On extremal problems of MPR-posets I

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A mathematical theory for the subject on ancestral character-state reconstructions under the maximum parsimony in phylogeny has been developing. We use the notations in [1] and [2]. Let Ω denote the set that may be either the set $\mathbf{R}$ of real numbers or the set $\mathbf{N}$ of nonnegative integers. Let $T = (V = V_O \cup V_H, E, \sigma)$ be an el-tree. A *reconstruction* on an el-tree T is an assignment $\lambda : V \to \Omega$ such that $\lambda|V_O$ (the restriction of λ to V_O) $= \sigma$. For each branch e in E of an el-tree T with a reconstruction λ, we define the *length* $l(e)$ of branch $e = \{u, v\}$ by $|\lambda(u) - \lambda(v)|$. Then the *length* $L(T|\lambda)$ of an el-tree T under the reconstruction λ is $\sum_{e \in E} l(e)$. A *Most-Parsimonious Reconstruction* denoted by MPR on an el-tree T is a reconstruction λ of which length is minimized. The set $S_u = \{\lambda(u) \mid \lambda \text{ is an MPR on } T\}$ of states is called the *MPR-set* of a node u. For an el-tree $T = (V_O \cup V_H, E, \sigma)$, we define a *rooted el-tree* $T^{(r)}$ rooted at any element r in V. The rooted el-tree $T^{(r)}$ is simply written T if it is understood. If r is an endnode, i.e., $r \in V_O$ and s is its unique child, we denote the rooted tree $T^{(r)}$ by (T_s, r).

Minaka has introduced the usual partial ordering on the set $\mathbf{Rmp}(T)$ of all MPRs on an el-tree, in order to investigate the relationships among the ACCTRAN reconstruction and the DELTRAN reconstruction which are considered to be more meaningful and useful MPRs in phylogeny, and other MPRs. The partially ordered set $(\mathbf{Rmp}(T), \leq)$ is called the *MPR-poset* or Minaka poset. Furthermore from a phylogenetic point of view, Minaka [3] has implicitly defined another partial ordering "a is ancestral to b" on a polarized transformation series, and then has introduced a partial ordering called "MPR partial order" on $\mathbf{Rmp}(T)$. The mathematically explicit definition is as follows. Let T be a rooted el-tree (T_s, r). For a and b in Ω, a partial ordering $a \leq_{\sigma(r)} b$ if and only if $\sigma(r) \leq a \leq b$ or $\sigma(r) \geq a \geq b$. For λ and μ in $\mathbf{Rmp}(T)$, a partial ordering $\lambda \leq_{\sigma(r)} \mu$ if and only if $\lambda(u) \leq_{\sigma(r)} \mu(u)$ for all u in V_H. The partially ordered set $(\mathbf{Rmp}(T), \leq_{\sigma(\mathrm{r})})$ is called a *$\sigma(r)$-version MPR-poset*. Let T be a rooted el-tree (T_s, r). We define a reconstruction λ on T by $\lambda(u) =$ the least element of the subposet $(S_u, \leq_{\sigma(r)})$ in $(\Omega, \leq_{\sigma(r)})$. This reconstruction λ is particularly written as $\lambda_{\min}^{<\sigma(r)>}$. From a lattice-theoretic point of view, we first have a question whether there exists the greatest element (or the least element) in the $\sigma(r)$-version MPR-poset or not. The answer is as follows. Let T be a rooted el-tree (T_s, r).

Theorem 1. *The reconstruction $\lambda_{\min}^{<\sigma(r)>}$ is the least element of $(\mathbf{Rmp}(T), \leq_{\sigma(r)})$.*

Theorem 2. *$(\mathbf{Rmp}(T), \leq_{\sigma(r)})$ has the greatest element if and only if for all $u \in V_H$, $\sigma(r) \leq \min(S_u)$ or $\sigma(r) \geq \max(S_u)$.*

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