Supplementary Material for "Variable Selection via Penalized Ridge Regression with Error-Prone Variables"

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In this supplementary materials, we present the technical proofs of theoretical results that are presented in the main paper.

Appendix A Derivation of (10)

We start our derivation on (7). The conditional expectation of (7), given $\mathbb{X}$ and $\mathbb{Y}$, is given by

$$E\left\{U^*(\beta)|\mathbb{Y},\mathbb{X}\right\} = -2\left[E\left\{\mathbb{W}|\mathbb{Y},\mathbb{X}\right\}\right]^\top \mathbb{Y} + 2\left[E\left\{\mathbb{W}^\top\mathbb{W}|\mathbb{Y},\mathbb{X}\right\}\right]\beta + 2V\beta.$$

To the end, we separately examine $E\left\{\mathbb{W}|\mathbb{Y},\mathbb{X}\right\}$ and $E\left\{\mathbb{W}^\top\mathbb{W}|\mathbb{Y},\mathbb{X}\right\}$.

Step 1: Derive $E\left\{\mathbb{W}|\mathbb{Y},\mathbb{X}\right\}$.

First, $E\left\{\mathbb{W}|\mathbb{Y},\mathbb{X}\right\}$ involves $E\left\{\mathbb{W}_{\mathrm{C}}|\mathbb{Y},\mathbb{X}\right\}$, $E\left\{\mathbb{W}_{\mathrm{B}}|\mathbb{Y},\mathbb{X}\right\}$, and $E\left\{\mathbb{W}_{\mathrm{D}}|\mathbb{Y},\mathbb{X}\right\}$. By (5), we can easily obtain that $E\left\{\mathbb{W}_{\mathrm{C}}|\mathbb{Y},\mathbb{X}\right\} = \mathbb{X}_{\mathrm{C}}$. For the binary random variables, by (5) with specification of Γ_{B} that is invertible, we have that $E\left\{(\mathbb{W}_{\mathrm{B}} - \mathbb{E}_{\mathrm{B}})\Gamma_{\mathrm{B}}^{-1}|\mathbb{Y},\mathbb{X}\right\} = \mathbb{X}_{\mathrm{B}}$. Finally, for the count variable, we have that $E\left\{\mathbb{W}_{\mathrm{D}} - \lambda 1_{n\times p_{\mathrm{D}}}|\mathbb{Y},\mathbb{X}\right\} = \mathbb{X}_{\mathrm{D}}$. Consequently, there exist two matrices $\Gamma^{-1} \triangleq \begin{pmatrix} I_{p_{\mathrm{C}}\times p_{\mathrm{C}}} & O_{p_{\mathrm{C}}\times p_{\mathrm{B}}} & O_{p_{\mathrm{C}}\times p_{\mathrm{D}}} \\ O_{p_{\mathrm{B}}\times p_{\mathrm{C}}} & \Gamma_{\mathrm{B}}^{-1} & O_{p_{\mathrm{B}}\times p_{\mathrm{D}}} \\ O_{p_{\mathrm{D}}\times p_{\mathrm{C}}} & O_{p_{\mathrm{D}}\times p_{\mathrm{B}}} & I_{p_{\mathrm{D}}\times p_{\mathrm{D}}} \end{pmatrix}$ and $\Lambda = (O_{n\times p_{\mathrm{C}}}, \mathbb{E}_{\mathrm{B}}, \lambda 1_{n\times p_{\mathrm{D}}})$, such that

$$E\left\{\mathbb{W}\Gamma^{-1} - \Lambda\middle|\mathbb{Y},\mathbb{X}\right\} = \mathbb{X}, \tag{A.1}$$

showing that error-prone variables $\mathbb{W}$ can be adjusted to the unobserved $\mathbb{X}$ by multiplying Γ^{-1} and subtracting Λ.

Step 2: Derive $E\left\{\mathbb{W}^\top\mathbb{W}|\mathbb{Y},\mathbb{X}\right\}$.

Next, we discuss $E\left\{\mathbb{W}^\top\mathbb{W}|\mathbb{Y},\mathbb{X}\right\}$. Noting that $\mathbb{W}^\top\mathbb{W}$ and $\mathbb{X}^\top\mathbb{X}$ are respectively written as

$$\mathbb{W}^\top\mathbb{W} = \begin{pmatrix} \mathbb{W}_{\mathrm{C}}^\top\mathbb{W}_{\mathrm{C}} & \mathbb{W}_{\mathrm{C}}^\top\mathbb{W}_{\mathrm{B}} & \mathbb{W}_{\mathrm{C}}^\top\mathbb{W}_{\mathrm{D}} \\ \mathbb{W}_{\mathrm{B}}^\top\mathbb{W}_{\mathrm{C}} & \mathbb{W}_{\mathrm{B}}^\top\mathbb{W}_{\mathrm{B}} & \mathbb{W}_{\mathrm{B}}^\top\mathbb{W}_{\mathrm{D}} \\ \mathbb{W}_{\mathrm{D}}^\top\mathbb{W}_{\mathrm{C}} & \mathbb{W}_{\mathrm{D}}^\top\mathbb{W}_{\mathrm{B}} & \mathbb{W}_{\mathrm{D}}^\top\mathbb{W}_{\mathrm{D}} \end{pmatrix} \text{ and } \mathbb{X}^\top\mathbb{X} = \begin{pmatrix} \mathbb{X}_{\mathrm{C}}^\top\mathbb{X}_{\mathrm{C}} & \mathbb{X}_{\mathrm{C}}^\top\mathbb{X}_{\mathrm{B}} & \mathbb{X}_{\mathrm{C}}^\top\mathbb{X}_{\mathrm{D}} \\ \mathbb{X}_{\mathrm{B}}^\top\mathbb{X}_{\mathrm{C}} & \mathbb{X}_{\mathrm{B}}^\top\mathbb{X}_{\mathrm{B}} & \mathbb{X}_{\mathrm{B}}^\top\mathbb{X}_{\mathrm{D}} \\ \mathbb{X}_{\mathrm{D}}^\top\mathbb{X}_{\mathrm{C}} & \mathbb{X}_{\mathrm{D}}^\top\mathbb{X}_{\mathrm{B}} & \mathbb{X}_{\mathrm{D}}^\top\mathbb{X}_{\mathrm{D}} \end{pmatrix}.$$

The conditional expectation is expressed as

$$E\left\{\mathbb{W}^\top\mathbb{W}|\mathbb{Y},\mathbb{X}\right\} = \begin{pmatrix} E\left\{\mathbb{W}_{\mathrm{C}}^\top\mathbb{W}_{\mathrm{C}}\middle|\mathbb{Y},\mathbb{X}\right\} & E\left\{\mathbb{W}_{\mathrm{C}}^\top\mathbb{W}_{\mathrm{B}}\middle|\mathbb{Y},\mathbb{X}\right\} & E\left\{\mathbb{W}_{\mathrm{C}}^\top\mathbb{W}_{\mathrm{D}}|\mathbb{Y},\mathbb{X}\right\} \\ E\left\{\mathbb{W}_{\mathrm{B}}^\top\mathbb{W}_{\mathrm{C}}\middle|\mathbb{Y},\mathbb{X}\right\} & E\left\{\mathbb{W}_{\mathrm{B}}^\top\mathbb{W}_{\mathrm{B}}\middle|\mathbb{Y},\mathbb{X}\right\} & E\left\{\mathbb{W}_{\mathrm{B}}^\top\mathbb{W}_{\mathrm{D}}|\mathbb{Y},\mathbb{X}\right\} \\ E\left\{\mathbb{W}_{\mathrm{D}}^\top\mathbb{W}_{\mathrm{C}}\middle|\mathbb{Y},\mathbb{X}\right\} & E\left\{\mathbb{W}_{\mathrm{D}}^\top\mathbb{W}_{\mathrm{B}}\middle|\mathbb{Y},\mathbb{X}\right\} & E\left\{\mathbb{W}_{\mathrm{D}}^\top\mathbb{W}_{\mathrm{D}}|\mathbb{Y},\mathbb{X}\right\} \end{pmatrix}. \quad \text{(A.2)}$$

We now examine components in the upper triangular of (A.2) since it is a symmetric matrix.

Step 2.1: Derive $E\left(\mathbb{W}_{\mathrm{C}}^\top\mathbb{W}_{\mathrm{C}}|\mathbb{Y},\mathbb{X}\right)$.

Under (5) and the definition of the conditional variance, $E\left(\mathbb{W}_{\mathrm{C}}^\top\mathbb{W}_{\mathrm{C}}|\mathbb{Y},\mathbb{X}\right) = \mathbb{X}_{\mathrm{C}}^\top\mathbb{X}_{\mathrm{C}} + \Sigma_\epsilon$, which gives that $E\left(\mathbb{W}_{\mathrm{C}}^\top\mathbb{W}_{\mathrm{C}} - \Sigma_\epsilon|\mathbb{Y},\mathbb{X}\right) = \mathbb{X}_{\mathrm{C}}^\top\mathbb{X}_{\mathrm{C}}$.

Step 2.2: Derive $E\left(\mathbb{W}_{\mathrm{B}}^\top\mathbb{W}_{\mathrm{B}}|\mathbb{Y},\mathbb{X}\right)$.

Recall that

$$\mathbb{W}_{\mathrm{B}} = \mathbb{X}_{\mathrm{B}}\Gamma_{\mathrm{B}} + \mathbb{E}_{\mathrm{B}}.$$

Then $\mathbb{W}_{\mathrm{B}}^\top\mathbb{W}_{\mathrm{B}}$ can be expressed as

$$\begin{aligned} \mathbb{W}_{\mathrm{B}}^\top\mathbb{W}_{\mathrm{B}} &= \Gamma_{\mathrm{B}}^\top\mathbb{X}_{\mathrm{B}}^\top\mathbb{X}_{\mathrm{B}}\Gamma_{\mathrm{B}} + 2\Gamma_{\mathrm{B}}^\top\mathbb{X}_{\mathrm{B}}^\top\mathbb{E}_{\mathrm{B}} + \mathbb{E}_{\mathrm{B}} \\ &= \Gamma_{\mathrm{B}}\mathbb{X}_{\mathrm{B}}^\top\mathbb{X}_{\mathrm{B}}\Gamma_{\mathrm{B}} + 2\Gamma_{\mathrm{B}}\Gamma_{\mathrm{B}}^{-1}\left(\mathbb{W}_{\mathrm{B}}^\top - \mathbb{E}_{\mathrm{B}}^\top\right)\mathbb{E}_{\mathrm{B}} + \mathbb{E}_{\mathrm{B}} \\ &= \Gamma_{\mathrm{B}}\mathbb{X}_{\mathrm{B}}^\top\mathbb{X}_{\mathrm{B}}\Gamma_{\mathrm{B}} + 2\mathbb{W}_{\mathrm{B}}^\top\mathbb{E}_{\mathrm{B}} - \mathbb{E}_{\mathrm{B}}, \end{aligned}$$

where the second step is due to that Γ_{B} is a symmetric matrix and the expression $\mathbb{X}_{\mathrm{B}} = \left(\mathbb{W}_{\mathrm{B}} - \mathbb{E}_{\mathrm{B}}\right)\Gamma_{\mathrm{B}}^{-1}$. Consequently, it suggests that

$$E\left\{\Gamma_{\mathrm{B}}^{-1}\left(\mathbb{W}_{\mathrm{B}}^\top\mathbb{W}_{\mathrm{B}} - 2\mathbb{W}_{\mathrm{B}}^\top\mathbb{E}_{\mathrm{B}} + \mathbb{E}_{\mathrm{B}}\right)\Gamma_{\mathrm{B}}^{-1}|\mathbb{Y},\mathbb{X}\right\} = \mathbb{X}_{\mathrm{B}}^\top\mathbb{X}_{\mathrm{B}}. \quad \text{(A.3)}$$

Step 2.3: Derive $E\left(\mathbb{W}_{\mathrm{D}}^\top\mathbb{W}_{\mathrm{D}}|\mathbb{Y},\mathbb{X}\right)$.

For the count random variable, $E\left(\mathbb{W}_{\mathrm{D}}^{\top}\mathbb{W}_{\mathrm{D}}|\mathbb{Y},\mathbb{X}\right) = \mathbb{X}_{\mathrm{D}}^{\top}\mathbb{X}_{\mathrm{D}} + n\lambda 1_{p_{\mathrm{D}}\times p_{\mathrm{D}}} + 2\lambda 1_{n\times p_{\mathrm{D}}}^{\top}\mathbb{X}_{\mathrm{D}}$ and $E\left(1_{n\times p_{\mathrm{D}}}^{\top}\mathbb{W}_{\mathrm{D}}|\mathbb{Y},\mathbb{X}\right) = \lambda 1_{n\times p_{\mathrm{D}}}^{\top}\mathbb{X} + n\lambda 1_{p_{\mathrm{D}}\times p_{\mathrm{D}}}$. They suggest that

$$E\Big(\mathbb{W}_{\mathrm{D}}^{\top}\mathbb{W}_{\mathrm{D}} + n\lambda 1_{p_{\mathrm{D}}\times p_{\mathrm{D}}} - 2\lambda 1_{n\times p_{\mathrm{D}}}^{\top}\mathbb{W}_{\mathrm{D}}|\mathbb{Y},\mathbb{X}\Big) = \mathbb{X}_{\mathrm{D}}^{\top}\mathbb{X}_{\mathrm{D}}.$$

Step 2.4: Derive $E\left(\mathbb{W}_{\mathrm{C}}^{\top}\mathbb{W}_{\mathrm{D}}|\mathbb{Y},\mathbb{X}\right)$.

Let $\mathbb{E}_{\mathrm{C}} \triangleq (\epsilon_{\mathrm{C}1},\cdots,\epsilon_{\mathrm{C}n})^{\top}$ and $\mathbb{E}_{\mathrm{D}} \triangleq (\epsilon_{\mathrm{D}1},\cdots,\epsilon_{\mathrm{D}n})^{\top}$ denote the $n\times p_{\mathrm{C}}$ and $n\times p_{\mathrm{D}}$ matrices, respectively, where $\epsilon_{\mathrm{C}i}$ and $\epsilon_{\mathrm{D}i}$ are the independent copies of ϵ_{C} and ϵ_{D} for $i=1,\cdots,n$, respectively. By two measurement error models for $\mathbb{W}_{\mathrm{C}}$ and $\mathbb{W}_{\mathrm{D}}$, we have that

$$\mathbb{W}_{\mathrm{C}}^{\top}\mathbb{W}_{\mathrm{D}} = \mathbb{X}_{\mathrm{C}}^{\top}\mathbb{X}_{\mathrm{D}} + \mathbb{E}_{\mathrm{C}}^{\top}\mathbb{X}_{\mathrm{D}} + \mathbb{X}_{\mathrm{C}}^{\top}\mathbb{E}_{\mathrm{D}} + \mathbb{E}_{\mathrm{C}}^{\top}\mathbb{E}_{\mathrm{D}}.$$

Then taking the conditional expectation gives that

$$E\left(\mathbb{W}_{\mathrm{C}}^{\top}\mathbb{W}_{\mathrm{D}}|\mathbb{Y},\mathbb{X}\right) = \mathbb{X}_{\mathrm{C}}^{\top}\mathbb{X}_{\mathrm{D}} + \lambda\mathbb{X}_{\mathrm{C}}^{\top}1_{n\times p_{\mathrm{D}}}.$$

In addition, the model (5) implies that $E\left(\mathbb{W}_{\mathrm{C}}|\mathbb{Y},\mathbb{X}\right) = \mathbb{X}_{\mathrm{C}}$. Thus, we obtain that

$$E\left(\mathbb{W}_{\mathrm{C}}^{\top}\mathbb{W}_{\mathrm{D}} - \lambda\mathbb{W}_{\mathrm{C}}^{\top}1_{n\times p_{\mathrm{D}}}|\mathbb{Y},\mathbb{X}\right) = \mathbb{X}_{\mathrm{C}}^{\top}\mathbb{X}_{\mathrm{D}}.$$

Step 2.5: Derive $E\left(\mathbb{W}_{\mathrm{C}}^{\top}\mathbb{W}_{\mathrm{B}}|\mathbb{Y},\mathbb{X}\right)$.

Similar to Step 2.4, applying two measurement error models for $\mathbb{W}_{\mathrm{C}}$ and $\mathbb{W}_{\mathrm{B}}$ and taking the conditional expectation give that

$$\begin{aligned} E\left(\mathbb{W}_{\mathrm{C}}^{\top}\mathbb{W}_{\mathrm{B}}|\mathbb{Y},\mathbb{X}\right) &= \mathbb{X}_{\mathrm{C}}^{\top}\left(\mathbb{X}_{\mathrm{B}}\Gamma_{\mathrm{B}} + \mathbb{E}_{\mathrm{B}}\right) \\ &= \mathbb{X}_{\mathrm{C}}^{\top}\mathbb{X}_{\mathrm{B}}\Gamma_{\mathrm{B}} + \mathbb{X}_{\mathrm{C}}^{\top}\mathbb{E}_{\mathrm{B}}. \end{aligned}$$

By $E\left(\mathbb{W}_{\mathrm{C}}|\mathbb{Y},\mathbb{X}\right) = \mathbb{X}_{\mathrm{C}}$, we conclude that

$$E\left\{(\mathbb{W}_{\mathrm{C}}^{\top}\mathbb{W}_{\mathrm{B}} - \mathbb{W}_{\mathrm{C}}^{\top}\mathbb{E}_{\mathrm{B}})\Gamma_{\mathrm{B}}^{-1}|\mathbb{Y},\mathbb{X}\right\} = \mathbb{X}_{\mathrm{C}}^{\top}\mathbb{X}_{\mathrm{B}}.$$

Step 2.6: Derive $E\left(\mathbb{W}_{\mathrm{B}}^{\top}\mathbb{W}_{\mathrm{D}}|\mathbb{Y},\mathbb{X}\right)$.

By the formulations of two measurement error models, we have that

$$E\left(\mathbb{W}_{\mathrm{B}}^{\top}\mathbb{W}_{\mathrm{D}}|\mathbb{Y},\mathbb{X}\right) = \Gamma_{\mathrm{B}}^{\top}\mathbb{X}_{\mathrm{B}}^{\top}\mathbb{X}_{\mathrm{D}} + \mathbb{E}_{\mathrm{B}}^{\top}\mathbb{X}_{\mathrm{D}} + \lambda\Gamma_{\mathrm{B}}^{\top}\mathbb{X}_{\mathrm{B}}^{\top}1_{n\times p_{\mathrm{D}}} + \lambda\mathbb{E}_{\mathrm{B}}^{\top}1_{n\times p_{\mathrm{D}}}.$$

Noting that $E\left\{\left(\mathbb{W}_{\mathrm{B}} - \mathbb{E}_{\mathrm{B}}\right)\Gamma_{\mathrm{B}}^{-1}|\mathbb{Y},\mathbb{X}\right\} = \mathbb{X}_{\mathrm{B}}$ and $E\left(\mathbb{W}_{\mathrm{D}} - \lambda 1_{n\times p_{\mathrm{D}}}|\mathbb{Y},\mathbb{X}\right) = \mathbb{X}_{\mathrm{D}}$, it implies that

$$E\left\{\Gamma_{\mathrm{B}}^{-1}(\mathbb{W}_{\mathrm{B}} - \mathbb{E}_{\mathrm{B}})^{\top}(\mathbb{W}_{\mathrm{D}} - \lambda 1_{n\times p_{\mathrm{D}}})|\mathbb{Y},\mathbb{X}\right\} = \mathbb{X}_{\mathrm{B}}^{\top}\mathbb{X}_{\mathrm{D}}.$$

Therefore, combining Steps 2.1-2.6 yield a new symmetric matrix, which is a function of $\mathbb{W}$ and $\theta = \{\Sigma_\epsilon, \lambda, \pi\}$:

$$\begin{aligned}
&\psi(\mathbb{W}, \theta) \\
= &\begin{pmatrix} \mathbb{W}_{\mathrm{C}}^\top \mathbb{W}_{\mathrm{C}} - \Sigma_\epsilon & (\mathbb{W}_{\mathrm{C}}^\top \mathbb{W}_{\mathrm{B}} - \mathbb{W}_{\mathrm{C}}^\top \mathbb{E}_{\mathrm{B}})\Gamma_{\mathrm{B}}^{-1} & \mathbb{W}_{\mathrm{C}}^\top \mathbb{W}_{\mathrm{D}} - \lambda \mathbb{W}_{\mathrm{C}} \\ & \Gamma_{\mathrm{B}}^{-1}\left(\mathbb{W}_{\mathrm{B}}^\top \mathbb{W}_{\mathrm{B}} - 2\mathbb{W}_{\mathrm{B}}^\top \mathbb{E}_{\mathrm{B}} + \mathbb{E}_{\mathrm{B}}\right)\Gamma_{\mathrm{B}}^{-1} & \Gamma_{\mathrm{B}}^{-1}(\mathbb{W}_{\mathrm{B}} - \mathbb{E}_{\mathrm{B}})^\top (\mathbb{W}_{\mathrm{D}} - \lambda 1_{n\times p_{\mathrm{D}}}) \\ & & \mathbb{W}_{\mathrm{D}}^\top \mathbb{W}_{\mathrm{D}} + n\lambda 1_{p_{\mathrm{D}}\times p_{\mathrm{D}}} - 2\lambda 1_{n\times p_{\mathrm{D}}}^\top \mathbb{W}_{\mathrm{D}} \end{pmatrix} \\
= &(\mathbb{W}\Gamma^{-1} - \Lambda)^\top (\mathbb{W}\Gamma^{-1} - \Lambda),
\end{aligned}$$

such that

$$E\left(\psi(\mathbb{W}, \theta) | \mathbb{Y}, \mathbb{X}\right) = \mathbb{X}^\top \mathbb{X}. \tag{A.4}$$

<u>**Step 3:**</u> Derive Equation (10).

