Supplementary Material to "Uniformly consistent proportion estimation for composite hypotheses via integral equations: 'the case of Gamma random variables'"

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Appendix B contains proofs related to Construction I, Appendix C proofs related to Construction II, Appendix D proofs related to the extension of Construction I, Appendix E a simulation study on the proposed estimators, and Appendix F how to adapt the constructions and estimators to the null sets $(-\infty, b]$ and $[a, b]$.

B Proofs Related to Construction I

B.1 Proof of Theorem 2

Recall $\tilde{c}_n(\theta) = \int x^n dG_\theta(x)$ and define

$$K_1^\dagger(t,x) = \frac{1}{2\pi\zeta_0} \int_a^b t dy \int_{-1}^1 \exp(-\iota t s y) \sum_{n=0}^\infty \frac{(\iota t s x \zeta_0 \tilde{a}_1)^n}{\tilde{a}_n n!} ds. \tag{53}$$

By assumption, $\tilde{c}_n(\theta) = \xi^n(\theta)\zeta(\theta)\tilde{a}_n = \zeta_0 \xi^n(\theta)\tilde{a}_n$, where $\zeta_0 \equiv \zeta \equiv 1$. So, $\mu(\theta) = \xi(\theta)\zeta(\theta)\tilde{a}_1 = \xi(\theta)\zeta_0\tilde{a}_1$ and

$$\begin{aligned}
\psi_1(t,\theta) &= \int K_1^\dagger(t,x)\, dG_\theta(x) \\
&= \frac{1}{2\pi\zeta_0} \int_a^b t dy \int_{-1}^1 \exp(-\iota t s y) \sum_{n=0}^\infty \frac{(\iota t s \zeta_0 \tilde{a}_1)^n}{\tilde{a}_n n!} \tilde{c}_n(\theta)\, ds \\
&= \frac{\zeta(\theta)}{2\pi\zeta_0} \int_a^b t dy \int_{-1}^1 \exp(-\iota t s y) \sum_{n=0}^\infty \frac{(\iota t s \zeta_0 \tilde{a}_1)^n}{n!} \xi^n(\theta)\, ds \\
&= \frac{1}{2\pi} \int_a^b t dy \int_{-1}^1 \exp[\iota t s \{\mu(\theta) - y\}]\, ds.
\end{aligned}$$

Since ψ_1 is real, $\psi_1 = \mathbb{E}\left\{\Re\left(K_1^\dagger\right)\right\}$. However,

$$K_1(t,x) = \Re\left\{K_1^\dagger(t,x)\right\} = \frac{1}{2\pi\zeta_0} \int_a^b t dy \int_{-1}^1 \sum_{n=0}^\infty \frac{(t s x \zeta_0 \tilde{a}_1)^n \cos(2^{-1} n\pi - t s y)}{\tilde{a}_n n!} ds.$$

Since $\mu(\theta)$ is smooth and strictly increasing in $\theta \in \Theta$, $a \le \mu \le b$ if and only if $\theta_a \le \theta \le \theta_b$. By Theorem 1, the pair (K, ψ) in (13) is as desired. □

B.2 Proof of Theorem 3

In order the present the proof, we quote Lemma 4 of Chen (2019) as follows: for a fixed $\sigma > 0$, let

$$\tilde{w}(z,x) = \sum_{n=0}^\infty \frac{(zx)^n}{n!\Gamma(\sigma+n)} \quad \text{for } z, x > 0. \tag{54}$$

If Z has CDF G_θ from the Gamma family with scale parameter σ, then

$$\mathbb{E}\left[\tilde{w}^2(z,Z)\right] \le C \left(\frac{z}{1-\theta}\right)^{3/4-\sigma} \exp\left(\frac{4z}{1-\theta}\right) \tag{55}$$

for positive and sufficiently large z.

Now we present the arguments. Firstly, we will obtain an upper bound for $\mathbb{V}\left\{\hat{\varphi}_m\left(t, \mathbf{z}\right)\right\}$. For Gamma family, $\zeta\left(\theta\right) \equiv \zeta_0 = 1$, $\tilde{a}_1 = \sigma$ and $\mu\left(\theta\right) = \sigma\xi\left(\theta\right)$. Define

$$w_1\left(t, x, y\right) = \Gamma\left(\sigma\right) \sum_{n=0}^{\infty} \frac{\left(tx\sigma\right)^n \cos\left(2^{-1}n\pi - ty\right)}{n!\Gamma\left(n+\sigma\right)} \quad \text{for } t \geq 0 \text{ and } x > 0,$$

and set $S_{1,m}\left(t, y\right) = m^{-1}\sum_{i=1}^m \left[w_1\left(t, z_i, y\right) - \mathbb{E}\left\{w_1\left(t, z_i, y\right)\right\}\right]$. Recall $\tilde{a}_n = \frac{\Gamma(n+\sigma)}{\Gamma(\sigma)}$. Then

$$K_1\left(t, x\right) = \frac{1}{2\pi}\int_a^b tdy \int_{-1}^1 w_1\left(ts, x, y\right) ds.$$

Define $\tilde{V}_{1,m} = \mathbb{V}\left\{\hat{\varphi}_{1,m}\left(t, \mathbf{z}\right)\right\}$, where $\hat{\varphi}_{1,m}\left(t, \mathbf{z}\right) = m^{-1}\sum_{i=1}^m K_1\left(t, z_i\right)$ and $\varphi_{1,m}\left(t, \boldsymbol{\theta}\right) = \mathbb{E}\left\{\hat{\varphi}_{1,m}\left(t, \mathbf{z}\right)\right\}$. Then, applying Hölder's inequality to $\int_{-1}^1 \left|S_{1,m}\left(ts, y\right)\right| ds$ and then to $\int_a^b dy \left[\left(\int_{-1}^1 \left|S_{1,m}\left(ts, y\right)\right|^2 ds\right)^{1/2}\right]$,

$$\begin{aligned}
\tilde{V}_{1,m} &= \mathbb{E}\left[\left\{\frac{1}{2\pi}\int_a^b tdy \int_{-1}^1 S_{1,m}\left(ts, y\right) ds\right\}^2\right] \\
&\leq \frac{t^2}{4\pi^2}\mathbb{E}\left[\left\{\int_a^b dy \left[\sqrt{2}\left(\int_{-1}^1 \left|S_{1,m}\left(ts, y\right)\right|^2 ds\right)^{1/2}\right]\right\}^2\right] \\
&= \frac{t^2}{2\pi^2}\mathbb{E}\left[\left\{\int_a^b \left(\int_{-1}^1 \left|S_{1,m}\left(ts, y\right)\right|^2 ds\right)^{1/2} dy\right\}^2\right] \\
&\leq \frac{t^2}{2\pi^2}\mathbb{E}\left[\left(b-a\right)\int_a^b \int_{-1}^1 \left|S_{1,m}\left(ts, y\right)\right|^2 dy\right] \\
&= \frac{\left(b-a\right)t^2}{2\pi^2}\mathbb{E}\left\{\int_a^b dy \int_{-1}^1 \left|S_{1,m}\left(ts, y\right)\right|^2 ds\right\},
\end{aligned}$$

i.e.,

$$\tilde{V}_{1,m} = \mathbb{E}\left[\left\{\frac{1}{2\pi}\int_a^b tdy \int_{-1}^1 S_{1,m}\left(ts, y\right) ds\right\}^2\right] \leq \frac{\left(b-a\right)t^2}{2\pi^2}\mathbb{E}\left\{\int_a^b dy \int_{-1}^1 \left|S_{1,m}\left(ts, y\right)\right|^2 ds\right\}. \tag{56}$$

Since $\left|w_1\left(t, x, y\right)\right| \leq \Gamma\left(\sigma\right)\tilde{w}\left(t\sigma, x\right)$ uniformly in $\left(t, x, y\right)$, the inequality (55) implies, for $t > 0$ sufficiently large,

$$\begin{aligned}
\tilde{V}_{1,m} &\leq Ct^2\mathbb{E}\left\{\int_a^b dy \int_{-1}^1 \left|S_{1,m}\left(ts, y\right)\right|^2 ds\right\} \leq \frac{Ct^2}{m^2}\sum_{i=1}^m \mathbb{E}\left[\tilde{w}^2\left(t\sigma, z_i\right)\right] \\
&\leq \frac{Ct^2}{m^2}\sum_{i=1}^m \left(\frac{t}{1-\theta_i}\right)^{3/4-\sigma} \exp\left(\frac{4t\sigma}{1-\theta_i}\right) \leq \frac{Ct^2}{m}V_{1,m},
\end{aligned}$$

where we recall $u_{3,m} = \min_{1\leq i\leq m}\left\{1-\theta_i\right\}$ and have set

$$V_{1,m} = \frac{1}{m}\exp\left(\frac{4t\sigma}{u_{3,m}}\right)\sum_{i=1}^m \left(\frac{t}{1-\theta_i}\right)^{3/4-\sigma}.$$

Recall for $\tau \in \left\{a, b\right\}$

$$K_{3,0}\left(t, x; \theta_\tau\right) = \frac{\Gamma\left(\sigma\right)}{\zeta_0}\int_{[-1,1]} \sum_{n=0}^{\infty} \frac{\left(-tsx\right)^n \cos\left\{2^{-1}\pi n + ts\xi\left(\theta_\tau\right)\right\}}{n!\Gamma\left(n+\sigma\right)}\omega\left(s\right) ds.$$

Define $\hat{\varphi}_{3,0,m}\left(t, \mathbf{z}; \tau\right) = m^{-1}\sum_{i=1}^m K_{3,0}\left(t, z_i; \theta_\tau\right)$ and $\varphi_{3,0,m}\left(t, \boldsymbol{\theta}; \tau\right) = \mathbb{E}\left\{\hat{\varphi}_{3,0,m}\left(t, \mathbf{z}; \tau\right)\right\}$. Then Theorem 8 of Chen (2019) implies, for $t > 0$ sufficiently large,

$$\mathbb{V}\left\{\hat{\varphi}_{3,0,m}\left(t, \mathbf{z}; \tau\right)\right\} \leq Cm^{-1}V_{0,m} \quad \text{with} \quad V_{0,m} = \frac{1}{m}\exp\left(\frac{4t}{u_{3,m}}\right)\sum_{i=1}^m \frac{t^{3/4-\sigma}}{\left(1-\theta_i\right)^{3/4-\sigma}}.$$

So, for $t > 0$ sufficiently large,

$$\mathbb{V}\left\{\hat{\varphi}_m\left(t, \mathbf{z}\right)\right\} \leq Cm^{-1}V_{0,m} + Ct^2m^{-1}V_{1,m} \leq Cm^{-1}\left(1+t^2\right)\tilde{V}_{1,m}^*, \tag{57}$$

where

$$\tilde{V}_{1,m}^{*}=\frac{1}{m}\exp\left(\frac{4t\max\left\{\sigma,1\right\}}{u_{3,m}}\right)\sum_{i=1}^{m}\left(\frac{t}{1-\theta_{i}}\right)^{3/4-\sigma}. \tag{58}$$

Secondly, we provide a uniform consistency class. If $\sigma\geq 3/4$, then (58) induces

$$\tilde{V}_{1,m}^{*}\leq t^{3/4-\sigma}\exp\left(\frac{4\max\left\{\sigma,1\right\}t}{u_{3,m}}\right)\left\Vert 1-\boldsymbol{\theta}\right\Vert_{\infty}^{\sigma-3/4} \tag{59}$$

for $t>0$ sufficiently large, where $\left\Vert 1-\boldsymbol{\theta}\right\Vert_{\infty}=\max_{1\leq i\leq m}\left(1-\theta_{i}\right)$. Let $\varepsilon>0$ be a constant and set $t_{m}=\left(4\max\left\{\sigma,1\right\}\right)^{-1}u_{3,m}\gamma\ln m$ for any fixed $\gamma\in(0,1)$. Then, (57) and (59) imply, for all m such that t_{m} is sufficiently large,

$$\Pr\left\{\frac{\left|\hat{\varphi}_{m}\left(t_{m},\mathbf{z}\right)-\varphi_{m}\left(t_{m},\boldsymbol{\theta}\right)\right|}{\pi_{1,m}}\geq\varepsilon\right\}\leq\frac{C\left\Vert 1-\boldsymbol{\theta}\right\Vert_{\infty}^{\sigma-3/4}}{\varepsilon^{2}m^{1-\gamma}\pi_{1,m}^{2}}\left(u_{3,m}\ln m\right)^{11/4-\sigma}. \tag{60}$$

Note that $\gamma=1$ can be set when $\sigma>11/4$ since $\lim_{m\rightarrow\infty}\left(\ln m\right)^{11/4-\sigma}=0$ for all such σ. In contrast, if $\sigma\leq 3/4$, then (58) implies

$$\tilde{V}_{1,m}^{*}\leq\left(\frac{t}{u_{3,m}}\right)^{3/4-\sigma}\exp\left(\frac{4t}{u_{3,m}}\right)$$

for all $t>0$ sufficiently large. Set $t_{m}=4^{-1}u_{3,m}\gamma\ln m$ for any fixed $\gamma\in(0,1)$. Then, for all m such that t_{m} is sufficiently large

$$\Pr\left\{\frac{\left|\hat{\varphi}_{m}\left(t_{m},\mathbf{z}\right)-\varphi_{m}\left(t_{m},\boldsymbol{\theta}\right)\right|}{\pi_{1,m}}\geq\varepsilon\right\}\leq\frac{C\left(\ln m\right)^{11/4-\sigma}u_{3,m}^{2}}{\varepsilon^{2}m^{1-\gamma}\pi_{1,m}^{2}}. \tag{61}$$

