Supplementary Material to the article "Posterior Contraction Rate and Asymptotic Bayes Optimality for One Group Global-Local Shrinkage Priors in Sparse Normal Means Problem"

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This document consists of two sections. In Section 1, we provide proofs of Theorem 1 and some auxiliary Lemmas related to MMLE provided in Section 2 of our main work. Section 2 provides the derivation of the posterior distribution of κ_i given X_i and τ. But before going into the technical details, let us restate the general class of one-group global-local priors that we are interested in this work:

$$\theta_i|\lambda_i, \tau \overset{ind}{\sim} \mathcal{N}(0, \lambda_i^2\tau^2), \lambda_i^2 \overset{ind}{\sim} \pi_1(\lambda_i^2), \tau \sim \pi_2(\tau), \tag{1.1}$$

where the local shrinkage parameter λ_i is modeled as,

$$\pi_1(\lambda_i^2) = K(\lambda_i^2)^{-a-1}L(\lambda_i^2), \tag{1.2}$$

where $K \in (0, \infty)$ is the constant of proportionality, a is a positive real number, and $L : (0, \infty) \to (0, \infty)$ is measurable non-constant slowly varying function, i.e., for any $\alpha > 0, \frac{L(\alpha x)}{L(x)} \to 1$ as $x \to \infty$. For the theoretical development of the paper, we consider slowly varying functions that satisfy Assumption1 ((**A1**) and **(A2)**) discussed in Section 2 of the main paper.

1 Proofs

1.1 Results related to the contraction rate of Empirical Bayes Procedure–plug-in Estimator

Proof of Theorem 1. For proving Theorem 1, first we need to show that, the total posterior variance using the empirical Bayes estimator of τ corresponding to this class of priors satisfies, as $n \to \infty$,

$$\sup_{\boldsymbol{\theta}_0 \in l_0[q_n]} \mathbb{E}_{\boldsymbol{\theta}_0} \sum_{i=1}^{n} Var(\theta_i|X_i, \widehat{\tau}) \lesssim q_n \log n, \tag{2}$$

where $Var(\theta_i|X_i, \widehat{\tau})$ denotes $Var(\theta_i|X_i, \tau)$ evaluated at $\tau = \widehat{\tau}$. We make use of Markov's inequality along with (2) and Theorem 2 of Ghosh and Chakrabarti (2017).
Now we move towards establishing (2). Let us define $\tilde{q}_n = \sum_{i=1}^{n} 1_{\{\theta_{0i} \neq 0\}}$. Thus, $\tilde{q}_n \leq q_n$. We then note

$$\mathbb{E}_{\boldsymbol{\theta}_0} \sum_{i=1}^{n} Var(\theta_i|X_i, \widehat{\tau}) = \sum_{i:\theta_{0i} \neq 0} \mathbb{E}_{\theta_{0i}} Var(\theta_i|X_i, \widehat{\tau}) + \sum_{i:\theta_{0i} = 0} \mathbb{E}_{\theta_{0i}} Var(\theta_i|X_i, \widehat{\tau}) \, . \tag{3}$$

We now prove that $\sum_{i:\theta_{0i} \neq 0} \mathbb{E}_{\theta_{0i}} Var(\theta_i|X_i, \widehat{\tau}) \lesssim \tilde{q}_n \log n$ and $\sum_{i:\theta_{0i} = 0} \mathbb{E}_{\theta_{0i}} Var(\theta_i|X_i, \widehat{\tau}) \lesssim q_n \log n$ in **Step-1** and **Step-2** below respectively. Combining these results we get the final result. Now let us prove **Step-1** and **Step-2**.

Proof of Step-1: Fix any $c > 1$ and choose $\rho > c$. Define $r_n = \sqrt{4a\rho^2 \log n}$. Fix any i such that $\theta_{0i} \neq 0$. We split $\mathbb{E}_{\theta_{0i}} Var(\theta_i|X_i, \widehat{\tau})$ as

$$\mathbb{E}_{\theta_{0i}} Var(\theta_i|X_i, \widehat{\tau}) = \mathbb{E}_{\theta_{0i}}[Var(\theta_i|X_i, \widehat{\tau}) 1_{\{|X_i| \leq r_n\}}] + \mathbb{E}_{\theta_{0i}}[Var(\theta_i|X_i, \widehat{\tau}) 1_{\{|X_i| > r_n\}}] \, . \tag{4}$$

Since for any fixed $x_i \in \mathbb{R}$ and $\tau > 0$, $Var(\theta_i|x_i, \tau) \leq 1 + x_i^2$.

$$\mathbb{E}_{\theta_{0i}}[Var(\theta_i|X_i, \widehat{\tau}) 1_{\{|X_i| \leq \sqrt{4a\rho^2 \log n}\}}] \leq 1 + 4a\rho^2 \log n \, . \tag{5}$$

Note that, for any fixed $x_i \in \mathbb{R}$, $x_i^2 E(\kappa_i^2|x_i, \tau) = x_i^2 E(\frac{1}{(1+\lambda_i^2\tau^2)^2}|x_i, \tau)$ is non-increasing in τ. Also using Lemma A.1 in Ghosh and Chakrabarti (2017), $Var(\theta_i|x_i, \tau) \leq 1 + x_i^2 E(\kappa_i^2|x_i, \tau)$. Using these two results,

$$Var(\theta_i|x_i, \widehat{\tau}) \leq 1 + x_i^2 E(\kappa_i^2|x_i, \widehat{\tau}) \leq 1 + x_i^2 E(\kappa_i^2|x_i, \frac{1}{n}) \text{ [Since } \widehat{\tau} \geq \frac{1}{n}] \, .$$

Using arguments similar to Lemma 3 of Ghosh and Chakrabarti (2017), one can show that for any fixed $\eta \in (0,1)$ and $\delta \in (0,1)$, the r.h.s. above can be bounded by a non-negative and measurable real-valued function $\tilde{h}(x_i, \tau, \eta, \delta)$ satisfying $\tilde{h}(x_i, \tau, \eta, \delta) = 1 + \tilde{h}_1(x_i, \tau) + \tilde{h}_2(x_i, \tau, \eta, \delta)$ with

$$\tilde{h}_1(x_i, \tau) = C_{**}\Big[x_i^2 \int_0^{\frac{x_i^2}{1+t_0}} \exp(-\frac{u}{2}) u^{a+\frac{1}{2}-1} du\Big]^{-1},$$

where C_{**} is a global constant independent of both x_i and τ and

$$\tilde{h}_2(x_i, \tau, \eta, \delta) = x_i^2 \frac{H(a, \eta, \delta)}{\Delta(\tau^2, \eta, \delta)} \tau^{-2a} e^{-\frac{\eta(1-\delta)x_i^2}{2}},$$

where $H(a, \eta, \delta) = \frac{(a+\frac{1}{2})(1-\eta\delta)^a}{K(\eta\delta)^{(a+\frac{1}{2})}}$ and $\Delta(\tau^2, \eta, \delta) = \frac{\int_{\frac{1}{\tau^2}(\frac{1}{\eta\delta}-1)}^{\infty} t^{-(a+\frac{3}{2})} L(t) dt}{(a+\frac{1}{2})^{-1}(\frac{1}{\tau^2}(\frac{1}{\eta\delta}-1))^{-(a+\frac{1}{2})}}$. Since, $\tilde{h}_1(x_i, \tau)$ is strictly decreasing in $|x_i|$ for any fixed τ, one has,

$$\sup_{|x_i| > \sqrt{4a\rho^2 \log n}} \tilde{h}_1(x_i, \frac{1}{n}) \leq C_{**}\Big[\rho^2 \log n \int_0^{\frac{4a\rho^2 \log n}{1+t_0}} \exp(-\frac{u}{2}) u^{a+\frac{1}{2}-1} du\Big]^{-1} \lesssim \frac{1}{\log n}.$$

Also noting that $\tilde{h}_2(x_i, \tau, \eta, \delta)$ is strictly decreasing in $|x_i| > C_1 = \sqrt{\frac{2}{\eta(1-\delta)}}$, for any $\rho > C_1$,

$$\sup_{|x_i| > \sqrt{4a\rho^2 \log n}} \tilde{h}_2(x_i, \frac{1}{n}, \eta, \delta) \leq 4a\rho^2 \frac{H(a, \eta, \delta)}{\Delta(\frac{1}{n^2}, \eta, \delta)} \log n \cdot n^{-2a(\frac{2\rho^2}{C_1^2}-1)}.$$

Using the definition of the slowly varying function $L(\cdot)$ and assumption (A1) in the right-hand side of above, we have,

$$\sup_{|x_i| > \sqrt{4a\rho^2 \log n}} \tilde{h}_2(x_i, \frac{1}{n}, \eta, \delta) = o(1) \,\text{as}\, n \to \infty.$$

Therefore, we have,

$$\sup_{|x_i| > \sqrt{4a\rho^2 \log n}} \tilde{h}(x_i, \frac{1}{n}, \eta, \delta) \leq 1 + \sup_{|x_i| > \sqrt{4a\rho^2 \log n}} \tilde{h}_1(x_i, \frac{1}{n}) + \sup_{|x_i| > \sqrt{4a\rho^2 \log n}} \tilde{h}_2(x_i, \frac{1}{n}, \eta, \delta) \lesssim 1.$$

Using above arguments, as $n \to \infty$,

$$\mathbb{E}_{\theta_{0i}}[Var(\theta_i|X_i, \widehat{\tau})1_{\{|X_i|>\sqrt{4a\rho^2 \log n}\}}] \lesssim 1 \,. \tag{6}$$

Combining (4),(5) and (6), we obtain

$$\mathbb{E}_{\theta_{0i}} Var(\theta_i|X_i, \widehat{\tau}) \lesssim \log n \,. \tag{7}$$

Noting that $\lesssim$ relation proved here actually holds uniformly in i's such that $\theta_{0i} \neq 0$, in that the corresponding constants in the upper bounds can be chosen to be the same for each i, we have

$$\sum_{i:\theta_{0i}\neq 0} \mathbb{E}_{\theta_{0i}} Var(\theta_i|X_i, \widehat{\tau}) \lesssim \tilde{q}_n \log n \,. \tag{8}$$

Proof of Step-2: Fix any i such that $\theta_{0i} = 0$. Define $v_n = \sqrt{c_1 \log n}$, where c_1 is defined in (5) of the main document.

