Proof of (14): For fixed $t > 0$, consider partition points of interval $[0, t]$: $0 = s_0 < s_1 < \cdots < s_N = t$, which have equal width $\Delta = \frac{t}{N}$ for each subinterval $[s_{j-1}, s_j]$, where $N$ is any large positive integer. Then, from (6.8) on Page 196 in Kalbfleisch and Prentice (2002) we have the following:

$$\int_0^t f(s \mid X^s) \, ds = o(1) + \sum_{k=0}^{N-1} \Delta \, f(s_k \mid X^{s_k})$$

(13)

$$= o(1) + \sum_{k=0}^{N-1} \left[ P\{s_k \leq T < s_k + \Delta \mid X^{s_k}\} + \Delta \, r_k \right]$$

(6.8) of K-P

$$= o(1) + \sum_{k=0}^{N-1} \left[ P\{s_k \leq T < s_k + \Delta \mid X^t\} + \Delta \, r_k \right]$$

(47)

where $r_k$ is the remainder term from (13) at each $s_k$. Letting $N \to \infty$, we know that (14) follows from (47) and the uniform convergence of (13). $\Box$

Proof of Lemma 1: For $\delta = 1$, we know

$$g(t, 1 \mid X^t) = \lim_{\Delta \to 0} \Delta^{-1} P\{t \leq V < t + \Delta, \delta = 1 \mid X^t\}$$

$$= \lim_{\Delta \to 0} \Delta^{-1} P\{t \leq T < t + \Delta, T \leq C \mid X^t\}.$$
Thus, the 1st equation of (16) follows from (13) and the following inequalities:

\[ P\{t \leq T < t + \Delta, T \leq C \mid X^t\} \leq P\{t \leq T < t + \Delta, t \leq C \mid X^t\} \]
\[ = P\{t \leq T < t + \Delta \mid X^t\} P\{t \leq C \mid X^t\} = P\{t \leq T < t + \Delta \mid X^t\} \hat{F}_C(t); \]
\[ P\{t \leq T < t + \Delta, T \leq C \mid X^t\} \geq P\{t \leq T < t + \Delta, t + \Delta \leq C \mid X^t\} \]
\[ = P\{t \leq T < t + \Delta \mid X^t\} P\{t + \Delta \leq C \mid X^t\} = P\{t \leq T < t + \Delta \mid X^t\} \hat{F}_C(t + \Delta). \]

For \( \delta = 0 \), we know

\[ g(t, 0 \mid X^t) = \lim_{\Delta \to 0} \Delta^{-1} P\{t \leq V < t + \Delta, \delta = 0 \mid X^t\} \]
\[ = \lim_{\Delta \to 0} \Delta^{-1} P\{t \leq C < t + \Delta, T > C \mid X^t\}. \]

Thus, the 2nd equation of (16) follows from (12)-(13) and the following inequalities:

\[ P\{t \leq C < t + \Delta, T > C \mid X^t\} \leq P\{t \leq C < t + \Delta, T \geq t \mid X^t\} \]
\[ = P\{t \leq C < t + \Delta \mid X^t\} P\{T \geq t \mid X^t\} = P\{t \leq C < t + \Delta\} \hat{F}(t \mid X^t); \]
\[ P\{t \leq C < t + \Delta, T > C \mid X^t\} \geq P\{t \leq C < t + \Delta, t + \Delta \leq T \mid X^t\} \]
\[ = P\{t \leq C < t + \Delta \mid X^t\} P\{t + \Delta \leq T \mid X^t\} = P\{t \leq C < t + \Delta\} \hat{F}(t + \Delta \mid X^t). \]