To determine a uniform consistency class, we only need to incorporate the speed of convergence of $\varphi_{m}\left(t,\boldsymbol{\mu}\right)$ to $\pi_{1,m}$. Recall for $\tau\in\{a,b\}$

$$\psi_{3,0}\left(t,\theta;\theta_{\tau}\right)=\int_{[-1,1]}\cos\left[ts\left\{\xi\left(\theta_{\tau}\right)-\xi\left(\theta\right)\right\}\right]\omega\left(s\right)ds,$$

which is exactly

$$\tilde{\psi}_{1,0}\left(t,\mu;\mu'\right)=\int_{[-1,1]}\omega\left(s\right)\cos\left\{ts\sigma^{-1}\left(\mu-\mu'\right)\right\}ds$$

that is defined by (44) (in Lemma 5) but evaluated at $\mu'=\tau$ since $\xi\left(\theta\right)=\left(1-\theta\right)^{-1}$ and $\mu\left(\theta\right)=\sigma\left(1-\theta\right)^{-1}=\sigma\xi\left(\theta\right)$. Recall $\tilde{u}_{3,m}=\min_{\tau\in\{a,b\}}\min_{\left\{j:\theta_{j}\neq\theta_{\tau}\right\}}\left|\xi\left(\theta_{\tau}\right)-\xi\left(\theta_{i}\right)\right|$,

$$\psi_{1}\left(t,\theta\right)=\int K_{1}\left(t,x\right)dG_{\theta}\left(x\right)=\frac{1}{\pi}\int_{\left(\mu\left(\theta\right)-b\right)t}^{\left(\mu\left(\theta\right)-a\right)t}\frac{\sin y}{y}dy,$$

and $u_{m}=\min_{\tau\in\{a,b\}}\min_{\left\{j:\mu_{j}\neq\tau\right\}}\left|\mu_{j}-\tau\right|$, the last of which is defined in Lemma 5. Noticing $\mu_{i}=\mu\left(\theta_{i}\right),i=1,\ldots,m$, we have $u_{m}=\sigma\tilde{u}_{3,m}$. Due to the differentiable, monotonic, bijection, $\mu\left(\theta\right)=\sigma\left(1-\theta\right)^{-1}$, between the mean $\mu=\mu\left(\theta\right)$ and natural parameter $\theta=\theta\left(\mu\right)$ for the Gamma family, Lemma 5 implies, when $tu_{m}=t\sigma\tilde{u}_{3,m}\geq 2$ and $t\left(b-a\right)\geq 2$,

$$\left|\frac{\varphi_{m}\left(t,\boldsymbol{\theta}\right)}{\pi_{1,m}}-1\right|\leq C\left(\frac{1}{t\pi_{1,m}}+\frac{1}{t\tilde{u}_{3,m}\pi_{1,m}}\right)\quad\text{for } t\sigma\tilde{u}_{3,m}\geq 2. \tag{62}$$

So, $\pi_{1,m}^{-1}\varphi_{m}\left(t_{m},\boldsymbol{\theta}\right)\rightarrow 1$ if $t_{m}^{-1}\left(1+\tilde{u}_{3,m}^{-1}\right)=o\left(\pi_{1,m}\right)$. Therefore, by (60) a uniform consistency class when $\sigma\geq 3/4$ is

$$\mathcal{Q}\left(\mathcal{F}\right)=\left\{\begin{array}{c}t_{m}=4^{-1}(\max\left\{\sigma,1\right\})^{-1}\gamma u_{3,m}\ln m,t_{m}^{-1}\left(1+\tilde{u}_{3,m}^{-1}\right)=o\left(\pi_{1,m}\right),\\ t_{m}\rightarrow\infty,\left\Vert 1-\boldsymbol{\theta}\right\Vert_{\infty}^{\sigma-3/4}t_{m}^{11/4-\sigma}=o\left(m^{1-\gamma}\pi_{1,m}^{2}\right)\end{array}\right\}$$

for each $\gamma\in(0,1)$, for which $\gamma=1$ can be set when $\sigma>11/4$, and by (61) a uniform consistency class when $\sigma\leq 3/4$ is

$$\mathcal{Q}\left(\mathcal{F}\right)=\left\{\begin{array}{c}t_{m}=4^{-1}\gamma u_{3,m}\ln m,t_{m}^{-1}\left(1+\tilde{u}_{3,m}^{-1}\right)=o\left(\pi_{1,m}\right),\\ t_{m}\rightarrow\infty,\left(\ln m\right)^{11/4-\sigma}u_{3,m}^{2}=o\left(m^{1-\gamma}\pi_{1,m}^{2}\right)\end{array}\right\}$$

for each $\gamma\in(0,1)$. $\square$

C Proofs Related to Construction II

C.1 Proof of Theorem 4

Recall $\tilde{c}_n(\theta) = \int x^n dG_\theta(x) = \zeta_0 \xi^n(\theta) \tilde{a}_n$ for the constant $\zeta_0 = 1$ and $\mu(\theta) = \tilde{c}_1$. Set

$$K_{4,0}^{\dagger}(t,x) = \frac{1}{2\pi\zeta_0} \int_0^t dy \int_{-1}^1 \exp(-\iota ysb) \sum_{n=0}^{\infty} \frac{(\iota ys)^n (\zeta_0 \tilde{a}_1 x)^{n+1}}{\tilde{a}_{n+1} n!} ds.$$

Then

$$K_{4,0}^{\dagger}(t,x) = \frac{1}{2\pi\zeta_0} \int_0^1 t dy \int_{-1}^1 \exp(-\iota tysb) \sum_{n=0}^{\infty} \frac{(\iota tys)^n (\zeta_0 \tilde{a}_1 x)^{n+1}}{\tilde{a}_{n+1} n!} ds. \tag{63}$$

Further,

$$\begin{aligned}
\int K_{4,0}^{\dagger}(t,x)\, dG_\theta(x) &= \frac{\zeta(\theta)}{2\pi\zeta_0} \int_0^t dy \int_{-1}^1 \exp(-\iota ysb) \sum_{n=0}^{\infty} \frac{(\iota ys)^n}{n!} (\zeta_0 \tilde{a}_1)^{n+1} \xi^{n+1}(\theta)\, ds \\
&= \frac{1}{2\pi} \int_0^t \mu(\theta)\, dy \int_{-1}^1 \exp(-\iota ysb) \exp(\iota ys\mu(\theta))\, ds \\
&= \frac{1}{2\pi} \int_0^t \mu(\theta)\, dy \int_{-1}^1 \exp[\iota ys \{\mu(\theta) - b\}]\, ds.
\end{aligned}$$

On the other hand, set

$$K_{4,1}^{\dagger}(t,x) = -\frac{1}{2\pi\zeta_0} \int_0^t dy \int_{-1}^1 b \exp(-\iota ysb) \sum_{n=0}^{\infty} \frac{(\iota ys)^n (\zeta_0 \tilde{a}_1 x)^n}{\tilde{a}_n n!} ds.$$

Then

$$K_{4,1}^{\dagger}(t,x) = -\frac{1}{2\pi\zeta_0} \int_0^1 t dy \int_{-1}^1 b \exp(-\iota tysb) \sum_{n=0}^{\infty} \frac{(\iota tys)^n (\zeta_0 \tilde{a}_1 x)^n}{\tilde{a}_n n!} ds. \tag{64}$$

Further,

$$\begin{aligned}
\int K_{4,1}^{\dagger}(t,x)\, dG_\theta(x) &= -\frac{b\zeta(\theta)}{2\pi\zeta_0} \int_0^t dy \int_{-1}^1 \exp(-\iota ysb) \sum_{n=0}^{\infty} \frac{(\iota ys)^n}{n!} (\zeta_0 \tilde{a}_1)^n \xi^n(\theta)\, ds \\
&= -\frac{b}{2\pi} \int_0^t dy \int_{-1}^1 \exp(-\iota ysb) \exp(\iota ys\mu(\theta))\, ds \\
&= -\frac{b}{2\pi} \int_0^t dy \int_{-1}^1 \exp[\iota ys \{\mu(\theta) - b\}]\, ds.
\end{aligned}$$

Set $K_1^{\dagger}(t,x) = K_{4,0}^{\dagger}(t,x) + K_{4,1}^{\dagger}(t,x)$. Then

$$K_1^{\dagger}(t,x) = \frac{1}{2\pi\zeta_0} \int_0^1 t dy \int_{-1}^1 \exp(-\iota tysb) \sum_{n=0}^{\infty} \frac{(\iota tys)^n (\zeta_0 \tilde{a}_1 x)^n}{n!} \left(\frac{\zeta_0 \tilde{a}_1 x}{\tilde{a}_{n+1}} - \frac{b}{\tilde{a}_n} \right) ds.$$

and

$$\psi_1(t,\theta) = \int K_1^{\dagger}(t,x)\, dG_\theta(x) = \frac{1}{2\pi} \int_0^t \{\mu(\theta) - b\}\, dy \int_{-1}^1 \exp[\iota ys \{\mu(\theta) - b\}]\, ds.$$

Since $\psi_1(t,\theta)$ is real-valued, we also have $\psi_1(t,\theta) = \int K_1(t,x)\, dG_\theta(x)$, where

$$\begin{aligned}
K_1(t,x) &= \Re\left\{ K_1^{\dagger}(t,x) \right\} \\
&= \frac{1}{2\pi\zeta_0} \int_0^1 t dy \int_{-1}^1 \sum_{n=0}^{\infty} \cos\left(2^{-1}\pi n - tysb\right) \frac{(tys)^n (\zeta_0 \tilde{a}_1 x)^n}{n!} \left(\frac{\zeta_0 \tilde{a}_1 x}{\tilde{a}_{n+1}} - \frac{b}{\tilde{a}_n} \right) ds.
\end{aligned}$$

Now set $K(t,x) = 2^{-1} - K_1(t,x) - 2^{-1} K_{3,0}(t,x;\theta_b)$ with

$$K_{3,0}(t,x;\theta_b) = \frac{1}{\zeta_0} \int_{[-1,1]} \sum_{n=0}^{\infty} \frac{(-tsx)^n \cos\left\{\frac{\pi}{2} n + ts\xi(\theta_b)\right\}}{\tilde{a}_n n!} \omega(s)\, ds$$

given by Theorem 1. Then

$$\psi(t,\theta)=\int K(t,x)\,dG_\theta(x)=2^{-1}-\int_0^t\{\mu(\theta)-b\}\,dy\int_{-1}^1\exp[\iota ys\{\mu(\theta)-b\}]\,ds$$
$$-2^{-1}\int_{[-1,1]}\cos[ts\{\xi(\theta_b)-\xi(\theta)\}]\,\omega(s)\,ds.$$

By Theorem 1 the pair (K,ψ) in (18) is as desired. □

C.2 Proof of Theorem 5

We need the following:

Lemma 8 *For a fixed $\sigma>0$, let*

$$\tilde{w}_2(t,x)=\Gamma(\sigma)\sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{x^{n+1}}{\Gamma(\sigma+n+1)}\text{ for } t,x>0.$$

If Z has CDF G_θ from the Gamma family with scale parameter σ, then

$$\mathbb{E}\left[\tilde{w}_2^2(z,Z)\right]\le\frac{Cz^{3/4-\sigma}}{(1-\theta)^{11/4-\sigma}}\exp\left(\frac{8z/\sqrt{2}}{1-\theta}\right)\tag{65}$$

for positive and sufficiently large z.

The proof of Lemma 8 is provided in Section C.3. Now we provide the arguments. First, we obtain an upper bound on $\mathbb{V}\{\hat{\varphi}_m(t,\mathbf{z})\}$. Note that $\zeta_0=1$ and $\tilde{a}_1=\sigma$. For $y\in[0,1]$ and $t,x>0$, define

$$w_{3,1}(t,x,y)=\Gamma(\sigma)\sum_{n=0}^{\infty}\cos\left(2^{-1}\pi n-tyb\right)\frac{(ty)^n}{n!}\frac{(\sigma x)^{n+1}}{\Gamma(\sigma+n+1)}$$

and

$$w_{3,2}(t,x,y)=\Gamma(\sigma)\sum_{n=0}^{\infty}\cos\left(2^{-1}\pi n-tyb\right)\frac{(ty)^n(\sigma x)^n}{n!\Gamma(\sigma+n)}.$$

Then, uniformly for $s\in[-1,1]$ and $y\in[0,1]$,

$$|w_{3,1}(ts,x,y)|\le\tilde{w}_{3,1}(t\sigma,x)=\sigma\Gamma(\sigma)\sum_{n=0}^{\infty}\frac{|t\sigma|^n}{n!}\frac{|x|^{n+1}}{\Gamma(\sigma+n+1)}\tag{66}$$

and

$$|w_{3,2}(ts,x,y)|\le\tilde{w}_{3,2}(t\sigma,x)=\Gamma(\sigma)\sum_{n=0}^{\infty}\frac{|t\sigma|^n|x|^n}{n!\Gamma(\sigma+n)}.\tag{67}$$

Notice $\tilde{a}_n=\Gamma(n+\sigma)/\Gamma(\sigma)$. Recall the functions $K_{4,0}^{\dagger}(t,x)$ and $K_{4,1}^{\dagger}(t,x)$ defined by (63) and (64) in the proof of Theorem 4 such that $K_1(t,x)=\Re\left\{K_{4,0}^{\dagger}(t,x)\right\}+\Re\left\{K_{4,1}^{\dagger}(t,x)\right\}$. Let $K_{4,0}(t,x)=\Re\left\{K_{4,0}^{\dagger}(t,x)\right\}$ and $K_{4,1}(t,x)=\Re\left\{K_{4,1}^{\dagger}(t,x)\right\}$. Then

$$K_{4,0}(t,x)=\frac{1}{2\pi}\int_0^1 tdy\int_{-1}^1 w_{3,1}(ts,x,y)\,dy$$

and

$$K_{4,1}(t,x)=\frac{-b}{2\pi}\int_0^1 tdy\int_{-1}^1 w_{3,2}(ts,x,y)\,dy.$$

Set $\hat{S}_{3,m,1}(ts,y,\mathbf{z})=m^{-1}\sum_{i=1}^m w_{3,1}(ts,z_i,y)$, $\hat{S}_{3,m,2}(ts,y,\mathbf{z})=-bm^{-1}\sum_{i=1}^m w_{3,2}(ts,z_i,y)$ and

$$\hat{S}_{3,m}(ts,y,\mathbf{z})=\hat{S}_{3,m,1}(ts,y,\mathbf{z})+\hat{S}_{3,m,2}(ts,y,\mathbf{z}).$$