Consequently, by (A.1) and (A.4), the corrected score function $U_c(\beta)$ is given by

$$U_c(\beta) = -2(\mathbb{W}\Gamma^{-1} - \Lambda)^\top \mathbb{Y} + 2\psi(\mathbb{W}, \theta)\beta + 2V\beta,$$

so that

$$\begin{aligned} E\left\{U_c(\beta) | \mathbb{Y}, \mathbb{X}\right\} &= -2\mathbb{X}^\top \mathbb{Y} + 2\mathbb{X}^\top \mathbb{X}\beta + 2V\beta \\ &\triangleq U(\beta), \end{aligned}$$

where $U(\beta)$ is the score function under the unobserved covariate $\mathbb{X}$. □

Appendix B Proofs of Main Theorems

B.1 Proof of Theorem 1

We divide this proof into two parts.

Part 1:

Define $H(\Omega) = \mathbb{W}\left(\psi(\mathbb{W}, \theta) + \Omega^{-1}\right)^{-1} \mathbb{W}^\top$. Noting that two constraints in the minimization problem (13) is equivalent to $\sum_{j=1}^{p} |\omega_j| \leq \tau$. In this way, by the Lagrange multiplier, the constrained minimization problem (13) can be expressed as (B.1),

$$\langle \mathbb{Y} - H(\Omega)\mathbb{Y}, \mathbb{Y} - H(\Omega)\mathbb{Y}\rangle + \xi \sum_{j=1}^{p} |\omega_j|, \tag{B.1}$$

where ξ is the tuning parameter.

Let $Z = \sqrt{n}(\Omega - \Omega_0)$, and let z_j with $j = 1, \cdots, p$ denote the corresponding diagonal entries. Define a function

$$\begin{aligned}
\Psi(Z) &= \left\langle \mathbb{Y} - H\left(\Omega_0 + \frac{Z}{\sqrt{n}}\right)\mathbb{Y}, \mathbb{Y} - H\left(\Omega_0 + \frac{Z}{\sqrt{n}}\right)\mathbb{Y} \right\rangle - \langle \mathbb{Y} - H(\Omega_0)\mathbb{Y}, \mathbb{Y} - H(\Omega_0)\mathbb{Y} \rangle \\
&\quad + \xi\left(\sum_{j=1}^{p} |\omega_j + z_j| - \sum_{j=1}^{p} |\omega_j|\right) \\
&= -2\mathbb{Y}^\top \left\{ H\left(\Omega_0 + \frac{Z}{\sqrt{n}}\right) - H(\Omega_0) \right\} \mathbb{Y} \\
&\quad + \mathbb{Y}^\top \left\{ H\left(\Omega_0 + \frac{Z}{\sqrt{n}}\right) H\left(\Omega_0 + \frac{Z}{\sqrt{n}}\right) - H(\Omega_0)H(\Omega_0) \right\} \mathbb{Y} \\
&\quad + \xi\left(\sum_{j=1}^{p} |\omega_j + \frac{z_j}{\sqrt{n}}| - \sum_{j=1}^{p} |\omega_j|\right) \\
&\triangleq A_1 + A_2 + A_3. \qquad \text{(B.2)}
\end{aligned}$$

Let $\widehat{Z} = \sqrt{n}(\widehat{\Omega} - \Omega_0)$, then it is clear that $\widehat{Z} = \underset{Z}{\operatorname{argmin}}\, \Psi(Z)$ and its existence holds due to Condition (C1).

To the end, we examine A_1, A_2, and A_3 separately.

<u>**Step 1:**</u> Examine A_1.

Note that

$$\begin{aligned}
& H\left(\Omega_0 + \frac{Z}{\sqrt{n}}\right) - H(\Omega_0) \\
&= \mathbb{W}\left\{ \big(\psi(\mathbb{W}, \theta)\big) + \left(\Omega_0 + \frac{Z}{\sqrt{n}}\right)^{-1} \right\}^{-1} \mathbb{W}^\top - \mathbb{W}\left\{ \big(\psi(\mathbb{W}, \theta)\big) + \Omega_0^{-1} \right\}^{-1} \mathbb{W}^\top \\
&= \mathbb{W}\left[\left\{ \big(\psi(\mathbb{W}, \theta)\big) + \left(\Omega_0 + \frac{Z}{\sqrt{n}}\right)^{-1} \right\}^{-1} - \left\{ \big(\psi(\mathbb{W}, \theta)\big) + \Omega_0^{-1} \right\}^{-1} \right] \mathbb{W}^\top. \qquad \text{(B.3)}
\end{aligned}$$

In addition, the inverse matrix $\left\{ \big(\psi(\mathbb{W}, \theta)\big) + \left(\Omega_0 + \frac{Z}{\sqrt{n}}\right)^{-1} \right\}^{-1}$ can be expressed as

$$\begin{aligned}
& \left\{ \big(\psi(\mathbb{W}, \theta)\big) + \left(\Omega_0 + \frac{Z}{\sqrt{n}}\right)^{-1} \right\}^{-1} \\
&= \left[\big(\psi(\mathbb{W}, \theta)\big) \left\{ I_{p\times p} + \big(\psi(\mathbb{W}, \theta)\big)^{-1} \left(\Omega_0 + \frac{Z}{\sqrt{n}}\right)^{-1} \right\} \right]^{-1} \\
&= \left\{ I_{p\times p} + \big(\psi(\mathbb{W}, \theta)\big)^{-1} \left(\Omega_0 + \frac{Z}{\sqrt{n}}\right)^{-1} \right\}^{-1} \{\psi(\mathbb{W}, \theta)\}^{-1}, \qquad \text{(B.4)}
\end{aligned}$$

which is valid since $\psi(\mathbb{W},\theta)$ is invertible and $\left\{\psi(\mathbb{W},\theta)\right\}^{-1}$ exists due to Condition (C2). Then applying the Neumann series to the first inverse matrix in (B.4) gives

$$\begin{aligned}
&\left\{I_{p\times p}+\big(\psi(\mathbb{W},\theta)\big)^{-1}\left(\Omega_0+\frac{Z}{\sqrt{n}}\right)^{-1}\right\}^{-1}\\
=\;&\left\{I_{p\times p}+\big(\psi(\mathbb{W},\theta)\big)^{-1}\Omega_0^{-1}\right\}^{-1}\\
&+\frac{1}{\sqrt{n}}\big(\psi(\mathbb{W},\theta)\big)^{-1}\left\{I_{p\times p}+\big(\psi(\mathbb{W},\theta)\big)^{-1}\Omega_0^{-1}\right\}^{-2}\Omega_0^{-2}Z\\
&+\frac{1}{n}\big(\psi(\mathbb{W},\theta)\big)^{-1}\left\{I_{p\times p}+\big(\psi(\mathbb{W},\theta)\big)^{-1}\Omega_0^{-1}\right\}^{-3}\Omega_0^{-4}\big(\psi(\mathbb{W},\theta)\big)^{-1}Z^2\\
&+o(n^{-1}).
\end{aligned}\tag{B.5}$$

Combining (B.3), (B.4), and (B.5) gives

$$\begin{aligned}
A_1 \;=\;& \frac{1}{\sqrt{n}}\mathbb{Y}^\top\mathbb{W}\big(\psi(\mathbb{W},\theta)\big)^{-1}\left\{I_{p\times p}+\big(\psi(\mathbb{W},\theta)\big)^{-1}\Omega_0^{-1}\right\}^{-2}\Omega_0^{-2}Z\big(\psi(\mathbb{W},\theta)\big)^{-1}\mathbb{W}^\top\mathbb{Y}\\
&+\Big[\frac{1}{n}\mathbb{Y}^\top\mathbb{W}\big(\psi(\mathbb{W},\theta)\big)^{-1}\left\{I_{p\times p}+\big(\psi(\mathbb{W},\theta)\big)^{-1}\Omega_0^{-1}\right\}^{-3}\Omega_0^{-4}\big(\psi(\mathbb{W},\theta)\big)^{-1}\\
&\cdot Z^2\big(\psi(\mathbb{W},\theta)\big)^{-1}\mathbb{W}^\top\mathbb{Y}\Big]+o(n^{-1})\\
=\;&\frac{1}{\sqrt{n}}\widetilde{\beta}^\top\left\{I_{p\times p}+\big(\psi(\mathbb{W},\theta)\big)^{-1}\Omega_0^{-1}\right\}^{-2}\Omega_0^{-2}Z\widetilde{\beta}\\
&+\frac{1}{n}\widetilde{\beta}^\top\left\{I_{p\times p}+\big(\psi(\mathbb{W},\theta)\big)^{-1}\Omega_0^{-1}\right\}^{-3}\Omega_0^{-4}\big(\psi(\mathbb{W},\theta)\big)^{-1}Z^2\widetilde{\beta}+o(n^{-1})\\
=\;&\frac{1}{\sqrt{n}}\beta^\top\left\{I_{p\times p}+\big(\psi(\mathbb{W},\theta)\big)^{-1}\Omega_0^{-1}\right\}^{-2}\Omega_0^{-2}Z\big(\widetilde{\beta}-\beta\big)\\
&+\frac{1}{\sqrt{n}}\beta^\top\left\{I_{p\times p}+\big(\psi(\mathbb{W},\theta)\big)^{-1}\Omega_0^{-1}\right\}^{-2}\Omega_0^{-2}Z\beta\\
&+\frac{1}{n}\beta^\top\left\{I_{p\times p}+\big(\psi(\mathbb{W},\theta)\big)^{-1}\Omega_0^{-1}\right\}^{-3}\Omega_0^{-4}\big(\psi(\mathbb{W},\theta)\big)^{-1}Z^2\beta+o(n^{-1})
\end{aligned}\tag{B.6}$$

where $\widetilde{\beta}=\big(\psi(\mathbb{W},\theta)\big)^{-1}\mathbb{W}^\top\mathbb{Y}$ is the least squared estimator from the corrected score function, i.e., (10) without the last term $2V\beta$, and the last step is due to the consistency of the least squared estimator.

Step 2: Examine A_2.

Note that $H\left(\Omega_0+\frac{Z}{\sqrt{n}}\right)H\left(\Omega_0+\frac{Z}{\sqrt{n}}\right)$ in A_2 can be written as

$$\begin{aligned}
& H\left(\Omega_0+\frac{Z}{\sqrt{n}}\right)H\left(\Omega_0+\frac{Z}{\sqrt{n}}\right) \\
= \; & \mathbb{W}\left\{\big(\psi(\mathbb{W},\theta)\big)+\left(\Omega_0+\frac{Z}{\sqrt{n}}\right)^{-1}\right\}^{-1}\psi(\mathbb{W},\theta)\left\{\big(\psi(\mathbb{W},\theta)\big)+\left(\Omega_0+\frac{Z}{\sqrt{n}}\right)^{-1}\right\}^{-1}\mathbb{W}^\top \\
= \; & \mathbb{W}\left\{\big(\psi(\mathbb{W},\theta)\big)+\left(\Omega_0+\frac{Z}{\sqrt{n}}\right)^{-1}\right\}^{-1}\mathbb{W}^\top \\
& -\mathbb{W}\left\{\big(\psi(\mathbb{W},\theta)\big)+\left(\Omega_0+\frac{Z}{\sqrt{n}}\right)^{-1}\right\}^{-1}\left(\Omega_0+\frac{Z}{\sqrt{n}}\right)^{-1}\left\{\big(\psi(\mathbb{W},\theta)\big)+\left(\Omega_0+\frac{Z}{\sqrt{n}}\right)^{-1}\right\}^{-1}\mathbb{W}^\top \\
\triangleq \; & A_{21}+A_{22}. \qquad \text{(B.7)}
\end{aligned}$$

For A_{21}, it is equal to $H\left(\Omega_0+\frac{Z}{\sqrt{n}}\right)$, then the relevant derivations follow the similar procedures in Step 1.

For A_{22}, by (B.4), we have

$$\begin{aligned}
A_{22} \;=\; & \mathbb{W}^\top\big(\psi(\mathbb{W},\theta)\big)^{-1}\left\{I_{p\times p}+\big(\psi(\mathbb{W},\theta)\big)^{-1}\left(\Omega_0+\frac{Z}{\sqrt{n}}\right)^{-1}\right\}^{-1}\left(\Omega_0+\frac{Z}{\sqrt{n}}\right)^{-1} \\
& \cdot\left\{I_{p\times p}+\big(\psi(\mathbb{W},\theta)\big)^{-1}\left(\Omega_0+\frac{Z}{\sqrt{n}}\right)^{-1}\right\}^{-1}\big(\psi(\mathbb{W},\theta)\big)^{-1}\mathbb{W}. \qquad \text{(B.8)}
\end{aligned}$$

Then following similar derivations of (B.5) and (B.6) and combining (B.7) and (B.8), A_2 can be expressed as

$$\begin{aligned}
A_2 \;=\; & A_1+\frac{1}{\sqrt{n}}\widetilde{\beta}^\top G_1(Z)\widetilde{\beta}+\frac{1}{n}\widetilde{\beta}G_2(Z)\widetilde{\beta}+o(n^{-1}) \\
=\; & A_1+\frac{1}{\sqrt{n}}\beta^\top G_1(Z)\big(\widetilde{\beta}-\beta\big)+\frac{1}{\sqrt{n}}\beta^\top G_1(Z)\beta+\frac{1}{n}\beta G_2(Z)\beta+o(n^{-1}) \qquad \text{(B.9)}
\end{aligned}$$

with

$$G_1(Z)=2\left[-\Omega_0^{-2}+\left\{I_{p\times p}+\big(\psi(\mathbb{W},\theta)\big)^{-1}\Omega_0^{-1}\right\}^{-3}\Omega_0^{-3}\big(\psi(\mathbb{W},\theta)\big)^{-1}\right]Z$$

and

$$\begin{aligned}
G_2(Z) \;=\; & \Big[\Omega_0^{-3}\left\{I_{p\times p}+\big(\psi(\mathbb{W},\theta)\big)^{-1}\Omega_0^{-1}\right\}^{-3} \\
& +3\big(\psi(\mathbb{W},\theta)\big)^{-1}\left\{I_{p\times p}+\big(\psi(\mathbb{W},\theta)\big)^{-1}\Omega_0^{-1}\right\}^{-4}\big(\psi(\mathbb{W},\theta)\big)^{-1}\Big]Z^2.
\end{aligned}$$

Step 3: Examine A_3.

Applying the similar derivations of Theorem 1 in Chen and Yi (2021a), we have

$$\xi\left(\sum_{j=1}^{p}\left|\omega_j+\frac{z_j}{\sqrt{n}}\right|-\sum_{j=1}^{p}\left|\omega_j\right|\right)=\frac{\xi}{\sqrt{n}}\sum_{j=1}^{p}\left\{\text{sign}(\omega_j)z_j+o_p(1)\right\}. \tag{B.10}$$

By Condition (C3) that $\frac{\xi}{\sqrt{n}}\to 0$ as $n\to\infty$, (B.10) is equal to $o_p(1)$.

Therefore, combining (B.6), (B.9), and (B.10) with (B.2) gives

$$\begin{aligned} n\Psi(Z) &= 2\beta^\top\left\{I_{p\times p}+\big(\psi(\mathbb{W},\theta)\big)^{-1}\Omega_0^{-1}\right\}^{-2}\Omega_0^{-2}Z\sqrt{n}\big(\widetilde{\beta}-\beta\big) \\ &\quad +2\beta^\top\left\{I_{p\times p}+\big(\psi(\mathbb{W},\theta)\big)^{-1}\Omega_0^{-1}\right\}^{-3}\Omega_0^{-4}\big(\psi(\mathbb{W},\theta)\big)^{-1}Z^2\beta \\ &\quad +\beta^\top G_1(Z)\sqrt{n}\big(\widetilde{\beta}-\beta\big)+\beta G_2(Z)\beta+o(1) \\ &\xrightarrow{d} \Psi_0(Z) \end{aligned}$$

as $n\to\infty$, where $\Psi_0(Z)$ follows a normal distribution because $\sqrt{n}\big(\widetilde{\beta}-\beta\big)$ converges in distribution to a normal distribution by the property of the least squared estimator. Moreover, note that Condition (C1) ensures that $\Psi(Z)$ has the minimizer. Hence, by the the argmin continuous mapping (Chen and Yi 2021a), we have that as $n\to\infty$,

$$\sqrt{n}\big(\widehat{\Omega}-\Omega_0\big)=\underset{Z}{\text{argmin}}\,\Psi(Z)\xrightarrow{d}\underset{Z}{\text{argmin}}\,\Psi_0(Z). \tag{B.11}$$

In particular, (B.11) also indicates $\sqrt{n}\big(\widehat{\Omega}-\Omega_0\big)=O_p(1)$, or $\big(\widehat{\Omega}-\Omega_0\big)=O_p(n^{-1/2})=o_p(1)$.

Part 2:

Next, to show $P(\widehat{\mathcal{A}}=\mathcal{A})=1$ as $n\to\infty$, it is equivalent to check $P(\widehat{\omega}_k\neq 0)=0$ if $\omega_k=0$ for $k=1,\cdots,p$.

Suppose that $k\notin\mathcal{A}$ and $k\in\widehat{\mathcal{A}}$, i.e., $\widehat{\omega}_k\neq 0$. Let $\widehat{z}$ denote a vector with the kth component being $\widehat{z}_k=\text{sign}(\widehat{\omega}_k)$ if $\widehat{\omega}_k\neq 0$ or $|\widehat{z}_k|<1$ otherwise. Applying the KKT condition to (B.1) with some derivations, we have

$$\begin{aligned} &-\beta^\top\mathbb{W}^\top\mathbb{Y}\Omega_0^{-1}-2\beta^\top(\psi(\mathbb{W},\theta))^{-1}\beta\Omega_0^{-2}\big(\widehat{\Omega}-\Omega_0\big)-\Omega_0^{-2}(\psi(\mathbb{W},\theta)+\Omega_0^{-1})^{-2}+\xi\widehat{z}+o_p(1) \\ \propto\ &-2\beta^\top(\psi(\mathbb{W},\theta))^{-1}\beta\Omega_0^{-2}\big(\widehat{\Omega}-\Omega_0\big)+\xi\widehat{z}+o_p(1) \\ =\ &0. \end{aligned}$$

Since $\widehat{\omega}_k>0$, it shows $\text{sign}(\widehat{\omega}_k)>0$. On the other hand, we have $\widehat{\omega}_k-\omega_{k,0}=o_p(1)$ due to (B.11). It indicates the kth entry $2\beta_k^2(\psi(\mathbb{W},\theta))_{kk}^{-1}\omega_{k,0}^{-2}\big(\widehat{\omega}_k-\omega_{k,0}\big)=o_p(1)$. Then for the kth entry, we have

$$0<P(\widehat{\omega}_k\neq 0)<P\big\{2\beta_k^2(\psi(\mathbb{W},\theta))_{kk}^{-1}\omega_{k,0}^{-2}\big(\widehat{\omega}_k-\omega_{k,0}\big)=\xi\widehat{z}_k\big\}=0.$$

Therefore, we have $P(\widehat{\omega}_k\neq 0)=0$ for $\omega_k=0$. $\square$

B.2 Proof of Theorem 2

Write $\widehat{\beta}(\widehat{\Omega})-\beta_0=\left\{\widehat{\beta}(\widehat{\Omega})-\beta(\widehat{\Omega})\right\}+\left\{\beta(\widehat{\Omega})-\beta(\Omega_0)\right\}\triangleq J_1+J_2$, where $\beta(\widehat{\Omega})$ and $\beta_0\equiv\beta(\Omega_0)$ have already been defined in Section 3.4. For J_1, $\widehat{\beta}(\widehat{\Omega})$ is asymptotic unbiased to $\beta(\widehat{\Omega})$ due to the discussion in Section 3.4 and the continue mapping theorem. Then $J_1 \xrightarrow{p} 0$ as $n\to\infty$. In addition, for J_2, applying Theorem 1 and the continue mapping theorem yields that $J_2 \xrightarrow{p} 0$ as $n\to\infty$. Therefore, the consistency of $\widehat{\beta}(\widehat{\Omega})$ is proved.