**Proof of Lemma 2:** From (18), for any \( t \in [t_i, t_i, V_i] \) we have for \( t_i = 0 \):

\[ -\int_0^t \lambda_0(s) e^{\beta_0 \tilde{Z}_i(s)} ds = -\int_{t_i, j_i}^t \lambda_0(s) e^{\beta_0 \tilde{Z}_i(s)} ds - \sum_{j=1}^{J_i} \int_{t_i, j_i}^{t_{i,j}} \lambda_0(s) e^{\beta_0 \tilde{Z}_i(s)} ds \]
\[ = -\int_{t_i, j_i}^t \lambda_0(s) e^{\beta_0 \tilde{Z}_i(t_i, j_i)} ds - \sum_{j=1}^{J_i} \int_{t_i, j_i}^{t_{i,j}} \lambda_0(s) e^{\beta_0 \tilde{Z}_i(t_i, j_i, j_i)} ds \]
\[ = -e^{\beta_0 \tilde{Z}_i(t_i, j_i)} \int_{t_i, j_i}^t \frac{f_0(s)}{F_0(s)} ds - \sum_{j=1}^{J_i} e^{\beta_0 \tilde{Z}_i(t_i, j_i, j_i)} \int_{t_i, j_i}^{t_{i,j}} \frac{f_0(s)}{F_0(s)} ds \]
\[ = e^{\beta_0 \tilde{Z}_i(t_i, j_i)} \ln \left( \frac{\hat{F}_0(t)}{F_0(t_i, j_i)} \right) + \sum_{j=1}^{J_i} e^{\beta_0 \tilde{Z}_i(t_i, j_i, j_i)} \ln \left( \frac{\hat{F}_0(t_{i,j})}{F_0(t_i, j_i, j_i)} \right). \]
Thus, for $\hat{z}_i(0) = 0$ and $c_{ij} = e^{\beta_0 \hat{z}_i(t_{ij})}$ with $c_{i0} = 1$ and $c_i = c_{iI}$, (19) can be written as

$$F(t \mid X_i^t) = \exp \left\{ c_{iI} \ln \left( \frac{\bar{F}_0(t)}{\bar{F}_0(t_{iI})} \right) + \sum_{j=1}^{J_i} c_{i,j-1} \ln \left( \frac{\bar{F}_0(t_{ij})}{\bar{F}_0(t_{i,j-1})} \right) \right\}$$

$$= \left( \frac{\bar{F}_0(t)}{\bar{F}_0(t_{iI})} \right)^{c_{iI}} \prod_{j=1}^{J_i} \left( \frac{\bar{F}_0(t_{ij})}{\bar{F}_0(t_{i,j-1})} \right)^{c_{i,j-1}}$$

$$= \left( \frac{\bar{F}_0(t)}{\bar{F}_0(t_{iI})} \right)^{c_{iI}} \left( \frac{\bar{F}_0(t_{i1})}{\bar{F}_0(t_{00})} \right)^{c_{00}} \times \left( \frac{\bar{F}_0(t_{i2})}{\bar{F}_0(t_{01})} \right)^{c_{1}} \times \cdots \times \left( \frac{\bar{F}_0(t_{i,J_i})}{\bar{F}_0(t_{i,J_{i-1}})} \right)^{c_{i,J_i-1}}$$

$$= \left[ \bar{F}_0(t) \right]^{c_{iI}} \prod_{j=1}^{J_i} \left[ \bar{F}_0(t_{ij}) \right]^{c_{i,j-1}-c_{ij}} = \left[ \bar{F}_0(t) \right]^{c_{iI}} \prod_{j=1}^{J_i} \left[ \bar{F}_0(t_{ij}) \right]^{c_{i,j-1}-c_{ij}}.$$ 

Equation (20) is obtained by $f(t \mid X_i^t) = -\frac{d}{dt}(\bar{F}(t \mid X_i^t)).$ \(\square\)

**Proof of (21):** From Lemma 2, we have the following:

$$\prod_{i=1}^{n} \left[ f(V_i \mid X_i^{V_i}) \right]^{\delta_i} \left[ \bar{F}(V_i \mid X_i^{V_i}) \right]^{1-\delta_i}$$

$$= \prod_{i=1}^{n} \left( c_i f_0(V_i) [\bar{F}_0(V_i)]^{c_{iI}} / \prod_{j=1}^{J_i} [\bar{F}_0(t_{ij})]^{c_{i,j-1}-c_{ij}} \right)^{\delta_i} \left( [\bar{F}_0(V_i)]^{c_i} \prod_{j=1}^{J_i} [\bar{F}_0(t_{ij})]^{c_{i,j-1}-c_{ij}} \right)^{1-\delta_i}$$

$$= \prod_{i=1}^{n} \left[ c_i f_0(V_i) \right]^{\delta_i} \left[ \bar{F}_0(V_i) \right]^{c_{iI}-\delta_i} \left( \prod_{j=1}^{J_i} [\bar{F}_0(t_{ij})]^{c_{i,j-1}-c_{ij}} \right). \square$$