Recall $\hat{\varphi}_{1,m}(t,\mathbf{z})=m^{-1}\sum_{i=1}^m K_1(t,z_i)$ and $\varphi_{1,m}(t,\boldsymbol{\theta})=m^{-1}\sum_{i=1}^m\mathbb{E}\{K_1(t,z_i)\}$. Then

$$\begin{aligned}\hat{\varphi}_{1,m}(t,\mathbf{z}) &= m^{-1}\sum_{i=1}^{m} K_1(t,z_i) = m^{-1}\sum_{i=1}^{m} K_{4,0}(t,z_i) + m^{-1}\sum_{i=1}^{m} K_{4,1}(t,z_i)\\ &= m^{-1}\sum_{i=1}^{m}\frac{1}{2\pi}\int_0^1 tdy\int_{-1}^{1} w_{3,1}(ts,z_i,y)\,dy + m^{-1}\sum_{i=1}^{m}\frac{-b}{2\pi}\int_0^1 tdy\int_{-1}^{1} w_{3,2}(ts,x,y)\,dy\\ &= \frac{t}{2\pi}\int_0^1 dy\int_{-1}^{1}\left[\hat{S}_{3,m,1}(ts,y,\mathbf{z})\,dy + \hat{S}_{3,m,2}(ts,y,\mathbf{z})\right] ds\\ &= \frac{t}{2\pi}\int_0^1 dy\int_{-1}^{1}\hat{S}_{3,m}(ts,y,\mathbf{z})\,ds,\end{aligned}$$

and setting

$$\begin{cases}\mathit{\Delta}_{3,m,j}(ts,y,\mathbf{z}) = \hat{S}_{3,m,j}(ts,y,\mathbf{z}) - \mathbb{E}\left(\hat{S}_{3,m,j}(ts,y,\mathbf{z})\right), j=1,2\\ \mathit{\Delta}_{3,m}(ts,y,\mathbf{z}) = \hat{S}_{3,m}(ts,y,\mathbf{z}) - \mathbb{E}\left(\hat{S}_{3,m}(ts,y,\mathbf{z})\right)\end{cases}$$

gives

$$\begin{aligned}\hat{\varphi}_{1,m}(t,\mathbf{z}) - \varphi_{1,m}(t,\boldsymbol{\theta}) &= \frac{t}{2\pi}\int_0^1 dy\int_{-1}^{1}\mathit{\Delta}_{3,m,1}(ts,y,\mathbf{z})\,ds + \frac{t}{2\pi}\int_0^1 dy\int_{-1}^{1}\mathit{\Delta}_{3,m,2}(ts,y,\mathbf{z})\,ds\\ &= \frac{t}{2\pi}\int_0^1 dy\int_{-1}^{1}\mathit{\Delta}_{3,m}(ts,y,\mathbf{z})\,ds.\end{aligned}$$

Therefore, using the same technique that obtained (56), we get

$$\begin{aligned}\mathbb{V}\{\hat{\varphi}_{1,m}(t,\mathbf{z})\} &\le \frac{t^2}{2\pi^2}\mathbb{E}\left(\int_0^1 dy\int_{-1}^{1}\mathit{\Delta}^2_{3,m}(ts,y,\mathbf{z})\,ds\right)\\ &= \frac{t^2}{2\pi^2}\int_0^1 dy\int_{-1}^{1}\mathbb{E}\left(\mathit{\Delta}^2_{3,m}(ts,y,\mathbf{z})\right) ds\\ &\le \frac{t^2}{\pi^2}\int_0^1 dy\int_{-1}^{1}\left[\mathbb{E}\left(\mathit{\Delta}^2_{3,m,1}(ts,y,\mathbf{z})\right) + \mathbb{E}\left(\mathit{\Delta}^2_{3,m,2}(ts,y,\mathbf{z})\right)\right] ds,\end{aligned} \tag{68}$$

where to obtain the first inequality in (68) we have used the fact (due to Hõlder's inequality)

$$\begin{aligned}&\mathbb{E}\left[\left(\int_{a_1}^{b_1} dy\int_{a_2}^{b_2}|X(s,y)|\,ds\right)^2\right]\\ &\le \mathbb{E}\left[\left(\int_{a_1}^{b_1} dy\sqrt{b_2-a_2}\left[\int_{a_2}^{b_2}|X(s,y)|^2\,ds\right]^{1/2}\right)^2\right]\\ &\le \prod\nolimits_{j=1}^{2}(b_j-a_j)\,\mathbb{E}\left(\int_{a_1}^{b_1} dy\int_{a_2}^{b_2}|X(s,y)|^2\,ds\right)\\ &= \prod\nolimits_{j=1}^{2}(b_j-a_j)\left(\int_{a_1}^{b_1} dy\int_{a_2}^{b_2}\mathbb{E}\left(|X(s,y)|^2\right) ds\right)\end{aligned} \tag{69}$$

for a random variable $X(s,y)$ with parameters (s,y) and finite constants $a_j < b_j$ with $j=1,2$, and to obtain the second inequality in (68) we have used the fact $(a_* + b_*)^2 \le 2a_*^2 + 2b_*^2$ for $a_*, b_* \in \mathbb{R}$. Note that $\mathbb{V}\left(\hat{S}_{3,m,j}(ts,y,\mathbf{z})\right) = \mathbb{E}\left(\mathit{\Delta}^2_{3,m,j}(ts,y,\mathbf{z})\right)$ for $j=1,2$. By the inequalities (66), (67), (55) and Lemma 8, we have, for $t>0$ sufficiently large,

$$\begin{aligned}\mathbb{V}\left(\hat{S}_{3,m,1}(ts,y,\mathbf{z})\right) &\le \frac{1}{m^2}\sum_{i=1}^{m}\mathbb{E}\left\{\tilde{w}^2_{3,1}(t\sigma,z_i)\right\} \le \frac{C}{m^2}\sum_{i=1}^{m}\frac{t^{3/4-\sigma}}{(1-\theta_i)^{11/4-\sigma}}\exp\left(\frac{8\sigma t/\sqrt{2}}{1-\theta_i}\right)\\ &\le V_{3,1,m} = \frac{C}{m^2}\exp\left(\frac{8\sigma t/\sqrt{2}}{u_{3,m}}\right)\sum_{i=1}^{m}\frac{t^{3/4-\sigma}}{(1-\theta_i)^{11/4-\sigma}}\end{aligned} \tag{70}$$

and

$$\begin{aligned}\mathbb{V}\left(\hat{S}_{3,m,2}(ts,y,\mathbf{z})\right) &\le \frac{b^2}{m^2}\sum_{i=1}^{m}\mathbb{E}\left\{\tilde{w}^2_{3,2}(t\sigma,z_i)\right\} \le \frac{b^2}{m^2}\sum_{i=1}^{m}\frac{t^{3/4-\sigma}}{(1-\theta_i)^{3/4-\sigma}}\exp\left(\frac{4\sigma t}{1-\theta_i}\right)\\ &\le V_{3,2,m} = \frac{C}{m^2}\exp\left(\frac{4\sigma t}{u_{3,m}}\right)\sum_{i=1}^{m}\frac{t^{3/4-\sigma}}{(1-\theta_i)^{3/4-\sigma}},\end{aligned} \tag{71}$$

where $u_{3,m} = \min_{1 \le i \le m} \{1 - \theta_i\}$. Combining (68), (70) and (71) gives, for $t > 0$ sufficiently large,

$$\mathbb{V}\left\{\hat{\varphi}_{1,m}\left(t,\mathbf{z}\right)\right\} \le \frac{Ct^{11/4-\sigma}}{m^2} \exp\left(\frac{4\sqrt{2}\sigma t}{u_{3,m}}\right) \sum_{i=1}^{m} l\left(\theta_i,\sigma\right)$$

where

$$l\left(\theta_i,\sigma\right) = \max\left\{\left(1-\theta_i\right)^{\sigma-11/4}, \left(1-\theta_i\right)^{\sigma-3/4}\right\}. \tag{72}$$

Recall

$$K_{3,0}\left(t,x;\theta_b\right) = \frac{\Gamma\left(\sigma\right)}{\zeta_0} \int_{[-1,1]} \sum_{n=0}^{\infty} \frac{\left(-tsx\right)^n \cos\left\{2^{-1}\pi n + ts\xi\left(\theta_b\right)\right\}}{n!\Gamma\left(n+\sigma\right)} \omega\left(s\right) ds.$$

and $\hat{\varphi}_{3,0,m}\left(t,\mathbf{z};\theta_b\right) = m^{-1}\sum_{i=1}^{m} K_{3,0}\left(t,z_i;\theta_b\right)$ and $\varphi_{3,0,m}\left(t,\boldsymbol{\theta};\tau\right) = \mathbb{E}\left\{\hat{\varphi}_{3,0,m}\left(t,\mathbf{z};\theta_b\right)\right\}$. Then Theorem 8 of Chen (2019) asserts, for $t > 0$ sufficiently large,

$$\mathbb{V}\left\{\hat{\varphi}_{3,0,m}\left(t,\mathbf{z};\theta_b\right)\right\} \le \frac{C}{m^2} \exp\left(\frac{4t}{u_{3,m}}\right) \sum_{i=1}^{m} \frac{t^{3/4-\sigma}}{\left(1-\theta_i\right)^{3/4-\sigma}}.$$

Recall $K\left(t,x\right) = 2^{-1} - K_1\left(t,x\right) - 2^{-1}K_{3,0}\left(t,x;\theta_b\right)$. Then, for $t > 0$ sufficiently large,

$$\begin{aligned}\mathbb{V}\left\{\hat{\varphi}_m\left(t,\mathbf{z}\right)\right\} &\le 2\mathbb{V}\left\{\hat{\varphi}_{1,m}\left(t,\mathbf{z}\right)\right\} + 2^{-1}\mathbb{V}\left\{\hat{\varphi}_{3,0,m}\left(t,\mathbf{z};\theta_b\right)\right\} \\ &\le V_{3,m} = \frac{Ct^{11/4-\sigma}}{m^2} \exp\left(\frac{4t\max\left\{1,\sqrt{2}\sigma\right\}}{u_{3,m}}\right) \sum_{i=1}^{m} l\left(\theta_i,\sigma\right),\end{aligned} \tag{73}$$

and

$$V_{3,m} \le V_{3,m}^{*} = \frac{Ct^{11/4-\sigma}}{m^2 u_{3,m}^2} \exp\left(\frac{4t\max\left\{1,\sqrt{2}\sigma\right\}}{u_{3,m}}\right) \sum_{i=1}^{m} \left(1-\theta_i\right)^{\sigma-3/4} \tag{74}$$

since $l\left(\theta_i,\sigma\right)$ in (72) is upper bounded by $Cu_{3,m}^{-2}\left(1-\theta_i\right)^{\sigma-3/4}$ regardless of whether $\liminf_{m\to\infty} u_{3,m} = 0$ or not.

Secondly, we provide a uniform consistency class. When $\sigma \ge 3/4$, then (73) and (74) imply

$$V_{3,m}^{*} \le V_{3,m}^{*\dagger} = \frac{Ct^{11/4-\sigma}}{m u_{3,m}^2} \exp\left(\frac{4t\max\left\{1,\sqrt{2}\sigma\right\}}{u_{3,m}}\right) \left\|1-\boldsymbol{\theta}\right\|_{\infty}^{\sigma-3/4},$$

and that setting $t_m = \left(4\max\left\{1,\sqrt{2}\sigma\right\}\right)^{-1} u_{3,m}\gamma \ln m$ for any fixed $\gamma \in (0,1)$ gives

$$\Pr\left\{\left|\frac{\hat{\varphi}_m\left(t_m,\mathbf{z}\right) - \varphi_m\left(t_m,\boldsymbol{\theta}\right)}{\pi_{1,m}}\right| \ge \varepsilon\right\} \le \frac{Cu_{3,m}^{3/4-\sigma}\left(\ln m\right)^{11/4-\sigma}}{\pi_{1,m}^2 m^{1-\gamma}\varepsilon^2} \left\|1-\boldsymbol{\theta}\right\|_{\infty}^{\sigma-3/4},$$

both for t and t_m sufficiently large. Note that $\gamma = 1$ can be set when $\sigma > 11/4$ since $\lim_{m\to\infty}\left(\ln m\right)^{11/4-\sigma} = 0$ for all such σ. In contrast, when $\sigma \le 3/4$, then (73) and (74) imply

$$V_{3,m}^{*} \le \tilde{V}_{3,m}^{**} = \frac{Ct^{11/4-\sigma}}{m u_{3,m}^2} \exp\left(\frac{4t\max\left\{1,\sqrt{2}\sigma\right\}}{u_{3,m}}\right) u_{3,m}^{\sigma-3/4},$$

and that choosing the same sequence t_m for any fixed $\gamma \in (0,1)$ gives

$$\Pr\left\{\left|\frac{\hat{\varphi}_m\left(t_m,\mathbf{z}\right) - \varphi_m\left(t_m,\boldsymbol{\theta}\right)}{\pi_{1,m}}\right| \ge \varepsilon\right\} \le \frac{C\left(\ln m\right)^{11/4-\sigma}}{\pi_{1,m}^2 m^{1-\gamma}\varepsilon^2},$$

both for t and t_m sufficiently large.

Recall

$$\psi_{3,0}\left(t,\theta;\theta_b\right) = \int_{[-1,1]} \cos\left[ts\left\{\xi\left(\theta_b\right) - \xi\left(\theta\right)\right\}\right] \omega\left(s\right) ds,$$

which is exactly

$$\tilde{\psi}_{1,0}\left(t,\mu;\mu'\right) = \int_{[-1,1]} \omega\left(s\right) \cos\left\{ts\sigma^{-1}\left(\mu-\mu'\right)\right\} ds$$

that is defined by (44) (in Lemma 5) but evaluated at $\mu' = b$ since $\xi(\theta) = (1-\theta)^{-1}$ and $\mu(\theta) = \sigma(1-\theta)^{-1} = \sigma\xi(\theta)$. Also recall

$$\begin{cases} \psi(t,\theta) = 2^{-1} - \psi_1(t,\theta) - 2^{-1}\psi_{3,0}(t,\theta;\theta_b) \\ \psi_1(t,\theta) = \frac{1}{\pi}\int_0^t \frac{\sin\{(\mu(\theta)-b)y\}}{y}dy \end{cases},$$

$\check{u}_{3,m} = \min_{\{j:\theta_j\neq\theta_b\}} |\xi(\theta_b) - \xi(\theta_j)|$, and $\tilde{u}_m(b) = \min_{\{j:\mu_j\neq b\}} |\mu_j - b|$ (the last of which is defined in Lemma 6), where $\mu_i = \mu(\theta_i), i = 1,\dots,m$. We have $\tilde{u}_m(b) = \sigma\check{u}_{3,m}$. Due to the differentiable, monotonic, bijection, $\mu(\theta) = \sigma(1-\theta)^{-1}$, between the mean $\mu = \mu(\theta)$ and natural parameter $\theta = \theta(\mu)$ for the Gamma family, Lemma 6 implies, when $t\tilde{u}_m(b) = t\sigma\check{u}_{3,m} \geq 2$,

$$\left|\pi_{1,m}^{-1}\varphi_m(t,\boldsymbol{\theta}) - 1\right| \leq \frac{C}{t\check{u}_{3,m}\pi_{1,m}}.$$