Next, we justify the asymptotic distribution. Write $\sqrt{n}\left\{\widehat{\beta}(\widehat{\Omega})-\beta_0\right\}=\sqrt{n}\left\{\widehat{\beta}(\widehat{\Omega})-\widehat{\beta}(\Omega_0)\right\}+\sqrt{n}\left\{\widehat{\beta}(\Omega_0)-\beta_0\right\}$. The first term can be approximately expressed as

$$\sqrt{n}\left\{\widehat{\beta}(\widehat{\Omega})-\widehat{\beta}(\Omega_0)\right\}=\sqrt{n}\left(\psi(\mathbb{W},\theta)+\Omega_0^{-1}\right)^{-2}\Omega_0^{-1}\left(\widehat{\Omega}-\Omega_0\right)\mathbb{W}^\top\mathbb{Y},$$

which is equal to $o_p(1)$ due to Theorem 1. For the second term, by the standard derivations of the least squared estimator, we have that as $n\to\infty$,

$$\sqrt{n}\left\{\widehat{\beta}(\Omega_0)-\beta_0\right\}\xrightarrow{d} N(0,\sigma_\beta^2),\tag{B.12}$$

where $\sigma_\beta^2=\left(\psi(\mathbb{W},\theta)+\Omega_0^{-1}\right)^{-1}\psi(\mathbb{W},\theta)\left(\psi(\mathbb{W},\theta)+\Omega_0^{-1}\right)^{-1}\sigma_\delta^2$. Consequently, combining two terms gives the desired results as $n\to\infty$. □

B.3 Proof of Theorem 3

Note that $\Phi(\Omega)$ in (17) has a similar form to (B.7) without $\frac{Z}{\sqrt{n}}$ and $\mathbb{W}$ on both sides, we follow the similar idea of (B.7) to present this proof. Specifically, by (22), we again consider

$$\Psi_\mu(A) \;\triangleq\; \varphi\left(\Omega_0+\frac{A}{\sqrt{n}}\right)+\zeta\sum_{j=1}^{p}\left|\omega_j+\frac{a_j}{\sqrt{n}}\right|-\varphi\left(\Omega_0\right)-\zeta\sum_{j=1}^{p}\left|\omega_j\right|,\tag{B.13}$$

where $A=\sqrt{n}\big(\Omega-\Omega_0\big)$, a_j is the jth component of A for $j=1,\cdots,p$, and ζ is the tuning parameter. First, similar to (B.10), we have $\zeta\left(\sum\limits_{j=1}^{p}\left|\omega_j+\frac{a_j}{\sqrt{n}}\right|-\sum\limits_{j=1}^{p}\left|\omega_j\right|\right)=o_p(1)$ by Condition (C4). To the end, $\Psi_\mu(A)$ can be expressed as

$$\Psi_\mu(A)=R_1+R_2+R_3+o_p(1),\tag{B.14}$$

where

$$\begin{aligned}R_1 \;=\;& \sigma_\delta^2\left\|\mathbb{W}\left\{\psi(\mathbb{W},\theta)+\left(\Omega_0+\frac{A}{\sqrt{n}}\right)^{-1}\right\}^{-1}\left[\frac{\partial\mu\{\beta(\Omega_0)\}}{\partial\beta}\right]\right\|_2^2\\ &-\sigma_\delta^2\left\|\mathbb{W}\Big(\psi(\mathbb{W},\theta)+\Omega_0^{-1}\Big)^{-1}\left[\frac{\partial\mu\{\beta(\Omega_0)\}}{\partial\beta}\right]\right\|_2^2,\end{aligned}\tag{B.15}$$

$$R_2 = n\left\| \left\{ B\left(\Omega_0 + \frac{A}{\sqrt{n}}\right)\widehat{\beta}\right\}^\top \left[\frac{\partial\mu\{\beta(\Omega_0)\}}{\partial\beta}\right]\right\|_2^2, \tag{B.16}$$

and

$$R_3 = -\sigma_\delta^2\left\| \mathbb{W}\left(\psi(\mathbb{W},\theta)\right)^{-1} B\left(\Omega_0 + \frac{A}{\sqrt{n}}\right)^\top \left[\frac{\partial\mu\{\beta(\Omega_0)\}}{\partial\beta}\right]\right\|_2^2 \tag{B.17}$$

with existence and continuity of $\frac{\partial\mu}{\partial\beta}$ ensured by Condition (C5).

We now examine R_1, R_2, and R_3 separately. For (B.15), by the triangle inequality, we can obtain

$$\begin{aligned} & \sigma_\delta^2\left\| \mathbb{W}\left\{\psi(\mathbb{W},\theta) + \left(\Omega_0 + \frac{A}{\sqrt{n}}\right)^{-1}\right\}^{-1} \left[\frac{\partial\mu\{\beta(\Omega_0)\}}{\partial\beta}\right] \right. \\ - & \left. \mathbb{W}\Big(\psi(\mathbb{W},\theta) + \Omega_0^{-1}\Big)^{-1}\left[\frac{\partial\mu\{\beta(\Omega_0)\}}{\partial\beta}\right]\right\|_2^2 \\ \geq & \ R_1, \end{aligned}$$

or can be expressed as

$$\begin{aligned} R_1 \ = & \ \sigma_\delta^2\left\| \mathbb{W}\left\{\psi(\mathbb{W},\theta) + \left(\Omega_0 + \frac{A}{\sqrt{n}}\right)^{-1}\right\}^{-1} \left[\frac{\partial\mu\{\beta(\Omega_0)\}}{\partial\beta}\right] \right. \\ - & \left. \mathbb{W}\Big(\psi(\mathbb{W},\theta) + \Omega_0^{-1}\Big)^{-1}\left[\frac{\partial\mu\{\beta(\Omega_0)\}}{\partial\beta}\right]\right\|_2^2 - r_1 \end{aligned}$$

for some positive value r_1. After that, following the similar derivation of (B.4), we have

$$R_1 = \sigma_\delta^2\left\|\frac{1}{\sqrt{n}}\mathcal{R}(A)\left[\frac{\partial\mu\{\beta(\Omega_0)\}}{\partial\beta}\right]\right\|_2^2 - r_1, \tag{B.18}$$

where $\mathcal{R}(A) = \{\psi(\mathbb{W},\theta)\}^{-1}\Omega_0^{-2}\Big\{I_{p\times p}+\Big(\psi(\mathbb{W},\theta)\Big)^{-1}\Omega_0^{-1}\Big\}^{-1}A\Big(\psi(\mathbb{W},\theta)\Big)^{-1}$. Similarly, (B.16) can be re-written as

$$R_2 \ = \ \left\|\widehat{\beta}^\top\mathcal{R}(A)\psi(\mathbb{W},\theta)\left[\frac{\partial\mu\{\beta(\Omega_0)\}}{\partial\beta}\right]\right\|_2^2. \tag{B.19}$$

Moreover, for (B.17), it can be shown that

$$R_3 = \sigma_\delta^2\left\|\frac{1}{\sqrt{n}}\mathbb{W}\{\psi(\mathbb{W},\theta)\}^{-1}\mathcal{R}(A)\psi(\mathbb{W},\theta)\left[\frac{\partial\mu\{\beta(\Omega_0)\}}{\partial\beta}\right]\right\|_2^2. \tag{B.20}$$

Therefore, combining (B.14) with (B.18), (B.19), and (B.20), we have that $n\Psi_\mu(A)$ is formulated in terms of $\sqrt{n}\big(\widehat{\beta}-\beta\big)$, which converges in distribution to the normal distribution due to the property of the least squares estimate. Moreover, since $\Psi_\mu(A)$ is formulated as convex and quadratic form, by the similar derivation in Theorem 1, we have that as $n \to \infty$, $\sqrt{n}\big(\widehat{\Omega}_\mu - \Omega_0\big) = \underset{A}{\operatorname{argmin}}\, n\Psi_\mu(A) \xrightarrow{d} \underset{A}{\operatorname{argmin}}\, \Psi_{\mu,0}(A)$, where $\Psi_{\mu,0}(A)$ is the function of A in terms of the asymptotic distribution of $\sqrt{n}\big(\widehat{\beta} - \beta\big)$. Therefore, applying similar derivations of Parts 1 and 2 in Appendix B.1 yields the desired result. □

B.4 Proof of Theorem 4

Write $\widehat{\mu}(\widehat{\Omega}_\mu) - \mu_0 = \Big\{\widehat{\mu}(\widehat{\Omega}_\mu) - \mu(\widehat{\Omega}_\mu)\Big\} + \Big\{\mu(\widehat{\Omega}_\mu) - \mu(\Omega_0)\Big\} \triangleq K_1 + K_2$, where $\mu_0 \equiv \mu(\Omega_0)$ has already been defined in Section 4.2. For K_1, $\widehat{\mu}(\widehat{\Omega}_\mu)$ is asymptotically unbiased to $\mu(\widehat{\Omega}_\mu)$ due to the asymptotic unbiasness of $\widehat{\beta}(\widehat{\Omega}_\mu)$ for $\beta(\widehat{\Omega}_\mu)$ in Section 3.4 and the continue mapping theorem. Then $K_1 \xrightarrow{p} 0$ as $n \to \infty$. In addition, for K_2, applying Theorem 3 and the continue mapping theorem yields that $K_2 \xrightarrow{p} 0$ as $n \to \infty$. Therefore, the consistency of $\widehat{\mu}(\widehat{\Omega}_\mu)$ is proved.

On the other hand, due to the consistency of $\widehat{\beta}$ with Theorem 3 accommodated, Theorem 2 gives that, as $n \to \infty$,

$$\sqrt{n}\left\{\widehat{\beta}(\widehat{\Omega}_\mu) - \beta(\Omega_0)\right\} \xrightarrow{d} N(0, \sigma_\beta^2).$$

Then by (16) and the delta method, we obtain that

$$\sqrt{n}\left\{\widehat{\mu}(\widehat{\Omega}_\mu) - \mu_0\right\} \xrightarrow{d} N\left(0, \sigma_\beta^2 \left[\frac{\partial \mu\{\beta(\Omega_0)\}}{\partial \beta}\right]^\top \left[\frac{\partial \mu\{\beta(\Omega_0)\}}{\partial \beta}\right]\right).$$

as $n \to \infty$. □

Appendix C Additional Numerical Results

In this section, we summarize additional numerical results for simulation studies, where Tables C.1 and C.2 contain estimators of β under $p = 500$ and two scenarios, and Tables C.3-C.12 include estimators of various focus parameters under $p = 8$ or 500 and Scenarios I or II. According to Tables C.1 and C.2, we can see that the proposed method still outperforms the LASSO and EN methods with smaller biases and accurate variable selection. When measurement error effects Σ_ϵ or collinearity σ_δ^2 become severe, the LASSO and EN methods induce larger biases, while the proposed method looks stable. In addition, noting that the AIC, BIC, and Mallows's Cp methods have no estimation output in Tables C.1 and C.2 because large value p incurs longer and restrictive computation.

Tables C.3-C.12 summarize mean squared error (MSE), specificity (SPE), and sensitivity (SEN) for various focus parameters under $p = 8$ or 500, where 'Proposed' is the estimation result obtained by the proposed method in Section 4 of the main text, and other methods represent the direct implementation of estimator in Tables 1, 2, C.1, and C.2 to the focus parameter. Consistent with the findings in the main text, the proposed method provides the smallest MSE and largest SPE and SEN, which reflects that the proposed method has satisfactory performance in estimating the focus parameter in the presence of collinearity and measurement error.

Table C.1: Simulation results under Scenario I: estimation of β with $p = 500$

Σ_ϵ	σ_δ^2	Method	$n = 400$						$n = 800$					
			$\|\Delta_\beta\|_2$	var(β_1)	var(β_2)	var(β_5)	SPE	SEN	$\|\Delta_\beta\|_2$	var(β_1)	var(β_2)	var(β_5)	SPE	SEN
0.15	0.50	Proposed	0.013	0.004	0.007	0.001	1.000	0.998	0.001	0.002	0.002	0.002	1.000	0.998
		AIC	–	–	–	–	–	–	–	–	–	–	–	–
		BIC	–	–	–	–	–	–	–	–	–	–	–	–
		CP	–	–	–	–	–	–	–	–	–	–	–	–
		LASSO	0.019	0.006	0.007	0.005	1.000	0.996	0.031	0.003	0.004	0.003	1.000	0.975
		EN	0.022	0.007	0.008	0.006	1.000	0.992	0.019	0.003	0.004	0.003	1.000	0.964
		CoCoLasso	0.013	0.009	0.010	0.006	1.000	0.902	0.012	0.003	0.003	0.003	1.000	0.909
		nclreg	0.013	0.009	0.008	0.007	1.000	0.809	0.010	0.003	0.003	0.004	1.000	0.813
		MUS	0.017	0.014	0.015	0.005	1.000	0.900	0.014	0.005	0.005	0.004	1.000	0.900
	0.75	Proposed	0.006	0.009	0.020	0.012	1.000	0.996	0.013	0.003	0.011	0.007	1.000	0.994
		AIC	–	–	–	–	–	–	–	–	–	–	–	–
		BIC	–	–	–	–	–	–	–	–	–	–	–	–
		CP	–	–	–	–	–	–	–	–	–	–	–	–
		LASSO	0.014	0.013	0.022	0.017	1.000	0.995	0.040	0.007	0.012	0.010	1.000	0.975
		EN	0.037	0.013	0.027	0.023	1.000	0.987	0.040	0.007	0.013	0.013	1.000	0.975
		CoCoLasso	0.025	0.011	0.024	0.016	1.000	0.912	0.023	0.004	0.018	0.008	1.000	0.907
		nclreg	0.022	0.012	0.024	0.018	1.000	0.803	0.019	0.005	0.016	0.009	1.000	0.804
		MUS	0.027	0.017	0.029	0.014	1.000	0.899	0.019	0.013	0.014	0.012	1.000	0.898
0.35	0.50	Proposed	0.021	0.010	0.012	0.011	1.000	0.997	0.004	0.002	0.007	0.006	1.000	0.996
		AIC	–	–	–	–	–	–	–	–	–	–	–	–
		BIC	–	–	–	–	–	–	–	–	–	–	–	–
		CP	–	–	–	–	–	–	–	–	–	–	–	–
		LASSO	0.064	0.012	0.014	0.011	1.000	0.984	0.033	0.006	0.008	0.006	1.000	0.953
		EN	0.052	0.013	0.017	0.016	1.000	0.975	0.029	0.006	0.008	0.008	1.000	0.968
		CoCoLasso	0.037	0.011	0.013	0.012	1.000	0.896	0.024	0.005	0.008	0.008	1.000	0.903
		nclreg	0.033	0.012	0.013	0.011	1.000	0.781	0.026	0.004	0.009	0.007	1.000	0.760
		MUS	0.024	0.015	0.014	0.014	0.999	0.900	0.026	0.005	0.008	0.008	1.000	0.896
	0.75	Proposed	0.054	0.010	0.051	0.011	1.000	0.996	0.010	0.003	0.021	0.006	1.000	0.994
		AIC	–	–	–	–	–	–	–	–	–	–	–	–
		BIC	–	–	–	–	–	–	–	–	–	–	–	–
		CP	–	–	–	–	–	–	–	–	–	–	–	–
		LASSO	0.069	0.039	0.085	0.058	1.000	0.985	0.142	0.015	0.032	0.026	1.000	0.955
		EN	0.078	0.037	0.082	0.046	1.000	0.969	0.030	0.016	0.034	0.030	1.000	0.981
		CoCoLasso	0.061	0.013	0.061	0.017	1.000	0.904	0.046	0.005	0.024	0.010	1.000	0.910
		nclreg	0.065	0.010	0.064	0.013	1.000	0.734	0.047	0.004	0.025	0.008	1.000	0.816
		MUS	0.070	0.016	0.068	0.015	1.000	0.887	0.044	0.006	0.024	0.009	1.000	0.890
0.65	0.50	Proposed	0.019	0.007	0.023	0.018	1.000	0.972	0.026	0.005	0.014	0.013	1.000	0.935
		AIC	–	–	–	–	–	–	–	–	–	–	–	–
		BIC	–	–	–	–	–	–	–	–	–	–	–	–
		CP	–	–	–	–	–	–	–	–	–	–	–	–
		LASSO	0.033	0.027	0.032	0.024	1.000	0.998	0.028	0.011	0.014	0.014	1.000	0.994
		EN	0.044	0.027	0.028	0.025	1.000	0.996	0.036	0.012	0.016	0.019	1.000	0.987
		CoCoLasso	0.024	0.012	0.028	0.021	1.000	0.908	0.027	0.010	0.016	0.016	1.000	0.908
		nclreg	0.026	0.011	0.027	0.021	1.000	0.740	0.030	0.007	0.016	0.015	1.000	0.703
		MUS	0.025	0.014	0.034	0.023	0.995	0.906	0.030	0.012	0.022	0.017	1.000	0.911
	0.75	Proposed	0.028	0.019	0.017	0.019	1.000	0.971	0.029	0.017	0.018	0.018	1.000	0.994
		AIC	–	–	–	–	–	–	–	–	–	–	–	–
		BIC	–	–	–	–	–	–	–	–	–	–	–	–
		CP	–	–	–	–	–	–	–	–	–	–	–	–
		LASSO	0.049	0.041	0.086	0.062	1.000	0.992	0.054	0.042	0.077	0.061	1.000	0.935
		EN	0.046	0.040	0.084	0.062	1.000	0.984	0.051	0.036	0.073	0.060	1.000	0.962
		CoCoLasso	0.036	0.022	0.028	0.023	1.000	0.913	0.035	0.021	0.025	0.210	1.000	0.916
		nclreg	0.039	0.023	0.029	0.025	1.000	0.737	0.037	0.026	0.027	0.020	1.000	0.707
		MUS	0.040	0.026	0.026	0.025	0.997	0.917	0.037	0.022	0.023	0.022	0.999	0.926

Table C.2: Simulation results under Scenario II: estimation of β with $p = 500$

Σ_ϵ	σ_δ^2	Method	$n = 400$						$n = 800$					
			$\|\Delta_\beta\|_2$	var(β_1)	var(β_2)	var(β_5)	SPE	SEN	$\|\Delta_\beta\|_2$	var(β_1)	var(β_2)	var(β_5)	SPE	SEN
0.15	0.50	Proposed	0.010	0.050	0.002	0.004	1.000	0.989	0.006	0.012	0.001	0.001	1.000	0.990
		AIC	–	–	–	–	–	–	–	–	–	–	–	–
		BIC	–	–	–	–	–	–	–	–	–	–	–	–
		CP	–	–	–	–	–	–	–	–	–	–	–	–
		LASSO	0.158	0.116	0.003	0.004	1.000	0.860	0.151	0.017	0.001	0.002	1.000	0.886
		EN	0.155	0.083	0.003	0.005	1.000	0.872	0.153	0.016	0.002	0.003	1.000	0.884
		CoCoLasso	0.124	0.053	0.005	0.007	1.000	0.737	0.121	0.022	0.003	0.003	1.000	0.726
		nclreg	0.122	0.056	0.005	0.007	1.000	0.754	0.106	0.022	0.003	0.003	1.000	0.780
		MUS	2.982	–	0.006	–	0.103	0.950	2.983	–	0.005	–	0.111	0.953
	0.75	Proposed	0.012	0.048	0.002	0.003	1.000	0.986	0.007	0.010	0.001	0.006	1.000	0.988
		AIC	–	–	–	–	–	–	–	–	–	–	–	–
		BIC	–	–	–	–	–	–	–	–	–	–	–	–
		CP	–	–	–	–	–	–	–	–	–	–	–	–
		LASSO	0.175	0.093	0.003	0.010	1.000	0.858	0.169	0.016	0.001	0.007	1.000	0.867
		EN	0.167	0.058	0.003	0.019	1.000	0.866	0.161	0.017	0.002	0.009	1.000	0.870
		CoCoLasso	0.130	0.050	0.007	0.007	1.000	0.772	0.125	0.018	0.002	0.009	1.000	0.745
		nclreg	0.134	0.053	0.006	0.007	0.993	0.797	0.127	0.018	0.002	0.008	1.000	0.774
		MUS	2.981	–	0.007	–	0.108	0.900	2.981	–	0.003	–	0.131	0.908
0.35	0.50	Proposed	0.013	0.053	0.003	0.005	1.000	0.990	0.006	0.010	0.001	0.003	1.000	0.992
		AIC	–	–	–	–	–	–	–	–	–	–	–	–
		BIC	–	–	–	–	–	–	–	–	–	–	–	–
		CP	–	–	–	–	–	–	–	–	–	–	–	–
		LASSO	0.182	0.121	0.003	0.007	1.000	0.855	0.178	0.019	0.002	0.004	1.000	0.862
		EN	0.179	0.073	0.004	0.012	1.000	0.863	0.175	0.020	0.002	0.005	1.000	0.864
		CoCoLasso	0.146	0.076	0.007	0.006	1.000	0.706	0.140	0.030	0.003	0.005	1.000	0.682
		nclreg	0.149	0.080	0.009	0.005	1.000	0.745	0.138	0.038	0.003	0.003	1.000	0.722
		MUS	2.984	–	0.006	–	0.075	0.800	2.981	–	0.005	–	0.115	0.811
	0.75	Proposed	0.017	0.056	0.003	0.014	1.000	0.985	0.009	0.020	0.001	0.011	1.000	0.988
		AIC	–	–	–	–	–	–	–	–	–	–	–	–
		BIC	–	–	–	–	–	–	–	–	–	–	–	–
		CP	–	–	–	–	–	–	–	–	–	–	–	–
		LASSO	0.201	0.092	0.004	0.022	1.000	0.830	0.196	0.024	0.002	0.023	1.000	0.852
		EN	0.197	0.083	0.005	0.023	1.000	0.855	0.190	0.021	0.002	0.027	1.000	0.856
		CoCoLasso	0.186	0.059	0.004	0.019	1.000	0.730	0.165	0.023	0.003	0.022	1.000	0.709
		nclreg	0.184	0.066	0.004	0.018	0.987	0.723	0.175	0.028	0.002	0.025	1.000	0.775
		MUS	2.983	–	0.007	–	0.094	0.796	2.981	–	0.005	–	0.121	0.809
0.65	0.50	Proposed	0.014	0.059	0.003	0.012	1.000	0.992	0.007	0.024	0.002	0.003	1.000	0.992
		AIC	–	–	–	–	–	–	–	–	–	–	–	–
		BIC	–	–	–	–	–	–	–	–	–	–	–	–
		CP	–	–	–	–	–	–	–	–	–	–	–	–
		LASSO	0.213	0.170	0.004	0.014	1.000	0.844	0.197	0.025	0.002	0.008	1.000	0.868
		EN	0.201	0.094	0.004	0.015	1.000	0.857	0.193	0.026	0.002	0.012	1.000	0.870
		CoCoLasso	0.118	0.111	0.006	0.018	1.000	0.684	0.084	0.032	0.005	0.011	1.000	0.915
		nclreg	0.111	0.103	0.007	0.015	1.000	0.738	0.095	0.033	0.004	0.012	1.000	0.743
		MUS	2.987	–	0.008	–	0.075	0.789	2.986	–	0.005	–	0.099	0.816
	0.75	Proposed	0.018	0.060	0.003	0.015	1.000	0.984	0.010	0.022	0.003	0.014	1.000	0.990
		AIC	–	–	–	–	–	–	–	–	–	–	–	–
		BIC	–	–	–	–	–	–	–	–	–	–	–	–
		CP	–	–	–	–	–	–	–	–	–	–	–	–
		LASSO	0.220	0.095	0.008	0.017	1.000	0.837	0.213	0.026	0.006	0.015	1.000	0.852
		EN	0.228	0.077	0.007	0.016	1.000	0.841	0.216	0.024	0.005	0.015	1.000	0.840
		CoCoLasso	0.130	0.065	0.006	0.017	0.987	0.739	0.150	0.024	0.004	0.016	0.980	0.673
		nclreg	0.126	0.066	0.006	0.016	1.000	0.735	0.145	0.026	0.004	0.016	1.000	0.706
		MUS	2.983	–	0.008	–	0.084	0.774	2.984	–	0.006	–	0.100	0.792