Case-1 First we consider the case when $a \in [\frac{1}{2}, 1)$. Again, we split $\mathbb{E}_{\theta_{0i}} Var(\theta_i|X_i, \widehat{\tau})$ as

$$\mathbb{E}_{\theta_{0i}} Var(\theta_i|X_i, \widehat{\tau}) = \mathbb{E}_{\theta_{0i}}[Var(\theta_i|X_i, \widehat{\tau})1_{\{|X_i|>v_n\}}] + \mathbb{E}_{\theta_{0i}}[Var(\theta_i|X_i, \widehat{\tau})1_{\{|X_i|\leq v_n\}}] \,. \tag{9}$$

Using $Var(\theta_i|x_i, \widehat{\tau}) \leq 1 + x_i^2$ and the identity $x^2\phi(x) = \phi(x) - \frac{d}{dx}[x\phi(x)]$, we obtain,

$$\mathbb{E}_{\theta_{0i}}[Var(\theta_i|X_i, \widehat{\tau})1_{\{|X_i|>v_n\}}] \leq 2\int_{\sqrt{c_1 \log n}}^{\infty} (1+x^2)\phi(x)dx \lesssim \sqrt{\log n} \cdot n^{-\frac{c_1}{2}} \,. \tag{10}$$

Now let us choose some $\gamma > 1$ such that $c_2\gamma - 1 > 1$. Next, we decompose the second term as follows:

$$\mathbb{E}_{\theta_{0i}}[Var(\theta_i|X_i, \widehat{\tau})1_{\{|X_i|\leq v_n\}}] = \mathbb{E}_{\theta_{0i}}[Var(\theta_i|X_i, \widehat{\tau})1_{\{\widehat{\tau}>\gamma\frac{q_n}{n}\}}1_{\{|X_i|\leq v_n\}}] + \mathbb{E}_{\theta_{0i}}[Var(\theta_i|X_i, \widehat{\tau})1_{\{\widehat{\tau}\leq\gamma\frac{q_n}{n}\}}1_{\{|X_i|\leq v_n\}}] \,. \tag{11}$$

Note that

$$\mathbb{E}_{\theta_{0i}}[Var(\theta_i|X_i, \widehat{\tau})1_{\{\widehat{\tau}>\gamma\frac{q_n}{n}\}}1_{\{|X_i|\leq v_n\}}] \leq (1 + c_1 \log n)\mathbb{P}_{\theta_0}[\widehat{\tau} > \gamma\frac{q_n}{n}, |X_i| \leq v_n]$$

$$\leq (1 + c_1 \log n)\mathbb{P}_{\theta_0}[\frac{1}{c_2 n}\sum_{j=1(\neq i)}^{n} 1_{\{|X_j|>\sqrt{c_1 \log n}\}} > \gamma\frac{q_n}{n}] \lesssim \frac{q_n}{n}\log n \,. \tag{12}$$

Inequality in the last line is due to employing similar arguments used for proving Lemma A.7 in van der pas et al. (2014).
We will now bound $\mathbb{E}_{\theta_{0i}}[Var(\theta_i|X_i,\widehat{\tau})1_{\{\widehat{\tau}\leq\gamma\frac{q_n}{n}\}}1_{\{|X_i|\leq\sqrt{c_1\log n}\}}]$. Since for any fixed $x_i \in \mathbb{R}$ and $\tau > 0$, $Var(\theta_i|x_i,\tau) \leq E(1-\kappa_i|x_i,\tau) + J(x_i,\tau)$ where $J(x_i,\tau) = x_i^2 E[(1-\kappa_i)^2|x_i,\tau]$. Since $E(1-\kappa_i|x_i,\tau)$ is non-decreasing in τ, so, $E(1-\kappa_i|x_i,\widehat{\tau}) \leq E(1-\kappa_i|x_i,\gamma\frac{q_n}{n})$ whenever $\widehat{\tau} \leq \gamma\frac{q_n}{n}$. Using Lemma 2 and A.2 of Ghosh and Chakrabarti (2017),

$$\begin{aligned}\mathbb{E}_{\theta_{0i}}[Var(\theta_i|X_i,\widehat{\tau})1_{\{\widehat{\tau}\leq\gamma\frac{q_n}{n}\}}1_{\{|X_i|\leq\sqrt{c_1\log n}\}}] &\lesssim (\frac{q_n}{n})^{2a}\int_0^{\sqrt{c_1\log n}} e^{\frac{x^2}{2}}e^{-\frac{x^2}{2}}dx \\ &= (\frac{q_n}{n})^{2a}\sqrt{c_1\log n}\,. \end{aligned} \tag{13}$$

Note that all these preceding arguments hold uniformly in i such that $\theta_{0i}=0$. Combining all these results, for $a \in [\frac{1}{2},1)$ using (9)-(13), we have,

$$\begin{aligned}\sum_{i:\theta_{0i}=0}\mathbb{E}_{\theta_{0i}}Var(\theta_i|X_i,\widehat{\tau}) &\lesssim (n-\tilde{q}_n)[\sqrt{\log n}\cdot n^{-\frac{c_1}{2}} + \frac{q_n}{n}\cdot\log n + (\frac{q_n}{n})^{2a}\sqrt{\log n}] \\ &\lesssim q_n\log n\,. \end{aligned} \tag{14}$$

The second inequality follows due to the fact that $\tilde{q}_n \leq q_n$ and $q_n = o(n)$ as $n \to \infty$.
Case-2 Now we assume $a \geq 1$ and split $\mathbb{E}_{\theta_{0i}}Var(\theta_i|X_i,\widehat{\tau})$ as

$$\mathbb{E}_{\theta_{0i}}Var(\theta_i|X_i,\widehat{\tau}) = \mathbb{E}_{\theta_{0i}}[Var(\theta_i|X_i,\widehat{\tau})1_{\{|X_i|>v_n\}}] + \mathbb{E}_{\theta_{0i}}[Var(\theta_i|X_i,\widehat{\tau})1_{\{|X_i|\leq v_n\}}], \tag{15}$$

where v_n is the same as defined in **Case-1**. Using exactly the same arguments used for the case $a \in [\frac{1}{2},1)$, we have the following for $a \geq 1$,

$$\mathbb{E}_{\theta_{0i}}[Var(\theta_i|X_i,\widehat{\tau})1_{\{|X_i|>\sqrt{c_1\log n}\}}] \lesssim \sqrt{\log n}\cdot n^{-\frac{c_1}{2}}\,. \tag{16}$$

and

$$\mathbb{E}_{\theta_{0i}}[Var(\theta_i|X_i,\widehat{\tau})1_{\{\widehat{\tau}>\gamma\frac{q_n}{n}\}}1_{\{|X_i|\leq\sqrt{c_1\log n}\}}] \lesssim \frac{q_n}{n}\log n\,. \tag{17}$$

For the part $\mathbb{E}_{\theta_{0i}}[Var(\theta_i|X_i,\widehat{\tau})1_{\{\widehat{\tau}\leq\gamma\frac{q_n}{n}\}}1_{\{|X_i|\leq\sqrt{c_1\log n}\}}]$, note that for fixed $x_i \in \mathbb{R}$ and any $\tau > 0$,

$$\begin{aligned}Var(\theta_i|x_i,\tau) &\leq E(1-\kappa_i|x_i,\tau) + x_i^2E[(1-\kappa_i)^2|x_i,\tau] \\ &\leq E(1-\kappa_i|x_i,\tau) + x_i^2E(1-\kappa_i|x_i,\tau) \\ &\leq E(1-\kappa_i|x_i,\tau)1_{\{|x_i|\leq 1\}} + 2x_i^2E(1-\kappa_i|x_i,\tau)\,. \end{aligned} \tag{18}$$

Since for fixed $x_i \in \mathbb{R}$ and any $\tau > 0$, $E(1-\kappa_i|x_i,\tau)$ is non-decreasing in τ, we have,

$$\mathbb{E}_{\theta_{0i}}[Var(\theta_i|X_i,\widehat{\tau})1_{\{\widehat{\tau}\leq\gamma\frac{q_n}{n}\}}1_{\{|X_i|\leq\sqrt{c_1\log n}\}}] \leq \mathbb{E}_{\theta_{0i}}[E(1-\kappa_i|X_i,\gamma\frac{q_n}{n})1_{\{|X_i|\leq 1\}}]+ 2\mathbb{E}_{\theta_{0i}}[X_i^2 E(1-\kappa_i|X_i,\gamma\frac{q_n}{n})1_{\{|X_i|\leq\sqrt{c_1\log n}\}}] \,. \tag{19}$$

For bounding the first term in the r.h.s. of (19), we use Lemma 1. We note that for any $\tau \in (0,1)$, $\frac{t\tau^2}{1+t\tau^2}\cdot\frac{1}{\sqrt{1+t\tau^2}}t^{-a-1} \leq \tau t^{-(a+\frac{1}{2})}$ and that $L(t)$ is bounded. Using the fact that the second term in the upper bound in Lemma 1 can be bounded

$$A_2 \leq \frac{2M}{(2a-1)}\tau e^{\frac{x_i^2}{2}} \,,$$

where A_2 has been defined in the proof of Lemma 1 of the main document. So, we have

$$\mathbb{E}_{\theta_{0i}}[E(1-\kappa_i|X_i,\gamma\frac{q_n}{n})1_{\{|X_i|\leq 1\}}] \lesssim \frac{q_n}{n}\int_0^1 e^{-\frac{x^2}{4}}dx + \frac{q_n}{n} \,.$$

Hence,

$$\mathbb{E}_{\theta_{0i}}[E(1-\kappa_i|X_i,\gamma\frac{q_n}{n})1_{\{|X_i|\leq 1\}}] \lesssim \frac{q_n}{n} \,. \tag{20}$$