So, $\pi_{1,m}^{-1}\varphi_m(t_m,\boldsymbol{\theta}) \to 1$ when $t_m^{-1}\check{u}_{3,m}^{-1} = o(\pi_{1,m})$. Therefore, a uniform consistency class is

$$\mathcal{Q}(\mathcal{F}) = \left\{ \begin{array}{c} t_m = \left(4\max\left\{1,\sqrt{2\sigma}\right\}\right)^{-1} u_{3,m}\gamma\ln m, t_m^{-1}\check{u}_{3,m}^{-1} = o(\pi_{1,m}), \\ t_m\to\infty, u_{3,m}^{3/4-\sigma}(\ln m)^{11/4-\sigma}\|1-\boldsymbol{\theta}\|_\infty^{\sigma-3/4} = o\left(\pi_{1,m}^2 m^{1-\gamma}\right) \end{array} \right\}$$

when $\sigma \geq 3/4$ for each $\gamma\in(0,1)$, for which $\gamma = 1$ can be set when $\sigma > 11/4$,, and it is

$$\mathcal{Q}(\mathcal{F}) = \left\{ \begin{array}{c} t_m = \left(4\max\left\{1,\sqrt{2\sigma}\right\}\right)^{-1} u_{3,m}\gamma\ln m, t_m^{-1}\check{u}_{3,m}^{-1} = o(\pi_{1,m}), \\ t_m\to\infty, (\ln m)^{11/4-\sigma} = o\left(\pi_{1,m}^2 m^{1-\gamma}\right) \end{array} \right\}$$

when $\sigma \leq 3/4$ for each $\gamma\in(0,1)$. □

C.3 Proof of Lemma 8

Recall (54), i.e.,

$$\tilde{w}(z,x) = \sum_{n=0}^{\infty}\frac{(zx)^n}{n!\Gamma(\sigma+n)} \quad \text{for } z,x>0.$$

From the proof of Lemma 4 of Chen (2019), we have

$$\tilde{w}(z,x) = (zx)^{\frac{1}{4}-\frac{\sigma}{2}}\exp\left(2\sqrt{zx}\right)\left[1+O\left\{(zx)^{-1}\right\}\right]$$

when $zx\to\infty$. So, when $zx\to\infty$,

$$\begin{aligned}\tilde{w}_2(z,x) &= \Gamma(\sigma)\sum_{n=0}^{\infty}\frac{z^n}{n!}\frac{x^{n+1}}{\Gamma(\sigma+n+1)} \\ &\leq \Gamma(\sigma)\,x\,(zx)^{\frac{1}{4}-\frac{\sigma}{2}}\exp\left(2\sqrt{zx}\right)\left[1+O\left\{(zx)^{-1}\right\}\right].\end{aligned}$$

Note $f_\theta(x) \leq C(1-\theta)^\sigma x^{\sigma-1}$ for all $\theta<1$ and $x>0$. Pick a constant $A>0$ such that

$$\tilde{w}_2(z,x) \leq 2\Gamma(\sigma)x(zx)^{\frac{1}{4}-\frac{\sigma}{2}}\exp\left(2\sqrt{zx}\right) \quad \text{for all } zx > A,$$

and define $A_{1,z} = \left[0, Az^{-1}\right]$ and $A_{2,z} = \left(Az^{-1},\infty\right)$ for each fixed $z>0$. Then, $\tilde{w}(z,x)\leq Ce^{zx} = O(1)$ and $\tilde{w}_2(z,x) \leq x\Gamma(\sigma)e^{zx} \leq Cx$ on the set $A_{1,z}$. Therefore,

$$\int_{A_{1,z}}\tilde{w}_2^2(z,x)\,dG_\theta(x) \leq C(1-\theta)^\sigma\int_{A_{1,z}}x^2x^{\sigma-1}dx \leq C(1-\theta)^\sigma z^{-(\sigma+2)}. \tag{75}$$

On the other hand,

$$\begin{aligned}\int_{A_{2,z}}\tilde{w}_2^2(z,x)\,dG_\theta(x) &\leq C\int_{A_{2,z}}x^2(zx)^{\frac{1}{2}-\sigma}\exp\left(4\sqrt{zx}\right)dG_\theta(x) \\ &= C\int_{A_{2,z}}x^2(zx)^{\frac{1}{2}-\sigma}\sum_{n=0}^{\infty}\frac{\left(4\sqrt{zx}\right)^n}{n!}dG_\theta(x) = Cz^{\frac{1}{2}-\sigma}B_3(z),\end{aligned} \tag{76}$$

where

$$B_3(z)=\sum_{n=0}^{\infty}\frac{4^n z^{n/2}}{n!}\tilde{c}^*_{2^{-1}(n+5)} \quad\text{and}\quad \tilde{c}^*_{2^{-1}(n+5)}=\int x^{2^{-1}(n+5)-\sigma}dG_\theta(x).$$

By the formula,

$$\frac{(1-\theta)^\sigma}{\Gamma(\sigma)}\int_0^\infty x^\beta e^{\theta x}x^{\sigma-1}e^{-x}dx=\frac{\Gamma(\beta+\sigma)}{\Gamma(\sigma)}\frac{(1-\theta)^\sigma}{(1-\theta)^{\beta+\sigma}} \quad \text{for } \alpha,\beta>0,$$

we have

$$\tilde{c}^*_{2^{-1}(n+5)}=\frac{\Gamma\left(2^{-1}n+2^{-1}\times 5\right)}{\Gamma(\sigma)}\frac{(1-\theta)^{\sigma-\frac{5}{2}}}{(1-\theta)^{2^{-1}n}}.$$

By Theorem 1 of Karatsuba (2001) that implies "Ramanujan's double inequality" as

$$\left(8x^3+4x^2+x+\frac{1}{100}\right)^{1/6}<\frac{\Gamma(x+1)}{\sqrt{\pi}\left(\frac{x}{e}\right)^x}<\left(8x^3+4x^2+x+\frac{1}{30}\right)^{1/6} \quad \text{for } x\geq 1$$

and which implies Stirling's formula,

$$\begin{aligned}\frac{\Gamma\left(\frac{n+5}{2}\right)}{n!}&\leq C\frac{\sqrt{\pi(n+3)}\left(\frac{n+3}{2}\right)^{\frac{n+3}{2}}}{e^{\frac{n+3}{2}}\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}\leq Ce^{\frac{n}{2}}2^{-\frac{n}{2}}\frac{(n+3)^{n/2}}{n^{n/2}}\frac{(n+3)^{3/2}}{n^{n/2}}\\&\leq Ce^{\frac{n}{2}}2^{-\frac{n}{2}}\frac{(n+3)^{7/4}}{n^{n/2}}\leq C2^{-\frac{n}{4}}\frac{1}{\sqrt{n!}},\forall n\geq 1.\end{aligned}$$

Therefore,

$$B_3(z)\leq C(1-\theta)^{\sigma-\frac{5}{2}}\sum_{n=0}^{\infty}\frac{4^n z^{n/2}2^{-n/4}}{(1-\theta)^{n/2}}\frac{1}{\sqrt{n!}}=C(1-\theta)^{\sigma-\frac{5}{2}}Q^*\left(\frac{16z/\sqrt{2}}{1-\theta}\right), \tag{77}$$

where $Q^*(z)=\sum_{n=0}^{\infty}\frac{z^{n/2}}{\sqrt{n!}}$. By definition (8.01) and identity (8.07) in Chapter 8 of Olver (1974),

$$Q^*(z)=\sqrt{2}(2\pi z)^{1/4}\exp\left(2^{-1}z\right)\left\{1+O\left(z^{-1}\right)\right\}. \tag{78}$$

Combining (76) through (78) gives

$$\int_{A_{2,z}}\tilde{w}_2^2(z,x)\,dG_\theta(x)\leq C(1-\theta)^{\sigma-\frac{5}{2}}z^{\frac{1}{2}-\sigma}\left(\frac{z}{1-\theta}\right)^{1/4}\exp\left(\frac{8z/\sqrt{2}}{1-\theta}\right)$$

for all positive and sufficiently large z. Recall (75). Thus, when $1-\theta>0$, $\sigma>0$ and z is positive and sufficiently large,

$$\begin{aligned}\mathbb{E}\left[\tilde{w}_2^2(z,Z)\right]&\leq\int_{A_{1,z}}\tilde{w}_2^2(z,x)\,dG_\theta(x)+\int_{A_{2,z}}\tilde{w}_2^2(z,x)\,dG_\theta(x)\\&\leq C\left\{(1-\theta)^\sigma z^{-(\sigma+2)}+\frac{z^{3/4-\sigma}}{(1-\theta)^{11/4-\sigma}}\exp\left(\frac{8z/\sqrt{2}}{1-\theta}\right)\right\}\\&\leq\frac{Cz^{3/4-\sigma}}{(1-\theta)^{11/4-\sigma}}\exp\left(\frac{8z/\sqrt{2}}{1-\theta}\right).\end{aligned}$$

□

D Proofs Related to the Extension

D.1 Proof of Theorem 6

Recall $\tilde{c}_n(\theta)=\int x^n dG_\theta(x)=\zeta_0\xi^n(\theta)\tilde{a}_n$ and $\mu(\theta)=\zeta_0\xi(\theta)\tilde{a}_1$ and $\zeta_0=1$. Define

$$K_1^\dagger(t,x)=\frac{t}{2\pi\zeta_0}\int_a^b\phi(y)\,dy\int_{-1}^{1}\exp(-\iota tsy)\sum_{n=0}^{\infty}\frac{(\iota tsx\zeta_0\tilde{a}_1)^n}{\tilde{a}_n n!}ds.$$

Then,

$$\begin{aligned}\psi_1(t,\theta) &= \int K_1^{\dagger}(t,x)\,dG_\theta(x) \\ &= \frac{t}{2\pi\zeta_0}\int_{-1}^{1}\hat{\phi}(ts)\sum_{n=0}^{\infty}\frac{(\iota ts)^n}{\tilde{a}_n n!}(\zeta_0\tilde{a}_1)^n\tilde{c}_n(\theta)\,ds \\ &= \frac{t}{2\pi}\int_a^b\phi(y)\,dy\int_{-1}^{1}\exp(-\iota tsy)\sum_{n=0}^{\infty}\frac{(\iota ts)^n}{n!}\mu^n(\theta)\,ds \\ &= \frac{t}{2\pi}\int_a^b\phi(y)\,dy\int_{-1}^{1}\exp[\iota ts\{\mu(\theta)-y\}]\,ds.\end{aligned}$$

Since ψ_1 is real, $\psi_1=\mathbb{E}\left\{\Re\left(K_1^{\dagger}\right)\right\}$. However,

$$K_1(t,x)=\Re\left\{K_1^{\dagger}(t,x)\right\}=\frac{t}{2\pi\zeta_0}\int_a^b\phi(y)\,dy\int_{-1}^{1}\sum_{n=0}^{\infty}\frac{(tsx\zeta_0\tilde{a}_1)^n\cos\left(2^{-1}n\pi-tsy\right)}{\tilde{a}_n n!}ds.$$

By Theorem 1, the pair (K,ψ) in (29) is as desired. □

D.2 Proof of Theorem 7

The proof uses almost identical arguments as those for the proof of Theorem 3. Take $t>0$. Recall

$$K_1(t,x)=\frac{t}{2\pi}\int_a^b\phi(y)\,dy\int_{-1}^{1}\sum_{n=0}^{\infty}\frac{(tsx\tilde{a}_1)^n\cos\left(2^{-1}n\pi-tsy\right)}{\tilde{a}_n n!}ds$$

with $\tilde{a}_1=\sigma$ and $\tilde{a}_n=\Gamma(n+\sigma)/\Gamma(\sigma)$ and

$$\psi_1(t,\theta)=\int K_1(t,x)\,dG_\theta(x)=\mathcal{D}_\phi(t,\mu(\theta);a,b)=\frac{1}{\pi}\int_a^b\frac{\sin\{(\mu(\theta)-y)\,t\}}{\mu(\theta)-y}\phi(y)\,dy.$$

Take $t>0$ to be sufficiently large. Recall the following from the proof of Theorem 3:

$$w_1(t,x,y)=\Gamma(\sigma)\sum_{n=0}^{\infty}\frac{(tx\sigma)^n\cos\left(2^{-1}n\pi-ty\right)}{n!\,\Gamma(n+\sigma)}\quad\text{for } t\geq 0 \text{ and } x>0,$$

and $S_{1,m}(t,y)=m^{-1}\sum_{i=1}^{m}[w_1(t,z_i,y)-\mathbb{E}\{w_1(t,z_i,y)\}]$. Then

$$K_1(t,x)=\frac{t}{2\pi}\int_a^b\phi(y)\,dy\int_{-1}^{1}w_1(ts,x,y)\,ds$$

and

$$\hat{\varphi}_{1,m}(t,\mathbf{z})-\mathbb{E}(\hat{\varphi}_{1,m}(t,\mathbf{z}))=\frac{t}{2\pi}\int_a^b\phi(y)\,dy\int_{-1}^{1}S_{1,m}(ts,y)\,ds.$$

So,

$$\mathbb{V}\{\hat{\varphi}_{1,m}(t,\mathbf{z})\}\leq\|\phi\|_\infty^2\,\tilde{V}_{1,m},\tag{79}$$

where as in the proof of Theorem 3

$$\tilde{V}_{1,m}=\mathbb{E}\left[\left\{\frac{1}{2\pi}\int_a^b tdy\int_{-1}^{1}S_{1,m}(ts,y)\,ds\right\}^2\right]\leq\frac{Ct^2}{m}\frac{1}{m}\exp\left(\frac{4t\sigma}{u_{3,m}}\right)\sum_{i=1}^{m}\left(\frac{t}{1-\theta_i}\right)^{3/4-\sigma}$$

and $u_{3,m}=\min_{1\leq i\leq m}\{1-\theta_i\}$.

