Combinatorial Structures of Associated Polyhedra of Set functions

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A set function is a function defined on the power set of a set. That is, it is of the form $\varphi : 2^{\Omega} \to \boldsymbol{R}$. We assume $\varphi(\emptyset) = 0$.

We focus on the combinatorial structures of faces of the polyhedra associated with set functions. The polyhedron associated with a set function φ is

$$P(\varphi) = \{p \in \boldsymbol{R}^{\Omega} | \sum_{x \in G} p(x) \leq \varphi(G) \text{ for all } G \subset \Omega\}.$$

The combinatorial structure means how the faces of $P(\varphi)$ intersect.

The associated polyhedron is not fully described by its combinatorial structure of its faces. But some problems on set functions can be discussed only on the combinatorial structures of their faces. For example, it is known that if we know the combinatorial structure of the associated polyhedron of a set function, we can know whether it is submodular or not.

We represent the combinatorial structures of polyhedra by (marked) polyhedral subdivisions of the simplex $\{p \in \mathbf{R}^{\Omega} | \sum_{x \in \Omega} p(x) = 1\}$. At that time, the available vertices of polyhedral subdivisions are fixed. The combinatorial structure of the associated polyhedron of a set function is represented by a coherent polyhedral subdivision. Conversely, for a coherent polyhedral subdivision, there exists a set function such that the combinatorial structure of the polyhedron associated with the set function is represented by that subdivision.

It is known that the all coherent subdivisions forms a lattice with respect to refinement and that such a lattice can be embodied as the face lattice of the secondary polytope.

We consider the two problems in terms of polyhedral subdivisions, namely, the sandwich problem and the extremal problem for monotone submodular functions.

The sandwich condition can be expressed by the condition to polyhedral subdivisions. We show that for two functions φ_1 and φ_2 such that $-\varphi_1(G) \leq \varphi_2(G)$ for all $G \subset \Omega$ and such that the subdivisions of φ_1 and φ_2 have a common refinement, there exists a vector $p \in R^{\Omega}$ such that $-\varphi_1(G) \leq p(G) \leq \varphi_2(G)$ for all $G \subset \Omega$. It is a generalization of the submodular version of sandwich theorem.

The extremal problem of monotone submodular functions is described as follows. The all monotone submodular functions form a convex cone. The condition for a monotone submodular function to be extremal in the cone of all monotone submodular functions can be expressed in terms of subdivisions. We show that a monotone submodular function φ is extremal in the cone of submodular functions if and only if it is maximal except the trivial element in the lattice.