Table C.3: Simulation results under Scenario I: estimation of μ_2 with $p = 8$

Σ_ϵ	σ_δ^2	Method	$n = 400$			$n = 800$		
			MSE	SPE	SEN	MSE	SPE	SEN
0.15	0.50	Proposed	0.081	1.000	0.918	0.128	1.000	0.962
		AIC	1.054	1.000	0.454	1.047	1.000	0.344
		BIC	1.034	1.000	0.635	1.013	1.000	0.546
		CP	1.054	1.000	0.455	1.048	1.000	0.345
		LASSO	1.037	1.000	0.436	1.059	1.000	0.386
		EN	1.077	1.000	0.325	1.066	1.000	0.306
		CoCoLasso	0.598	1.000	0.631	0.799	1.000	0.586
		nclreg	0.472	1.000	0.558	0.493	1.000	0.566
		MUS	0.303	0.501	0.664	0.358	0.501	0.665
	0.75	Proposed	0.194	1.000	0.907	0.125	1.000	0.962
		AIC	1.056	1.000	0.433	1.058	1.000	0.347
		BIC	1.017	1.000	0.626	1.016	1.000	0.542
		CP	1.055	1.000	0.436	1.059	1.000	0.348
		LASSO	1.162	1.000	0.421	1.082	1.000	0.374
		EN	1.219	1.000	0.304	1.094	1.000	0.297
		CoCoLasso	0.356	1.000	0.587	0.254	1.000	0.533
		nclreg	0.515	1.000	0.576	0.465	1.000	0.506
		MUS	0.556	0.754	0.576	0.276	0.768	0.568
0.35	0.50	Proposed	0.087	1.000	0.733	0.149	1.000	0.824
		AIC	1.133	1.000	0.313	1.281	1.000	0.258
		BIC	1.069	1.000	0.476	1.245	1.000	0.361
		CP	1.132	1.000	0.315	1.281	1.000	0.259
		LASSO	1.117	1.000	0.327	1.269	1.000	0.296
		EN	1.176	1.000	0.266	1.273	1.000	0.244
		CoCoLasso	0.158	1.000	0.565	0.165	1.000	0.513
		nclreg	0.163	1.000	0.536	0.182	1.000	0.458
		MUS	0.157	0.507	0.665	0.173	0.501	0.667
	0.75	Proposed	0.237	0.982	0.725	0.162	1.000	0.817
		AIC	1.489	1.000	0.315	1.237	1.000	0.252
		BIC	1.203	1.000	0.475	1.226	1.000	0.360
		CP	1.482	1.000	0.317	1.237	1.000	0.254
		LASSO	1.624	1.000	0.334	1.227	1.000	0.305
		EN	1.637	1.000	0.268	1.232	1.000	0.245
		CoCoLasso	1.074	1.000	0.528	1.090	1.000	0.477
		nclreg	1.128	1.000	0.494	1.171	1.000	0.472
		MUS	1.103	0.899	0.561	1.092	0.957	0.556
0.65	0.50	Proposed	0.103	1.000	0.768	0.084	1.000	0.772
		AIC	1.396	1.000	0.264	1.985	1.000	0.203
		BIC	1.266	1.000	0.384	1.890	1.000	0.308
		CP	1.396	1.000	0.266	1.985	1.000	0.203
		LASSO	1.381	1.000	0.297	1.966	1.000	0.258
		EN	1.432	1.000	0.237	2.105	1.000	0.200
		CoCoLasso	0.243	1.000	0.620	0.622	1.000	0.583
		nclreg	0.670	1.000	0.490	0.878	1.000	0.460
		MUS	0.759	0.550	0.659	0.727	0.523	0.663
	0.75	Proposed	0.118	1.000	0.755	0.085	0.978	0.771
		AIC	1.792	1.000	0.248	5.752	1.000	0.201
		BIC	1.800	1.000	0.377	5.503	1.000	0.311
		CP	1.794	1.000	0.249	5.752	1.000	0.202
		LASSO	1.777	1.000	0.281	5.609	1.000	0.263
		EN	1.781	1.000	0.221	12.543	1.000	0.205
		CoCoLasso	0.898	1.000	0.586	0.731	1.000	0.523
		nclreg	0.888	1.000	0.444	0.899	1.000	0.432
		MUS	0.761	0.959	0.533	0.744	0.991	0.519

Table C.4: Simulation results under Scenario I: estimation of μ_3 with $p = 8$

Σ_ϵ	σ_δ^2	Method	$n = 400$			$n = 800$		
			MSE	SPE	SEN	MSE	SPE	SEN
0.15	0.50	Proposed	0.286	1.000	0.920	0.152	1.000	0.957
		AIC	1.099	1.000	0.432	1.088	1.000	0.339
		BIC	1.079	1.000	0.627	1.028	1.000	0.541
		CP	1.099	1.000	0.434	1.088	1.000	0.339
		LASSO	1.078	1.000	0.432	1.114	1.000	0.376
		EN	1.129	1.000	0.323	1.120	1.000	0.288
		CoCoLasso	0.692	1.000	0.677	0.761	1.000	0.612
		nclreg	0.864	1.000	0.604	0.589	1.000	0.602
		MUS	0.949	1.000	0.830	0.763	1.000	0.832
	0.75	Proposed	0.175	1.000	0.941	0.134	1.000	0.972
		AIC	1.042	1.000	0.445	1.075	1.000	0.353
		BIC	0.984	1.000	0.620	1.028	1.000	0.538
		CP	1.041	1.000	0.450	1.074	1.000	0.355
		LASSO	1.097	1.000	0.421	1.104	1.000	0.377
		EN	1.142	1.000	0.322	1.115	1.000	0.295
		CoCoLasso	0.364	1.000	0.582	0.300	1.000	0.535
		nclreg	0.317	1.000	0.594	0.183	1.000	0.510
		MUS	0.982	1.000	0.658	0.849	1.000	0.645
0.35	0.50	Proposed	0.109	1.000	0.784	0.111	1.000	0.859
		AIC	1.284	1.000	0.303	1.243	1.000	0.251
		BIC	1.244	1.000	0.464	1.211	1.000	0.359
		CP	1.283	1.000	0.305	1.243	1.000	0.252
		LASSO	1.270	1.000	0.324	1.233	1.000	0.304
		EN	1.317	1.000	0.265	1.240	1.000	0.243
		CoCoLasso	0.229	1.000	0.594	0.352	1.000	0.522
		nclreg	0.435	1.000	0.552	0.331	1.000	0.464
		MUS	0.481	1.000	0.729	0.276	1.000	0.733
	0.75	Proposed	0.132	0.982	0.768	0.104	1.000	0.860
		AIC	1.486	1.000	0.309	1.265	1.000	0.246
		BIC	1.352	1.000	0.469	1.229	1.000	0.359
		CP	1.486	1.000	0.310	1.265	1.000	0.248
		LASSO	1.533	1.000	0.332	1.248	1.000	0.302
		EN	1.559	1.000	0.266	1.258	1.000	0.242
		CoCoLasso	1.087	1.000	0.534	1.193	1.000	0.478
		nclreg	0.417	1.000	0.496	0.603	1.000	0.472
		MUS	0.394	1.000	0.594	0.616	1.000	0.571
0.65	0.50	Proposed	1.076	1.000	0.756	0.781	1.000	0.729
		AIC	1.525	1.000	0.218	2.031	1.000	0.203
		BIC	1.394	1.000	0.233	1.700	1.000	0.310
		CP	1.523	1.000	0.219	2.023	1.000	0.203
		LASSO	1.501	1.000	0.223	1.769	1.000	0.265
		EN	1.573	1.000	0.219	1.819	1.000	0.197
		CoCoLasso	0.815	1.000	0.660	0.903	1.000	0.608
		nclreg	1.105	1.000	0.512	0.965	1.000	0.466
		MUS	1.344	1.000	0.709	1.096	1.000	0.692
	0.75	Proposed	1.229	1.000	0.748	0.733	0.971	0.744
		AIC	2.096	0.290	0.189	1.557	1.000	0.206
		BIC	1.774	0.290	0.278	1.500	1.000	0.308
		CP	2.093	0.290	0.190	1.558	1.000	0.207
		LASSO	2.059	0.290	0.209	1.528	1.000	0.263
		EN	2.222	0.290	0.168	1.565	1.000	0.206
		CoCoLasso	0.752	1.000	0.609	0.586	1.000	0.529
		nclreg	1.765	1.000	0.546	0.835	1.000	0.532
		MUS	1.386	1.000	0.546	0.931	1.000	0.522

Table C.5: Simulation results under Scenario I: estimation of μ_1 with $p = 500$

Σ_ϵ	σ_δ^2	Method	$n = 400$			$n = 800$		
			MSE	SPE	SEN	MSE	SPE	SEN
0.15	0.50	Proposed	0.057	0.994	0.979	0.033	1.000	0.995
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.084	1.000	0.785	1.065	1.000	0.771
		EN	1.069	1.000	0.657	1.045	1.000	0.670
		CoCoLasso	0.064	1.000	0.910	0.048	1.000	0.915
		nclreg	0.063	1.000	0.864	0.061	1.000	0.875
		MUS	0.064	1.000	0.906	0.047	1.000	0.906
	0.75	Proposed	0.099	0.998	0.960	0.035	1.000	0.988
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.063	1.000	0.682	1.029	1.000	0.664
		EN	1.040	1.000	0.624	1.018	1.000	0.621
		CoCoLasso	0.100	1.000	0.911	0.079	1.000	0.911
		nclreg	0.105	1.000	0.888	0.060	1.000	0.890
		MUS	0.102	1.000	0.905	0.059	1.000	0.909
0.35	0.50	Proposed	0.068	0.964	0.942	0.048	1.000	0.982
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.152	1.000	0.732	1.075	1.000	0.699
		EN	1.127	1.000	0.636	1.057	1.000	0.632
		CoCoLasso	0.089	1.000	0.903	0.067	1.000	0.901
		nclreg	0.097	1.000	0.853	0.103	1.000	0.847
		MUS	0.077	1.000	0.916	0.097	1.000	0.910
	0.75	Proposed	0.303	1.000	0.963	0.128	0.988	0.923
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.067	1.000	0.642	1.049	1.000	0.631
		EN	1.046	1.000	0.593	1.043	1.000	0.598
		CoCoLasso	0.490	1.000	0.897	0.449	1.000	0.900
		nclreg	0.360	1.000	0.870	0.253	1.000	0.850
		MUS	0.376	1.000	0.893	0.353	1.000	0.904
0.65	0.50	Proposed	0.096	1.000	0.913	0.079	1.000	0.932
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.185	1.000	0.703	1.068	1.000	0.677
		EN	1.152	1.000	0.646	1.055	1.000	0.630
		CoCoLasso	0.105	1.000	0.908	0.108	1.000	0.901
		nclreg	0.102	1.000	0.816	0.106	1.000	0.847
		MUS	0.103	1.000	0.896	0.107	1.000	0.916
	0.75	Proposed	0.468	1.000	0.913	0.240	1.000	0.921
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.733	1.000	0.624	1.148	1.000	0.610
		EN	1.431	1.000	0.581	1.137	1.000	0.575
		CoCoLasso	0.403	1.000	0.906	0.310	1.000	0.911
		nclreg	0.688	1.000	0.806	0.324	1.000	0.880
		MUS	0.549	1.000	0.903	0.346	1.000	0.882

Table C.6: Simulation results under Scenario I: estimation of μ_2 with $p = 500$

Σ_ϵ	σ_δ^2	Method	$n = 400$			$n = 800$		
			MSE	SPE	SEN	MSE	SPE	SEN
0.15	0.50	Proposed	0.085	0.866	0.922	0.085	0.908	0.923
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.068	0.707	0.802	1.037	0.781	0.798
		EN	1.073	0.891	0.697	1.034	0.923	0.708
		CoCoLasso	0.663	1.000	0.907	0.266	1.000	0.913
		nclreg	1.011	1.000	0.860	0.314	1.000	0.873
		MUS	0.913	0.500	0.906	0.814	0.500	0.910
	0.75	Proposed	0.307	0.996	0.918	0.329	0.995	0.923
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.061	0.961	0.731	1.023	0.989	0.710
		EN	1.049	0.993	0.672	1.015	1.000	0.666
		CoCoLasso	0.427	1.000	0.910	0.855	1.000	0.911
		nclreg	1.036	1.000	0.885	0.740	1.000	0.890
		MUS	0.980	0.709	0.905	0.883	0.738	0.900
0.35	0.50	Proposed	0.036	0.988	0.912	0.019	0.978	0.922
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.112	0.855	0.765	1.066	0.949	0.741
		EN	1.104	0.945	0.682	1.061	0.988	0.683
		CoCoLasso	1.002	1.000	0.900	0.529	1.000	0.901
		nclreg	1.004	1.000	0.851	0.847	1.000	0.845
		MUS	1.019	0.502	0.896	1.048	0.500	0.884
	0.75	Proposed	0.104	1.000	0.908	0.103	1.000	0.909
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.059	0.998	0.694	1.029	1.000	0.680
		EN	1.042	0.999	0.650	1.026	1.000	0.640
		CoCoLasso	0.929	1.000	0.897	0.715	1.000	0.900
		nclreg	1.027	1.000	0.869	0.963	1.000	0.850
		MUS	1.060	0.882	0.875	0.981	0.948	0.886
0.65	0.50	Proposed	0.286	1.000	0.986	0.090	1.000	0.980
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.182	0.933	0.750	1.098	0.990	0.721
		EN	1.166	0.979	0.680	1.096	0.998	0.666
		CoCoLasso	0.739	1.000	0.904	1.045	1.000	0.898
		nclreg	1.063	1.000	0.814	0.956	1.000	0.846
		MUS	1.087	0.525	0.906	1.088	0.507	0.876
	0.75	Proposed	0.476	1.000	0.976	0.312	1.000	0.981
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	4.658	1.000	0.668	4.373	1.000	0.659
		EN	3.207	1.000	0.622	4.334	1.000	0.616
		CoCoLasso	0.969	1.000	0.905	1.138	1.000	0.911
		nclreg	1.311	1.000	0.806	1.509	1.000	0.880
		MUS	1.263	0.955	0.915	1.284	0.984	0.914

Table C.7: Simulation results under Scenario I: estimation of μ_3 with $p = 500$

Σ_ϵ	σ_δ^2	Method	$n = 400$			$n = 800$		
			MSE	SPE	SEN	MSE	SPE	SEN
0.15	0.50	Proposed	0.356	0.999	0.986	0.555	1.000	0.923
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.211	1.000	0.855	1.345	1.000	0.692
		EN	1.198	1.000	0.714	1.376	1.000	0.667
		CoCoLasso	0.801	1.000	0.910	0.967	1.000	0.915
		nclreg	0.581	1.000	0.864	1.109	1.000	0.875
		MUS	0.692	1.000	0.918	0.929	1.000	0.918
	0.75	Proposed	0.155	0.999	0.963	1.051	1.000	0.971
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.293	1.000	0.735	1.391	1.000	0.669
		EN	1.230	1.000	0.667	1.404	1.000	0.633
		CoCoLasso	0.754	1.000	0.911	1.109	1.000	0.911
		nclreg	0.733	1.000	0.888	1.108	1.000	0.890
		MUS	0.735	1.000	0.907	0.931	1.000	0.910
0.35	0.50	Proposed	0.412	1.000	0.931	0.597	1.000	0.967
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.259	1.000	0.787	1.126	1.000	0.755
		EN	1.216	1.000	0.696	1.126	1.000	0.688
		CoCoLasso	1.009	1.000	0.903	0.928	1.000	0.901
		nclreg	0.810	1.000	0.853	1.005	1.000	0.847
		MUS	0.924	1.000	0.908	0.917	1.000	0.911
	0.75	Proposed	0.723	1.000	0.963	0.159	1.000	0.925
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.636	1.000	0.692	1.749	1.000	0.681
		EN	1.671	1.000	0.645	1.724	1.000	0.640
		CoCoLasso	0.906	1.000	0.897	0.845	1.000	0.900
		nclreg	1.001	1.000	0.870	0.947	1.000	0.850
		MUS	0.954	1.000	0.905	0.931	1.000	0.909
0.65	0.50	Proposed	0.263	1.000	0.955	0.627	1.000	0.915
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.868	1.000	0.757	1.788	1.000	0.725
		EN	1.855	1.000	0.689	1.743	1.000	0.672
		CoCoLasso	0.824	1.000	0.908	0.957	1.000	0.901
		nclreg	0.840	1.000	0.816	0.862	1.000	0.847
		MUS	0.837	1.000	0.888	0.823	1.000	0.890
	0.75	Proposed	0.660	1.000	0.960	0.872	1.000	0.933
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.100	1.00	0.666	2.068	1.000	0.658
		EN	3.234	1.000	0.621	3.652	1.000	0.612
		CoCoLasso	0.693	1.000	0.906	0.941	1.000	0.911
		nclreg	0.719	1.000	0.806	0.984	1.000	0.880
		MUS	0.727	1.000	0.895	0.951	1.000	0.884

Table C.8: Simulation results under Scenario II: estimation of μ_2 with $p = 8$

Σ_ϵ	σ_δ^2	Method	$n = 400$			$n = 800$		
			MSE	SPE	SEN	MSE	SPE	SEN
0.15	0.50	Proposed	0.103	1.000	0.983	0.095	1.000	0.983
		AIC	2.144	1.000	0.465	2.129	1.000	0.472
		BIC	2.138	1.000	0.470	2.127	1.000	0.475
		CP	2.150	1.000	0.463	2.134	1.000	0.472
		LASSO	2.135	1.000	0.493	2.123	1.000	0.508
		EN	2.130	1.000	0.494	2.119	1.000	0.510
		CoCoLasso	1.516	1.000	0.535	1.488	1.000	0.496
		nclreg	1.163	1.000	0.578	1.475	1.000	0.560
		MUS	3.000	0.000	0.856	2.900	0.000	0.855
	0.75	Proposed	0.109	1.000	0.979	0.100	1.000	0.982
		AIC	2.152	1.000	0.460	2.147	1.000	0.469
		BIC	2.150	1.000	0.464	2.148	1.000	0.470
		CP	2.171	1.000	0.459	2.158	1.000	0.462
		LASSO	2.144	1.000	0.488	2.137	1.000	0.496
		EN	2.145	1.000	0.488	2.131	1.000	0.503
		CoCoLasso	1.365	1.000	0.689	1.287	1.000	0.652
		nclreg	1.504	1.000	0.528	1.435	1.000	0.476
		MUS	3.010	0.000	0.853	3.007	0.000	0.843
0.35	0.50	Proposed	0.108	1.000	0.983	0.105	1.000	0.984
		AIC	2.146	1.000	0.470	2.133	1.000	0.475
		BIC	2.141	1.000	0.473	2.130	1.000	0.475
		CP	2.149	1.000	0.466	2.145	1.000	0.471
		LASSO	2.130	1.000	0.502	2.128	1.000	0.511
		EN	2.132	1.000	0.505	2.128	1.000	0.513
		CoCoLasso	1.008	1.000	0.588	1.036	1.000	0.555
		nclreg	1.014	1.000	0.610	1.035	1.000	0.572
		MUS	2.980	0.000	0.891	2.993	0.000	0.881
	0.75	Proposed	0.112	1.000	0.980	0.110	1.000	0.980
		AIC	2.160	1.000	0.456	2.153	1.000	0.461
		BIC	2.162	1.000	0.453	2.153	1.000	0.463
		CP	2.168	1.000	0.450	2.161	1.000	0.458
		LASSO	2.144	1.000	0.485	2.137	1.000	0.493
		EN	2.140	1.000	0.487	2.135	1.000	0.498
		CoCoLasso	1.628	1.000	0.556	1.461	1.000	0.528
		nclreg	2.000	1.000	0.552	1.479	1.000	0.506
		MUS	3.006	0.000	0.882	2.956	0.000	0.873
0.65	0.50	Proposed	0.114	1.000	0.983	0.105	1.000	0.984
		AIC	2.167	1.000	0.470	2.160	1.000	0.475
		BIC	2.164	1.000	0.475	2.161	1.000	0.473
		CP	2.175	1.000	0.453	2.173	1.000	0.466
		LASSO	2.155	1.000	0.488	2.148	1.000	0.495
		EN	2.152	1.000	0.490	2.148	1.000	0.495
		CoCoLasso	1.936	1.000	0.689	1.935	1.000	0.652
		nclreg	2.002	1.000	0.590	2.112	1.000	0.568
		MUS	3.100	0.000	0.899	3.088	0.000	0.880
	0.75	Proposed	0.114	1.000	0.983	0.105	1.000	0.984
		AIC	2.167	1.000	0.470	2.160	1.000	0.475
		BIC	2.164	1.000	0.475	2.161	1.000	0.473
		CP	2.175	1.000	0.453	2.173	1.000	0.466
		LASSO	2.155	1.000	0.488	2.148	1.000	0.495
		EN	2.152	1.000	0.490	2.148	1.000	0.495
		CoCoLasso	1.276	1.000	0.570	1.447	1.000	0.547
		nclreg	2.003	1.000	0.516	2.038	1.000	0.546
		MUS	3.120	0.000	0.893	3.108	0.000	0.878