For the second term in the r.h.s. of (19) we shall use the upper bound of $E(1-\kappa_i|x_i,\tau)$ of the form (7) of the main article and hence,

$$\mathbb{E}_{\theta_{0i}}[X_i^2 E(1-\kappa_i|X_i,\gamma\frac{q_n}{n})1_{\{|X_i|\leq\sqrt{c_1\log n}\}}] \lesssim \frac{q_n}{n}\int_0^{\sqrt{c_1\log n}} x^2 e^{\frac{x^2}{4}}\phi(x)dx + \int_0^{\sqrt{c_1\log n}}\int_1^\infty \frac{t(\gamma\frac{q_n}{n})^2}{1+t(\gamma\frac{q_n}{n})^2}\cdot\frac{1}{\sqrt{1+t(\gamma\frac{q_n}{n})^2}}t^{-a-1}L(t)e^{\frac{x^2}{2}\cdot\frac{t(\gamma\frac{q_n}{n})^2}{1+t(\gamma\frac{q_n}{n})^2}}x^2\phi(x)dtdx \,. \tag{21}$$

Note that the first integral is bounded by a constant. Using Fubini's theorem and the transformation $y = \frac{x}{\sqrt{1+t(\gamma\frac{q_n}{n})^2}}$, the second integral becomes

$$\frac{1}{\sqrt{2\pi}}\int_1^\infty t^{-a-1}L(t)t(\gamma\frac{q_n}{n})^2\Big(\int_0^{\sqrt{\frac{c_1\log n}{1+t(\gamma\frac{q_n}{n})^2}}} y^2 e^{-\frac{y^2}{2}}dy\Big)dt \,.$$

We handle the above integral separately for $a = 1$ and $a > 1$. For $a > 1$, using the boundedness of $L(t)$ it is easy to show that,

$$\frac{1}{\sqrt{2\pi}} \int_1^\infty t^{-a-1} L(t) t (\gamma \frac{q_n}{n})^2 \Big(\int_0^{\sqrt{\frac{c_1 \log n}{1 + t(\gamma \frac{q_n}{n})^2}}} y^2 e^{-\frac{y^2}{2}} dy \Big) dt \lesssim \frac{q_n}{n} \,. \tag{22}$$

For $a = 1$ note that,

$$\int_1^\infty t^{-a-1} L(t) t (\gamma \frac{q_n}{n})^2 \Big(\int_0^{\sqrt{\frac{c_1 \log n}{1 + t(\gamma \frac{q_n}{n})^2}}} y^2 e^{-\frac{y^2}{2}} dy \Big) dt$$

$$\leq (\gamma \frac{q_n}{n})^2 \sqrt{2\pi} \int_1^{\frac{c_1 \log n}{(\gamma \frac{q_n}{n})^2 (2\pi)^{\frac{1}{3}}}} \frac{1}{t} L(t) dt + (\sqrt{c_1 \log n})^3 (\gamma \frac{q_n}{n})^2 \int_{\frac{c_1 \log n}{(\gamma \frac{q_n}{n})^2 (2\pi)^{\frac{1}{3}}}}^\infty \frac{L(t)}{t} \cdot \frac{1}{(t(\gamma \frac{q_n}{n})^2)^{\frac{3}{2}}} dt \,. \tag{23}$$

Here the division in the range of t in (23) occurs due to the fact the integral $(\int_0^{\sqrt{\frac{c_1 \log n}{t(\gamma \frac{q_n}{n})^2}}} y^2 e^{-\frac{y^2}{2}} dy)$ can be bounded by $(\frac{c_1 \log n}{t(\gamma \frac{q_n}{n})^2})^{\frac{3}{2}}$ when $t \geq \frac{c_1 \log n}{(\gamma \frac{q_n}{n})^2 (2\pi)^{\frac{1}{3}}}$ and by $\sqrt{2\pi}$ when $t \leq \frac{c_1 \log n}{(\gamma \frac{q_n}{n})^2 (2\pi)^{\frac{1}{3}}}$. For the first term in (23) with the boundedness of $L(t)$,

$$(\gamma \frac{q_n}{n})^2 \int_1^{\frac{c_1 \log n}{(\gamma \frac{q_n}{n})^2 (2\pi)^{\frac{1}{3}}}} \frac{1}{t} L(t) dt \leq (\gamma \frac{q_n}{n})^2 M \log \left(\frac{c_1 \log n}{(\gamma \frac{q_n}{n})^2 (2\pi)^{\frac{1}{3}}} \right) .$$

Hence for sufficiently large n with $q_n \propto n^\beta, 0 < \beta < 1$,

$$(\gamma \frac{q_n}{n})^2 \int_1^{\frac{c_1 \log n}{(\gamma \frac{q_n}{n})^2 (2\pi)^{\frac{1}{3}}}} \frac{1}{t} L(t) dt \lesssim \frac{q_n}{n} \sqrt{\log n} \,. \tag{24}$$

Now for the second term in (23) again using the boundedness of $L(t)$,

$$(\sqrt{c_1 \log n})^3 (\gamma \frac{q_n}{n})^2 \int_{\frac{c_1 \log n}{(\gamma \frac{q_n}{n})^2 (2\pi)^{\frac{1}{3}}}}^\infty t \cdot t^{-2} L(t) \cdot \frac{1}{(t(\gamma \frac{q_n}{n})^2)^{\frac{3}{2}}} dt \lesssim \frac{q_n}{n} \,. \tag{25}$$

So, using (23)-(25), for $a = 1$,

$$\int_1^\infty t^{-a-1} L(t) t (\gamma \frac{q_n}{n})^2 \Big(\int_0^{\sqrt{\frac{c_1 \log n}{1 + t(\gamma \frac{q_n}{n})^2}}} y^2 e^{-\frac{y^2}{2}} dy \Big) dt \lesssim \frac{q_n}{n} \sqrt{\log n} \,. \tag{26}$$

With the help of (22) and (26), we have, for $a \geq 1$

$$\int_1^\infty t^{-a-1}L(t)t(\gamma\frac{q_n}{n})^2\Big(\int_0^{\sqrt{\frac{c_1\log n}{1+t(\gamma\frac{q_n}{n})^2}}} y^2 e^{-\frac{y^2}{2}}dy\Big)dt \lesssim \frac{q_n}{n}\sqrt{\log n}\,. \tag{27}$$

Combining these facts, we finally have

$$\mathbb{E}_{\theta_{0i}}[X_i^2 E(1-\kappa_i|X_i,\gamma\frac{q_n}{n})1_{\{|X_i|\leq\sqrt{c_1\log n}\}}] \lesssim \frac{q_n}{n}\sqrt{\log n}\,. \tag{28}$$

Note that all these preceding arguments hold uniformly in i such that $\theta_{0i}=0$. Combining all these results, for $a \geq 1$, using (15)-(28), we have,

$$\begin{aligned}\sum_{i:\theta_{0i}=0}\mathbb{E}_{\theta_{0i}}Var(\theta_i|X_i,\widehat{\tau}) &\lesssim (n-\tilde{q}_n)[\sqrt{\log n}\cdot n^{-\frac{c_1}{2}}+\frac{q_n}{n}\cdot\log n+\frac{q_n}{n}\sqrt{\log n}] \\ &\lesssim q_n\log n\,. \end{aligned}\tag{29}$$

Using (14) and (29), for $a \geq \frac{1}{2}$, we have,

$$\sum_{i:\theta_{0i}=0}\mathbb{E}_{\theta_{0i}}Var(\theta_i|X_i,\widehat{\tau}) \lesssim q_n\log n\,. \tag{30}$$

Now using (3), (8) and (30), for sufficiently large n,

$$\mathbb{E}_{\boldsymbol{\theta}_0}\sum_{i=1}^n Var(\theta_i|X_i,\widehat{\tau}) \lesssim q_n\log n\,.$$

Finally, taking supremum over all $\boldsymbol{\theta}_0 \in l_0[q_n]$, the result is obtained.

□

Remark 1. *Note that, we have used different bounds on $Var(\theta_i|X_i,\tau)$ for two different ranges of a. When $a \in [\frac{1}{2},1)$, we have used the upper bound of $J(X_i,\tau)$ provided by Ghosh and Chakrabarti (2017). However using the same arguments when $a \geq 1$ yields an upper bound on $Var(\theta_i|X_i,\tau)$ such that $\mathbb{E}_{\theta_{0i}}[Var(\theta_i|X_i,\widehat{\tau})1_{\{\widehat{\tau}\leq\gamma\frac{q_n}{n}\}}1_{\{|X_i|\leq\sqrt{c_1\log n}\}}]$ exceeds near minimax rate. As a result, we need to come up with a sharper upper bound, and hence (18) and Lemma 1 of the main document come in very handy.*

1.2 Lemmas related to MMLE

Lemma 1. *The derivative of the log-likelihood function is of the form*

$$\frac{d}{d\tau}M_\tau(\mathbf{X}) = \frac{1}{\tau}\sum_{i=1}^{n} m_\tau(x_i), \tag{31}$$

where

$$m_\tau(x) = x^2\frac{J_{\alpha+1,a}(x) - J_{\alpha+2,a}(x)}{I_{\alpha,a}(x)} - \frac{J_{\alpha+1,a}(x)}{I_{\alpha,a}(x)}, \tag{32}$$

and $I_{\alpha,a}(x)$ and for $k = 1, 2$, $J_{\alpha+k,a}(x)$ are defined as follows

$$I_{\alpha,a}(x) = \int_0^1 e^{\frac{x^2 z}{2}} z^{\alpha-1}(1-z)^{a-\frac{1}{2}}\left(\frac{1}{\tau^2 + (1-\tau^2)z}\right)^{a+\alpha} dz \tag{33}$$

and

$$J_{\alpha+k,a}(x) = \int_0^1 e^{\frac{x^2 z}{2}} z^{\alpha+k-1}(1-z)^{a-\frac{1}{2}}\left(\frac{1}{\tau^2 + (1-\tau^2)z}\right)^{a+\alpha} dz. \tag{34}$$