From the proof of Theorem 3, recall, for $\tau\in\{a,b\}$,

$$K_{3,0}(t,x;\theta_\tau)=\Gamma(\sigma)\int_{[-1,1]}\sum_{n=0}^{\infty}\frac{(-tsx)^n\cos\left\{2^{-1}\pi n+ts\xi(\theta_\tau)\right\}}{n!\,\Gamma(n+\sigma)}\omega(s)\,ds$$

and $\hat{\varphi}_{3,0,m}(t,\mathbf{z};\tau)=m^{-1}\sum_{i=1}^{m}K_{3,0}(t,z_i;\theta_\tau)$. Since

$$\begin{cases}K(t,x)=K_1(t,x)-2^{-1}\{\phi(a)K_{3,0}(t,x;\theta_a)+\phi(b)K_{3,0}(t,x;\theta_b)\}\\ \psi(t,\mu)=\psi_1(t,\mu)-2^{-1}\{\phi(a)\psi_{3,0}(t,\mu;\theta_a)+\phi(b)\psi_{3,0}(t,\mu;\theta_b)\}\end{cases},$$

then the bound derived in the proof of Theorem 3, i.e.,

$$\mathbb{V}\left\{\hat{\varphi}_{3,0,m}\left(t,\mathbf{z};\tau\right)\right\} \leq Cm^{-1}V_{0,m} \quad \text{with} \quad V_{0,m} = \frac{1}{m}\exp\left(\frac{4t}{u_{3,m}}\right)\sum_{i=1}^{m}\frac{t^{3/4-\sigma}}{\left(1-\theta_i\right)^{3/4-\sigma}}$$

together with (79), implies

$$\mathbb{V}\left\{\hat{\varphi}_m\left(t,\mathbf{z}\right)\right\} \leq V_{2,m}^{\dagger} = \frac{C\left\|\phi\right\|_{\infty}^{2}\left(1+t^2\right)}{m^2}\exp\left(\frac{4t\max\left\{\sigma,1\right\}}{u_{3,m}}\right)\sum_{i=1}^{m}\left(\frac{t}{1-\theta_i}\right)^{3/4-\sigma}. \tag{80}$$

So, when $\sigma \geq 3/4$, (80) implies

$$\Pr\left\{\frac{\left|\hat{\varphi}_m\left(t_m,\mathbf{z}\right)-\varphi_m\left(t_m,\boldsymbol{\theta}\right)\right|}{\breve{\pi}_{0,m}} \geq \varepsilon\right\} \leq \frac{C\left\|1-\boldsymbol{\theta}\right\|_{\infty}^{\sigma-3/4}}{\varepsilon^2 m^{1-\gamma}\breve{\pi}_{0,m}^{2}}\left(u_{3,m}\ln m\right)^{11/4-\sigma} \tag{81}$$

by setting $t_m = 4^{-1}\left(\max\left\{\sigma,1\right\}\right)^{-1}u_{3,m}\gamma\ln m$ for any fixed $\gamma \in (0,1)$, whereas, when $\sigma \leq 3/4$, (80) implies

$$\Pr\left\{\frac{\left|\hat{\varphi}_m\left(t_m,\mathbf{z}\right)-\varphi_m\left(t_m,\boldsymbol{\theta}\right)\right|}{\breve{\pi}_{0,m}} \geq \varepsilon\right\} \leq \frac{C\left(\ln m\right)^{11/4-\sigma}u_{3,m}^{2}}{\varepsilon^2 m^{1-\gamma}\breve{\pi}_{0,m}^{2}} \tag{82}$$

by setting $t_m = 4^{-1}u_{3,m}\gamma\ln m$ for any fixed $\gamma \in (0,1)$, both for t_m sufficiently large. Note that $\gamma = 1$ can be set in (81) when $\sigma > 11/4$ since $\lim_{m\to\infty}\left(\ln m\right)^{11/4-\sigma} = 0$ for all such σ.

Finally, recall $\tilde{u}_{3,m} = \min_{\tau\in\{a,b\}}\min_{\{j:\theta_j\neq\theta_\tau\}}\left|\xi\left(\theta_\tau\right)-\xi\left(\theta_i\right)\right|$ and for $\tau \in \{a,b\}$

$$\psi_{3,0}\left(t,\theta;\theta_\tau\right) = \int_{[-1,1]}\cos\left[ts\left\{\xi\left(\theta_\tau\right)-\xi\left(\theta\right)\right\}\right]\omega\left(s\right)ds,$$

which is exactly

$$\tilde{\psi}_{1,0}\left(t,\mu;\mu'\right) = \int_{[-1,1]}\omega\left(s\right)\cos\left\{ts\sigma^{-1}\left(\mu-\mu'\right)\right\}ds$$

that is defined by (44) (in Lemma 5) but evaluated at $\mu' = \tau$ since $\xi\left(\theta\right) = \left(1-\theta\right)^{-1}$ and $\mu\left(\theta\right) = \sigma\left(1-\theta\right)^{-1} = \sigma\xi\left(\theta\right)$. Further, $u_m = \sigma\tilde{u}_{3,m}$. So, Lemma 7, i.e.,

$$\left|\breve{\pi}_{0,m}^{-1}\varphi_m\left(t,\boldsymbol{\mu}\right)-1\right| \leq \frac{C}{t\breve{\pi}_{0,m}}\left(1+\left\|\phi\right\|_{1,\infty}+\frac{1}{u_m}\right)$$

becomes

$$\left|\breve{\pi}_{0,m}^{-1}\varphi_m\left(t,\boldsymbol{\theta}\right)-1\right| \leq \frac{C}{t\breve{\pi}_{0,m}}\left(1+\left\|\phi\right\|_{1,\infty}+\frac{1}{\tilde{u}_{3,m}}\right). \tag{83}$$

Thus, $\breve{\pi}_{0,m}^{-1}\varphi_m\left(t_m,\boldsymbol{\mu}\right) \to 1$ when $t_m^{-1}\left(1+\tilde{u}_{3,m}^{-1}\right) = o\left(\breve{\pi}_{0,m}\right)$. Since (81), (82) and (83) asymptotically are identical to (60), (61) and (62) respectively, we obtain from (81), (82) and (83) the claimed uniform consistency class for $\sigma \geq 3/4$ and $\sigma \leq 3/4$ respectively, for which $\gamma = 1$ can be set when $\sigma > 11/4$. □

E Simulation study

We will present a simulation study on the proposed estimators, with a comparison to the "MR" estimator of Meinshausen and Rice (2006) or Storey's estimator of Storey et al. (2004) for the case of a one-sided null. For one-sided null $\Theta_0 = (-\infty, b) \cap U$, when X_0 is an observation from a random variable X with CDF F_μ, $\mu \in U$, its one-sided p-value is computed as $1 - F_b\left(X_0\right)$. We will not include a comparison with the two estimators of Dickhaus (2013); Hoang and Dickhaus (2021b,a), since it is not an aim here to investigate for Gamma random variables whether the definition of randomized p-value of Dickhaus (2013); Hoang and Dickhaus (2021b,a) leads to valid randomized p-values that can be practically computed.

We numerically implement the solution (ψ, K) in two cases as follows: (a) if ψ or K is defined by a univariate integral, then the univariate integral is approximated by a Riemann sum based on an equally spaced partition with norm 0.01 of the corresponding domain of integration; (b) if ψ or K is defined by a double integral, then the double integral is computed as an iterated integral, for which each univariate integral is computed as if it were case (a). We choose norm 0.01 for a partition so as to reduce a bit the computational complexity of the proposed estimators when the number of hypotheses to test is very large. However, we will not explore here how much more accurate these estimators can be when finer partitions are used to obtain the Riemman sums, or explore here which density function $\omega(s)$ on

$[-1,1]$ should be used to give the best performances to the proposed estimators among all continuous densities on $[-1,1]$ that are of bounded variation. By default, we will choose the triangular density $\omega(s) = (1-|s|)\,1_{[-1,1]}(s)$, since numerical evidence in Jin (2008); Chen (2019) shows that this density leads to good performances of the proposed estimators for the setting of a point null.

The MR estimator (designed for continuous p-values) is implemented as follows: let the ascendingly ordered p-values be $p_{(1)} < p_{(2)} < \cdots < p_{(m)}$ for $m > 4$, set $b_m^* = m^{-1/2}\sqrt{2\ln\ln m}$, and define

$$q_i^* = \left(1 - p_{(i)}\right)^{-1}\left\{im^{-1} - p_{(i)} - b_m^*\sqrt{p_{(i)}\left(1-p_{(i)}\right)}\right\};$$

then $\hat{\pi}_{1,m}^{\mathsf{MR}} = \min\left\{1, \max\left\{0, \max_{2\le i\le m-2} q_i^*\right\}\right\}$ is the MR estimator. Storey's estimator will be implemented by the qvalue package (version 2.14.1) via the 'pi0.method=smoother' option. All simulations will be done with R version 3.5.0.

For an estimator $\hat{\pi}_{1,m}$ of $\pi_{1,m}$ or an estimator $\hat{\pi}_{0,m}$ of $\tilde{\pi}_{0,m}$, its accuracy is measured by the excess $\tilde{\delta}_m = \hat{\pi}_{1,m}\pi_{1,m}^{-1} - 1$ or $\tilde{\delta}_m = \hat{\pi}_{0,m}\tilde{\pi}_{0,m}^{-1} - 1$. For each experiment, the mean μ_m^* and standard deviation σ_m^* of $\tilde{\delta}_m$ is estimated from independent realizations of the experiment. Among two estimators, the one that has smaller σ_m^* is taken to be more stable, and the one that has both smaller σ_m^* and smaller $|\mu_m^*|$ is better. In each boxplot in each figure of simulation results to be presented later, the horizontal bar has been programmed to represent the mean of the quantity being plotted and the black dots represent the outliers from the quantity being plotted.

E.1 Simulation design and results

For $a < b$, let $\mathsf{U}(a,b)$ be the uniform distribution on the closed interval $[a,b]$. When implementing the estimator in Theorem 2 or Theorem 4, the power series in the definition of K in (13) or (18) is replaced by the partial sum of its first 26 terms, i.e., the power series is truncated at $n = 25$. However, the double integral in K in (13) or (18) has to be approximated by a Riemann sum (using the scheme described in the beginning of Appendix E) for each z_i for a total of m times. This greatly increases the computational complexity of applying K to $\{z_i\}_{i=1}^m$ when m is very large. So, we only consider 4 values for m, i.e., $m = 10^3$, 5×10^3, 10^4 or 5×10^4, together with 2 sparsity levels $\pi_{1,m} = 0.2$ (indicating the "dense regime") or $(\ln\ln m)^{-1}$ (indicating the "moderately sparse regime"). We set $\sigma = 4$ for the simulated Gamma random variables. The speed of the proposed estimators $t_m = \sqrt{0.25\sigma^{-1}u_{3,m}\ln m}$ (i.e., $\gamma = 1$ is set for t_m) for a bounded null and $t_m = 2^{-5/4}\sigma^{-1/2}\sqrt{u_{3,m}\ln m}$ (i.e., $\gamma = 1$ is set for t_m) for a one-side null, both with $u_{3,m} = 0.2/\ln\ln m$, so that the consistency conditions in Theorem 3 and Theorem 5 are satisfied. The simulated data are generated as follows:

- Scenario I "estimating $\pi_{1,m}$ for a bounded null": set $\theta_a = 0$, $\theta_b = 0.35$, $\theta_* = -0.2$ and $\theta^* = 0.55$; generate m_0 θ_i's independently from $\mathsf{U}(\theta_a + u_{3,m}, \theta_b - u_{3,m})$, m_{11} θ_i's independently from $\mathsf{U}(\theta_b + u_{3,m}, \theta^*)$, and m_{11} θ_i's independently from $\mathsf{U}(\theta_*, \theta_a - u_{3,m})$, where $m_{11} = \max\{1, \lfloor 0.5m_1\rfloor - \lfloor m/\ln\ln m\rfloor\}$ and $\lfloor x\rfloor$ is the integer part of $x \in \mathbb{R}$; set half of the remaining $m - m_0 - 2m_{11}$ θ_i's to be θ_a, and the rest to be θ_b.
- Scenario II "estimating $\pi_{1,m}$ for a one-sided null": generate m_0 μ_i's independently from $\mathsf{U}(\theta_*, \theta_b - u_{3,m})$, and $\lfloor 0.9m_1\rfloor$ μ_i's independently from $\mathsf{U}(\theta_b + u_{3,m}, \theta^*)$; set the rest θ_i's to be θ_b.

Each triple of $(m, \pi_{1,m}, \Theta_0)$ determines an experiment, and there are 16 experiments in total. Each experiment is repeated independently 100 times. The assessment method for an estimator $\hat{\pi}_{1,m}$ of $\pi_{1,m}$ is again based on the mean and standard deviation of the excess $\tilde{\delta}_m = \hat{\pi}_{1,m}\pi_{1,m}^{-1} - 1$. As mentioned earlier in this section, to numerically approximate K and hence the new estimators (since they are defined by integrals and power series), we computed a 26-term partial sum of each of those power series and computed Riemann sums based on an equally spaced one-dimensional domain partition with norm 0.01 for those integrals. Namely, the new estimators are implemented by this scheme of numerical approximation, which we call "numerical versions". Even with this simple approximation scheme of relatively low computational complexity, the sequential nature of computing an approximation to K and evaluating this approximation at each of the m observations via "for" loops and the sequential nature of repeating an experiment via a "for" loop took much time, and the simulations took around 45 days to complete on a computer with 8-core CPU and 64GB of RAM. Note that this numerical implementation/approximation causes numerical error and that the simulations are for the "numerical versions" of the new estimators rather than the new estimators themselves.

Figure F.1 visualizes the simulation results, for which Storey's estimator is not shown since it is always 0 for all experiments in Scenario II, and Table 1 provides numerical summaries that complement the visualizations in Figure F.1. Please note again that these results are for the numerical implementation, i.e., numerical approximation, of the new estimators, rather than the new estimators themselves, even though the interpretations of the results will be for the new estimators. The following three observations can be made: (i) for estimating the alternative proportion for a one-sided null, the proposed estimator is more accurate than the MR estimator, is very stable, and in the dense regime shows a clear trend of convergence towards consistency. In contrast, the MR estimator is always very close to 0 regardless of the sparsity regime for $\pi_{1,m}$, either failing to detect the existence of alternative hypotheses or very inaccurately estimating the alternative proportion. This largely explains why for a one-side null in the dense regime, our new estimators have slightly larger standard deviation than the MR estimator (because the latter almost always gives an estimate that is very close to 0). (ii) for estimating the alternative proportion for a bounded null, the proposed estimator is stable and reasonably accurate, and in the dense regime shows a clear trend of convergence towards consistency. (iii) the proposed estimator seems to be much more accurate in the moderately sparse regime than in the dense regime. We remark that the accuracy and speed of convergence of the proposed estimators can be improved by employing more accurate Riemann sums for the integrals and more accurate partial sums of the power series in the computation of the matching function than currently used. (iv) non-asymptotically the new estimator $\hat{\varphi}_m(t, \boldsymbol{\mu})$ of the proportion of false nulls $\pi_{1,m}$ often over-estimates $\pi_{1,m}$, meaning that its dual $\hat{\psi}_m(t, \boldsymbol{\mu})$, which estimates the proportion of true nulls $\pi_{0,m} = 1 - \pi_{1,m}$, usually under-estimates $\pi_{0,m}$. In terms of false discovery rate (FDR) control in nonasymptotic settings, an adaptive FDR procedure that uses the new estimators $\hat{\psi}_m(t, \boldsymbol{\mu})$ may fail to maintain a prespecified nominal FDR, even though such a procedure may have larger power compared to its non-adaptive counterparts.

