^2}\right)H_3(t)\ .$$

PROOF. Let the state be in (n_1, n_2, n_3) at time t and consider the product of the numbers of three species. By i), ii) and iii) the state is in

$$\begin{array}{lll}(n_1+1, n_2-1, n_3) & \text{with probability} & \dfrac{2n_1n_2}{N^2}\\[2ex] (n_1, n_2+1, n_3-1) & \quad\prime\prime & \dfrac{2n_2n_3}{N^2}\\[2ex] (n_1-1, n_2, n_3+1) & \quad\prime\prime & \dfrac{2n_3n_1}{N^2}\\[2ex] (n_1, n_2, n_3) & \quad\prime\prime & \dfrac{n_1^2+n_2^2+n_3^2}{N^2}\ .\end{array}$$

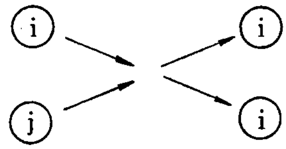
So the expectation of the product is

$$\left(1-2\frac{{}_3C_2}{N^2}\right)n_1n_2n_3\ .$$

2. On an odd number of species

On an odd number of species, we can determine equivalence relation as the case of three species. Then we can consider a similar Markov chain and derive the similar relation.

DEFINITION. If the collision between i and j is represented as the following, we say that i is stronger than j.



DEFINITION. There are $2k+1$ species in a box. If each species is stronger than k species but weaker than another k species, we say that all species are equivalent.

It is obvious that the system with an even number of species can not satisfy the above equivalence.

i) There are $2k+1$ species in a box whose numbers are $n_1, n_2, \cdots, n_{2k+1}$, with

$$\sum_{i=1}^{2k+1} n_i = N .$$

ii) All species are equivalent.

iii) We assume that in a unit time one collision occurs, and assume the uniform distribution of colliding pairs. For simplifying the formulation we permit a collision with itself.

iv) The random variable $H_{2k+1}(t)$ means the product of the numbers of $2k+1$ species.

THEOREM 2. *If* i), ii), iii) *and* iv) *are satisfied, we can derive the following relation* $\mathrm{E}\,(H_{2k+1}(t+1) \mid H_{2k+1}(t)) = (1-2({}_{2k+1}C_2/N^2))H_{2k+1}(t)$.

PROOF. Let the state be in $(n_1, n_2, \cdots, n_{2k+1})$ at time t and consider the product of the numbers of $2k+1$ species. By the above i), ii), iii), iv) the state at time $t+1$ is in

$$(n_1, n_2, \cdots, n_i+1, \cdots, n_j-1, \cdots, n_{2k+1})$$

$$\text{with probability } \frac{2n_i n_j}{N^2}$$

and remains unchanged with probability $\sum_{i=1}^{2k+1} n_i^2/N^2$. So the expectation of the product is $(1-2({}_{2k+1}C_2/N^2))n_1 n_2 \cdots n_{2k+1}$.

3. On an even number of species

DEFINITION. If all species become equivalent by addition of an appropriate species, we say that all species are quasi-equivalent.

To simplify the discussion we consider the case of six species. If all species are quasi-equivalent, we can represent the relation among the species by Fig. 2, without loss of generality. If we take away any one of two species 1 or 6, all species become equivalent.

In general case of $2k$ species we can find two species i and j such that all species become equivalent if we take away any one of these two.

	1	2	3	4	5	6
1	0	1	−1	1	−1	1
2	−1	0	1	−1	1	−1
3	1	−1	0	1	−1	1
4	−1	1	−1	0	1	−1
5	1	−1	1	−1	0	1
6	−1	1	−1	1	−1	0

This means species 2 is stronger than species 3.

Fig. 2

One of these two is stronger than $k-1$ species and weaker than another k species. We name this species "weakest species".

We consider the following system.

i) There are $2k$ species in a box composed of $n_1, n_2, \cdots, n_{2k}$ individuals with $\sum_{i=1}^{2k} n_i = N$.

ii) All species are quasi-equivalent.