Proof. Since, $X_i|\theta_i \overset{ind}{\sim} \mathcal{N}(\theta_i, 1)$ and $\theta_i|\lambda_i, \tau \overset{ind}{\sim} \mathcal{N}(0, \lambda_i^2\tau^2)$, the marginal density of X_i given τ is of the form

$$\begin{aligned}
\psi_\tau(x) &= K\int_0^\infty \frac{e^{-\frac{1}{2}\frac{x^2}{(1+\lambda^2\tau^2)}}}{\sqrt{1+\lambda^2\tau^2}\sqrt{2\pi}}(\lambda^2)^{-a-1}L(\lambda^2)d\lambda^2 \\
&= K\frac{\tau^{2a}}{\sqrt{2\pi}}\int_0^1 e^{-\frac{x^2(1-z)}{2}} z^{-a-1}(1-z)^{a-\frac{1}{2}} L\left(\frac{1}{\tau^2}\frac{z}{1-z}\right)dz,
\end{aligned}$$

where equality in the second line follows due to the substitution $1 - z = (1 + \lambda^2\tau^2)^{-1}$. Next, noting that $L(t) = (1 + \frac{1}{t})^{-(a+\alpha)}$,

$$\begin{aligned}
\psi_\tau(x) &= K\tau^{2a}\int_0^1 \frac{e^{-\frac{x^2(1-z)}{2}}}{\sqrt{2\pi}} z^{\alpha-1}(1-z)^{a-\frac{1}{2}}\left[\frac{1}{\tau^2 + (1-\tau^2)z}\right]^{a+\alpha}dz \\
&= K\tau^{2a} I_{\alpha,a}(x)\phi(x),
\end{aligned} \tag{35}$$

where $\phi(x)$ denotes the standard normal density. Using (35),

$$\begin{aligned}\frac{\dot{\psi}_\tau}{\psi_\tau} &= \frac{2a\tau^{2a-1}I_{\alpha,a}(x) + \tau^{2a}\dot{I}_{\alpha,a}(x)}{\tau^{2a}I_{\alpha,a}(x)} \\ &= \frac{2aI_{\alpha,a}(x) + \tau\dot{I}_{\alpha,a}(x)}{\tau I_{\alpha,a}(x)} \\ &= \frac{\int_0^1 e^{\frac{x^2z}{2}} z^{\alpha-1}(1-z)^{a-\frac{1}{2}}(\frac{1}{N(z)})^{a+\alpha+1}[2aN(z) - 2\tau^2(\alpha+a)(1-z)]dz}{\tau I_{\alpha,a}(x)},\end{aligned} \tag{36}$$

where $N(z) = \tau^2 + (1-\tau^2)z$. On the other hand, using integration by parts,

$$x^2(J_{\alpha+1,a}(x) - J_{\alpha+2,a}(x)) = \int_0^1 x^2 e^{\frac{x^2z}{2}} z^{\alpha}(1-z)^{a+\frac{1}{2}}(\frac{1}{N(z)})^{a+\alpha}dz$$
$$= \int_0^1 e^{\frac{x^2z}{2}} \frac{z^{\alpha-1}(1-z)^{a-\frac{1}{2}}}{(N(z))^{a+\alpha+1}}[N(z)\{-2\alpha(1-z) + 2(a+\frac{1}{2})z\} + 2z(1-z)(1-\tau^2)(\alpha+a)]dz.$$

As a consequence of this,

$$\begin{aligned}&x^2(J_{\alpha+1,a}(x) - J_{\alpha+2,a}(x)) - J_{\alpha+1,a}(x) \\ &= \int_0^1 e^{\frac{x^2z}{2}} \frac{z^{\alpha-1}(1-z)^{a-\frac{1}{2}}}{(N(z))^{a+\alpha+1}}[N(z)\{-2\alpha(1-z) + 2az\} + 2(\alpha+a)(1-z)(N(z)-\tau^2)]dz.\end{aligned} \tag{37}$$

On Simplification the r.h.s. of (37) matches with the numerator of (36) and completes the proof. ☐

Lemma 2. *Let κ_τ be the solution to the equation $\frac{e^{x^2/2}}{x^2/2} = \frac{1}{\tau}$ where $0 < \tau < 1$. Choose any $B \geq 1$. There exist functions R_τ with $\sup_x |R_\tau(x)| = O(\tau^{\frac{1}{2}})$ as $\tau \to 0$, such that, for $\alpha \geq \frac{1}{2}$,*

$$I_{\alpha,\frac{1}{2}}(x) = \begin{cases} \big(\frac{K^{-1}}{\tau} + \left(\frac{x^2}{2}\right)^{\frac{1}{2}} \int_1^{\frac{x^2}{2}} e^v v^{-\frac{3}{2}}dv\big)(1+R_\tau(x)), \textit{uniformly in } |x| \leq B\kappa_\tau, \\ \left(\frac{x^2}{2}\right)^{\frac{1}{2}} \int_1^{\frac{x^2}{2}} e^v v^{-\frac{3}{2}}dv(1+R_\tau(x)), \textit{uniformly in } |x| > B\kappa_\tau. \end{cases} \tag{38}$$

Further, given $\epsilon_\tau \to 0$*, there exist functions* S_τ *with* $\sup_{x \geq 1/\epsilon_\tau} |S_\tau(x)| = O(\tau^{\frac{1}{2}} + \epsilon_\tau^2)$*, such that as* $\tau \to 0$*,*

$$I_{\alpha,\frac{1}{2}}(x) = \frac{e^{x^2/2}}{x^2/2}(1 + S_\tau(x)), \tag{39}$$

where $K = \frac{\Gamma(\frac{1}{2}+\alpha)}{\sqrt{\pi}\Gamma(\alpha)}$*.*

Proof. We consider the cases separately when $|x| \leq B\kappa_\tau$ and $|x| > B\kappa_\tau$. Note, as observed in van der pas et al. (2017), $\kappa_\tau \sim \zeta_\tau + \frac{2\log\zeta_\tau}{\zeta_\tau}$ as $\tau \to 0$ where $\zeta_\tau = \sqrt{2\log(\frac{1}{\tau})}$.

Case 1:- When $|x| \leq B\kappa_\tau$, the range of the integration in $I_{\alpha,\frac{1}{2}}(x)$ is divided into three parts, namely,

$$I_1 = \int_0^\tau e^{\frac{x^2 z}{2}} z^{\alpha-1}\left(\frac{1}{\tau^2 + (1-\tau^2)z}\right)^{\frac{1}{2}+\alpha} dz,$$

$$I_2 = \int_\tau^{(\frac{2}{x^2})\wedge 1} e^{\frac{x^2 z}{2}} z^{\alpha-1}\left(\frac{1}{\tau^2 + (1-\tau^2)z}\right)^{\frac{1}{2}+\alpha} dz$$

and

$$I_3 = \int_{(\frac{2}{x^2})\wedge 1}^1 e^{\frac{x^2 z}{2}} z^{\alpha-1}\left(\frac{1}{\tau^2 + (1-\tau^2)z}\right)^{\frac{1}{2}+\alpha} dz,$$

where $y_1 \wedge y_2$ denotes the minimum of y_1 and y_2. Next, making the substitution $z = u\tau^2$ in I_1, we have

$$I_1 = \tau^{-1}\int_0^{\frac{1}{\tau}} u^{\alpha-1}(1 + u(1-\tau^2))^{-(\frac{1}{2}+\alpha)} e^{\frac{x^2\tau^2 u}{2}} du. \tag{40}$$

Next, define

$$I_1^* = \tau^{-1}\int_0^{\frac{1}{\tau}} u^{\alpha-1}(1 + u(1-\tau^2))^{-(\frac{1}{2}+\alpha)} du.$$

Our target is to show that $\frac{I_1 - I_1^*}{I_1^*} \to 0$ as $\tau \to 0$. Now following the argument same as that used in Lemma C.9 of van der pas et al. (2017), for $|x| \leq B\kappa_\tau$, the exponent in the integral tends to 1, uniformly in $u \leq \frac{1}{\tau}$. Since, for any $y \geq 0$, $e^y - 1 \leq ye^y$,

$$I_1 - I_1^* = \tau^{-1} \int_0^{\frac{1}{\tau}} u^{\alpha-1}(1 + u(1-\tau^2))^{-(\frac{1}{2}+\alpha)}[e^{\frac{x^2\tau^2 u}{2}} - 1]du$$
$$\lesssim x^2 \tau e^{\frac{x^2\tau}{2}} \int_0^{\frac{1}{\tau}} u^{-\frac{1}{2}} du \lesssim \tau^{\frac{1}{2}} \log(\frac{1}{\tau})(1 + o(1)).$$

Now we want to find the asymptotic order of I_1^* Towards this, observe that, replacing $\frac{1}{1+(1-\tau^2)u}$ by $\frac{1}{(1+u)(1-\tau^2)}$ provides a multiplicative error of the order $1 + O(\tau^2)$, i.e., $\frac{1}{1+(1-\tau^2)u} = \frac{1}{(1+u)(1-\tau^2)}(1 + O(\tau^2))$. Since,

$$\int_0^\infty u^{\alpha-1}(1+u)^{-(\alpha+\frac{1}{2})}du = \int_0^\infty u^{-\frac{3}{2}}(1 + \frac{1}{u})^{-\frac{1}{2}-\alpha}du = K^{-1} \tag{41}$$

and

$$\int_{\frac{1}{\tau}}^\infty u^{\alpha-1}(1+u)^{-(\alpha+\frac{1}{2})}du \lesssim \tau^{\frac{1}{2}}, \tag{42}$$

we have as $\tau \to 0$,

$$I_1^* = \frac{K^{-1}}{\tau}[1 + O(\tau^{\frac{1}{2}})],$$

uniformly in $|x| \leq B\kappa_\tau$. This implies

$$0 < \frac{I_1 - I_1^*}{I_1^*} \lesssim \tau^{\frac{3}{2}} \log(\frac{1}{\tau})(1 + o(1)),$$

and hence $\frac{I_1 - I_1^*}{I_1^*} \to 0$ as $\tau \to 0$. Combining all these arguments along with (41) and (42), we obtain

$$I_1 = \frac{K^{-1}}{\tau}[1 + O(\tau^{\frac{1}{2}})], \tag{43}$$

uniformly in $|x| \leq B\kappa_\tau$. Moving towards the second integral, first, we make a transformation $\frac{x^2 z}{2} = v$ and hence

$$I_2 = \left(\frac{x^2}{2}\right)^{\frac{1}{2}} \int_{\frac{x^2\tau}{2}}^{\frac{x^2}{2}\wedge 1} e^v v^{\alpha-1}\left(\frac{\tau^2 x^2}{2} + (1-\tau^2)v\right)^{-(\frac{1}{2}+\alpha)} dv.$$