Now let us explain why in the moderately sparse regime, i.e., $\pi_{1,m} = 1/\ln(\ln m)$, Figure F.1 does not provide numerical evidence that our "New" estimators are consistent but does not undermine our rigorous theory, and why in the dense regime, i.e., $\pi_{1,m} = 0.2$, Figure F.1 provides numerical evidence that our "New" estimators are consistent but not with a fast enough speed of convergence. Recall that we have truncated the power series (that define K and our "New" estimators) to a 26-term finite sum and used Riemannian sums of one-dimensional partition norm 0.01 to approximate integrals (that define K and our "New" estimators) when implementing these estimators, which gives the actual "New" estimators as, e.g., $\hat{\pi}_{1,m}^{\dagger,\text{New}}$. Let $\hat{\pi}_{1,m}^{\text{New}}$ specifically denote our "New" estimators. Then, the numerical error of the "New" estimators $\hat{\pi}_{1,m}^{\text{new}}$ is $\tilde{e}_m^{\text{New}} = \hat{\pi}_{1,m}^{\text{New}} - \hat{\pi}_{1,m}^{\dagger,\text{New}}$.

Recall $\tilde{\delta}_m = \hat{\pi}_{1,m}/\pi_{1,m} - 1$, where $\hat{\pi}_{1,m}$ is an estimate of $\pi_{1,m}$, and that $\tilde{\delta}_m$ converges to 0 as $m \to \infty$ is equivalent to the consistency of the estimator $\hat{\pi}_{1,m}$. Due to our numerical approximation, for our "New" estimator, $\tilde{\delta}_m$ is actually computed as

$$\tilde{\delta}_m = \frac{\hat{\pi}_{1,m}^{\dagger,\text{New}}}{\pi_{1,m}} - 1 = \frac{\hat{\pi}_{1,m}^{\text{New}}}{\pi_{1,m}} - \frac{\tilde{e}_m^{\text{New}}}{\pi_{1,m}} - 1.$$

Since our theory has rigorously proved that $\hat{\pi}_{1,m}^{\text{New}}/\pi_{1,m} - 1$ converges to 0 in probability as $m \to \infty$, we see the actual $\tilde{\delta}_m$ computed for our "New" estimator $\hat{\pi}_{1,m}^{\text{New}}$, as given above, satisfies

$$\tilde{\delta}_m \approx \frac{\tilde{e}_m^{\text{New}}}{\pi_{1,m}} \quad \text{with high probability for large } m.$$

However, the numerical error $\tilde{e}_m^{\text{New}}$ may not converge to 0 as $m \to \infty$. So, in the dense regime when $\pi_{1,m} = 0.2$, we will see a trend of convergence for $\tilde{\delta}_m$ as m increases. But such a convergence may stall if $\tilde{e}_m^{\text{New}}$ does not decrease with m. Non-monotone decreasing or not small enough $\tilde{e}_m^{\text{New}}$ also creates a feeling that $|\tilde{\delta}_m|$ is larger for the "New" estimator than the "MR" estimator, which is not true for all m but may be true for small m. This is exactly what happened for the dense regime in Figure F.1. In contrast, in the moderately sparse regime when $\pi_{1,m} = 1/\ln(\ln m)$, $\pi_{1,m}$ monotonically decreases as m increases and $\pi_{1,m}$ converges to 0 as $m \to \infty$. So, when $\tilde{e}_m^{\text{New}}$ is not of smaller order than $\pi_{1,m} = 1/\ln(\ln m)$ as m increases, we may see the actual $\tilde{\delta}_m$ on average increases with m. This is exactly what happened for the moderately sparse regime in Figure F.1.

Unless we increase the numerical precision or equivalently reduce the numerical error $\tilde{e}_m^{\text{New}}$ (dynamically also with respect to m), increasing m but keeping the current numerical approximation scheme as described earlier will not allow us to see a clear trend of convergence of our "New" estimators in the moderately sparse regime where $\lim_{m\to\infty} \pi_{1,m} = 0$. In fact, how to rigorously and precisely control the numerical error when truncating a power series and using Riemann sums to approximate integrals when implementing our proposed estimator requires very delicate analysis, may well form another manuscript, and unfortunately cannot be fully numerically explored in this work. Nevertheless, for practical applications where we do not need to repeat an experiment many times as is done in the simulations here, we recommend keeping as many terms and using as fine partitions as one's computational recourses allow when respectively truncating the power series and forming Riemann sums that numerically approximate the definitions of the new estimators.

F Estimators for closed or half-closed nulls and their consistency

Let us discuss how to adapt the constructions, the estimators, their concentration inequalities, and their consistency results to estimating the proportion $\pi_{1,m}$ when the null hypotheses are closed or half-closed sets. For the Gamma family, the mean parameter μ is a function of the natural parameter θ, such that $\mu = \mu(\theta) = \sigma(1-\theta)^{-1}$ for $\theta < 1$, and this function is differentiable and strictly monotone with inverse $\theta = \theta(\mu) = 1 - \sigma\mu^{-1}$. Here $\sigma > 0$ is the scale parameter. Further, $\xi(\theta) = (1-\theta)^{-1}$, $\xi(\theta) = \sigma^{-1}\mu(\theta)$, $\mu_i = \mu(\theta_i)$ and θ_τ is such that $\mu(\theta_\tau) = \tau$ for $\tau \in \{a, b\}$.

Recall

$$\begin{cases} \psi_{3,0}(t,\theta;\theta') = \int_{[-1,1]} \cos[ts\{\xi(\theta') - \xi(\theta)\}]\,\omega(s)\,ds \\ \tilde{\psi}_{1,0}(t,\mu;\mu') = \int_{[-1,1]} \omega(s) \cos\{ts\sigma^{-1}(\mu - \mu')\}\,ds \end{cases},$$

where $\tilde{\psi}_{1,0}(t,\mu;\mu')$ is defined in Section A.4. We see that $\mu = \mu(\theta) = \sigma(1-\theta)^{-1}$ implies $\psi_{3,0}(t,\theta;\theta') = \tilde{\psi}_{1,0}(t,\mu;\mu')$. This fact will be used in our discussion on the quantity $\varphi_m(t,\boldsymbol{\theta})$ or its equivalent $\varphi_m(t,\boldsymbol{\mu})$, where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_m)$, $\theta_i = \theta(\mu_i)$, $\theta(\boldsymbol{\mu}) = (\theta(\mu_1), \ldots, \theta(\mu_m))$ and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_m)$.

F.1 The case of a bounded null

When $\Theta_0 = [a,b]$, we can just set

$$\begin{cases} K(t,x) = K_1(t,x) + 2^{-1}\{K_{3,0}(t,x;\theta_a) + K_{3,0}(t,x;\theta_b)\} \\ \psi(t,\theta) = \psi_1(t,\theta) + 2^{-1}\{\psi_{3,0}(t,\theta;\theta_a) + \psi_{3,0}(t,\theta;\theta_b)\} \end{cases} \tag{84}$$

in comparison to the construction when $\Theta_0 = (a,b)$ as

$$\begin{cases} K(t,x) = K_1(t,x) - 2^{-1}\{K_{3,0}(t,x;\theta_a) + K_{3,0}(t,x;\theta_b)\} \\ \psi(t,\theta) = \psi_1(t,\theta) - 2^{-1}\{\psi_{3,0}(t,\theta;\theta_a) + \psi_{3,0}(t,\theta;\theta_b)\} \end{cases}. \tag{85}$$

The definitions of the estimator and its expectation for either $\Theta_0 = (a,b)$ or $\Theta_0 = [a,b]$ remain identical as

$$\hat{\varphi}_m(t,\mathbf{z}) = m^{-1}\sum_{i=1}^m \{1 - K(t,z_i)\} \quad \text{and} \quad \varphi_m(t,\boldsymbol{\theta}) = m^{-1}\sum_{i=1}^m \{1 - \psi(t,\mu_i)\}.$$

When $\Theta_0 = (a,b)$, in the proofs for the estimator $\hat{\varphi}_m(t,\mathbf{z})$, we have used $e_{1,m}(t) := \hat{\varphi}_{1,m}(t,\mathbf{z}) - \varphi_{1,m}(t,\boldsymbol{\theta})$, where

$$\hat{\varphi}_{1,m}(t,\mathbf{z}) = m^{-1}\sum_{i=1}^m K_1(t,z_i) \quad \text{and} \quad \varphi_{1,m}(t,\boldsymbol{\theta}) = \mathbb{E}\{\hat{\varphi}_{1,m}(t,\mathbf{z})\},$$

$e_{3,0,m}(t,\tau) := \hat{\varphi}_{3,0,m}(t,\mathbf{z};\tau) - \varphi_{3,0,m}(t,\boldsymbol{\theta};\tau)$, $\tau \in \{a,b\}$, where

$$\hat{\varphi}_{3,0,m}(t,\mathbf{z};\tau) = m^{-1}\sum_{i=1}^m K_{3,0}(t,z_i;\theta_\tau) \quad \text{and} \quad \varphi_{3,0,m}(t,\boldsymbol{\theta};\tau) = \mathbb{E}\{\hat{\varphi}_{3,0,m}(t,\mathbf{z};\tau)\},$$

$e_m(t) := \hat{\varphi}_m(t,\mathbf{z}) - \varphi_m(t,\boldsymbol{\theta})$ and

$$e_m(t) = -e_{1,m}(t) + 2^{-1}e_{3,0,m}(t,a) + 2^{-1}e_{3,0,m}(t,b). \tag{86}$$

Further, to upper bound the variance of $-e_m(t)$, which is also the variance of $e_m(t)$, we have upper bounded the variances of $e_{1,m}(t)$, $e_{3,0,m}(t,a)$ and $e_{3,0,m}(t,b)$ individually, and then directly replaced each variance in each summand on the right-hand side of the inequality

$$\mathbb{V}[e_m(t)] \leq 2\mathbb{V}\{e_{1,m}(t)\} + \mathbb{V}[e_{3,0,m}(t,a)] + \mathbb{V}[e_{3,0,m}(t,b)] \tag{87}$$

with these individual variance upper bounds. In addition, concentration of $|e_m(t)|$ is derived by Chebyshev's inequality based on the upper bound for the variance of $e_m(t)$.

In the setting of the closed null $\Theta_0 = [a,b]$, (84) implies that (86) becomes

$$e_m(t) = \hat{\varphi}_m(t,\mathbf{z}) - \varphi_m(t,\boldsymbol{\theta}) = -e_{1,m}(t) - 2^{-1}e_{3,0,m}(t,a) - 2^{-1}e_{3,0,m}(t,b). \tag{88}$$

However, (87) remains valid for $e_m(t)$ in (88). So, the upper bound on the variance of $e_m(t)$ and the concentration of $|e_m(t)|$ we have derived for the setting $\Theta_0 = (a,b)$ and construction (85) remain valid for $e_m(t)$ and $|e_m(t)|$ for the setting $\Theta_0 = [a,b]$ and construction (84).

Now let us discuss $\varphi_m(t,\boldsymbol{\mu})$, which is equivalent to $\varphi_m(t,\boldsymbol{\theta})$ due to the mapping $\mu = \mu(\theta) = \sigma(1-\theta)^{-1}$ for $\theta < 1$. When $\Theta_0 = (a,b)$,

$$\varphi_m(t,\boldsymbol{\mu}) = 1 - \varphi_{1,m}(t,\boldsymbol{\mu}) + 2^{-1}\varphi_{1,0,m}(t,\boldsymbol{\mu};a) + 2^{-1}\varphi_{1,0,m}(t,\boldsymbol{\mu};b) = \sum_{i=1}^5 \widetilde{d}_{1,m} \tag{89}$$

and

$$\pi_{1,m}^{-1}\varphi_m(t,\boldsymbol{\mu})-1=\pi_{1,m}^{-1}\widetilde{d}_{1,m}-1+\pi_{1,m}^{-1}\widetilde{d}_{2,m}+\pi_{1,m}^{-1}\widetilde{d}_{3,m}+\pi_{1,m}^{-1}\widetilde{d}_{4,m}+\pi_{1,m}^{-1}\widetilde{d}_{5,m},$$

where

$$\begin{cases}\widetilde{d}_{1,m}=1-m^{-1}\sum_{\{j:\mu_j\in(a,b)\}}\psi_1(t,\mu_j)\\ \widetilde{d}_{2,m}=-m^{-1}\sum_{\{j:\mu_j=a\}}\psi_1(t,\mu_j)+2^{-1}m^{-1}\sum_{\{j:\mu_j=a\}}\tilde{\psi}_{1,0}(t,\mu_j;a)\\ \widetilde{d}_{3,m}=-m^{-1}\sum_{\{j:\mu_j=b\}}\psi_1(t,\mu_j)+2^{-1}m^{-1}\sum_{\{j:\mu_j=b\}}\tilde{\psi}_{1,0}(t,\mu_j;b)\\ \widetilde{d}_{4,m}=2^{-1}m^{-1}\sum_{\{j:\mu_j\neq a\}}\tilde{\psi}_{1,0}(t,\mu_j;a)+2^{-1}m^{-1}\sum_{\{j:\mu_j\neq b\}}\tilde{\psi}_{1,0}(t,\mu_j;b)\\ \widetilde{d}_{5,m}=-m^{-1}\sum_{\{j:\mu_j<a\}}\psi_1(t,\mu_j)-m^{-1}\sum_{\{j:\mu_j>b\}}\psi_1(t,\mu_j)\end{cases}.$$

Further, when $\Theta_0=(a,b)$, to upper bound $\left|\pi_{1,m}^{-1}\varphi_m(t,\boldsymbol{\mu})-1\right|$, we have replaced each $\left|\widetilde{d}_{j,m}\right|, 2\le j\le 5$ by its upper bound $\hat{d}_{j,m}, 2\le j\le 5$ and replaced $\left|\pi_{1,m}^{-1}\widetilde{d}_{1,m}-1\right|$ by its upper bound $\hat{d}_{0,m}$ in the inequality

$$\left|\pi_{1,m}^{-1}\varphi_m(t,\boldsymbol{\mu})-1\right|\le\left|\pi_{1,m}^{-1}\widetilde{d}_{1,m}-1\right|+\pi_{1,m}^{-1}\left|\widetilde{d}_{2,m}\right|+\pi_{1,m}^{-1}\left|\widetilde{d}_{3,m}\right|+\pi_{1,m}^{-1}\left|\widetilde{d}_{4,m}\right|+\pi_{1,m}^{-1}\left|\widetilde{d}_{5,m}\right|. \quad (90)$$