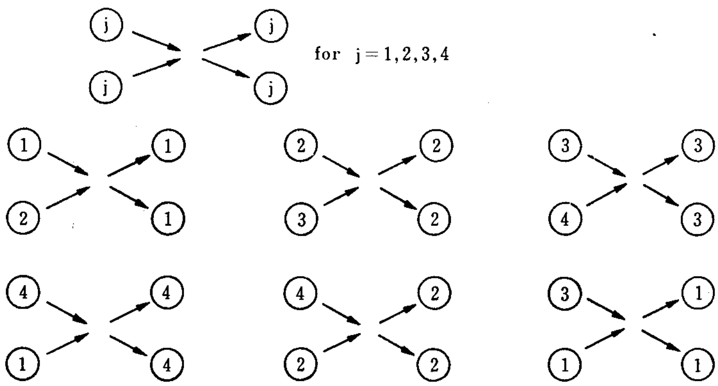
iii) We assume that in a unit time one collision occurs, and assume the uniform distribution of colliding pairs. For simplifying the formulation we permit a collision with itself.

iv) The random variable $H_{i,2k}(t)$ means the product of $2k-1$ species excluding weakest species i.

THEOREM 3. *If the above* i), ii), iii) *and* iv) *are satisfied, and if the number of weakest species i is represented by random variable $X_i(t)$, the following relation holds.*

$$\mathrm{E}\,(H_{i,2k}(t+1)\,|\,H_{i,2k}(t)) = \left(1 + \frac{2X_i(t) - {}_{2k-1}C_2}{N^2}\right) H_{i,2k}(t)\ .$$

PROOF. To simplify the discussion we consider the case of four species. The collision rule can be represented by the following



Let the state be (n_1, n_2, n_3, n_4), then the state at time $t+1$ is

$$(n_1+1, n_2-1, n_3, n_4) \quad \text{with probability} \quad \frac{2n_1n_2}{N^2}$$

$$(n_1, n_2+1, n_3-1, n_4) \quad \text{"} \quad \frac{2n_2n_3}{N^2}$$

$$(n_1, n_2, n_3+1, n_4-1) \quad \text{"} \quad \frac{2n_3n_4}{N^2}$$

$$(n_1-1, n_2, n_3, n_4+1) \quad \text{"} \quad \frac{2n_4n_1}{N^2}$$

$$(n_1, n_2+1, n_3, n_4-1) \quad \text{"} \quad \frac{2n_2n_4}{N^2}$$

$$(n_1+1, n_2, n_3-1, n_4) \quad \text{"} \quad \frac{2n_1n_3}{N^2} .$$

So the expectation of the product is

$$\left(1+\frac{2n_4-{}_{2k-1}C_2}{N^2}\right)n_1n_2n_3 .$$

So

$$\mathrm{E}\,(H_3'(t+1)\,|\,H_3'(t))=\left(1+\frac{2X_4(t)-{}_3C_2}{N^2}\right)H_3'(t) .$$

And we can prove

$$\mathrm{E}\,(H_{2k-1}'(t+1)\,|\,H_{2k-1}'(t))=\left(1+\frac{2X_i(t)-{}_{2k-1}C_2}{N^2}\right)H_{2k-1}'(t) .$$

4. Conclusion

Let there be $2s+1$ species and all species are equivalent, the state transition is governed by Theorem 2. In an appropriate time, one species ruins and all other species become quasi-equivalent. Then the system is governed by Theorem 3. In an appropriate time species i will be ruined and all species will be equivalent and system will consist of $2s-1$ species. Then the system will be governed by Theorem 2. In this way Theorems 2 and 3 will be alternately applied, and at last one species will remain.