Table C.9: Simulation results under Scenario II: estimation of μ_3 with $p = 8$

Σ_ϵ	σ^2_δ	Method	$n = 400$			$n = 800$		
			MSE	SPE	SEN	MSE	SPE	SEN
0.15	0.50	Proposed	0.129	1.000	0.966	0.115	1.000	0.983
		AIC	2.236	1.000	0.430	2.230	1.000	0.443
		BIC	2.228	1.000	0.437	2.220	1.000	0.465
		CP	2.245	1.000	0.426	2.234	1.000	0.432
		LASSO	2.211	1.000	0.474	2.203	1.000	0.489
		EN	2.199	1.000	0.490	2.197	1.000	0.495
		CoCoLasso	1.391	1.000	0.600	1.395	1.000	0.570
		nclreg	1.902	1.000	0.666	1.236	1.000	0.656
		MUS	2.445	0.294	0.954	2.543	0.319	0.962
	0.75	Proposed	0.133	1.000	0.960	0.119	1.000	0.981
		AIC	2.250	1.000	0.430	2.246	1.000	0.440
		BIC	2.247	1.000	0.436	2.241	1.000	0.456
		CP	2.253	1.000	0.423	2.249	1.000	0.430
		LASSO	2.226	1.000	0.473	2.220	1.000	0.485
		EN	2.230	1.000	0.486	2.221	1.000	0.488
		CoCoLasso	1.397	1.000	0.569	1.405	1.000	0.535
		nclreg	1.048	1.000	0.612	1.047	1.000	0.556
		MUS	2.419	0.302	0.954	2.493	0.349	0.959
0.35	0.50	Proposed	0.132	1.000	0.961	0.113	1.000	0.977
		AIC	2.244	1.000	0.430	2.240	1.000	0.443
		BIC	2.240	1.000	0.435	2.240	1.000	0.450
		CP	2.256	1.000	0.421	2.243	1.000	0.429
		LASSO	2.220	1.000	0.470	2.214	1.000	0.482
		EN	2.220	1.000	0.477	2.211	1.000	0.484
		CoCoLasso	1.782	1.000	0.549	1.691	1.000	0.510
		nclreg	1.000	1.000	0.580	1.507	1.000	0.550
		MUS	2.399	0.291	0.908	2.436	0.349	0.908
	0.75	Proposed	0.136	1.000	0.958	0.115	1.000	0.975
		AIC	2.250	1.000	0.432	2.245	1.000	0.440
		BIC	2.250	1.000	0.434	2.247	1.000	0.447
		CP	2.261	1.000	0.417	2.258	1.000	0.420
		LASSO	2.242	1.000	0.465	2.234	1.000	0.479
		EN	2.242	1.000	0.467	2.231	1.000	0.480
		CoCoLasso	1.208	1.000	0.517	1.156	1.000	0.568
		nclreg	1.121	1.000	0.496	1.054	1.000	0.554
		MUS	2.342	0.318	0.898	2.366	0.363	0.894
0.65	0.50	Proposed	0.132	1.000	0.965	0.126	1.000	0.970
		AIC	2.251	1.000	0.434	2.248	1.000	0.440
		BIC	2.250	1.000	0.436	2.247	1.000	0.446
		CP	2.267	1.000	0.429	2.255	1.000	0.435
		LASSO	2.238	1.000	0.468	2.230	1.000	0.475
		EN	2.235	1.000	0.470	2.230	1.000	0.478
		CoCoLasso	1.156	1.000	0.866	1.171	1.000	0.824
		nclreg	1.164	1.000	0.558	1.195	1.000	0.524
		MUS	2.430	0.296	0.898	2.421	0.360	0.900
	0.75	Proposed	0.137	1.000	0.958	0.131	1.000	0.969
		AIC	2.260	1.000	0.430	2.255	1.000	0.438
		BIC	2.259	1.000	0.433	2.255	1.000	0.440
		CP	2.271	1.000	0.421	2.262	1.000	0.430
		LASSO	2.244	1.000	0.460	2.238	1.000	0.467
		EN	2.243	1.000	0.463	2.237	1.000	0.470
		CoCoLasso	1.733	1.000	0.726	1.289	1.000	0.692
		nclreg	1.487	1.000	0.644	1.380	1.000	0.690
		MUS	2.242	0.320	0.900	2.350	0.367	0.911

Table C.10: Simulation results under Scenario II: estimation of μ_1 with $p = 500$

Σ_ϵ	σ_δ^2	Method	$n = 400$			$n = 800$		
			MSE	SPE	SEN	MSE	SPE	SEN
0.15	0.50	Proposed	0.043	0.998	0.983	0.035	1.000	0.995
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.148	1.000	0.793	2.125	1.000	0.801
		EN	2.139	1.000	0.795	2.120	1.000	0.803
		CoCoLasso	0.246	1.000	0.813	0.214	1.000	0.836
		nclreg	0.232	1.000	0.885	0.218	1.000	0.884
		MUS	1.000	0.000	0.909	1.000	0.000	0.911
	0.75	Proposed	0.049	0.998	0.980	0.038	1.000	0.995
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.156	1.000	0.774	2.150	1.000	0.785
		EN	2.150	1.000	0.778	2.148	1.000	0.789
		CoCoLasso	0.275	1.000	0.839	0.271	1.000	0.817
		nclreg	0.292	1.000	0.869	0.268	1.000	0.868
		MUS	1.100	0.000	0.895	1.033	0.000	0.904
0.35	0.50	Proposed	0.047	0.996	0.980	0.039	1.000	0.991
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.153	1.000	0.774	2.146	1.000	0.783
		EN	2.152	1.000	0.775	2.148	1.000	0.780
		CoCoLasso	0.279	1.000	0.802	0.195	1.000	0.808
		nclreg	0.275	1.000	0.855	0.178	1.000	0.840
		MUS	1.033	0.000	0.880	1.020	0.000	0.869
	0.75	Proposed	0.052	0.993	0.981	0.046	0.995	0.990
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.170	1.000	0.758	2.159	1.000	0.775
		EN	2.168	1.000	0.761	2.162	1.000	0.774
		CoCoLasso	0.246	1.000	0.822	0.195	1.000	0.816
		nclreg	0.228	1.000	0.830	0.192	1.000	0.854
		MUS	1.052	0.000	0.873	1.044	0.000	0.889
0.65	0.50	Proposed	0.048	0.993	0.980	0.043	1.000	0.988
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.167	1.000	0.760	2.155	1.000	0.776
		EN	2.164	1.000	0.764	2.154	1.000	0.778
		CoCoLasso	0.231	1.000	0.980	0.246	1.000	0.653
		nclreg	0.210	1.000	0.838	0.232	1.000	0.828
		MUS	1.066	0.000	0.863	1.060	0.000	0.859
	0.75	Proposed	0.055	0.990	0.980	0.050	0.994	0.983
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.180	1.000	0.752	2.169	1.000	0.763
		EN	2.177	1.000	0.756	2.169	1.000	0.762
		CoCoLasso	0.229	1.000	0.954	0.215	1.000	0.725
		nclreg	0.233	1.000	0.835	0.205	1.000	0.818
		MUS	1.070	0.000	0.855	1.063	0.000	0.868

Table C.11: Simulation results under Scenario II: estimation of μ_2 with $p = 500$

Σ_ϵ	σ_δ^2	Method	$n = 400$			$n = 800$		
			MSE	SPE	SEN	MSE	SPE	SEN
0.15	0.50	Proposed	0.060	0.995	0.985	0.050	1.000	0.993
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.189	1.000	0.793	2.165	1.000	0.790
		EN	2.169	1.000	0.803	2.164	1.000	0.790
		CoCoLasso	0.790	1.000	0.806	0.502	1.000	0.828
		nclreg	0.623	1.000	0.877	0.356	1.000	0.876
		MUS	2.000	0.000	0.901	2.010	0.000	0.900
	0.75	Proposed	0.064	0.995	0.982	0.056	1.000	0.990
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.193	1.000	0.790	2.170	1.000	0.794
		EN	2.177	1.000	0.797	2.171	1.000	0.793
		CoCoLasso	0.955	1.000	0.834	0.986	1.000	0.812
		nclreg	0.726	1.000	0.864	0.900	1.000	0.863
		MUS	2.012	0.000	0.910	2.018	0.000	0.907
0.35	0.50	Proposed	0.063	0.993	0.984	0.054	1.000	0.990
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.201	1.000	0.775	2.189	1.000	0.788
		EN	2.198	1.000	0.778	2.183	1.000	0.790
		CoCoLasso	0.746	1.000	0.794	0.763	1.000	0.800
		nclreg	0.744	1.000	0.847	0.769	1.000	0.832
		MUS	2.022	0.000	0.913	2.017	0.000	0.909
	0.75	Proposed	0.068	0.990	0.981	0.060	0.994	0.986
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.236	1.000	0.753	2.221	1.000	0.765
		EN	2.230	1.000	0.755	2.224	1.000	0.766
		CoCoLasso	0.723	1.000	0.817	0.769	1.000	0.811
		nclreg	0.768	1.000	0.825	0.003	0.750	0.850
		MUS	2.034	0.000	0.908	2.023	0.000	0.912
0.65	0.50	Proposed	0.064	0.995	0.986	0.056	1.000	0.991
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.228	1.000	0.750	2.219	1.000	0.769
		EN	2.230	1.000	0.750	2.213	1.000	0.778
		CoCoLasso	0.661	1.000	0.970	0.785	1.000	0.646
		nclreg	0.739	1.000	0.829	0.735	1.000	0.822
		MUS	2.041	0.000	0.890	2.036	0.000	0.895
	0.75	Proposed	0.070	0.991	0.983	0.064	0.996	0.985
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.278	0.988	0.723	2.266	0.988	0.741
		EN	2.273	0.991	0.728	2.260	0.989	0.750
		CoCoLasso	0.666	1.000	0.947	0.749	1.000	0.718
		nclreg	0.738	1.000	0.832	0.736	1.000	0.816
		MUS	2.107	0.000	0.878	2.102	0.000	0.883

Table C.12: Simulation results under Scenario II: estimation of μ_3 with $p = 500$

Σ_ϵ	σ^2_δ	Method	$n = 400$			$n = 800$		
			MSE	SPE	SEN	MSE	SPE	SEN
0.15	0.50	Proposed	0.103	0.998	0.990	0.095	1.000	0.991
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.301	0.975	0.756	2.295	0.970	0.751
		EN	2.303	0.964	0.754	2.276	0.974	0.767
		CoCoLasso	0.379	1.000	0.813	0.189	1.000	0.836
		nclreg	0.310	1.000	0.885	0.167	1.000	0.884
		MUS	1.046	0.116	0.900	1.042	0.144	0.900
	0.75	Proposed	0.108	0.998	0.989	0.100	1.000	0.991
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.334	0.970	0.751	2.320	0.970	0.757
		EN	2.330	0.968	0.755	2.317	0.972	0.762
		CoCoLasso	0.345	1.000	0.839	0.334	1.000	0.817
		nclreg	0.385	1.000	0.869	0.205	1.000	0.868
		MUS	1.047	0.125	0.893	1.001	0.179	0.895
0.35	0.50	Proposed	0.105	0.998	0.990	0.099	1.000	0.992
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.358	0.965	0.749	2.350	0.970	0.752
		EN	2.355	0.965	0.750	2.348	0.974	0.755
		CoCoLasso	0.393	1.000	0.802	0.237	1.000	0.808
		nclreg	0.247	1.000	0.855	0.240	1.000	0.840
		MUS	1.042	0.106	0.883	0.998	0.172	0.871
	0.75	Proposed	0.110	0.995	0.988	0.104	0.995	0.990
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.396	0.951	0.728	2.388	0.966	0.730
		EN	2.390	0.954	0.730	2.384	0.967	0.734
		CoCoLasso	0.315	1.000	0.822	0.336	1.000	0.816
		nclreg	0.311	1.000	0.830	0.320	1.000	0.854
		MUS	1.033	0.133	0.879	0.998	0.181	0.886
0.65	0.50	Proposed	0.113	0.995	0.987	0.107	0.996	0.990
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.403	0.947	0.711	2.388	0.950	0.720
		EN	2.396	0.954	0.720	2.384	0.957	0.725
		CoCoLasso	0.311	1.000	0.980	0.626	1.000	0.653
		nclreg	0.408	1.000	0.838	0.534	1.000	0.828
		MUS	1.077	0.111	0.890	1.061	0.148	0.901
	0.75	Proposed	0.118	0.990	0.984	0.109	0.994	0.990
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.425	0.933	0.707	2.418	0.947	0.715
		EN	2.421	0.935	0.710	2.423	0.947	0.722
		CoCoLasso	0.462	1.000	0.954	0.651	1.000	0.725
		nclreg	0.499	1.000	0.835	0.530	1.000	0.818
		MUS	1.024	0.126	0.887	1.033	0.150	0.896

Appendix D Numerical Results for The Main/Validation Data

In this section, we report the estimator of β and μ_j in the presence of the validation data. Following the discussion in Section 3.3, we use the validation data $\mathcal{V}$ generated in Section 5.1 to estimate θ, and denote the estimator as $\widehat{\theta}$. After that, we repeat the estimation procedures (13) and (21) to estimate β and μ_j for $j = 1, 2, 3$ under the main data $\mathcal{M}$, respectively. Numerical results under Scenarios I and II and different values of p are summarized in Tables D.1-D.16.

Parallel with the finding in Section 5.2 and Appendix C, we observe that the estimators of β and μ_j look satisfactory when θ is estimated by the validation sample $\mathcal{V}$. Moreover, we can see that the proposed method still outperforms other existing methods. It concludes that the proposed method is valid to estimate β and the focus parameter μ_j under the main/validation study design.

Table D.1: Simulation results under Scenario I and the main/validation data: estimation of β with $p = 8$

Σ_ϵ	σ_δ^2	Method	$n = 400$						$n = 800$					
			$\|\Delta_\beta\|_2$	var(β_1)	var(β_2)	var(β_5)	SPE	SEN	$\|\Delta_\beta\|_2$	var(β_1)	var(β_2)	var(β_5)	SPE	SEN
0.15	0.50	Proposed	0.003	0.005	0.006	0.005	1.000	0.905	0.001	0.003	0.003	0.003	1.000	0.913
		AIC	0.010	0.032	0.008	0.006	1.000	0.523	0.010	0.018	0.005	0.005	1.000	0.516
		BIC	0.008	0.031	0.007	0.005	1.000	0.744	0.008	0.018	0.005	0.005	1.000	0.739
		CP	0.014	0.032	0.008	0.006	1.000	0.510	0.011	0.018	0.005	0.005	1.000	0.507
		LASSO	0.007	0.006	0.006	0.006	1.000	0.740	0.006	0.003	0.004	0.003	1.000	0.735
		EN	0.005	0.006	0.007	0.007	1.000	0.757	0.005	0.003	0.004	0.004	1.000	0.755
		CoCoLasso	0.006	0.010	0.010	0.010	1.000	0.634	0.005	0.009	0.007	0.010	1.000	0.562
		nclreg	0.004	0.008	0.010	0.007	1.000	0.695	0.003	0.003	0.004	0.004	1.000	0.587
		MUS	0.008	0.007	0.007	0.006	1.000	0.894	0.012	0.006	0.006	0.006	1.000	0.899
	0.75	Proposed	0.003	0.006	0.014	0.012	1.000	0.905	0.001	0.004	0.007	0.007	1.000	0.913
		AIC	0.014	0.033	0.023	0.021	1.000	0.517	0.014	0.018	0.011	0.010	1.000	0.513
		BIC	0.013	0.033	0.023	0.018	1.000	0.730	0.012	0.017	0.011	0.009	1.000	0.729
		CP	0.016	0.033	0.023	0.021	1.000	0.503	0.017	0.018	0.011	0.010	1.000	0.507
		LASSO	0.013	0.014	0.026	0.018	1.000	0.738	0.013	0.006	0.012	0.010	1.000	0.731
		EN	0.011	0.014	0.027	0.023	1.000	0.741	0.010	0.006	0.012	0.012	1.000	0.736
		CoCoLasso	0.012	0.007	0.016	0.014	1.000	0.545	0.010	0.010	0.011	0.008	1.000	0.503
		nclreg	0.010	0.007	0.014	0.013	1.000	0.585	0.006	0.005	0.007	0.007	1.000	0.497
		MUS	0.014	0.008	0.018	0.015	1.000	0.745	0.010	0.006	0.009	0.008	1.000	0.731
0.35	0.50	Proposed	0.006	0.009	0.008	0.011	1.000	0.911	0.004	0.004	0.005	0.005	1.000	0.911
		AIC	0.015	0.040	0.016	0.018	1.000	0.520	0.015	0.019	0.008	0.008	1.000	0.516
		BIC	0.013	0.041	0.025	0.017	1.000	0.711	0.011	0.019	0.008	0.008	1.000	0.709
		CP	0.018	0.040	0.026	0.018	1.000	0.516	0.014	0.019	0.009	0.008	1.000	0.504
		LASSO	0.011	0.012	0.014	0.011	1.000	0.725	0.012	0.005	0.008	0.006	1.000	0.721
		EN	0.012	0.012	0.016	0.013	1.000	0.730	0.011	0.005	0.008	0.007	1.000	0.726
		CoCoLasso	0.016	0.011	0.010	0.012	1.000	0.546	0.010	0.012	0.010	0.009	1.000	0.485
		nclreg	0.012	0.010	0.009	0.012	1.000	0.560	0.005	0.005	0.005	0.005	1.000	0.504
		MUS	0.013	0.014	0.014	0.013	1.000	0.880	0.008	0.012	0.012	0.012	1.000	0.890
	0.75	Proposed	0.008	0.009	0.010	0.008	1.000	0.908	0.006	0.004	0.004	0.005	1.000	0.910
		AIC	0.020	0.040	0.013	0.037	1.000	0.516	0.018	0.020	0.012	0.021	1.000	0.508
		BIC	0.020	0.037	0.013	0.026	1.000	0.654	0.018	0.020	0.012	0.018	1.000	0.667
		CP	0.024	0.040	0.013	0.037	1.000	0.507	0.023	0.020	0.012	0.021	1.000	0.507
		LASSO	0.017	0.035	0.062	0.068	1.000	0.709	0.016	0.015	0.033	0.033	1.000	0.715
		EN	0.017	0.036	0.066	0.084	1.000	0.711	0.014	0.015	0.034	0.038	1.000	0.716
		CoCoLasso	0.016	0.014	0.011	0.013	1.000	0.678	0.013	0.006	0.006	0.005	1.000	0.648
		nclreg	0.014	0.013	0.013	0.012	1.000	0.689	0.012	0.005	0.005	0.007	1.000	0.638
		MUS	0.014	0.015	0.017	0.015	1.000	0.689	0.019	0.013	0.013	0.013	1.000	0.659
0.65	0.50	Proposed	0.010	0.019	0.029	0.025	1.000	0.908	0.009	0.008	0.012	0.013	1.000	0.913
		AIC	0.031	0.047	0.034	0.028	1.000	0.517	0.030	0.024	0.022	0.020	1.000	0.510
		BIC	0.028	0.051	0.034	0.026	1.000	0.598	0.021	0.024	0.022	0.018	1.000	0.609
		CP	0.037	0.047	0.034	0.028	1.000	0.511	0.034	0.024	0.022	0.020	1.000	0.506
		LASSO	0.024	0.022	0.036	0.029	1.000	0.698	0.022	0.013	0.017	0.017	1.000	0.711
		EN	0.022	0.024	0.031	0.028	1.000	0.704	0.021	0.013	0.017	0.019	1.000	0.716
		CoCoLasso	0.016	0.021	0.030	0.027	1.000	0.580	0.014	0.008	0.016	0.016	1.000	0.496
		nclreg	0.012	0.021	0.031	0.026	1.000	0.539	0.018	0.007	0.016	0.014	1.000	0.477
		MUS	0.019	0.024	0.034	0.033	0.999	0.871	0.018	0.012	0.022	0.021	1.000	0.883
	0.75	Proposed	0.014	0.028	0.057	0.029	1.000	0.905	0.010	0.026	0.051	0.047	1.000	0.910
		AIC	0.037	0.072	0.066	0.036	1.000	0.514	0.034	0.028	0.053	0.057	1.000	0.508
		BIC	0.035	0.059	0.065	0.035	1.000	0.569	0.033	0.027	0.052	0.050	1.000	0.573
		CP	0.040	0.072	0.066	0.036	1.000	0.503	0.040	0.028	0.053	0.057	1.000	0.500
		LASSO	0.029	0.029	0.079	0.036	1.000	0.664	0.025	0.048	0.107	0.151	1.000	0.673
		EN	0.028	0.027	0.064	0.033	1.000	0.678	0.025	0.049	0.108	0.159	1.000	0.676
		CoCoLasso	0.016	0.031	0.042	0.032	1.000	0.617	0.019	0.034	0.067	0.055	1.000	0.650
		nclreg	0.018	0.033	0.062	0.031	1.000	0.647	0.017	0.035	0.067	0.056	1.000	0.599
		MUS	0.020	0.035	0.066	0.035	0.999	0.653	0.021	0.033	0.063	0.052	1.000	0.623