In case $\Theta_0=[a,b]$, (89) becomes

$$\begin{aligned}\varphi_m(t,\boldsymbol{\mu})&=1-\varphi_{1,m}(t,\boldsymbol{\mu})-2^{-1}\varphi_{1,0,m}(t,\boldsymbol{\mu};a)-2^{-1}\varphi_{1,0,m}(t,\boldsymbol{\mu};b)\\&=\widetilde{d}_{1,m}+\widetilde{d}^*_{2,m}+\widetilde{d}^*_{3,m}-\widetilde{d}_{4,m}+\widetilde{d}_{5,m}\end{aligned}$$

and

$$\left|\pi_{1,m}^{-1}\varphi_m(t,\boldsymbol{\mu})-1\right|\le\left|\pi_{1,m}^{-1}\left(\widetilde{d}_{1,m}+\widetilde{d}^*_{2,m}+\widetilde{d}^*_{3,m}\right)-1\right|+\pi_{1,m}^{-1}\left|\widetilde{d}_{4,m}\right|+\pi_{1,m}^{-1}\left|\widetilde{d}_{5,m}\right| \quad (91)$$

where

$$\begin{cases}\widetilde{d}^*_{2,m}=-m^{-1}\sum_{\{j:\mu_j=a\}}\psi_1(t,\mu_j)-2^{-1}m^{-1}\sum_{\{j:\mu_j=a\}}\tilde{\psi}_{1,0}(t,\mu_j;a)\\ \widetilde{d}^*_{3,m}=-m^{-1}\sum_{\{j:\mu_j=b\}}\psi_1(t,\mu_j)-2^{-1}m^{-1}\sum_{\{j:\mu_j=b\}}\tilde{\psi}_{1,0}(t,\mu_j;b)\end{cases}.$$

However,

$$\pi_{1,m}^{-1}\left|\widetilde{d}_{1,m}+\widetilde{d}^*_{2,m}+\widetilde{d}^*_{3,m}-1\right|\le\hat{d}_{0,m}+\pi_{1,m}^{-1}\hat{d}_{2,m}+\pi_{1,m}^{-1}\hat{d}_{3,m}.$$

So, the upper bound for $\left|\pi_{1,m}^{-1}\varphi_m(t,\boldsymbol{\mu})-1\right|$ in (90) when $\Theta_0=(a,b)$ is also an upper bound for $\left|\pi_{1,m}^{-1}\varphi_m(t,\boldsymbol{\mu})-1\right|$ in (91) when $\Theta_0=[a,b]$.

Therefore, all results we have derived for the estimator $\hat{\varphi}_m(t,\mathbf{z})$ when $\Theta_0=(a,b)$ for the construction (85) remain valid for the estimator $\hat{\varphi}_m(t,\mathbf{z})$ when $\Theta_0=[a,b]$ for the construction (84).

F.2 The case of a one-sided null

When $\Theta_0=(-\infty,b]$, we can just set

$$\begin{cases}K(t,x)=2^{-1}-K_1(t,x)+2^{-1}K_{3,0}(t,x;\theta_b)\\ \psi(t,\theta)=2^{-1}-\psi_1(t,\theta)+2^{-1}\psi_{3,0}(t,\theta;\theta_b)\end{cases} \quad (92)$$

in comparison to the construction when $\Theta_0=(-\infty,b)$ as

$$\begin{cases}K(t,x)=2^{-1}-K_1(t,x)-2^{-1}K_{3,0}(t,x;\theta_b)\\ \psi(t,\theta)=2^{-1}-\psi_1(t,\theta)-2^{-1}\psi_{3,0}(t,\theta;\theta_b)\end{cases}. \quad (93)$$

The definitions of the estimator and its expectation for either $\Theta_0=(-\infty,b]$ or $\Theta_0=(-\infty,b)$ remain identical as

$$\hat{\varphi}_m(t,\mathbf{z})=m^{-1}\sum_{i=1}^m\{1-K(t,z_i)\}\quad\text{and}\quad\varphi_m(t,\boldsymbol{\mu})=m^{-1}\sum_{i=1}^m\{1-\psi(t,\mu_i)\}.$$

We will reuse the definitions of $e_m(t)$, $e_{1,m}(t)$ and $e_{3,0,m}(t,b)$ introduced previously in Section F.1. Then $e_m(t)=e_{1,m}(t)+2^{-1}e_{3,m,0}(t,b)$ when $\Theta_0=(-\infty,b)$ becomes

$$e_m(t)=e_{1,m}(t)-2^{-1}e_{3,0,m}(t,b)\ \text{ when } \Theta_0=(-\infty,b].$$

Again, to upper bound the variance of $-e_m(t)$, which is also the variance of $e_m(t)$, we have upper bounded the variances of $e_{1,m}(t)$ and $e_{3,0,m}(t,b)$ individually, and then directly replaced each variance in each summand on the right-hand side of the inequality

$$\mathbb{V}[e_m(t)]\le 2\mathbb{V}\{e_{1,m}(t)\}+2^{-1}\mathbb{V}[e_{3,0,m}(t,b)] \quad (94)$$

with these individual variance upper bounds. In addition, concentration of $|e_m(t)|$ is derived by Chebyshev's inequality based on the upper bound for the variance of $e_m(t)$. However, (94) remains valid for $e_m(t)$ when $\Theta_0 = (-\infty, b]$. So, the upper bound on the variance of $e_m(t)$ and the concentration of $|e_m(t)|$ we have derived for the setting $\Theta_0 = (-\infty, b)$ remain valid for $e_m(t)$ and $|e_m(t)|$ for the setting $\Theta_0 = (-\infty, b]$.

Now let us discuss $\varphi_m(t, \boldsymbol{\mu})$, which is equivalent to $\varphi_m(t, \boldsymbol{\theta})$ due to the mapping $\mu = \mu(\theta) = \sigma(1-\theta)^{-1}$ for $\theta < 1$. When $\Theta_0 = (-\infty, b)$, we have

$$\varphi_m(t, \boldsymbol{\mu}) = 2^{-1} + \varphi_{1,m}(t, \boldsymbol{\mu}) + 2^{-1}\varphi_{1,0,m}(t, \boldsymbol{\mu}; b) = \bar{d}_{1,m} + \bar{d}_{2,m} + \bar{d}_{3,m} + \bar{d}_{4,m},$$

where

$$\begin{cases} \bar{d}_{1,m} = m^{-1}\sum_{\{i:\mu_i>b\}}\left(2^{-1} + \psi_1(t, \mu_i)\right) \\ \bar{d}_{2,m} = m^{-1}\sum_{\{i:\mu_i=b\}}\left(2^{-1} + \psi_1(t, \mu_i) + 2^{-1}\tilde{\psi}_{1,0}(t, \mu_i; b)\right) \\ \bar{d}_{3,m} = m^{-1}\sum_{\{i:\mu_i<b\}}\left(2^{-1} + \psi_1(t, \mu_i)\right) \\ \bar{d}_{4,m} = 2^{-1}m^{-1}\sum_{\{i:\mu_i\neq b\}}\tilde{\psi}_{1,0}(t, \mu_i; b) \end{cases}$$

and specifically $\bar{d}_{2,m} = m^{-1}\sum_{\{i:\mu_i=b\}} 1$. Further, when $\Theta_0 = (-\infty, b)$, to upper bound $\left|\pi_{1,m}^{-1}\varphi_m(t, \boldsymbol{\mu}) - 1\right|$, we have replaced each $\left|\bar{d}_{j,m}\right|, 3 \leq j \leq 4$ by its upper bound and replaced $\left|\pi_{1,m}^{-1}\left(\bar{d}_{1,m} + \bar{d}_{2,m}\right) - 1\right|$ by its upper bound in the inequality

$$\left|\pi_{1,m}^{-1}\varphi_m(t, \boldsymbol{\mu}) - 1\right| \leq \left|\pi_{1,m}^{-1}\left(\bar{d}_{1,m} + \bar{d}_{2,m}\right) - 1\right| + \pi_{1,m}^{-1}\left|\bar{d}_{3,m}\right| + \pi_{1,m}^{-1}\left|\bar{d}_{4,m}\right|,$$

where

$$\pi_{1,m}^{-1}\left(\bar{d}_{1,m} + \bar{d}_{2,m}\right) - 1 = \pi_{1,m}^{-1}m^{-1}\sum_{\{i:\mu_i>b\}}\left(\psi_1(t, \mu_i) - 2^{-1}\right). \tag{95}$$

Specifically, the upper bound on $\left|\pi_{1,m}^{-1}\left(\bar{d}_{1,m} + \bar{d}_{2,m}\right) - 1\right|$ is directly based on the inequality

$$\left|\psi_1(t, \mu_i) - 2^{-1}\right| \leq 2(t\tilde{u}_m)^{-1} \text{ for } \mu_i > b,$$

where $\tilde{u}_m = \min_{\{j:\mu_j\neq b\}}|\mu_j - b|$.

In contrast, when $\Theta_0 = (-\infty, b]$, we have

$$\varphi_m(t, \boldsymbol{\mu}) = 2^{-1} + \varphi_{1,m}(t, \boldsymbol{\mu}) - 2^{-1}\varphi_{1,0,m}(t, \boldsymbol{\mu}; b) = \bar{d}_{1,m} + \bar{d}_{2,m}^{*} + \bar{d}_{3,m} - \bar{d}_{4,m},$$

and

$$\left|\pi_{1,m}^{-1}\varphi_m(t, \boldsymbol{\mu}) - 1\right| \leq \left|\pi_{1,m}^{-1}\bar{d}_{1,m} - 1\right| + \pi_{1,m}^{-1}\left|\bar{d}_{3,m}\right| + \pi_{1,m}^{-1}\left|\bar{d}_{4,m}\right|,$$

where

$$\bar{d}_{2,m}^{*} = m^{-1}\sum_{\{i:\mu_i=b\}}\left(2^{-1} + \psi_1(t, \mu_i) - 2^{-1}\tilde{\psi}_{1,0}(t, \mu_i; b)\right) = 0.$$

However, again

$$\pi_{1,m}^{-1}\bar{d}_{1,m} - 1 = \pi_{1,m}^{-1}m^{-1}\sum_{\{i:\mu_i>b\}}\left(\psi_1(t, \mu_i) - 2^{-1}\right),$$

whose right-hand side is identical for that of (95). Therefore, the upper bound we have derived for $\left|\pi_{1,m}^{-1}\varphi_m(t, \boldsymbol{\mu}) - 1\right|$ when $\Theta_0 = (-\infty, b)$ is also an upper bound for $\left|\pi_{1,m}^{-1}\varphi_m(t, \boldsymbol{\mu}) - 1\right|$ when $\Theta_0 = (-\infty, b]$.

Therefore, results we have derived for the estimator $\hat{\varphi}_m(t, \mathbf{z})$ when $\Theta_0 = (-\infty, b)$ for the construction (93) remain valid for the estimator $\hat{\varphi}_m(t, \mathbf{z})$ when $\Theta_0 = (-\infty, b]$ for the construction (92).

F.3 The case of the extensions

When $\Theta_0 = [a, b]$, we can just set

$$\begin{cases} K(t, x) = K_1(t, x) + 2^{-1}\{\phi(a)K_{3,0}(t, x; a) + \phi(b)K_{3,0}(t, x; b)\} \\ \psi(t, \mu) = \psi_1(t, \mu) + 2^{-1}\{\phi(a)\psi_{3,0}(t, \mu; a) + \phi(b)\psi_{3,0}(t, \mu; b)\} \end{cases} \tag{96}$$

in comparison to the construction for $\Theta_0 = (a, b)$ as

$$\begin{cases} K(t, x) = K_1(t, x) - 2^{-1}\{\phi(a)K_{3,0}(t, x; a) + \phi(b)K_{3,0}(t, x; b)\} \\ \psi(t, \mu) = \psi_1(t, \mu) - 2^{-1}\{\phi(a)\psi_{3,0}(t, \mu; a) + \phi(b)\psi_{3,0}(t, \mu; b)\} \end{cases}. \tag{97}$$

Again the definitions of the estimator and its expectation for either $\Theta_0 = (a, b)$ or $\Theta_0 = [a, b]$ remain identical as

$$\hat{\varphi}_m(t, \mathbf{z}) = m^{-1}\sum_{i=1}^{m} K(t, z_i) \quad \text{and } \varphi_m(t, \boldsymbol{\mu}) = m^{-1}\sum_{i=1}^{m}\psi(t, \mu_i),$$

and we will reuse the definitions of $e_m(t)$, $e_{1,m}(t)$ and $e_{3,0,m}(t,b)$ introduced previously in Section F.1.

For $\Theta_0=(a,b)$ we have

$$e_m(t)=\hat{\varphi}_m(t,\mathbf{z})-\varphi_m(t,\boldsymbol{\mu})=e_{1,m}(t)-2^{-1}\phi(a)\,e_{3,0,m}(t,a)-2^{-1}\phi(a)\,e_{3,0,m}(t,b),$$

whereas for $\Theta_0=[a,b]$ we have

$$e_m(t)=\hat{\varphi}_m(t,\mathbf{z})-\varphi_m(t,\boldsymbol{\mu})=e_{1,m}(t)+2^{-1}\phi(a)\,e_{3,0,m}(t,a)+2^{-1}\phi(a)\,e_{3,0,m}(t,b).$$

Since we have employed the inequality

$$\mathbb{V}[e_m(t)]\leq 2\mathbb{V}\{e_{1,m}(t)\}+\|\omega\|_\infty^2\,\mathbb{V}[e_{1,0,m}(t,a)]+\|\omega\|_\infty^2\,\mathbb{V}[e_{1,0,m}(t,b)] \tag{98}$$

and then directly replaced each variance in each summand on the right-hand side of (98) by their individual upper bounds, the upper bound we have obtained on for the variance of $e_m(t)$ when $\Theta_0=(a,b)$ is also an upper bound for the variance of $e_m(t)$ when $\Theta_0=[a,b]$. Further, since concentration inequalities for $e_m(t)$ when $\Theta_0=(a,b)$ have been derived by Chebyshev's inequality based on the upper bound for the variance of $e_m(t)$ when $\Theta_0=(a,b)$, these concentration inequalities are also valid for $e_m(t)$ when $\Theta_0=[a,b]$.