Now, we bound $\frac{\tau^2 x^2}{2} + (1-\tau^2)v$ below by $(1-\tau^2)v$ and observe that the upper limit of the range of integration can be bounded by 1 irrespective of whether $\frac{x^2}{2} \leq 1$ or not. Hence, we can show that

$$I_2 \lesssim \tau^{-\frac{1}{2}}$$

and this contributes negligibly compared to I_1. Finally, for I_3, again after the same transformation

$$I_3 = \left(\frac{x^2}{2}\right)^{\frac{1}{2}} \int_1^{\frac{x^2}{2}} e^v v^{\alpha-1} \left(\frac{\tau^2 x^2}{2} + (1-\tau^2)v\right)^{-(\frac{1}{2}+\alpha)} dv.$$

Here observe that, the integral contributes nothing when $\frac{x^2}{2} \leq 1$, hence we are only interested when $\frac{x^2}{2} > 1$. Next, we define

$$I_3^* = \left(\frac{x^2}{2}\right)^{\frac{1}{2}} \int_1^{\frac{x^2}{2}} e^v v^{-\frac{3}{2}} dv.$$

Now our target is to show that the difference between I_3 and I_3^* is negligible compared to I_3^* as $\tau \to 0$. In order to prove that, first note,

$$\left(\frac{1}{\frac{\tau^2 x^2}{2} + (1-\tau^2)v}\right)^{\frac{1}{2}+\alpha} \leq \left(\frac{1}{v}\right)^{\frac{1}{2}+\alpha} \left[1 + \frac{\tau^2(v+x^2)}{v(1-\tau^2)}\right]^{\frac{1}{2}+\alpha}. \tag{44}$$

Now, first, consider the case when $\alpha + \frac{1}{2}$ is a positive integer. Hence using the Binomial theorem, we have

$$\left(\frac{1}{\frac{\tau^2 x^2}{2} + (1-\tau^2)v}\right)^{\frac{1}{2}+\alpha} - v^{-(\frac{1}{2}+\alpha)} \leq v^{-(\frac{1}{2}+\alpha)} \sum_{j=1}^{\frac{1}{2}+\alpha} \binom{\frac{1}{2}+\alpha}{j} \left[\frac{\tau^2}{(1-\tau^2)}\left(1+\frac{x^2}{v}\right)\right]^j$$

Next, observing that for $1 \leq v \leq \frac{x^2}{2}, 2 \leq \frac{x^2}{v} \leq x^2$, for $|x| \leq B\kappa_\tau$, the difference between I_3 and I_3^* can be bounded as

$$I_3 - I_3^* \lesssim \left(\frac{\tau^2}{1-\tau^2}\right) \log\left(\frac{1}{\tau}\right)(1+o(1))\left(\frac{x^2}{2}\right)^{\frac{1}{2}} \int_1^{\frac{x^2}{2}} e^v v^{-\frac{3}{2}} dv,$$

which implies

$$I_3 - I_3^* \lesssim \left(\frac{\tau^2}{1-\tau^2}\right) \log\left(\frac{1}{\tau}\right)(1+o(1))I_3^*.$$

On the other hand, observe that,

$$I_3 - I_3^* = \left(\frac{x^2}{2}\right)^{\frac{1}{2}} \int_1^{\frac{x^2}{2}} e^v v^{-\frac{3}{2}} \left[\left(\frac{\tau^2 x^2}{2} + 1 - \tau^2\right)^{-(\alpha+\frac{1}{2})} - 1 \right] dv$$
$$\gtrsim \left[\left(\frac{\tau^2 \zeta_\tau^2}{2}(1+o(1)) + 1 - \tau^2\right)^{-(\alpha+\frac{1}{2})} - 1 \right] I_3^*.$$

These two bounds ensure $\frac{I_3 - I_3^*}{I_3^*} \to 0$ as $\tau \to 0$ since $I_3^* \sim \frac{e^{x^2/2}}{x^2/2}$ by Lemma C.8 of van der pas et al. (2017). Next, consider the case when $\alpha + \frac{1}{2}$ is a fraction. When $\alpha + \frac{1}{2}$ is a fraction, then there exists another fraction $b > 0$ such that $\alpha + \frac{1}{2} + b$ is a positive integer. Hence, in this case, $\left[1 + \frac{\tau^2(v+x^2)}{v(1-\tau^2)}\right]^{\frac{1}{2}+\alpha} \leq \left[1 + \frac{\tau^2(v+x^2)}{v(1-\tau^2)}\right]^{\alpha+\frac{1}{2}+b}$. Now, applying exactly the same set of arguments on $\alpha + \frac{1}{2} + b$ in place of $\alpha + \frac{1}{2}$, we again can show that, $\frac{I_3 - I_3^*}{I_3^*} \to 0$ as $\tau \to 0$. This completes the proof for $|x| \leq B\kappa_\tau$.

Case 2:- When $|x| > B\kappa_\tau$, choose any $A \in (0, 1)$. In this case, the range of the integration in $I_{\alpha,\frac{1}{2}}(x)$ is divided into two parts, namely,

$$I_4 = \int_0^A e^{\frac{x^2 z}{2}} z^{\alpha-1} \left(\frac{1}{\tau^2 + (1-\tau^2)z}\right)^{\frac{1}{2}+\alpha} dz$$

and

$$I_5 = \int_A^1 e^{\frac{x^2 z}{2}} z^{\alpha-1} \left(\frac{1}{\tau^2 + (1-\tau^2)z}\right)^{\frac{1}{2}+\alpha} dz.$$

Note that

$$I_4 \leq e^{\frac{x^2 A}{2}} \left(\frac{1}{\tau^2}\right)^{\frac{1}{2}+\alpha} \int_0^A z^{\alpha-1} dz$$
$$\lesssim \tau^{-1-2\alpha} e^{\frac{x^2 A}{2}}.$$

One can choose $B \geq 1$ and $A \in (0, 1)$ such that $B^2(1-A) > 2\alpha + \frac{3}{2}$, which implies $I_4 \ll \tau^{\frac{1}{2}} \frac{e^{x^2/2}}{x^2/2}$, and hence the contribution is negligible compared to the second term in the expression of $I_{\alpha,\frac{1}{2}}(x)$ as given in (38). Finally, for I_5, we use that, for $z \geq A$, $\frac{1}{\tau^2+(1-\tau^2)z} = \frac{1}{z}[1+O(\tau^2)]$. This implies I_5 is of the form

$$I_5 = \int_A^1 z^{-\frac{3}{2}} e^{\frac{x^2 z}{2}} dz [1+O(\tau^2)]^{\frac{1}{2}+\alpha}. \tag{45}$$

Next, note that, with the transformation $\frac{x^2 z}{2} = v$,

$$\begin{aligned} \int_A^1 z^{-\frac{3}{2}} e^{\frac{x^2 z}{2}} dz &= \left(\frac{x^2}{2}\right)^{\frac{1}{2}} \int_{\frac{x^2 A}{2}}^{\frac{x^2}{2}} e^v v^{-\frac{3}{2}} dv \\ &= \left(\frac{x^2}{2}\right)^{\frac{1}{2}} \left[\int_1^{\frac{x^2}{2}} - \int_1^{\frac{x^2 A}{2}} \right] e^v v^{-\frac{3}{2}} dv. \end{aligned} \tag{46}$$

Now, using the first assertion of Lemma C.8 of van der pas et al. (2017), the second integral is bounded above by a multiple of $(x^2/2)^{-1} e^{x^2 A/2}$, which is negligible compared to the first (this is of the order of $(x^2/2)^{-1} e^{x^2/2}$). Hence, combining (45) and (46), we immediately have

$$I_5 = \left(\frac{x^2}{2}\right)^{\frac{1}{2}} \int_1^{\frac{x^2}{2}} e^v v^{-\frac{3}{2}} dv \big[1+O(\tau^2)\big]. \tag{47}$$

Combining (43) and (47), the r.h.s. of (38) is established.
On the other hand, expanding the integral given in (38) with the help of Lemma C.8 of van der pas et al. (2017) provides (39). □

Lemma 3. *There exist functions $R_{\tau,1}$ with $\sup_x |R_{\tau,1}(x)| = O(\tau^{\frac{1}{2}})$ as $\tau \to 0$, such that for $\alpha \geq \frac{1}{2}$,*

$$J_{\alpha+1,\frac{1}{2}}(x) = \left(\frac{x^2}{2}\right)^{-\frac{1}{2}} \int_0^{\frac{x^2}{2}} e^v v^{-\frac{1}{2}} dv (1+R_{\tau,1}(x)) \lesssim (1 \wedge x^{-2}) e^{\frac{x^2}{2}} \tag{48}$$

and

$$J_{\alpha+1,\frac{1}{2}}(x) - J_{\alpha+2,\frac{1}{2}}(x) = \left(\frac{x^2}{2}\right)^{-\frac{1}{2}} \int_0^{\frac{x^2}{2}} e^v v^{-\frac{1}{2}} \left(1 - \frac{2v}{x^2}\right) dv (1+R_{\tau,1}(x)) \lesssim (1 \wedge x^{-4}) e^{\frac{x^2}{2}}. \tag{49}$$