Table D.2: Simulation results under Scenario I and the main/validation data: estimation of β with $p = 500$

Σ_ϵ	σ_δ^2	Method	$n = 400$						$n = 800$					
			$\|\Delta_\beta\|_2$	var(β_1)	var(β_2)	var(β_5)	SPE	SEN	$\|\Delta_\beta\|_2$	var(β_1)	var(β_2)	var(β_5)	SPE	SEN
0.15	0.50	Proposed	0.008	0.011	0.003	0.005	1.000	0.997	0.005	0.013	0.001	0.002	1.000	0.998
		AIC	–	–	–	–	–	–	–	–	–	–	–	–
		BIC	–	–	–	–	–	–	–	–	–	–	–	–
		CP	–	–	–	–	–	–	–	–	–	–	–	–
		LASSO	0.023	0.040	0.003	0.004	1.000	0.980	0.022	0.017	0.001	0.002	1.000	0.978
		EN	0.022	0.037	0.003	0.005	1.000	0.982	0.020	0.018	0.001	0.003	1.000	0.984
		CoCoLasso	0.015	0.016	0.004	0.006	1.000	0.906	0.018	0.016	0.008	0.006	1.000	0.896
		nclreg	0.012	0.016	0.005	0.006	1.000	0.787	0.014	0.015	0.005	0.004	1.000	0.803
		MUS	0.019	0.015	0.005	0.008	1.000	0.900	0.014	0.018	0.005	0.005	1.000	0.906
	0.75	Proposed	0.010	0.005	0.003	0.001	1.000	0.995	0.008	0.004	0.001	0.005	1.000	0.997
		AIC	–	–	–	–	–	–	–	–	–	–	–	–
		BIC	–	–	–	–	–	–	–	–	–	–	–	–
		CP	–	–	–	–	–	–	–	–	–	–	–	–
		LASSO	0.030	0.046	0.003	0.011	1.000	0.980	0.026	0.017	0.001	0.008	1.000	0.975
		EN	0.029	0.041	0.003	0.019	1.000	0.980	0.026	0.017	0.001	0.010	1.000	0.977
		CoCoLasso	0.023	0.011	0.011	0.009	1.000	0.911	0.021	0.006	0.004	0.010	1.000	0.896
		nclreg	0.022	0.014	0.010	0.008	1.000	0.768	0.023	0.005	0.003	0.011	1.000	0.796
		MUS	0.027	0.017	0.008	0.010	1.000	0.908	0.016	0.007	0.006	0.009	1.000	0.872
0.35	0.50	Proposed	0.007	0.035	0.003	0.007	1.000	0.996	0.004	0.012	0.002	0.004	1.000	0.998
		AIC	–	–	–	–	–	–	–	–	–	–	–	–
		BIC	–	–	–	–	–	–	–	–	–	–	–	–
		CP	–	–	–	–	–	–	–	–	–	–	–	–
		LASSO	0.029	0.052	0.003	0.008	1.000	0.978	0.027	0.017	0.002	0.004	1.000	0.973
		EN	0.028	0.040	0.003	0.010	1.000	0.980	0.025	0.017	0.002	0.006	1.000	0.980
		CoCoLasso	0.014	0.040	0.006	0.011	1.000	0.901	0.014	0.013	0.006	0.006	1.000	0.893
		nclreg	0.017	0.041	0.005	0.013	1.000	0.781	0.015	0.016	0.007	0.005	1.000	0.723
		MUS	0.018	0.044	0.005	0.014	0.999	0.900	0.018	0.019	0.007	0.006	1.000	0.897
	0.75	Proposed	0.011	0.014	0.004	0.019	1.000	0.994	0.008	0.015	0.002	0.011	1.000	0.997
		AIC	–	–	–	–	–	–	–	–	–	–	–	–
		BIC	–	–	–	–	–	–	–	–	–	–	–	–
		CP	–	–	–	–	–	–	–	–	–	–	–	–
		LASSO	0.035	0.059	0.004	0.038	1.000	0.970	0.033	0.019	0.002	0.020	1.000	0.974
		EN	0.032	0.045	0.004	0.035	1.000	0.974	0.032	0.020	0.002	0.021	1.000	0.975
		CoCoLasso	0.017	0.015	0.013	0.021	1.000	0.902	0.016	0.016	0.005	0.017	1.000	0.888
		nclreg	0.019	0.015	0.012	0.025	1.000	0.723	0.018	0.017	0.005	0.016	1.000	0.793
		MUS	0.018	0.016	0.007	0.025	1.000	0.907	0.016	0.022	0.005	0.012	1.000	0.917
0.65	0.50	Proposed	0.012	0.015	0.005	0.013	1.000	0.993	0.010	0.015	0.002	0.008	1.000	0.995
		AIC	–	–	–	–	–	–	–	–	–	–	–	–
		BIC	–	–	–	–	–	–	–	–	–	–	–	–
		CP	–	–	–	–	–	–	–	–	–	–	–	–
		LASSO	0.041	0.064	0.005	0.016	1.000	0.966	0.038	0.024	0.002	0.010	1.000	0.970
		EN	0.037	0.055	0.005	0.021	1.000	0.971	0.034	0.024	0.002	0.011	1.000	0.975
		CoCoLasso	0.015	0.021	0.013	0.016	1.000	0.897	0.014	0.016	0.005	0.011	1.000	0.884
		nclreg	0.016	0.017	0.012	0.015	1.000	0.757	0.015	0.017	0.006	0.011	1.000	0.690
		MUS	0.017	0.020	0.014	0.019	0.995	0.893	0.016	0.022	0.007	0.012	1.000	0.910
	0.75	Proposed	0.016	0.020	0.005	0.018	1.000	0.988	0.012	0.023	0.002	0.028	1.000	0.990
		AIC	–	–	–	–	–	–	–	–	–	–	–	–
		BIC	–	–	–	–	–	–	–	–	–	–	–	–
		CP	–	–	–	–	–	–	–	–	–	–	–	–
		LASSO	0.052	0.081	0.046	0.067	1.000	0.953	0.045	0.026	0.003	0.076	1.000	0.967
		EN	0.049	0.078	0.049	0.059	1.000	0.962	0.045	0.026	0.003	0.079	1.000	0.965
		CoCoLasso	0.024	0.021	0.012	0.021	1.000	0.896	0.020	0.025	0.005	0.031	1.000	0.900
		nclreg	0.028	0.023	0.010	0.020	1.000	0.739	0.022	0.026	0.006	0.036	1.000	0.732
		MUS	0.025	0.025	0.006	0.025	0.991	0.897	0.020	0.026	0.008	0.032	1.000	0.906

Table D.3: Simulation results under Scenario II and the main/validation data: estimation of β with $p = 8$

Σ_ϵ	σ_δ^2	Method	$n = 400$						$n = 800$					
			$\\|\Delta_\beta\\|_2$	var(β_1)	var(β_2)	var(β_5)	SPE	SEN	$\\|\Delta_\beta\\|_2$	var(β_1)	var(β_2)	var(β_5)	SPE	SEN
0.15	0.50	Proposed	0.043	0.005	0.006	0.005	1.000	0.886	0.036	0.002	0.002	0.003	1.000	0.898
		AIC	0.593	0.033	0.007	0.006	1.000	0.483	0.588	0.015	0.004	0.005	1.000	0.480
		BIC	0.514	0.033	0.007	0.006	1.000	0.495	0.509	0.015	0.003	0.005	1.000	0.490
		CP	0.603	0.033	0.008	0.006	1.000	0.479	0.596	0.015	0.004	0.005	1.000	0.471
		LASSO	0.177	0.006	0.008	0.007	1.000	0.508	0.158	0.003	0.004	0.003	1.000	0.514
		EN	0.169	0.007	0.010	0.006	1.000	0.512	0.158	0.003	0.004	0.003	1.000	0.516
		CoCoLasso	0.072	0.006	0.006	0.007	1.000	0.605	0.053	0.003	0.004	0.004	1.000	0.551
		nclreg	0.069	0.007	0.007	0.006	1.000	0.684	0.054	0.004	0.004	0.004	1.000	0.626
		MUS	2.935	–	0.008	–	0.230	0.798	2.939	–	0.006	–	0.273	0.800
	0.75	Proposed	0.048	0.008	0.012	0.012	1.000	0.885	0.039	0.005	0.008	0.005	1.000	0.895
		AIC	0.611	0.034	0.003	0.011	1.000	0.475	0.605	0.015	0.002	0.008	1.000	0.480
		BIC	0.578	0.034	0.003	0.009	1.000	0.488	0.567	0.015	0.002	0.005	1.000	0.491
		CP	0.620	0.034	0.003	0.011	1.000	0.466	0.616	0.015	0.002	0.008	1.000	0.470
		LASSO	0.203	0.014	0.021	0.018	1.000	0.500	0.188	0.008	0.012	0.016	1.000	0.511
		EN	0.198	0.015	0.022	0.025	1.000	0.503	0.180	0.008	0.012	0.014	1.000	0.511
		CoCoLasso	0.138	0.013	0.021	0.022	1.000	0.546	0.136	0.011	0.010	0.011	1.000	0.498
		nclreg	0.132	0.013	0.015	0.019	1.000	0.601	0.136	0.007	0.010	0.011	1.000	0.556
		MUS	2.926	–	0.021	–	0.265	0.799	2.933	–	0.011	–	0.277	0.799
0.35	0.50	Proposed	0.048	0.009	0.006	0.010	1.000	0.883	0.037	0.005	0.005	0.005	1.000	0.895
		AIC	1.620	0.041	0.010	0.019	1.000	0.479	1.613	0.019	0.007	0.008	1.000	0.484
		BIC	1.614	0.045	0.009	0.017	1.000	0.482	1.609	0.018	0.006	0.008	1.000	0.487
		CP	1.633	0.041	0.010	0.019	1.000	0.473	1.626	0.019	0.007	0.008	1.000	0.477
		LASSO	1.205	0.011	0.013	0.018	1.000	0.500	1.206	0.005	0.006	0.006	1.000	0.506
		EN	1.200	0.013	0.015	0.017	1.000	0.501	1.203	0.005	0.008	0.007	1.000	0.506
		CoCoLasso	0.075	0.014	0.012	0.013	1.000	0.534	0.077	0.007	0.006	0.006	1.000	0.595
		nclreg	0.093	0.011	0.013	0.015	1.000	0.507	0.082	0.006	0.006	0.006	1.000	0.555
		MUS	2.934	–	0.011	–	0.217	0.789	2.941	–	0.008	–	0.247	0.810
	0.75	Proposed	0.046	0.018	0.039	0.029	1.000	0.880	0.039	0.011	0.016	0.019	1.000	0.892
		AIC	1.731	0.039	0.003	0.035	1.000	0.450	1.717	0.020	0.022	0.021	1.000	0.473
		BIC	1.709	0.039	0.003	0.025	1.000	0.463	1.702	0.020	0.022	0.021	1.000	0.479
		CP	1.739	0.039	0.003	0.035	1.000	0.447	1.728	0.020	0.022	0.021	1.000	0.450
		LASSO	1.443	0.027	0.069	0.046	1.000	0.486	1.398	0.017	0.046	0.029	1.000	0.498
		EN	1.438	0.025	0.065	0.042	1.000	0.493	1.390	0.018	0.047	0.036	1.000	0.500
		CoCoLasso	0.150	0.024	0.051	0.032	1.000	0.567	0.189	0.012	0.021	0.021	1.000	0.534
		nclreg	0.147	0.024	0.057	0.035	1.000	0.562	0.146	0.015	0.023	0.023	1.000	0.506
		MUS	2.935	–	0.062	–	0.221	0.800	2.938	–	0.031	–	0.238	0.813
0.65	0.50	Proposed	0.048	0.023	0.029	0.028	1.000	0.879	0.040	0.011	0.009	0.015	1.000	0.895
		AIC	1.793	0.050	0.035	0.031	1.000	0.466	1.778	0.023	0.012	0.020	1.000	0.470
		BIC	1.789	0.053	0.034	0.034	1.000	0.479	1.762	0.023	0.012	0.010	1.000	0.480
		CP	1.795	0.050	0.035	0.030	1.000	0.458	1.789	0.023	0.012	0.020	1.000	0.465
		LASSO	1.538	0.025	0.034	0.029	1.000	0.497	1.511	0.013	0.012	0.023	1.000	0.509
		EN	1.527	0.031	0.033	0.030	1.000	0.495	1.509	0.012	0.014	0.022	1.000	0.509
		CoCoLasso	0.074	0.024	0.024	0.030	1.000	0.676	0.088	0.012	0.013	0.016	1.000	0.578
		nclreg	0.081	0.026	0.029	0.030	1.000	0.540	0.096	0.013	0.014	0.018	1.000	0.498
		MUS	2.938	–	0.032	–	0.201	0.793	2.938	–	0.014	–	0.247	0.805
	0.75	Proposed	0.050	0.048	0.023	0.024	1.000	0.870	0.049	0.025	0.027	0.029	1.000	0.886
		AIC	1.837	0.054	0.027	0.029	1.000	0.459	1.810	0.029	0.033	0.068	1.000	0.464
		BIC	1.824	0.079	0.025	0.025	1.000	0.462	1.799	0.028	0.033	0.058	1.000	0.472
		CP	1.850	0.054	0.027	0.029	1.000	0.446	1.846	0.029	0.033	0.068	1.000	0.457
		LASSO	1.611	0.097	0.033	0.033	1.000	0.483	1.596	0.041	0.042	0.039	1.000	0.489
		EN	1.589	0.083	0.031	0.036	1.000	0.488	1.588	0.037	0.040	0.036	1.000	0.492
		CoCoLasso	0.145	0.031	0.024	0.027	0.999	0.634	0.142	0.033	0.031	0.034	1.000	0.631
		nclreg	0.146	0.068	0.028	0.025	1.000	0.534	0.141	0.035	0.033	0.032	1.000	0.579
		MUS	2.930	–	0.032	–	0.217	0.774	2.933	–	0.031	–	0.244	0.808

Table D.4: Simulation results under Scenario II and the main/validation data: estimation of β with $p = 500$

Σ_ϵ	σ_δ^2	Method	$n = 400$						$n = 800$					
			$\Vert\Delta_\beta\Vert_2$	var(β_1)	var(β_2)	var(β_5)	SPE	SEN	$\Vert\Delta_\beta\Vert_2$	var(β_1)	var(β_2)	var(β_5)	SPE	SEN
0.15	0.50	Proposed	0.012	0.014	0.003	0.005	1.000	0.992	0.008	0.012	0.001	0.002	1.000	0.995
		AIC	–	–	–	–	–	–	–	–	–	–	–	–
		BIC	–	–	–	–	–	–	–	–	–	–	–	–
		CP	–	–	–	–	–	–	–	–	–	–	–	–
		LASSO	0.167	0.063	0.003	0.005	1.000	0.873	0.165	0.060	0.001	0.003	1.000	0.880
		EN	0.160	0.040	0.004	0.006	1.000	0.878	0.157	0.037	0.001	0.003	1.000	0.880
		CoCoLasso	0.123	0.033	0.006	0.006	1.000	0.840	0.084	0.021	0.001	0.004	1.000	0.820
		nclreg	0.137	0.037	0.006	0.005	1.000	0.781	0.105	0.023	0.001	0.003	1.000	0.740
		MUS	2.986	–	0.006	–	0.081	0.910	2.982	–	0.004	–	0.110	0.900
	0.75	Proposed	0.015	0.023	0.009	0.006	1.000	0.990	0.011	0.020	0.001	0.007	1.000	0.993
		AIC	–	–	–	–	–	–	–	–	–	–	–	–
		BIC	–	–	–	–	–	–	–	–	–	–	–	–
		CP	–	–	–	–	–	–	–	–	–	–	–	–
		LASSO	0.174	0.069	0.011	0.017	1.000	0.866	0.170	0.061	0.001	0.007	1.000	0.875
		EN	0.173	0.067	0.012	0.015	1.000	0.870	0.167	0.056	0.001	0.011	1.000	0.878
		CoCoLasso	0.079	0.054	0.011	0.012	1.000	0.839	0.077	0.031	0.003	0.010	1.000	0.844
		nclreg	0.101	0.061	0.014	0.012	1.000	0.730	0.118	0.032	0.002	0.008	1.000	0.754
		MUS	2.977	–	0.016	–	0.106	0.891	2.980	–	0.004	–	0.012	0.906
0.35	0.50	Proposed	0.016	0.018	0.004	0.008	1.000	0.990	0.013	0.014	0.002	0.004	1.000	0.992
		AIC	–	–	–	–	–	–	–	–	–	–	–	–
		BIC	–	–	–	–	–	–	–	–	–	–	–	–
		CP	–	–	–	–	–	–	–	–	–	–	–	–
		LASSO	0.183	0.034	0.007	0.010	1.000	0.858	0.177	0.019	0.002	0.006	1.000	0.866
		EN	0.180	0.035	0.005	0.013	1.000	0.861	0.169	0.020	0.002	0.004	1.000	0.875
		CoCoLasso	0.140	0.034	0.006	0.010	1.000	0.808	0.125	0.013	0.004	0.005	1.000	0.805
		nclreg	0.133	0.034	0.005	0.009	0.983	0.821	0.124	0.017	0.003	0.005	1.000	0.809
		MUS	2.986	–	0.006	–	0.076	0.890	2.987	–	0.004	–	0.089	0.896
	0.75	Proposed	0.020	0.012	0.003	0.012	1.000	0.986	0.017	0.019	0.002	0.011	1.000	0.989
		AIC	–	–	–	–	–	–	–	–	–	–	–	–
		BIC	–	–	–	–	–	–	–	–	–	–	–	–
		CP	–	–	–	–	–	–	–	–	–	–	–	–
		LASSO	0.201	0.059	0.010	0.023	1.000	0.833	0.195	0.042	0.002	0.019	1.000	0.846
		EN	0.198	0.054	0.008	0.019	1.000	0.845	0.194	0.041	0.002	0.021	1.000	0.850
		CoCoLasso	0.135	0.040	0.006	0.012	1.000	0.811	0.137	0.033	0.003	0.012	1.000	0.810
		nclreg	0.137	0.044	0.006	0.015	1.000	0.769	0.140	0.038	0.003	0.014	1.000	0.729
		MUS	2.983	–	0.007	–	0.083	0.883	2.981	–	0.005	–	0.100	0.875
0.65	0.50	Proposed	0.021	0.019	0.005	0.018	1.000	0.988	0.018	0.017	0.002	0.007	1.000	0.990
		AIC	–	–	–	–	–	–	–	–	–	–	–	–
		BIC	–	–	–	–	–	–	–	–	–	–	–	–
		CP	–	–	–	–	–	–	–	–	–	–	–	–
		LASSO	0.213	0.039	0.010	0.025	1.000	0.828	0.204	0.022	0.002	0.019	1.000	0.832
		EN	0.209	0.035	0.008	0.021	1.000	0.831	0.202	0.023	0.003	0.017	1.000	0.836
		CoCoLasso	0.140	0.024	0.007	0.023	1.000	0.776	0.123	0.020	0.006	0.012	1.000	0.872
		nclreg	0.142	0.026	0.009	0.023	0.950	0.737	0.130	0.020	0.003	0.012	1.000	0.749
		MUS	2.988	–	0.010	–	0.066	0.886	2.986	–	0.005	–	0.093	0.890
	0.75	Proposed	0.024	0.022	0.017	0.023	1.000	0.988	0.020	0.020	0.002	0.025	1.000	0.989
		AIC	–	–	–	–	–	–	–	–	–	–	–	–
		BIC	–	–	–	–	–	–	–	–	–	–	–	–
		CP	–	–	–	–	–	–	–	–	–	–	–	–
		LASSO	0.230	0.027	0.028	0.034	1.000	0.810	0.221	0.023	0.002	0.034	1.000	0.825
		EN	0.227	0.024	0.026	0.030	1.000	0.814	0.218	0.024	0.004	0.037	1.000	0.826
		CoCoLasso	0.139	0.023	0.020	0.024	1.000	0.902	0.122	0.025	0.005	0.030	1.000	0.847
		nclreg	0.157	0.024	0.021	0.026	0.983	0.713	0.145	0.023	0.006	0.032	1.000	0.685
		MUS	2.986	–	0.020	–	0.071	0.865	2.995	–	0.007	–	0.027	0.872

Table D.5: Simulation results under Scenario I and the main/validation data: estimation of μ_1 with $p = 8$