5. An example of computer simulation

Let there be 5 species each with 20 individuals and all species be equivalent, the change of the system is governed by the law which we

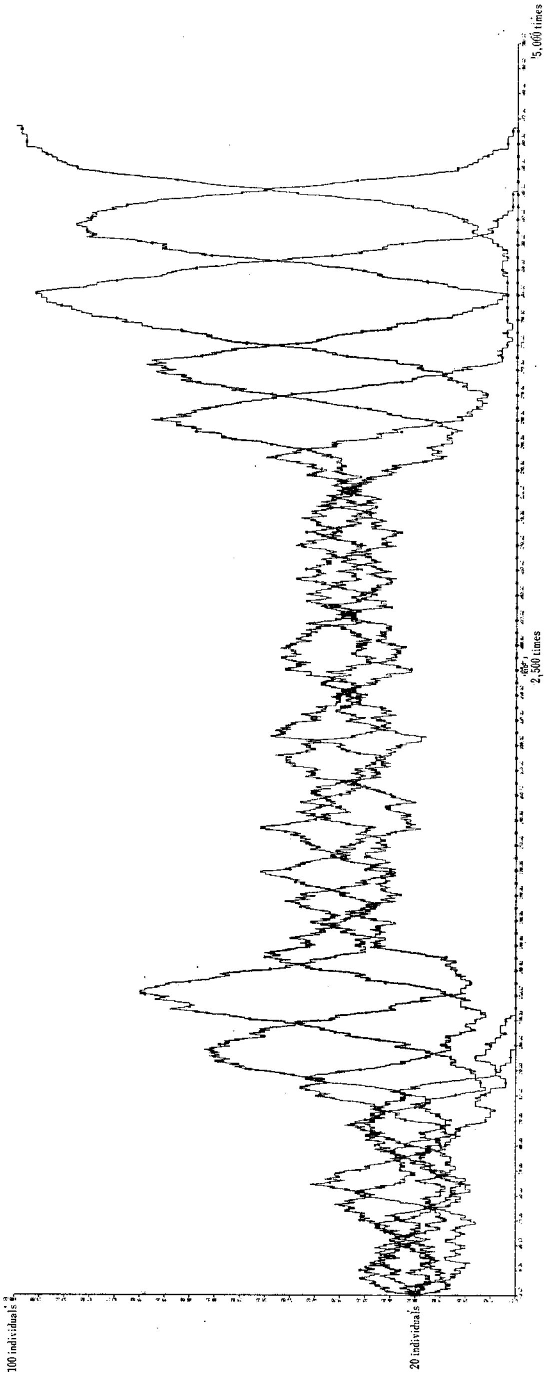


Fig. 3(a)

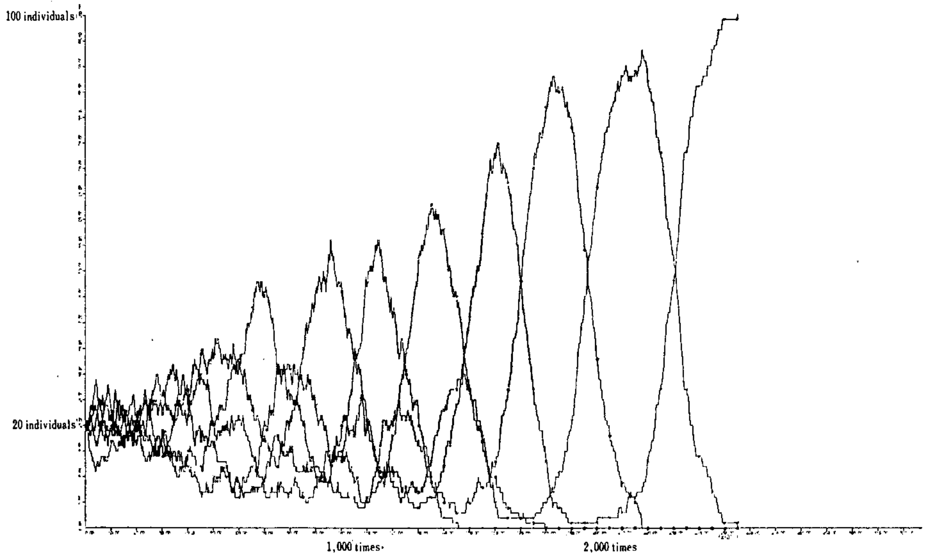


Fig. 3(b)

have discussed in Section 4. The computer simulation of this case is in Fig. 3.

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