Proof. Recall that $J_{\alpha+1,\frac{1}{2}}(x)$ is defined as

$$J_{\alpha+1,\frac{1}{2}}(x) = \int_0^1 e^{\frac{x^2 z}{2}} z^{\alpha} \left(\frac{1}{\tau^2 + (1-\tau^2)z} \right)^{\frac{1}{2}+\alpha} dz. \tag{50}$$

Next, we split the range of the integration into $[0, \tau]$ and $[\tau, 1]$. Note that, the contribution of the first integral obtained from (50) is bounded by

$$\int_0^{\tau} e^{\frac{x^2 z}{2}} z^{\alpha} \left(\frac{1}{\tau^2 + (1-\tau^2)z} \right)^{\frac{1}{2}+\alpha} dz \lesssim e^{\frac{x^2 \tau}{2}} \tau^{\frac{1}{2}}. \tag{51}$$

On the other hand, note that, when $z \geq \tau$, $\frac{1}{\tau^2+(1-\tau^2)z} = \frac{1}{z}[1 + O(\tau)]$. Therefore, we have

$$\begin{aligned} \int_{\tau}^1 e^{\frac{x^2 z}{2}} z^{\alpha} \left(\frac{1}{\tau^2 + (1-\tau^2)z} \right)^{\frac{1}{2}+\alpha} dz &= \int_{\tau}^1 e^{\frac{x^2 z}{2}} z^{-\frac{1}{2}} dz [1 + O(\tau)]^{\frac{1}{2}+\alpha} \\ &\gtrsim \exp(\frac{x^2}{2}\tau)(1 - \tau^{\frac{1}{2}})[1 + O(\tau)]. \end{aligned} \tag{52}$$

Combining (51) and (52) we see that the contribution of the first integral is negligible compared to the second one. Hence, one has

$$J_{\alpha+1,\frac{1}{2}}(x) = \int_{\tau}^1 e^{\frac{x^2 z}{2}} z^{-\frac{1}{2}} dz (1 + O(\tau) + O(\tau^{\frac{1}{2}})).$$

Also note that, $\tau = O(\tau^{\frac{1}{2}})$ as $\tau \to 0$. Observe that, (51) and (52) also imply that,

$$\int_{\tau}^1 e^{\frac{x^2 z}{2}} z^{-\frac{1}{2}} dz = \int_0^1 e^{\frac{x^2 z}{2}} z^{-\frac{1}{2}} dz [1 + O(\tau^{\frac{1}{2}})] \to 0 \text{ as } \tau \to 0.$$

As a result,

$$J_{\alpha+1,\frac{1}{2}}(x) = \int_0^1 e^{\frac{x^2 z}{2}} z^{-\frac{1}{2}} dz (1 + O(\tau^{\frac{1}{2}})).$$

The equality in (48) follows due to the change of variable $\frac{x^2 z}{2} = v$. The second assertion in (48) is due to exactly the same set of arguments used in Lemma C.10 of van der pas et al. (2017).
Proof of (49) follows using a similar set of arguments. □

Lemma 4. *The function $x \mapsto m_\tau(x)$ is symmetric about 0 and non-decreasing in $[0,\infty)$ with*
(i) $-2\alpha \le m_\tau(x) \le 2a$, *for all* $x \in \mathbb{R}$ *and* $\tau \in (0,1)$.
(ii) $|m_\tau(x)| \lesssim \tau e^{x^2/2}(x^{-2} \wedge 1)$, *as* $\tau \to 0$, *for every* x.

Proof. The symmetric behavior follows from the definition of $m_\tau(x)$ as given in (31).
For monotonicity, using (36) of Lemma 1, it readily follows that

$$\begin{aligned}
m_\tau(x) &= 2a + \tau \frac{\dot{I}_{\alpha,a(x)}}{I_{\alpha,a}(x)} \\
&= 2a - 2\tau^2(a+\alpha)\frac{\int_0^1 e^{\frac{x^2 z}{2}} z^{\alpha-1}(1-z)^{a+\frac{1}{2}}(\frac{1}{N(z)})^{a+\alpha+1}dz}{\int_0^1 e^{\frac{x^2 z}{2}} z^{\alpha-1}(1-z)^{a-\frac{1}{2}}(\frac{1}{N(z)})^{a+\alpha}dz} \qquad (53)\\
&= 2a + 2\tau^2(a+\alpha)\frac{\int_0^1 e^{\frac{x^2 z}{2}} (\frac{z-1}{\tau^2+(1-\tau^2)z}) z^{\alpha-1}(1-z)^{a-\frac{1}{2}}(\frac{1}{N(z)})^{a+\alpha}dz}{\int_0^1 z^{\alpha-1} e^{\frac{x^2 z}{2}}(1-z)^{a-\frac{1}{2}}(\frac{1}{N(z)})^{a+\alpha}dz} \\
&= 2a + 2\tau^2(a+\alpha)\int_0^1 \left(\frac{z-1}{\tau^2+(1-\tau^2)z}\right) g_x(z)dz,
\end{aligned}$$

where $z \mapsto g_x(z)$ is a probability density function on $[0,1]$ with $g_x(z) \propto z^{\alpha-1}e^{\frac{x^2 z}{2}}(1-z)^{a-\frac{1}{2}}\big(\frac{1}{N(z)}\big)^{a+\alpha}$. Next, following the same set of arguments as used in Lemma C.7 of van der pas et al. (2017), the proof of monotonicity of $m_\tau(x)$ follows. We now prove statements (i) and (ii) of the lemma.
(i) The upper bound is obvious by using (53). For the lower bound, note that

$$\tau \frac{\dot{I}_{\alpha,a(x)}}{I_{\alpha,a}(x)} = -2(\alpha+a)\int_0^1 \left(\frac{(1-z)\tau^2}{\tau^2+(1-\tau^2)z}\right) g_x(z)dz.$$

Since, for any $0 \le z \le 1$, $\tau^2 + (1-\tau^2)z \ge \tau^2$ implies $\frac{(1-z)\tau^2}{\tau^2+(1-\tau^2)z} \le 1$, the lower bound follows from it.
(ii) Using the definition of $m_\tau(x)$ as given in (31) followed by the triangle inequality, an upper bound on $|m_\tau(x)|$ is obtained as

$$|m_\tau(x)| \le x^2 \frac{|J_{\alpha+1,a}(x) - J_{\alpha+2,a}(x)|}{I_{\alpha,a}(x)} + \frac{|J_{\alpha+1,a}(x)|}{I_{\alpha,a}(x)}.$$

The assertion is proved by using (48) and (49) of Lemma 3, (38) of Lemma 2 and finally noting that $\frac{e^{x^2/2}}{x^2/2} \ge \frac{1}{\tau}$ for $|x| \ge B\kappa_\tau$ for $B \ge 1$ as $\tau \to 0$. □

As an immediate consequence of Lemma 4, we have the following corollary.

Corollary 1. *Let $X \sim \mathcal{N}(\theta, 1)$. Then, as $\tau \to 0$,*

$$E_\theta m_\tau^2(X) = \begin{cases} O(\tau \zeta_\tau^{-2}), |\theta| \lesssim \zeta_\tau^{-1}, \\ O(\tau^{\frac{1}{16}} \zeta_\tau^{-2}), |\theta| \leq \frac{\zeta_\tau}{4}. \end{cases}$$

Proof. Noting that the upper bound of the absolute value of $m_\tau(x)$ as obtained in (ii) of Lemma 4 matches that of (vii) of Lemma C.7 of van der pas et al. (2017), the proof is immediate using the same set of arguments used in Lemma C.5 of van der pas et al. (2017). □

Lemma 5. *Let $X \sim \mathcal{N}(\theta, 1)$. For $|\theta| \lesssim \zeta_\tau^{-1}$, and $\tau \leq \tau_1 < \tau_2$ and $\tau_2 \to 0$,*

$$E_\theta \left(\frac{\zeta_{\tau_1}}{\tau_1} m_{\tau_1}(X) - \frac{\zeta_{\tau_2}}{\tau_2} m_{\tau_2}(X) \right)^2 \lesssim (\tau_2 - \tau_1)^2 \tau_1^{-3}. \tag{54}$$

Further, for $|\theta| \leq \frac{\zeta_\tau}{4}$, and $\epsilon = \frac{1}{16}$, and $\tau \leq \tau_1 < \tau_2$ and $\tau_2 \to 0$,

$$E_\theta \left(\frac{\zeta_{\tau_1}}{\tau_1^\epsilon} m_{\tau_1}(X) - \frac{\zeta_{\tau_2}}{\tau_2^\epsilon} m_{\tau_2}(X) \right)^2 \lesssim (\tau_2 - \tau_1)^2 \tau_1^{-2-\epsilon}. \tag{55}$$

Proof. Using Lemma C.11 of van der pas et al. (2017) with $V_\tau = \frac{\zeta_\tau}{\tau} m_\tau(X)$, the l.h.s. of (54) can be upper bounded as

$$\begin{aligned} &E_\theta \left(\frac{\zeta_{\tau_1}}{\tau_1} m_{\tau_1}(X) - \frac{\zeta_{\tau_2}}{\tau_2} m_{\tau_2}(X) \right)^2 \leq (\tau_2 - \tau_1)^2 \sup_{\tau \in [\tau_1, \tau_2]} E_\theta \left(\frac{\zeta_\tau}{\tau} \dot{m}_\tau(X) - \frac{\zeta_\tau + \zeta_\tau^{-1}}{\tau^2} m_\tau(X) \right)^2 \\ &\leq 2(\tau_2 - \tau_1)^2 \left[\sup_{\tau \in [\tau_1, \tau_2]} E_\theta \left(\frac{\zeta_\tau}{\tau} \dot{m}_\tau(X) \right)^2 + \sup_{\tau \in [\tau_1, \tau_2]} E_\theta \left(\frac{\zeta_\tau + \zeta_\tau^{-1}}{\tau^2} m_\tau(X) \right)^2 \right]. \end{aligned} \tag{56}$$

Note that, with the help of the first part of Corollary 1, the second term in the r.h.s. of (56) is bounded above by a constant times $\sup_{\tau \in [\tau_1, \tau_2]} \tau^{-3} \lesssim \tau_1^{-3}$. Hence, in order to show that (54) holds, it is enough to show that the first term in the r.h.s. of (54) is bounded above by a constant times τ_1^{-3}.