Σ_ϵ	σ_δ^2	Method	$n = 400$			$n = 800$		
			MSE	SPE	SEN	MSE	SPE	SEN
0.15	0.50	Proposed	1.226	1.000	0.811	1.174	1.000	0.830
		AIC	4.478	1.000	0.503	4.465	1.000	0.495
		BIC	4.469	1.000	0.504	4.453	1.000	0.506
		CP	4.475	1.000	0.479	4.471	1.000	0.482
		LASSO	4.138	1.000	0.531	4.125	1.000	0.528
		EN	4.121	1.000	0.539	4.120	1.000	0.534
		CoCoLasso	4.011	1.000	0.664	4.074	1.000	0.591
		nclreg	4.007	1.000	0.678	4.011	1.000	0.607
		MUS	3.329	1.000	0.710	3.325	1.000	0.714
	0.75	Proposed	1.240	1.000	0.808	1.221	1.000	0.821
		AIC	4.495	1.000	0.497	4.487	1.000	0.502
		BIC	4.488	1.000	0.500	4.474	1.000	0.510
		CP	4.497	1.000	0.489	4.489	1.000	0.493
		LASSO	4.155	1.000	0.526	4.131	1.000	0.530
		EN	4.150	1.000	0.530	4.130	1.000	0.537
		CoCoLasso	4.043	1.000	0.544	4.119	1.000	0.511
		nclreg	4.036	1.000	0.577	4.000	1.000	0.502
		MUS	3.972	1.000	0.532	3.920	1.000	0.522
0.35	0.50	Proposed	1.238	1.000	0.804	1.210	1.000	0.811
		AIC	4.501	1.000	0.488	4.485	1.000	0.477
		BIC	4.497	1.000	0.497	4.486	1.000	0.480
		CP	4.506	1.000	0.476	4.490	1.000	0.475
		LASSO	4.303	1.000	0.516	4.295	1.000	0.521
		EN	4.289	1.000	0.520	4.290	1.000	0.527
		CoCoLasso	4.076	1.000	0.567	4.043	1.000	0.505
		nclreg	4.048	1.000	0.607	4.089	1.000	0.504
		MUS	3.394	1.000	0.706	3.377	1.000	0.713
	0.75	Proposed	1.245	1.000	0.804	1.231	1.000	0.808
		AIC	4.523	1.000	0.480	4.512	1.000	0.482
		BIC	4.511	1.000	0.483	4.506	1.000	0.485
		CP	4.536	1.000	0.475	4.523	1.000	0.479
		LASSO	4.375	1.000	0.504	4.360	1.000	0.513
		EN	4.366	1.000	0.510	4.358	1.000	0.514
		CoCoLasso	4.051	1.000	0.506	4.046	1.000	0.571
		nclreg	4.078	1.000	0.495	4.048	1.000	0.545
		MUS	4.358	1.000	0.492	4.348	1.000	0.471
0.65	0.50	Proposed	1.241	1.000	0.805	1.233	1.000	0.811
		AIC	4.520	1.000	0.486	4.509	1.000	0.480
		BIC	4.513	1.000	0.490	4.508	1.000	0.484
		CP	4.534	1.000	0.471	4.530	1.000	0.478
		LASSO	4.356	1.000	0.508	4.340	1.000	0.516
		EN	4.338	1.000	0.511	4.340	1.000	0.520
		CoCoLasso	4.038	1.000	0.648	4.028	1.000	0.558
		nclreg	4.039	1.000	0.536	4.118	1.000	0.572
		MUS	3.464	1.000	0.694	3.447	1.000	0.702
	0.75	Proposed	1.248	1.000	0.801	1.240	1.000	0.811
		AIC	4.536	1.000	0.479	4.530	1.000	0.470
		BIC	4.535	1.000	0.480	4.528	1.000	0.474
		CP	4.540	1.000	0.461	4.537	1.000	0.468
		LASSO	4.393	1.000	0.497	4.366	1.000	0.506
		EN	4.389	1.000	0.500	4.360	1.000	0.508
		CoCoLasso	4.063	1.000	0.538	4.098	1.000	0.476
		nclreg	4.077	1.000	0.556	4.116	1.000	0.502
		MUS	4.436	1.000	0.467	4.423	1.000	0.445

Table D.6: Simulation results under Scenario I and the main/validation data: estimation of μ_2 with $p = 8$

Σ_ϵ	σ_δ^2	Method	$n = 400$			$n = 800$		
			MSE	SPE	SEN	MSE	SPE	SEN
0.15	0.50	Proposed	0.075	1.000	0.923	0.068	1.000	0.960
		AIC	1.060	1.000	0.560	1.047	1.000	0.544
		BIC	1.054	1.000	0.563	1.040	1.000	0.556
		CP	1.070	1.000	0.545	1.051	1.000	0.548
		LASSO	1.033	1.000	0.586	1.029	1.000	0.590
		EN	1.038	1.000	0.585	1.026	1.000	0.593
		CoCoLasso	0.916	1.000	0.624	0.953	1.000	0.572
		nclreg	1.011	1.000	0.627	1.047	1.000	0.575
		MUS	1.197	0.502	0.663	1.146	0.500	0.666
	0.75	Proposed	0.078	1.000	0.920	0.072	1.000	0.954
		AIC	1.080	1.000	0.557	1.069	1.000	0.560
		BIC	1.075	1.000	0.561	1.064	1.000	0.562
		CP	1.093	1.000	0.540	1.088	1.000	0.550
		LASSO	1.050	1.000	0.581	1.047	1.000	0.593
		EN	1.048	1.000	0.582	1.046	1.000	0.593
		CoCoLasso	1.015	1.000	0.537	1.019	1.000	0.511
		nclreg	1.005	1.000	0.556	1.033	1.000	0.496
		MUS	1.499	0.826	0.563	1.478	0.858	0.562
0.35	0.50	Proposed	0.078	1.000	0.921	0.074	1.000	0.957
		AIC	1.069	1.000	0.558	1.061	1.000	0.550
		BIC	1.066	1.000	0.560	1.064	1.000	0.555
		CP	1.078	1.000	0.551	1.072	1.000	0.545
		LASSO	1.042	1.000	0.570	1.038	1.000	0.582
		EN	1.040	1.000	0.573	1.036	1.000	0.583
		CoCoLasso	1.073	1.000	0.546	1.062	1.000	0.498
		nclreg	1.112	1.000	0.578	1.254	1.000	0.496
		MUS	1.083	0.519	0.663	1.878	0.504	0.667
	0.75	Proposed	0.083	1.000	0.921	0.079	1.000	0.948
		AIC	1.092	1.000	0.536	1.080	1.000	0.525
		BIC	1.090	1.000	0.540	1.080	1.000	0.526
		CP	1.088	1.000	0.540	1.084	1.000	0.531
		LASSO	1.075	1.000	0.565	1.060	1.000	0.557
		EN	1.074	1.000	0.567	1.062	1.000	0.556
		CoCoLasso	1.262	1.000	0.503	1.424	1.000	0.470
		nclreg	1.300	1.000	0.490	1.566	1.000	0.445
		MUS	1.258	0.938	0.553	1.226	0.975	0.541
0.65	0.50	Proposed	0.086	1.000	0.919	0.079	1.000	0.943
		AIC	1.080	1.000	0.550	1.076	1.000	0.563
		BIC	1.076	1.000	0.553	1.077	1.000	0.560
		CP	1.089	1.000	0.541	1.083	1.000	0.549
		LASSO	1.062	1.000	0.562	1.058	1.000	0.577
		EN	1.059	1.000	0.563	1.056	1.000	0.578
		CoCoLasso	1.063	1.000	0.614	1.058	1.000	0.546
		nclreg	1.071	1.000	0.519	1.076	1.000	0.466
		MUS	1.943	0.543	0.658	1.779	0.531	0.663
	0.75	Proposed	0.089	1.000	0.916	0.084	1.000	0.940
		AIC	1.091	1.000	0.546	1.087	1.000	0.552
		BIC	1.088	1.000	0.548	1.086	1.000	0.556
		CP	1.097	1.000	0.533	1.093	1.000	0.541
		LASSO	1.075	1.000	0.559	1.070	1.000	0.566
		EN	1.075	1.000	0.560	1.071	1.000	0.568
		CoCoLasso	1.125	1.000	0.530	1.118	1.000	0.575
		nclreg	1.133	1.000	0.455	1.116	1.000	0.502
		MUS	1.842	0.970	0.535	1.696	0.993	0.517

Table D.7: Simulation results under Scenario I and the main/validation data: estimation of μ_3 with $p = 8$

Σ_ϵ	σ_δ^2	Method	$n = 400$			$n = 800$		
			MSE	SPE	SEN	MSE	SPE	SEN
0.15	0.50	Proposed	0.250	1.000	0.955	0.212	1.000	0.957
		AIC	1.106	1.000	0.540	1.096	1.000	0.539
		BIC	1.098	1.000	0.556	1.092	1.000	0.542
		CP	1.110	1.000	0.534	1.099	1.000	0.539
		LASSO	1.069	1.000	0.562	1.065	1.000	0.567
		EN	1.070	1.000	0.563	1.062	1.000	0.568
		CoCoLasso	0.996	1.000	0.664	0.962	1.000	0.591
		nclreg	0.990	1.000	0.678	0.986	1.000	0.607
		MUS	0.756	1.000	0.829	0.757	1.000	0.833
	0.75	Proposed	0.254	1.000	0.951	0.230	1.000	0.955
		AIC	1.117	1.000	0.539	1.106	1.000	0.544
		BIC	1.105	1.000	0.540	1.102	1.000	0.546
		CP	1.121	1.000	0.528	1.119	1.000	0.533
		LASSO	1.074	1.000	0.556	1.070	1.000	0.561
		EN	1.074	1.000	0.556	1.072	1.000	0.564
		CoCoLasso	1.008	1.000	0.544	1.036	1.000	0.511
		nclreg	1.059	1.000	0.577	1.065	1.000	0.502
		MUS	0.777	1.000	0.621	0.848	1.000	0.609
0.35	0.50	Proposed	0.261	1.000	0.954	0.249	1.000	0.960
		AIC	1.123	1.000	0.530	1.116	1.000	0.535
		BIC	1.120	1.000	0.540	1.112	1.000	0.543
		CP	1.127	1.000	0.528	1.119	1.000	0.531
		LASSO	1.088	1.000	0.557	1.085	1.000	0.562
		EN	1.087	1.000	0.556	1.082	1.000	0.567
		CoCoLasso	1.064	1.000	0.567	1.028	1.000	0.505
		nclreg	1.040	1.000	0.607	1.039	1.000	0.504
		MUS	0.837	1.000	0.823	0.829	1.000	0.832
	0.75	Proposed	0.268	1.000	0.950	0.251	1.000	0.953
		AIC	1.135	1.000	0.522	1.129	1.000	0.527
		BIC	1.132	1.000	0.526	1.127	1.000	0.533
		CP	1.138	1.000	0.520	1.135	1.000	0.525
		LASSO	1.092	1.000	0.549	1.090	1.000	0.555
		EN	1.090	1.000	0.550	1.090	1.000	0.557
		CoCoLasso	1.085	1.000	0.506	1.038	1.000	0.471
		nclreg	1.074	1.000	0.495	1.043	1.000	0.445
		MUS	0.764	1.000	0.574	0.739	1.000	0.549
0.65	0.50	Proposed	0.273	1.000	0.949	0.269	1.000	0.957
		AIC	1.142	1.000	0.522	1.130	1.000	0.530
		BIC	1.140	1.000	0.527	1.132	1.000	0.533
		CP	1.147	1.000	0.519	1.139	1.000	0.526
		LASSO	1.101	1.000	0.540	1.098	1.000	0.554
		EN	1.103	1.000	0.540	1.092	1.000	0.556
		CoCoLasso	1.116	1.000	0.648	1.050	1.000	0.558
		nclreg	1.118	1.000	0.536	1.110	1.000	0.472
		MUS	0.738	1.000	0.810	0.736	1.000	0.819
	0.75	Proposed	0.279	1.000	0.945	0.271	1.000	0.952
		AIC	1.150	1.000	0.518	1.143	1.000	0.526
		BIC	1.148	1.000	0.520	1.143	1.000	0.530
		CP	1.156	1.000	0.512	1.152	1.000	0.522
		LASSO	1.120	1.000	0.538	1.106	1.000	0.543
		EN	1.117	1.000	0.540	1.109	1.000	0.545
		CoCoLasso	1.124	1.000	0.638	1.108	1.000	0.676
		nclreg	1.122	1.000	0.556	1.119	1.000	0.502
		MUS	1.079	1.000	0.545	1.048	1.000	0.519

Table D.8: Simulation results under Scenario I and the main/validation data: estimation of μ_1 with $p = 500$

Σ_ϵ	σ_δ^2	Method	$n = 400$			$n = 800$		
			MSE	SPE	SEN	MSE	SPE	SEN
0.15	0.50	Proposed	0.048	0.992	0.989	0.043	1.000	0.991
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.119	1.000	0.793	1.105	1.000	0.791
		EN	1.112	1.000	0.797	1.105	1.000	0.796
		CoCoLasso	1.036	1.000	0.894	1.013	1.000	0.903
		nclreg	1.037	1.000	0.863	1.017	1.000	0.872
		MUS	0.968	1.000	0.896	0.947	1.000	0.896
	0.75	Proposed	0.052	0.993	0.990	0.047	1.000	0.990
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.133	1.000	0.790	1.125	1.000	0.790
		EN	1.130	1.000	0.794	1.123	1.000	0.792
		CoCoLasso	1.054	1.000	0.898	1.014	1.000	0.895
		nclreg	1.061	1.000	0.861	1.016	1.000	0.892
		MUS	1.034	1.000	0.904	1.026	0.072	0.902
0.35	0.50	Proposed	0.046	1.000	0.988	0.043	1.000	0.990
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.127	1.000	0.785	1.119	1.000	0.788
		EN	1.129	1.000	0.787	1.120	1.000	0.791
		CoCoLasso	1.042	1.000	0.907	1.038	1.000	0.883
		nclreg	1.115	1.000	0.845	1.062	1.000	0.869
		MUS	0.942	1.000	0.886	0.911	1.000	0.891
	0.75	Proposed	0.055	1.000	0.985	0.050	1.000	0.990
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.140	1.000	0.773	1.136	1.000	0.776
		EN	1.140	1.000	0.775	1.132	1.000	0.780
		CoCoLasso	1.048	1.000	0.900	1.035	1.000	0.908
		nclreg	1.049	1.000	0.870	1.085	1.000	0.881
		MUS	1.039	1.000	0.893	1.027	1.000	0.910
0.65	0.50	Proposed	0.060	1.000	0.984	0.055	1.000	0.987
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.175	1.000	0.767	1.168	1.000	0.771
		EN	1.172	1.000	0.770	1.170	1.000	0.772
		CoCoLasso	1.108	1.000	0.912	1.097	1.000	0.901
		nclreg	1.165	1.000	0.763	1.169	1.000	0.822
		MUS	0.918	1.000	0.906	0.948	1.000	0.906
	0.75	Proposed	0.071	1.000	0.980	0.068	1.000	0.984
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.193	1.000	0.756	1.184	1.000	0.763
		EN	1.190	1.000	0.758	1.180	1.000	0.767
		CoCoLasso	1.129	1.000	0.909	1.136	1.000	0.887
		nclreg	1.156	1.000	0.803	1.162	1.000	0.8001
		MUS	1.047	1.000	0.913	1.045	1.000	0.912

Table D.9: Simulation results under Scenario I and the main/validation data: estimation of μ_2 with $p = 500$

Σ_ϵ	σ_δ^2	Method	$n = 400$			$n = 800$		
			MSE	SPE	SEN	MSE	SPE	SEN
0.15	0.50	Proposed	0.070	0.951	0.950	0.068	0.958	0.953
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.134	0.823	0.802	1.127	0.831	0.816
		EN	1.130	0.830	0.807	1.124	0.839	0.818
		CoCoLasso	1.067	1.000	0.891	0.947	1.000	0.901
		nclreg	1.073	1.000	0.858	0.956	1.000	0.869
		MUS	1.746	0.500	0.906	1.455	0.500	0.906
	0.75	Proposed	0.078	0.950	0.945	0.071	0.958	0.953
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.308	0.820	0.793	1.296	0.831	0.816
		EN	1.303	0.820	0.797	1.294	0.839	0.818
		CoCoLasso	1.139	1.000	0.897	1.108	1.000	0.895
		nclreg	1.231	1.000	0.860	1.235	1.000	0.892
		MUS	1.160	0.790	0.905	1.283	0.857	0.907
0.35	0.50	Proposed	0.083	0.950	0.952	0.077	0.960	0.955
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.158	0.820	0.808	1.143	0.833	0.814
		EN	1.154	0.824	0.806	1.140	0.836	0.816
		CoCoLasso	1.142	1.000	0.905	1.132	1.000	0.883
		nclreg	1.147	1.000	0.843	1.135	1.000	0.868
		MUS	1.742	0.504	0.899	1.335	0.501	0.903
	0.75	Proposed	0.091	0.948	0.950	0.085	0.953	0.951
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.175	0.816	0.797	1.166	0.828	0.805
		EN	1.177	0.818	0.796	1.164	0.830	0.809
		CoCoLasso	1.145	1.000	0.899	1.139	1.000	0.908
		nclreg	1.149	1.000	0.870	1.141	1.000	0.881
		MUS	1.179	0.879	0.895	1.165	0.990	0.905
0.65	0.50	Proposed	0.097	0.934	0.946	0.088	0.957	0.955
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.873	0.808	0.786	1.751	0.813	0.794
		EN	1.875	0.803	0.792	1.747	0.821	0.802
		CoCoLasso	1.254	1.000	0.909	1.261	1.000	0.899
		nclreg	1.522	1.000	0.762	1.518	1.000	0.822
		MUS	1.656	0.531	0.884	1.320	0.512	0.886
	0.75	Proposed	0.112	0.939	0.941	0.094	0.947	0.951
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.954	0.775	0.773	1.932	0.789	0.776
		EN	1.948	0.780	0.777	1.928	0.793	0.783
		CoCoLasso	1.301	1.000	0.908	1.471	1.000	0.887
		nclreg	1.729	1.000	0.803	1.711	1.000	0.800
		MUS	1.531	0.953	0.879	1.202	0.970	0.884

Table D.10: Simulation results under Scenario I and the main/validation data: estimation of μ_3 with $p = 500$

Σ_ϵ	σ_δ^2	Method	$n = 400$			$n = 800$		
			MSE	SPE	SEN	MSE	SPE	SEN
0.15	0.50	Proposed	0.313	1.000	0.983	0.310	1.000	0.983
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.195	1.000	0.840	1.190	1.000	0852
		EN	1.188	1.000	0.843	1.176	1.000	0.856
		CoCoLasso	0.535	1.000	0.894	0.349	1.000	0.903
		nclreg	0.551	1.000	0.863	0.362	1.000	0.872
		MUS	0.518	1.000	0.898	0.479	1.000	0.908
	0.75	Proposed	0.320	1.000	0.980	0.316	1.000	0.986
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.211	1.000	0.837	1.205	1.000	0844
		EN	1.214	1.000	0.839	1.206	1.000	0.846
		CoCoLasso	0.617	1.000	0.898	0.452	1.000	0.895
		nclreg	0.618	1.000	0.861	0.486	1.000	0.892
		MUS	0.592	1.000	0.906	0.641	0.972	0.872
0.35	0.50	Proposed	0.333	1.000	0.979	0.327	1.000	0.980
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.224	1.000	0.826	1.219	1.000	0832
		EN	1.218	1.000	0.831	1.217	1.000	0.836
		CoCoLasso	0.645	1.000	0.907	0.689	1.000	0.883
		nclreg	0.721	1.000	0.845	0.736	1.000	0.869
		MUS	0.631	1.000	0.910	0.625	1.000	0.908
	0.75	Proposed	0.345	1.000	0.973	0.340	1.000	0.977
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.369	1.000	0.811	1.355	1.000	0824
		EN	1.368	1.000	0.813	1.357	1.000	0.826
		CoCoLasso	0.716	1.000	0.900	0.728	1.000	0.908
		nclreg	0.811	1.000	0.870	0.806	1.000	0.881
		MUS	0.811	1.000	0.875	0.732	1.000	0.889
0.65	0.50	Proposed	0.351	1.000	0.973	0.344	1.000	0.977
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.403	1.000	0.808	1.395	1.000	0811
		EN	1.408	1.000	0.810	1.397	1.000	0.815
		CoCoLasso	0.965	1.000	0.912	0.991	1.000	0.901
		nclreg	1.075	1.000	0.763	1.062	1.000	0.822
		MUS	0.957	1.000	0.915	0.913	1.000	0.918
	0.75	Proposed	0.366	1.000	0.957	0.359	1.000	0.965
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.445	1.000	0.784	1.438	1.000	0798
		EN	1.440	1.000	0.790	1.427	1.000	0.802
		CoCoLasso	1.028	1.000	0.909	1.075	1.000	0.887
		nclreg	1.137	1.000	0.803	1.126	1.000	0.800
		MUS	1.046	1.000	0.895	1.089	1.000	0.884

Table D.11: Simulation results under Scenario II and the main/validation data: estimation of μ_1 with $p = 8$

