Integral polyhedra associated with submodular functions defined on $\{0,1,2\}^S$

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Abstract

A generalization of submodular set functions, named *greedy-type* function, is introduced. A greedy-type function is a function from $\{0,1,2\}^S$ to $\mathbb{R}$ which satisfies:

(G1) $f(2p) = 2f(p)$ for every $p \in \{0,1\}^S$,

(G2) $f(p) + f(q) = f(p \vee q) + f(p \wedge q)$ for any $p, q \in \{0,1,2\}^S$,

(G3) $f(\chi^X + \chi^Y) + f(\chi^X + \chi^Z) \geq f(\chi^Y + \chi^Z) + f(2\chi^X)$ for any $X, Y, Z \subseteq S$ with $X \subsetneq Y \subsetneq Z$,

where

- S is a nonempty finite set,
- $\chi^X \in \{0,1\}^S$ denotes the characteristic vector of $X \subseteq S$,
- $(p \vee q)_e = \max\{p_e, q_e\}, (p \wedge q)_e = \min\{p_e, q_e\} (e \in S)$.

The polyhedron associated with a greedy-type function $f : \{0,1,2\}^S \to \mathbb{R}$ is

$$\{x \in \mathbb{R}^S \mid \sum_{e \in S} p_e x_e \leq f(p) \text{ for every } p \in \{0,1,2\}^S\}$$

and is denoted by $P(f)$. The face structure of $P(f)$ is characterized with maximal chains of a certain partial order defined on $\{0,1,2\}^S$. A dual algorithm, whose validity is ensured by this characterization, maximizing a linear function $c^T x$ over $P(f)$ is proposed. Through this algorithm integrality of $P(f)$ is shown for integer-valued function of greedy-type.

Certain types of bipartite network flows which give a rise to greedy-type functions will also be discussed.