For the first term, observe that using (32) with $a = \frac{1}{2}$,

$$\dot{m}_\tau(x) = \frac{I_{\alpha,\frac{1}{2}}(x)[(x^2-1)\dot{J}_{\alpha+1,\frac{1}{2}}(x) - x^2\dot{J}_{\alpha+2,\frac{1}{2}}(x)] - \dot{I}_{\alpha,\frac{1}{2}}(x)m_\tau(x)I_{\alpha,\frac{1}{2}}(x)}{(I_{\alpha,\frac{1}{2}}(x))^2}$$
$$= (x^2-1)\frac{\dot{J}_{\alpha+1,\frac{1}{2}}(x)}{I_{\alpha,\frac{1}{2}}(x)} - x^2\frac{\dot{J}_{\alpha+2,\frac{1}{2}}(x)}{I_{\alpha,\frac{1}{2}}(x)} - \frac{\dot{I}_{\alpha,\frac{1}{2}}(x)}{I_{\alpha,\frac{1}{2}}(x)}m_\tau(x). \tag{57}$$

Now, note that, using the definition of $J_{\alpha+1,\frac{1}{2}}(x)$, $\dot{J}_{\alpha+1,\frac{1}{2}}(x) = 2\tau(\alpha+\frac{1}{2})(H_{\alpha+2,\frac{1}{2}}(x) - H_{\alpha+1,\frac{1}{2}}(x))$ and $\dot{J}_{\alpha+2,\frac{1}{2}}(x) = 2\tau(\alpha+\frac{1}{2})(H_{\alpha+3,\frac{1}{2}}(x) - H_{\alpha+2,\frac{1}{2}}(x))$ where $H_{\alpha+k,\frac{1}{2}}(x) = \int_0^1 e^{\frac{x^2 z}{2}} z^{\alpha+k-1}(1-z)^{a-\frac{1}{2}}\left(\frac{1}{\tau^2+(1-\tau^2)z}\right)^{\alpha+\frac{3}{2}} dz$. Next, note that, $H_{\alpha+k,\frac{1}{2}}(x)$ is a decreasing function of k. Also, observe that, $H_{\alpha+1,\frac{1}{2}}(x) \leq I_{\alpha,\frac{1}{2}}(x)$. Finally, for the third term in the r.h.s. of (57), the definition of $I_{\alpha,\frac{1}{2}}(x)$ implies

$$-\frac{\dot{I}_{\alpha,\frac{1}{2}}(x)}{I_{\alpha,\frac{1}{2}}(x)} \leq 2\tau(\alpha+\frac{1}{2}).\frac{1}{\tau^2}.$$

Combining all these arguments provides an upper bound for the r.h.s. of (57) as

$$\dot{m}_\tau(x) \leq 2\tau(\alpha+\frac{1}{2})[1+x^2+\frac{1}{\tau^2}m_\tau(x)].$$

As a consequence of this,

$$E_\theta \dot{m}_\tau^2(X) \lesssim \tau^2[1 + E_\theta X^4 + \frac{1}{\tau^4}E_\theta m_\tau^2(X)]. \tag{58}$$

Note that, in this case, $E_\theta X^4$ is bounded by a constant and from the first part of Corollary 1, $E_\theta m_\tau^2(X)$ is bounded by $\tau\zeta_\tau^{-2}$. This shows that $(\frac{\zeta_\tau}{\tau})^2 E_\theta \dot{m}_\tau^2(X)$ is bounded above by a multiple of τ^{-3} and (54) is established.
For proving (55), we again use Lemma C.11 of van der pas et al. (2017), but with $V_\tau = \frac{\zeta_\tau}{\tau^\epsilon}m_\tau(X)$. As a result, we have

$$E_\theta\left(\frac{\zeta_{\tau_1}}{\tau_1^\epsilon}m_{\tau_1}(X) - \frac{\zeta_{\tau_2}}{\tau_2^\epsilon}m_{\tau_2}(X)\right)^2$$
$$\leq 2(\tau_2-\tau_1)^2\left[\sup_{\tau\in[\tau_1,\tau_2]} E_\theta\left(\frac{\zeta_\tau}{\tau^\epsilon}\dot{m}_\tau(X)\right)^2 + \sup_{\tau\in[\tau_1,\tau_2]} E_\theta\left(\frac{\epsilon\zeta_\tau+\zeta_\tau^{-1}}{\tau^{1+\epsilon}}m_\tau(X)\right)^2\right].$$

Using the second part of Corollary 1, the second term in the r.h.s. of the above inequality is bounded above by a constant times $\tau^{-2-\epsilon}$. Next, we follow the same steps as before and obtain an expression same as (58). In this case, $E_\theta X^4$ is bounded by ζ_τ^4 and from second part of Corollary 1, $E_\theta m_\tau^2(X)$ is bounded by $\tau^\epsilon \zeta_\tau^{-2}$. This implies that $(\frac{\zeta_\tau}{\tau^{1+\epsilon}})^2 E_\theta \dot{m}_\tau^2(X)$ is bounded above by a multiple of $\tau^{-2-\epsilon}$ completing the proof of (55). □

Next, we provide a corollary which becomes very important to study the relationship between $m_\tau(X_i)$ and its expectation for zero means.

Corollary 2. *If the cardinality of $I_0 := \{i : \theta_{0,i} = 0\}$ tends to infinity, then*

$$\sup_{\frac{1}{n}\leq\tau\leq\frac{1}{\log n}} \frac{1}{|I_0|}|\sum_{i\in I_0} m_\tau(X_i) - \sum_{i\in I_0} E_{\theta_0} m_\tau(X_i)| \xrightarrow{P_{\theta_0}} 0.$$

Proof. The proof of this result follows using a similar set of arguments to those of Lemma C.6 of van der pas et al. (2017) with some modifications. The main difference lies in calculating the entropy of the process $G_n(\tau) = |I_0|^{-1}\sum_{i\in I_0} m_\tau(X_i)$. Here, instead of covering the interval $[\frac{1}{n}, 1]$ as given in Lemma C.6 of van der pas et al. (2017), we need to cover the interval $[\frac{1}{n}, \frac{1}{\log n}]$. We use dyadic rationals $[\frac{2^i}{n}, \frac{2^{i+1}}{n}]$ to cover this interval with $i = 0, 1, 2, \cdots, [\log_2(\frac{n}{\log n})]$. Next, we follow steps similar to Lemma C.6 of van der pas et al. (2017) along with Lemma 5. □

Lemma 6. *Let $X \sim \mathcal{N}(\theta, 1)$. Then as $\tau \to 0$*

$$E_\theta m_\tau(X) = \begin{cases} -\frac{2}{K^{-1}\sqrt{2\pi}}\frac{\tau}{\zeta_\tau}(1+o(1)), & |\theta| = o(\zeta_\tau^{-2}), \\ o(\tau^{\frac{1}{16}}\zeta_\tau^{-1}), & |\theta| \leq \frac{\zeta_\tau}{4}. \end{cases}$$

Proof. For proving the first assertion, following the steps of Proposition C.2 of van der pas et al. (2017), we can show that both $\int_{|x|\geq\kappa_\tau} m_\tau(x)\phi(x-\theta)dx = o(\frac{\tau}{\zeta_\tau})$ and
$\int_{\zeta_\tau\leq|x|\leq\kappa_\tau} m_\tau(x)\phi(x-\theta)dx = o(\frac{\tau}{\zeta_\tau})$,where $\kappa_\tau \sim \zeta_\tau + \frac{2\log\zeta_\tau}{\zeta_\tau}$ as $\tau \to 0$, where κ_τ is defined in Lemma 2. The remaining argument also follows using similar sets of arguments used in Proposition C.2 of van der pas et al. (2017) along with some algebraic manipulations. However, for the sake of completeness, we present all the steps.
Note that, from (38) of Lemma 2, when $\frac{x^2}{2} \leq 1$, $I_{\alpha,\frac{1}{2}}(x) = \frac{K^{-1}}{\tau}[1 + O(\sqrt{\tau})]$.

On the other hand, since, $\frac{e^{\frac{x^2}{2}}}{x^2}$ is increasing for large values of x and attains the value $\frac{\tau}{\zeta_\tau^2}$ at $x = \zeta_\tau$, by (38) of Lemma 2, $I_{\alpha,\frac{1}{2}}(x) = \frac{K^{-1}}{\tau}[1 + O(\frac{1}{\zeta_\tau^2})]$, when $1 \leq \frac{x^2}{2} \leq \log(\frac{1}{\tau})$. Combining these two facts, we get that, $I_{\alpha,\frac{1}{2}}(x) = \frac{K^{-1}}{\tau}[1 + O(\frac{1}{\zeta_\tau^2})]$, uniformly in $x \in (0, \zeta_\tau)$. Hence,

$$\int_{|x|\leq\zeta_\tau} m_\tau(x)\phi(x-\theta)dx = \int_0^{\zeta_\tau} \frac{x^2(J_{\alpha+1,\frac{1}{2}}(x) - J_{\alpha+2,\frac{1}{2}}(x)) - J_{\alpha+1,\frac{1}{2}}(x)}{\frac{K^{-1}}{\tau}}\phi(x)dx + R_\tau, \tag{59}$$

where the absolute value of R_τ is bounded by $\int_0^{\zeta_\tau} |x^2(J_{\alpha+1,\frac{1}{2}}(x) - J_{\alpha+2,\frac{1}{2}}(x)) - J_{\alpha+1,\frac{1}{2}}(x)|\phi(x)dx$ times $\sup_{|x|\leq\zeta_\tau} |\frac{\phi(x-\theta)}{I_{\alpha,\frac{1}{2}}(x)\phi(x)} - \frac{K}{\tau^{-1}}|$. By using Lemma 3, the integrand is bounded above by a constant for x near 0 and by a multiple of x^{-2} otherwise, which makes the integral to be bounded. Next, observe that,

$$\begin{aligned}\sup_{|x|\leq\zeta_\tau} |\frac{\phi(x-\theta)}{I_{\alpha,\frac{1}{2}}(x)\phi(x)} - \frac{K}{\tau^{-1}}| &= \sup_{|x|\leq\zeta_\tau} \frac{\tau K}{I_{\alpha,\frac{1}{2}}(x)}|\tau^{-1}K^{-1}(e^{x\theta-\frac{\theta^2}{2}}) - I_{\alpha,\frac{1}{2}}(x)| \\ &= \sup_{|x|\leq\zeta_\tau} \tau K|\frac{e^{x\theta-\frac{\theta^2}{2}}}{1+O(\frac{1}{\zeta_\tau^2})} - 1| \lesssim \tau[\frac{1}{\zeta_\tau^2} + e^{\zeta_\tau\theta-\frac{\theta^2}{2}} - 1].\end{aligned}$$