Σ_ϵ	σ_δ^2	Method	$n = 400$			$n = 800$		
			MSE	SPE	SEN	MSE	SPE	SEN
0.15	0.50	Proposed	1.441	1.000	0.827	1.424	1.000	0.841
		AIC	5.547	1.000	0.465	5.541	1.000	0.470
		BIC	5.549	1.000	0.471	5.543	1.000	0.472
		CP	5.566	1.000	0.460	5.360	1.000	0.463
		LASSO	5.308	1.000	0.512	5.298	1.000	0.519
		EN	5.311	1.000	0.515	5.286	1.000	0.520
		CoCoLasso	4.975	1.000	0.626	4.921	1.000	0.553
		nclreg	5.002	1.000	0.688	5.004	1.000	0.620
		MUS	5.040	0.000	0.800	5.038	0.000	0.833
	0.75	Proposed	1.466	1.000	0.823	1.450	1.000	0.829
		AIC	5.579	1.000	0.460	5.567	1.000	0.472
		BIC	5.573	1.000	0.467	5.567	1.000	0.475
		CP	5.584	1.000	0.453	5.379	1.000	0.456
		LASSO	5.366	1.000	0.509	5.345	1.000	0.513
		EN	5.360	1.000	0.509	5.337	1.000	0.514
		CoCoLasso	4.759	1.000	0.558	4.914	1.000	0.515
		nclreg	5.007	1.000	0.595	5.024	1.000	0.545
		MUS	5.047	0.000	0.785	5.042	0.000	0.780
0.35	0.50	Proposed	1.453	1.000	0.825	1.446	1.000	0.833
		AIC	5.562	1.000	0.458	5.558	1.000	0.463
		BIC	5.560	1.000	0.460	5.553	1.000	0.465
		CP	5.571	1.000	0.451	5.566	1.000	0.458
		LASSO	5.324	1.000	0.506	5.313	1.000	0.511
		EN	5.320	1.000	0.509	5.311	1.000	0.514
		CoCoLasso	5.127	1.000	0.517	5.119	1.000	0.497
		nclreg	5.001	1.000	0.596	5.004	1.000	0.530
		MUS	5.063	0.000	0.806	5.055	0.000	0.794
	0.75	Proposed	1.472	1.000	0.821	1.463	1.000	0.830
		AIC	5.584	1.000	0.450	5.571	1.000	0.457
		BIC	5.580	1.000	0.453	5.568	1.000	0.460
		CP	5.592	1.000	0.448	5.583	1.000	0.452
		LASSO	5.346	1.000	0.497	5.326	1.000	0.503
		EN	5.342	1.000	0.499	5.321	1.000	0.504
		CoCoLasso	5.141	1.000	0.504	5.134	1.000	0.546
		nclreg	5.065	1.000	0.494	5.016	1.000	0.545
		MUS	5.071	0.000	0.805	5.066	0.000	0.798
0.65	0.50	Proposed	1.471	1.000	0.820	1.466	1.000	0.828
		AIC	5.578	1.000	0.443	5.562	1.000	0.451
		BIC	5.576	1.000	0.450	5.563	1.000	0.455
		CP	5.585	1.000	0.421	5.577	1.000	0.428
		LASSO	5.351	1.000	0.481	5.339	1.000	0.489
		EN	5.348	1.000	0.486	5.341	1.000	0.491
		CoCoLasso	5.009	1.000	0.692	4.931	1.000	0.582
		nclreg	5.022	1.000	0.537	5.001	1.000	0.475
		MUS	5.065	0.000	0.794	5.048	0.000	0.798
	0.75	Proposed	1.476	1.000	0.816	1.473	1.000	0.824
		AIC	5.583	1.000	0.435	5.579	1.000	0.446
		BIC	5.579	1.000	0.438	5.574	1.000	0.445
		CP	5.592	1.000	0.414	5.583	1.000	0.417
		LASSO	5.360	1.000	0.477	5.351	1.000	0.482
		EN	5.357	1.000	0.478	5.347	1.000	0.485
		CoCoLasso	5.014	1.000	0.576	5.026	1.000	0.485
		nclreg	5.020	1.000	0.434	5.082	1.000	0.479
		MUS	5.073	0.000	0.790	5.066	0.000	0.795

Table D.12: Simulation results under Scenario II and the main/validation data: estimation of μ_2 with $p = 8$

Σ_ϵ	σ_δ^2	Method	$n = 400$			$n = 800$		
			MSE	SPE	SEN	MSE	SPE	SEN
0.15	0.50	Proposed	0.111	1.000	0.986	0.103	1.000	0.987
		AIC	2.178	1.000	0.465	2.166	1.000	0.470
		BIC	2.170	1.000	0.470	2.162	1.000	0.473
		CP	2.185	1.000	0.463	2.179	1.000	0.467
		LASSO	2.123	1.000	0.493	2.118	1.000	0.512
		EN	2.120	1.000	0.494	2.116	1.000	0.513
		CoCoLasso	2.073	1.000	0.544	2.026	1.000	0.485
		nclreg	2.004	1.000	0.597	2.019	1.000	0.538
		MUS	3.010	0.000	0.884	2.987	0.000	0.864
	0.75	Proposed	0.116	1.000	0.983	0.109	1.000	0.984
		AIC	2.185	1.000	0.461	2.174	1.000	0.466
		BIC	2.177	1.000	0.466	2.170	1.000	0.468
		CP	2.193	1.000	0.457	2.188	1.000	0.460
		LASSO	2.135	1.000	0.481	2.123	1.000	0.495
		EN	2.131	1.000	0.484	2.120	1.000	0.501
		CoCoLasso	2.135	1.000	0.494	2.082	1.000	0.553
		nclreg	2.128	1.000	0.518	2.079	1.000	0.478
		MUS	3.025	0.000	0.866	3.011	0.000	0.860
0.35	0.50	Proposed	0.117	1.000	0.984	0.112	1.000	0.984
		AIC	2.182	1.000	0.455	2.176	1.000	0.461
		BIC	2.177	1.000	0.457	2.170	1.000	0.463
		CP	2.190	1.000	0.448	2.179	1.000	0.456
		LASSO	2.131	1.000	0.489	2.126	1.000	0.493
		EN	2.128	1.000	0.490	2.121	1.000	0.495
		CoCoLasso	1.978	1.000	0.558	1.907	1.000	0.539
		nclreg	1.942	1.000	0.521	1.901	1.000	0.559
		MUS	3.032	0.000	0.891	2.993	0.000	0.876
	0.75	Proposed	0.121	1.000	0.980	0.117	1.000	0.985
		AIC	2.189	1.000	0.451	2.186	1.000	0.458
		BIC	2.186	1.000	0.453	2.182	1.000	0.460
		CP	2.195	1.000	0.443	2.194	1.000	0.428
		LASSO	2.140	1.000	0.482	2.138	1.000	0.486
		EN	2.137	1.000	0.485	2.135	1.000	0.489
		CoCoLasso	2.060	1.000	0.549	2.058	1.000	0.507
		nclreg	2.087	1.000	0.550	2.057	1.000	0.508
		MUS	3.046	0.000	0.889	3.038	0.000	0.881
0.65	0.50	Proposed	0.125	1.000	0.981	0.118	1.000	0.984
		AIC	2.195	1.000	0.450	2.189	1.000	0.457
		BIC	2.197	1.000	0.452	2.187	1.000	0.459
		CP	2.201	1.000	0.444	2.193	1.000	0.450
		LASSO	2.150	1.000	0.483	2.139	1.000	0.489
		EN	2.148	1.000	0.485	2.140	1.000	0.490
		CoCoLasso	1.721	1.000	0.542	1.692	1.000	0.454
		nclreg	1.902	1.000	0.572	1.832	1.000	0.523
		MUS	3.045	0.000	0.890	3.041	0.000	0.877
	0.75	Proposed	0.131	1.000	0.977	0.118	1.000	0.980
		AIC	2.210	1.000	0.446	2.209	1.000	0.450
		BIC	2.208	1.000	0.449	2.207	1.000	0.453
		CP	2.216	1.000	0.440	2.214	1.000	0.447
		LASSO	2.174	1.000	0.475	2.166	1.000	0.481
		EN	2.170	1.000	0.478	2.163	1.000	0.483
		CoCoLasso	2.013	1.000	0.452	1.941	1.000	0.671
		nclreg	2.007	1.000	0.501	1.846	1.000	0.468
		MUS	3.051	0.000	0.891	3.044	0.000	0.878

Table D.13: Simulation results under Scenario II and the main/validation data: estimation of μ_3 with $p = 8$

Σ_ϵ	σ_δ^2	Method	$n = 400$			$n = 800$		
			MSE	SPE	SEN	MSE	SPE	SEN
0.15	0.50	Proposed	0.135	1.000	0.963	0.126	1.000	0.970
		AIC	2.329	1.000	0.456	2.320	1.000	0.460
		BIC	2.328	1.000	0.453	2.317	1.000	0.465
		CP	2.345	1.000	0.447	2.334	1.000	0.452
		LASSO	2.276	1.000	0.489	2.268	1.000	0.497
		EN	2.278	1.000	0.490	2.266	1.000	0.502
		CoCoLasso	2.014	1.000	0.626	1.942	1.000	0.553
		nclreg	2.007	1.000	0.688	1.855	1.000	0.620
		MUS	2.445	0.279	0.917	2.456	0.345	0.919
	0.75	Proposed	0.141	1.000	0.960	0.137	1.000	0.966
		AIC	2.340	1.000	0.450	2.335	1.000	0.453
		BIC	2.336	1.000	0.450	2.330	1.000	0.455
		CP	2.355	1.000	0.441	2.343	1.000	0.438
		LASSO	2.281	1.000	0.485	2.275	1.000	0.489
		EN	2.280	1.000	0.488	2.273	1.000	0.492
		CoCoLasso	2.162	1.000	0.558	1.618	1.000	0.515
		nclreg	2.165	1.000	0.595	1.524	1.000	0.545
		MUS	2.388	0.301	0.907	2.397	0.348	0.906
0.35	0.50	Proposed	0.138	1.000	0.961	0.132	1.000	0.966
		AIC	2.340	1.000	0.454	2.335	1.000	0.458
		BIC	2.342	1.000	0.452	2.334	1.000	0.458
		CP	2.348	1.000	0.444	2.340	1.000	0.449
		LASSO	2.307	1.000	0.483	2.297	1.000	0.490
		EN	2.306	1.000	0.486	2.296	1.000	0.492
		CoCoLasso	1.802	1.000	0.517	1.594	1.000	0.497
		nclreg	1.792	1.000	0.596	1.609	1.000	0.530
		MUS	2.345	0.308	0.903	2.424	0.365	0.898
	0.75	Proposed	0.144	1.000	0.957	0.140	1.000	0.963
		AIC	2.353	1.000	0.438	2.347	1.000	0.455
		BIC	2.350	1.000	0.4440	2.344	1.000	0.456
		CP	2.357	1.000	0.431	2.350	1.000	0.443
		LASSO	2.310	1.000	0.479	2.306	1.000	0.484
		EN	2.311	1.000	0.480	2.303	1.000	0.486
		CoCoLasso	1.816	1.000	0.504	1.710	1.000	0.446
		nclreg	1.865	1.000	0.494	1.730	1.000	0.445
		MUS	2.368	0.313	0.893	2.383	0.353	0.889
0.65	0.50	Proposed	0.149	1.000	0.955	0.144	1.000	0.959
		AIC	2.414	1.000	0.449	2.398	1.000	0.452
		BIC	2.412	1.000	0.449	2.394	1.000	0.453
		CP	2.420	1.000	0.440	2.402	1.000	0.442
		LASSO	2.356	1.000	0.477	2.337	1.000	0.483
		EN	2.356	1.000	0.478	2.336	1.000	0.484
		CoCoLasso	1.658	1.000	0.692	1.744	1.000	0.582
		nclreg	1.821	1.000	0.537	1.819	1.000	0.475
		MUS	2.384	0.298	0.899	2.382	0.370	0.900
	0.75	Proposed	0.153	1.000	0.952	0.149	1.000	0.957
		AIC	2.430	1.000	0.444	2.416	1.000	0.450
		BIC	2.431	1.000	0.445	2.414	1.000	0.450
		CP	2.435	1.000	0.436	2.428	1.000	0.439
		LASSO	2.367	1.000	0.471	2.360	1.000	0.478
		EN	2.364	1.000	0.473	2.357	1.000	0.480
		CoCoLasso	1.935	1.000	0.576	1.945	1.000	0.485
		nclreg	1.972	1.000	0.534	1.942	1.000	0.579
		MUS	2.261	0.322	0.901	2.312	0.366	0.905

Table D.14: Simulation results under Scenario II and the main/validation data: estimation of μ_1 with $p = 500$

Σ_ϵ	σ_δ^2	Method	$n = 400$			$n = 800$		
			MSE	SPE	SEN	MSE	SPE	SEN
0.15	0.50	Proposed	0.036	1.000	0.985	0.030	1.000	0.990
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.166	1.000	0.805	2.154	1.000	0.810
		EN	2.159	1.000	0.807	2.152	1.000	0.813
		CoCoLasso	1.516	1.000	0.843	1.655	1.000	0.839
		nclreg	2.032	1.000	0.852	2.028	1.000	0.863
		MUS	2.231	0.000	0.900	2.210	0.000	0.903
	0.75	Proposed	0.040	1.000	0.983	0.038	1.000	0.880
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.175	1.000	0.796	2.169	1.000	0.804
		EN	2.170	1.000	0.797	2.166	1.000	0.808
		CoCoLasso	1.639	1.000	0.836	1.706	1.000	0.836
		nclreg	2.026	1.000	0.902	2.025	1.000	0.865
		MUS	2.249	0.000	0.903	2.228	0.000	0.907
0.35	0.50	Proposed	0.041	1.000	0.978	0.037	1.000	0.987
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.235	1.000	0.785	2.228	1.000	0.798
		EN	2.229	1.000	0.787	2.225	1.000	0.798
		CoCoLasso	1.832	1.000	0.804	2.259	1.000	0.793
		nclreg	2.206	1.000	0.868	2.168	1.000	0.870
		MUS	2.234	0.000	0.910	2.229	0.000	0.913
	0.75	Proposed	0.049	1.000	0.975	0.045	1.000	0.981
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.288	1.000	0.774	2.275	1.000	0.780
		EN	2.279	1.000	0.777	2.275	1.000	0.782
		CoCoLasso	1.783	1.000	0.812	2.169	1.000	0.800
		nclreg	2.242	1.000	0.820	2.211	1.000	0.858
		MUS	2.255	0.000	0.891	2.250	0.000	0.896
0.65	0.50	Proposed	0.050	1.000	0.974	0.046	1.000	0.980
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.293	1.000	0.763	2.288	1.000	0.771
		EN	2.290	1.000	0.768	2.285	1.000	0.774
		CoCoLasso	1.689	1.000	0.950	2.110	1.000	0.727
		nclreg	2.080	1.000	0.868	2.097	1.000	0.852
		MUS	2.308	0.000	0.889	2.295	0.000	0.891
	0.75	Proposed	0.057	1.000	0.968	0.052	1.000	0.977
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.318	1.000	0.731	2.299	1.000	0.746
		EN	2.320	1.000	0.736	2.294	1.000	0.750
		CoCoLasso	2.141	1.000	0.929	2.156	1.000	0.759
		nclreg	2.196	1.000	0.847	2.164	1.000	0.799
		MUS	2.314	0.000	0.870	2.307	0.000	0.882

Table D.15: Simulation results under Scenario II and the main/validation data: estimation of μ_2 with $p = 500$

Σ_ϵ	σ_δ^2	Method	$n = 400$			$n = 800$		
			MSE	SPE	SEN	MSE	SPE	SEN
0.15	0.50	Proposed	0.043	1.000	0.992	0.039	1.000	0.990
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	1.948	1.000	0.811	1.928	1.000	0.807
		EN	1.931	1.000	0.815	1.923	1.000	0.810
		CoCoLasso	1.349	1.000	0.833	1.396	1.000	0.831
		nclreg	1.599	1.000	0.845	1.530	1.000	0.854
		MUS	2.000	0.000	0.901	1.978	0.000	0.909
	0.75	Proposed	0.052	1.000	0.987	0.047	1.000	0.992
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.311	1.000	0.795	2.298	1.000	0.802
		EN	2.309	1.000	0.798	2.299	1.000	0.805
		CoCoLasso	1.927	1.000	0.830	1.664	1.000	0.832
		nclreg	1.990	1.000	0.895	1.630	1.000	0.857
		MUS	2.025	0.000	0.892	2.017	0.000	0.903
0.35	0.50	Proposed	0.049	1.000	0.989	0.044	1.000	0.990
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.296	1.000	0.801	2.290	1.000	0.812
		EN	2.287	1.000	0.810	2.282	1.000	0.816
		CoCoLasso	1.704	1.000	0.797	1.688	1.000	0.788
		nclreg	1.662	1.000	0.860	1.569	1.000	0.863
		MUS	2.044	0.000	0.887	2.029	0.000	0.894
	0.75	Proposed	0.060	1.000	0.982	0.056	1.000	0.987
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.453	1.000	0.789	2.436	1.000	0.793
		EN	2.449	1.000	0.792	2.430	1.000	0.798
		CoCoLasso	1.909	1.000	0.807	1.904	1.000	0.796
		nclreg	1.923	1.000	0.814	1.877	1.000	0.854
		MUS	2.050	0.000	0.873	2.041	0.000	0.886
0.65	0.50	Proposed	0.069	1.000	0.978	0.063	1.000	0.982
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.503	1.000	0.755	2.498	1.000	0.766
		EN	2.505	1.000	0.760	2.483	1.000	0.770
		CoCoLasso	1.873	1.000	0.941	1.846	1.000	0.721
		nclreg	1.935	1.000	0.860	1.851	1.000	0.846
		MUS	2.049	0.000	0.891	2.033	0.000	0.896
	0.75	Proposed	0.077	1.000	0.966	0.072	1.000	0.975
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.584	1.000	0.731	2.567	1.000	0.750
		EN	2.579	1.000	0.736	2.569	1.000	0.757
		CoCoLasso	1.983	1.000	0.922	1.915	1.000	0.753
		nclreg	2.014	1.000	0.843	1.915	1.000	0.797
		MUS	2.062	0.000	0.877	2.054	0.000	0.881

Table D.16: Simulation results under Scenario II and the main/validation data: estimation of μ_3 with $p = 500$

Σ_ϵ	σ_δ^2	Method	$n = 400$			$n = 800$		
			MSE	SPE	SEN	MSE	SPE	SEN
0.15	0.50	Proposed	0.095	1.000	0.993	0.088	1.000	0.990
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.288	0.974	0.773	2.283	0.975	0.780
		EN	2.284	0.976	0.775	2.279	0.975	0.782
		CoCoLasso	2.132	1.000	0.843	1.940	1.000	0.839
		nclreg	2.121	1.000	0.852	1.933	1.000	0.863
		MUS	2.082	0.107	0.900	2.021	0.155	0.904
	0.75	Proposed	0.103	1.000	0.989	0.097	1.000	0.986
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.303	0.970	0.768	2.298	0.975	0.775
		EN	2.297	0.970	0.770	2.293	0.975	0.778
		CoCoLasso	2.275	1.000	0.836	1.937	1.000	0.836
		nclreg	2.268	1.000	0.902	1.966	1.000	0.865
		MUS	1.986	0.136	0.910	1.976	0.018	0.906
0.35	0.50	Proposed	0.110	1.000	0.980	0.107	1.000	0.992
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.369	0.966	0.763	2.355	0.973	0.774
		EN	2.358	0.967	0.768	2.349	0.976	0.778
		CoCoLasso	2.192	1.000	0.804	2.174	1.000	0.793
		nclreg	2.195	1.000	0.868	2.183	1.000	0.870
		MUS	2.070	0.113	0.896	2.082	0.134	0.903
	0.75	Proposed	0.123	1.000	0.976	0.118	1.000	0.985
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.439	0.960	0.755	2.420	0.975	0.761
		EN	2.430	0.960	0.756	2.419	0.972	0.766
		CoCoLasso	2.325	1.000	0.812	2.262	1.000	0.800
		nclreg	2.256	1.000	0.820	2.238	1.000	0.858
		MUS	2.026	0.119	0.891	1.999	0.150	0.907
0.65	0.50	Proposed	0.128	1.000	0.977	0.121	1.000	0.988
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.543	0.963	0.745	2.536	0.975	0.751
		EN	2.528	0.965	0.748	2.533	0.979	0.756
		CoCoLasso	2.357	1.000	0.950	2.455	1.000	0.727
		nclreg	2.456	1.000	0.868	2.449	1.000	0.852
		MUS	2.095	0.099	0.880	2.065	0.139	0.885
	0.75	Proposed	0.137	1.000	0.962	0.134	1.000	0.979
		AIC	–	–	–	–	–	–
		BIC	–	–	–	–	–	–
		CP	–	–	–	–	–	–
		LASSO	2.746	0.952	0.725	2.726	0.975	0.733
		EN	2.739	0.957	0.728	2.735	0.979	0.736
		CoCoLasso	2.360	1.000	0.929	2.471	1.000	0.759
		nclreg	2.566	1.000	0.847	2.563	1.000	0.799
		MUS	2.162	0.107	0.873	2.187	0.040	0.869

Appendix E An Additional Plot for Real Data Analysis

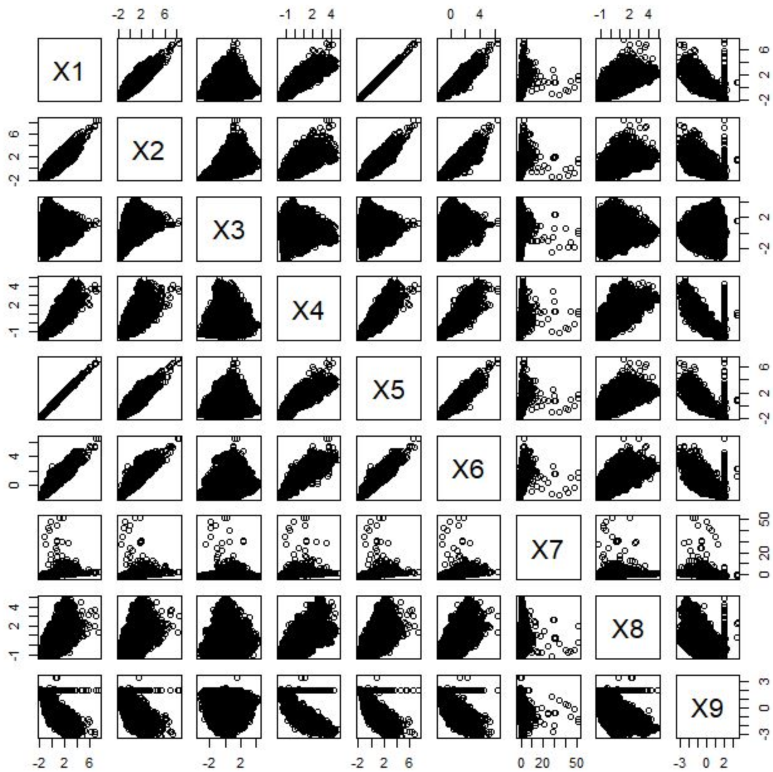

Figure E.1: A pairwise scatter plot for the covariates in the protein tertiary structures data in Section 5.3 of the main text