Now let us discuss $\varphi_m(t,\boldsymbol{\mu})$, which is equivalent to $\varphi_m(t,\boldsymbol{\theta})$ due to the mapping $\mu=\mu(\theta)=\sigma(1-\theta)^{-1}$ for $\theta<1$. When $\Theta_0=(a,b)$, we have

$$\varphi_m(t,\boldsymbol{\mu})=m^{-1}\sum_{i=1}^m\left[\psi_1(t,\mu_i)-2^{-1}\left\{\phi(a)\,\psi_{1,0}(t,\mu;a)+\phi(b)\,\tilde{\psi}_{1,0}(t,\mu;b)\right\}\right],$$

$\varphi_m(t,\boldsymbol{\mu})=\sum_{j=1}^5 d_{\phi,j}(t,\boldsymbol{\mu})$ and

$$\left|\breve{\pi}_{0,m}^{-1}\varphi_m(t,\boldsymbol{\mu})-1\right|\leq\left|\breve{\pi}_{0,m}^{-1}d_{\phi,1}(t,\boldsymbol{\mu})-1\right|+\sum_{j=2}^5\breve{\pi}_{0,m}^{-1}\left|d_{\phi,j}(t,\boldsymbol{\mu})\right|, \tag{99}$$

where

$$\begin{cases} d_{\phi,1}(t,\boldsymbol{\mu})=m^{-1}\sum_{\{i:\mu_i\in(a,b)\}}\psi_1(t,\mu_i)\\ d_{\phi,2}(t,\boldsymbol{\mu})=m^{-1}\sum_{\{i:\mu_i=a\}}\left(\psi_1(t,\mu_i)-2^{-1}\phi(a)\,\tilde{\psi}_{1,0}(t,\mu_i;a)\right)\\ d_{\phi,3}(t,\boldsymbol{\mu})=m^{-1}\sum_{\{i:\mu_i=b\}}\left(\psi_1(t,\mu_i)-2^{-1}\phi(b)\,\tilde{\psi}_{1,0}(t,\mu_i;b)\right)\\ d_{\phi,4}(t,\boldsymbol{\mu})=-m^{-1}\left(\sum_{\{i:\mu_i\neq a\}}+\sum_{\{i:\mu_i\neq b\}}\right)\left\{2^{-1}\left[\phi(a)\,\tilde{\psi}_{1,0}(t,\mu_i;a)+\phi(b)\,\tilde{\psi}_{1,0}(t,\mu_i;b)\right]\right\}\\ d_{\phi,5}(t,\boldsymbol{\mu})=m^{-1}\sum_{\{i:\mu_i<b\}}\psi_1(t,\mu_i)+m^{-1}\sum_{\{i:\mu_i>b\}}\psi_1(t,\mu_i)\end{cases}.$$

Further, when $\Theta_0=(a,b)$, to upper bound $\left|\breve{\pi}_{0,m}^{-1}\varphi_m(t,\boldsymbol{\mu})-1\right|$, we have replaced each $\left|d_{\phi,j}(t,\boldsymbol{\mu})\right|,2\leq j\leq 5$ by its upper bound $\hat{d}_{\phi,j}(t,\boldsymbol{\mu}),2\leq j\leq 5$ and replaced $\left|\breve{\pi}_{0,m}^{-1}d_{\phi,1}(t,\boldsymbol{\mu})-1\right|$ by its upper bound $\hat{d}_{\phi,0}(t,\boldsymbol{\mu})$ directly in (99).

When $\Theta_0=[a,b]$, we have

$$\varphi_m(t,\boldsymbol{\mu})=m^{-1}\sum_{i=1}^m\left[\psi_1(t,\mu_i)+2^{-1}\left\{\phi(a)\,\psi_{1,0}(t,\mu;a)+\phi(b)\,\tilde{\psi}_{1,0}(t,\mu;b)\right\}\right]$$

and

$$\varphi_m(t,\boldsymbol{\mu})=d_{\phi,1}(t,\boldsymbol{\mu})+d_{\phi,2}^*(t,\boldsymbol{\mu})+d_{\phi,3}^*(t,\boldsymbol{\mu})-d_{\phi,4}(t,\boldsymbol{\mu})+d_{\phi,5}(t,\boldsymbol{\mu})$$

where

$$\begin{cases} d_{\phi,2}^*(t,\boldsymbol{\mu})=m^{-1}\sum_{\{i:\mu_i=a\}}\left(\psi_1(t,\mu_i)+2^{-1}\phi(a)\,\tilde{\psi}_{1,0}(t,\mu_i;a)\right)\\ d_{\phi,3}^*(t,\boldsymbol{\mu})=m^{-1}\sum_{\{i:\mu_i=b\}}\left(\psi_1(t,\mu_i)+2^{-1}\phi(b)\,\tilde{\psi}_{1,0}(t,\mu_i;b)\right)\end{cases}.$$

Then

$$\begin{aligned}\left|\breve{\pi}_{0,m}^{-1}\varphi_m(t,\boldsymbol{\mu})-1\right|&\leq\left|\breve{\pi}_{0,m}^{-1}\left[d_{\phi,1}(t,\boldsymbol{\mu})+d_{\phi,2}^*(t,\boldsymbol{\mu})+d_{\phi,3}^*(t,\boldsymbol{\mu})\right]-1\right|\\&\quad+\breve{\pi}_{0,m}^{-1}\left|d_{\phi,4}(t,\boldsymbol{\mu})\right|+\breve{\pi}_{0,m}^{-1}\left|d_{\phi,5}(t,\boldsymbol{\mu})\right|.\end{aligned}$$

However,

$$\left|\breve{\pi}_{0,m}^{-1}\left[d_{\phi,1}(t,\boldsymbol{\mu})+d_{\phi,2}^*(t,\boldsymbol{\mu})+d_{\phi,3}^*(t,\boldsymbol{\mu})\right]-1\right|\leq\hat{d}_{\phi,0}(t,\boldsymbol{\mu})+\breve{\pi}_{0,m}^{-1}\hat{d}_{\phi,2}(t,\boldsymbol{\mu})+\breve{\pi}_{0,m}^{-1}\hat{d}_{\phi,3}(t,\boldsymbol{\mu}).$$

Therefore, the upper bound we have derived for $\left|\breve{\pi}_{0,m}^{-1}\varphi_m(t,\boldsymbol{\mu})-1\right|$ when $\Theta_0=(a,b)$ is also an upper bound for $\left|\breve{\pi}_{0,m}^{-1}\varphi_m(t,\boldsymbol{\mu})-1\right|$ when $\Theta_0=[a,b]$.

In summary, results we have derived for the estimator $\hat{\varphi}_m(t,\mathbf{z})$ when $\Theta_0=(a,b)$ for the construction (97) remain valid for the estimator $\hat{\varphi}_m(t,\mathbf{z})$ when $\Theta_0=[a,b]$ for the construction (96).

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m	Method	Sparsity	Null Type	$\hat{E}(\tilde{\delta}_m)$	$\hat{\sigma}(\tilde{\delta}_m)$
1000	MR	$\pi_{1,m} = 0.2$	One-sided null	-0.9954	0.0061
1000	New	$\pi_{1,m} = 0.2$	One-sided null	1.2245	0.0161
5000	MR	$\pi_{1,m} = 0.2$	One-sided null	-0.9973	0.0024
5000	New	$\pi_{1,m} = 0.2$	One-sided null	1.1659	0.0090
10000	MR	$\pi_{1,m} = 0.2$	One-sided null	-0.9969	0.0022
10000	New	$\pi_{1,m} = 0.2$	One-sided null	1.1397	0.0067
50000	MR	$\pi_{1,m} = 0.2$	One-sided null	-0.9964	0.0012
50000	New	$\pi_{1,m} = 0.2$	One-sided null	1.0782	0.0042
1000	MR	$\pi_{1,m} = 1/\ln(\ln m)$	One-sided null	-0.9558	0.0139
1000	New	$\pi_{1,m} = 1/\ln(\ln m)$	One-sided null	-0.0399	0.0077
5000	MR	$\pi_{1,m} = 1/\ln(\ln m)$	One-sided null	-0.9638	0.0063
5000	New	$\pi_{1,m} = 1/\ln(\ln m)$	One-sided null	0.0393	0.0046
10000	MR	$\pi_{1,m} = 1/\ln(\ln m)$	One-sided null	-0.9666	0.0050
10000	New	$\pi_{1,m} = 1/\ln(\ln m)$	One-sided null	0.0669	0.0036
50000	MR	$\pi_{1,m} = 1/\ln(\ln m)$	One-sided null	-0.9721	0.0021
50000	New	$\pi_{1,m} = 1/\ln(\ln m)$	One-sided null	0.1163	0.0023
1000	New	$\pi_{1,m} = 0.2$	Bounded null	3.0421	0.0045
5000	New	$\pi_{1,m} = 0.2$	Bounded null	2.8260	0.0034
10000	New	$\pi_{1,m} = 0.2$	Bounded null	2.7343	0.0028
50000	New	$\pi_{1,m} = 0.2$	Bounded null	2.5244	0.0018
1000	New	$\pi_{1,m} = 1/\ln(\ln m)$	Bounded null	0.5625	0.0019
5000	New	$\pi_{1,m} = 1/\ln(\ln m)$	Bounded null	0.6416	0.0016
10000	New	$\pi_{1,m} = 1/\ln(\ln m)$	Bounded null	0.6620	0.0014
50000	New	$\pi_{1,m} = 1/\ln(\ln m)$	Bounded null	0.6842	0.0010

Table 1 In the table, $\tilde{\delta}_m = \hat{\pi}_{1,m}/\pi_{1,m} - 1$ (where $\hat{\pi}_{1,m}$ is an estimate of $\pi_{1,m}$), $\hat{E}(\tilde{\delta}_m)$ is the sample mean of $\tilde{\delta}_m$, and $\hat{\sigma}(\tilde{\delta}_m)$ the sample standard deviation of $\tilde{\delta}_m$. When $\pi_{1,m} = 0.2$, our proposed estimators "New" show a clear trend of convergence to 0 as m increases. For $\pi_{1,m} = 1/\ln(\ln m)$ though, $\tilde{\delta}_m$ for our "New" estimators does not show a clear trend of convergence to 0 as m increases. However, this is an artifact of the numerical error when implementing our "New" estimators, as explained in Section E.1.

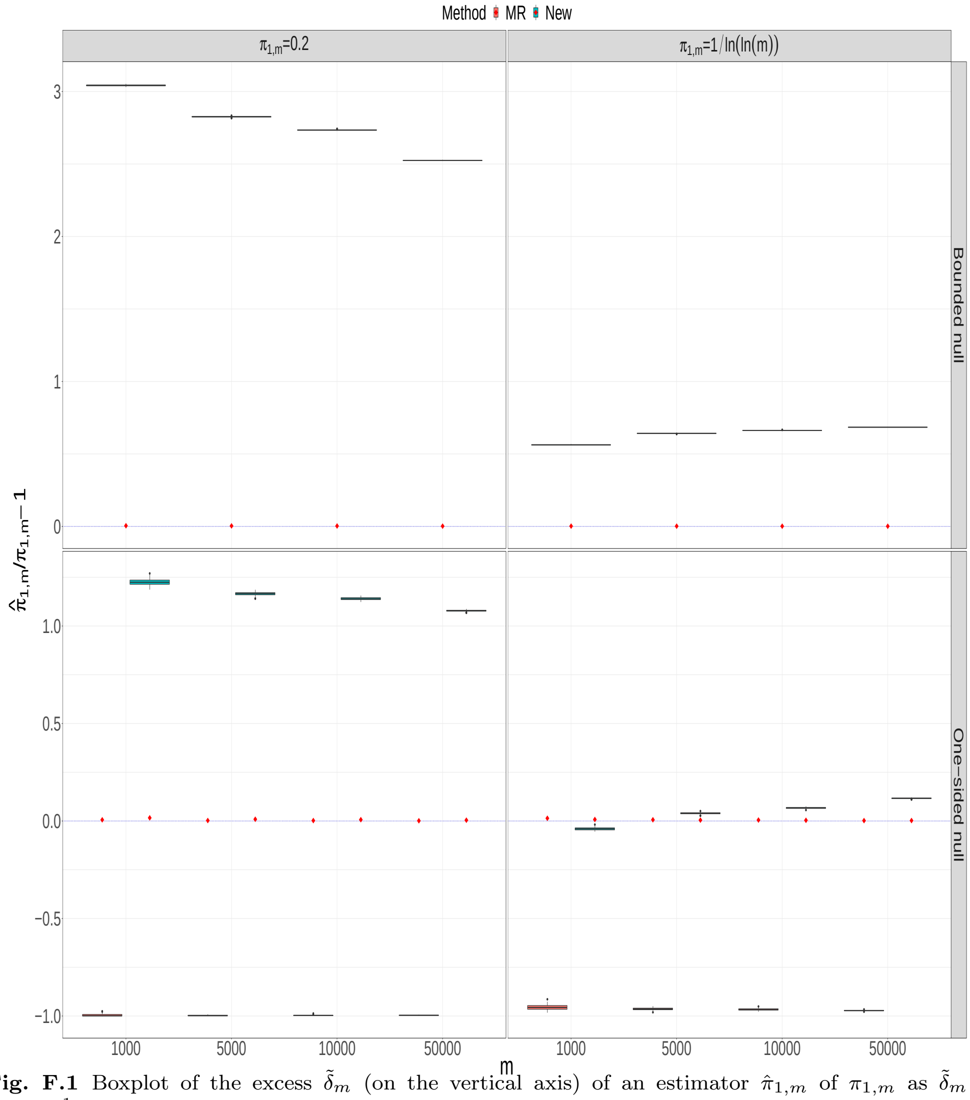

Fig. F.1 Boxplot of the excess $\tilde{\delta}_m$ (on the vertical axis) of an estimator $\hat{\pi}_{1,m}$ of $\pi_{1,m}$ as $\tilde{\delta}_m = \hat{\pi}_{1,m}\pi_{1,m}^{-1} - 1$. The thick horizontal line and the diamond in each boxplot are respectively the mean and standard deviation of $\tilde{\delta}_m$, and the dotted horizontal line is the reference for $\tilde{\delta}_m = 0$. An estimator with a narrower boxplot that is closer to the dotted horizontal line is better. All estimators have been applied to Gamma family. For the case of a one-side null, the right one for each pair of boxplots for each m is for the proposed estimator "New" and the left one is for the "MR" estimator. No simulation was done for the "MR" estimator for a bounded null. Note that the "Method" legend for boxplots is basically invisible in the subplots since each boxplot contains observations that vary so little and are hence very narrow vertically.