Observe that, for $|\theta| = o(\zeta_\tau^{-2})$, $\zeta_\tau|\theta| - \frac{\theta^2}{2} = o(\zeta_\tau^{-1})$ and using the fact $e^y - 1 \sim y$ as $y \to 0$, we have

$$\sup_{|x|\leq\zeta_\tau} |\frac{\phi(x-\theta)}{I_{\alpha,\frac{1}{2}}(x)\phi(x)} - \frac{K}{\tau^{-1}}| \lesssim \tau[\frac{1}{\zeta_\tau^2} + o(\zeta_\tau^{-1})] = o(\frac{\tau}{\zeta_\tau}).$$

These two arguments show that R_τ is negligible compared to $\frac{\tau}{\zeta_\tau}$.
Next, using Fubini's Theorem, the integral in (59) can be rewritten as

$$\begin{aligned}&\int_0^{\zeta_\tau} \frac{x^2(J_{\alpha+1,\frac{1}{2}}(x) - J_{\alpha+2,\frac{1}{2}}(x)) - J_{\alpha+1,\frac{1}{2}}(x)}{\frac{K^{-1}}{\tau}}\phi(x)dx \\ &= K\tau \int_0^1 \int_0^{\zeta_\tau} z^\alpha \left(\frac{1}{N(z)}\right)^{\frac{1}{2}+\alpha} e^{\frac{x^2 z}{2}}[x^2(1-z) - 1]\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\, dz \qquad (60) \\ &= K\tau \int_0^1 z^\alpha \left(\frac{1}{N(z)}\right)^{\frac{1}{2}+\alpha} \int_0^{\zeta_\tau} [x^2(1-z) - 1]\frac{e^{-\frac{x^2(1-z)}{2}}}{\sqrt{2\pi}}dx\, dz.\end{aligned}$$

Note that the inner integral becomes zero if the range of integration is $(0,\infty)$ instead of $(0,\zeta_\tau)$. Hence, we have

$$\begin{aligned}
&\int_0^{\zeta_\tau} \frac{x^2(J_{\alpha+1,\frac{1}{2}}(x) - J_{\alpha+2,\frac{1}{2}}(x)) - J_{\alpha+1,\frac{1}{2}}(x)}{\frac{K^{-1}}{\tau}}\phi(x)dx \\
&= -K\tau \int_0^1 z^\alpha \left(\frac{1}{N(z)}\right)^{\frac{1}{2}+\alpha} \int_{\zeta_\tau}^\infty [x^2(1-z)-1]\frac{e^{-\frac{x^2(1-z)}{2}}}{\sqrt{2\pi}}dx\, dz \\
&= -K\tau \int_0^1 z^\alpha \left(\frac{1}{N(z)}\right)^{\frac{1}{2}+\alpha} \frac{\zeta_\tau}{\sqrt{2\pi}} e^{-\frac{\zeta_\tau^2(1-z)}{2}}dz.
\end{aligned}$$

In the last line above, we use the fact $\int_y^\infty [(vb)^2-1]\phi(vb)dv = y\phi(yb)$. Next, similar to Proposition C.2 of van der pas et al. (2017), we split the range of integration in $(0,\frac{1}{2}]$ and $(\frac{1}{2},1)$. When $0 \le z \le \frac{1}{2}$, the absolute value of the integral is bounded above by

$$\begin{aligned}
K\tau \int_0^{\frac{1}{2}} z^\alpha \left(\frac{1}{N(z)}\right)^{\frac{1}{2}+\alpha} \frac{\zeta_\tau}{\sqrt{2\pi}} e^{-\frac{\zeta_\tau^2(1-z)}{2}}dz &\lesssim K\frac{e^{-\frac{\zeta_\tau^2}{4}}\tau\zeta_\tau}{\sqrt{2\pi}(1-\tau^2)^{\frac{1}{2}+\alpha}}\int_0^{\frac{1}{2}} z^{-\frac{1}{2}}dz \\
&= O(e^{-\frac{\zeta_\tau^2}{4}}\tau\zeta_\tau) = o(\frac{\tau}{\zeta_\tau}).
\end{aligned}$$

On the other hand, when $\frac{1}{2} \le z \le 1$, we again use, $\frac{1}{\tau^2+(1-\tau^2)z} = \frac{1}{z}[1+O(\tau^2)]$. This implies

$$\begin{aligned}
&-K\tau \int_{\frac{1}{2}}^1 z^\alpha \left(\frac{1}{N(z)}\right)^{\frac{1}{2}+\alpha} \frac{\zeta_\tau}{\sqrt{2\pi}} e^{-\frac{\zeta_\tau^2(1-z)}{2}}dz \\
&= -\frac{K\tau\zeta_\tau}{\sqrt{2\pi}} \int_{\frac{1}{2}}^1 z^{-\frac{1}{2}} e^{-\frac{\zeta_\tau^2(1-z)}{2}}dz[1+O(\tau^2)] \\
&= -\frac{K}{\sqrt{2\pi}}\frac{\tau}{\zeta_\tau}\int_0^{\frac{\zeta_\tau^2}{2}} e^{-\frac{u}{2}}\frac{1}{(1-\frac{u}{\zeta_\tau^2})^{\frac{1}{2}}}du[1+O(\tau^2)],
\end{aligned}$$

where the equality is due to the substitution $\zeta_\tau^2(1-z) = u$. Finally, following the same argument as used in Proposition C.2 of van der pas et al. (2017), the integral tends to $\int_0^\infty e^{-\frac{u}{2}}du = 2$, completing the proof of the first assertion. For proving the second statement, we follow the steps mentioned in Proposition C.2 of van der pas et al. (2017). □

2 Derivation of the posterior distribution of κ_i given X_i and τ

Recall that, the hierarchical formulation is given as

$$\begin{aligned} X_i|\theta_i &\overset{ind}{\sim} \mathcal{N}(\theta_i, 1) \\ \theta_i|\kappa_i &\overset{ind}{\sim} \mathcal{N}(0, \frac{1-\kappa_i}{\kappa_i}), \end{aligned} \tag{61}$$

where $\kappa_i = 1/(1+\lambda_i^2\tau^2)$. Due to the change of variable, prior distribution of κ_i given τ is of the form,

$$\pi(\kappa_i|\tau) \propto \kappa_i^{a-1}(1-\kappa_i)^{-a-1}L(\frac{1}{\tau^2}(\frac{1}{\kappa_i}-1)). \tag{62}$$

Hence, combining (61), (62) and using Bayes theorem, the joint posterior distribution of (θ_i, κ_i) given X_i and τ is obtained as

$$\begin{aligned} \pi(\theta_i, \kappa_i|X_i, \tau) &\propto (\frac{1-\kappa_i}{\kappa_i})^{-\frac{1}{2}} \exp[-\frac{\kappa_i\theta_i^2}{2(1-\kappa_i)} - \frac{(\theta_i - X_i)^2}{2}]\kappa_i^{a-1}(1-\kappa_i)^{-a-1}L(\frac{1}{\tau^2}(\frac{1}{\kappa_i}-1)) \\ &\propto \kappa_i^{a-\frac{1}{2}}(1-\kappa_i)^{-a-\frac{3}{2}}L(\frac{1}{\tau^2}(\frac{1}{\kappa_i}-1)) \exp[-\frac{\theta_i^2}{2(1-\kappa_i)} + \theta_i X_i]. \end{aligned} \tag{63}$$

Integrating out θ_i, from (63) the posterior distribution of κ_i is given by

$$\begin{aligned} \pi(\kappa_i|X_i, \tau) &\propto \kappa_i^{a-\frac{1}{2}}(1-\kappa_i)^{-a-\frac{3}{2}}L(\frac{1}{\tau^2}(\frac{1}{\kappa_i}-1)) \int_{-\infty}^{\infty} \exp[-\frac{\theta_i^2}{2(1-\kappa_i)} + \theta_i X_i]d\theta_i \\ &\propto \kappa_i^{a-\frac{1}{2}}(1-\kappa_i)^{-a-\frac{3}{2}}L(\frac{1}{\tau^2}(\frac{1}{\kappa_i}-1))(1-\kappa_i)^{\frac{1}{2}} \exp\frac{(1-\kappa_i)X_i^2}{2} \\ &= \kappa_i^{a-\frac{1}{2}}(1-\kappa_i)^{-a-1}L(\frac{1}{\tau^2}(\frac{1}{\kappa_i}-1)) \exp\frac{(1-\kappa_i)X_i^2}{2}. \end{aligned}$$

References

Ghosh. P., and Chakrabarti, A. (2017). Asymptotic optimality of one-group shrinkage priors in sparse high-dimensional problems. *Bayesian Analysis*, 12(4):1133–1161. 2, 3, 5, 8 van der Pas, S. L., Kleijn, B. J., and Van Der Vaart, A. W. (2014). The horseshoe estimator: Posterior concentration around nearly black vectors. *Electronic Journal of Statistics*, 8(2):2585–2618. 5

van der Pas, S. L., Szabo, B. and van der Vaart, A. W. (2017). Adap- tive posterior contraction rates for the horseshoe. *Electronic Journal of Statistics*, 11(2):3196–3225. 11, 14, 15, 16, 17, 18, 19, 20, 22