**Proof of Lemma 3:** Let $A_{kij} = I \{ V_k \leq t_{ij} < V_{k+1} \}$ with $V_{n+1} = \infty$ and let $d_{ij} = c_{i,j-1} - c_{ij}$ with $c_{ij}$’s given by (23). Since for $F$ given by (25), we have $F(t_{ij}) = 0$ for $0 \leq t_{ij} < V_1$, then the last component of (22) can be simplified as:

$$\prod_{i=1}^{n} \prod_{j=1}^{J_i} [\bar{F}(t_{ij})]^{d_{ij}} = \prod_{i=1}^{n} \prod_{j=1}^{J_i} \prod_{k=1}^{n} \left[ \bar{F}(t_{ij}) \right]^{A_{kij}d_{ij}}$$

$$= \prod_{k=1}^{n} \prod_{i=1}^{n} \prod_{j=1}^{J_i} \left[ \bar{F}(V_k) \right]^{A_{kij}} = \prod_{k=1}^{n} \left[ \bar{F}(V_k) \right]^{A_k} = \prod_{i=1}^{n} \left[ \bar{F}(V_i) \right]^{A_i},$$

3
where $A_k = \sum_{i=1}^{n} \sum_{j=1}^{J_k} A_{kij}d_{ij}$. Then, (22) can be written as:

$$L(\beta, F) = \prod_{i=1}^{n} [c_i dF(V_i)]^{\delta_i} [F(V_i)]^{d_i - \delta_i}$$

(48)

where for $A_i = \sum_{k=1}^{n} \sum_{j=1}^{J_k} A_{ikj}d_{kj}$, we have for $c_i$'s given in (23),

$$d_i = c_i + A_i = c_i + \sum_{k=1}^{n} \sum_{j=1}^{J_k} A_{ikj}d_{kj} = c_i + \sum_{k=1}^{n} \sum_{j=1}^{J_k} (c_{k,j-1} - c_{kj}) I\{V_i \leq t_{kj} < V_{i+1}\}.$$  

Thus, (26) follows from (25) and (48). □

**Proof of Theorem 1:** Let $a_i = \frac{p_i}{b_i}$ with $b_i = \sum_{j=i}^{n} p_j$, then we have $b_{i+1} = (b_i - p_i)$, $b_1 = 1$, $b_{n+1} = p_{n+1}$ and $(1 - a_i) = \frac{b_{i+1}}{b_i}$. From $\prod_{i=1}^{n} (1 - a_i) = b_{n+1}$ and

$$\prod_{i=1}^{n} a_i^{d_i}(1-a_i)^{-h_i} = \left(\prod_{i=1}^{n} a_i^{d_i}\right) \prod_{i=1}^{n} \left(\frac{b_{i+1}}{b_i}\right)^{-h_i} = \left(\prod_{i=1}^{n} a_i^{d_i}\right) b_{n+1}^{n-h_n} \left(\prod_{i=1}^{n} b_i^{d_i}\right) = \left(\prod_{i=1}^{n} b_i^{d_i}\right) b_{n+1}^{n-h_n},$$

where $h_i = d_1 + \cdots + d_i$, we obtain the following for (26):

$$L(\beta, p) = \prod_{i=1}^{n} [c_i p_i]^{\delta_i} (b_{i+1})^{d_i - \delta_i} = \prod_{i=1}^{n} [c_i p_i]^{\delta_i} (b_i - p_i)^{d_i - \delta_i}$$

$$= \prod_{i=1}^{n} c_i^{\delta_i} p_i^{d_i} \left(\frac{1 - a_i}{a_i}\right)^{d_i - \delta_i} = \left(\prod_{i=1}^{n} c_i^{\delta_i} a_i^{d_i}(1-a_i)^{-h_i}\right) \prod_{i=1}^{n} \left(\frac{1 - a_i}{a_i}\right)^{d_i - \delta_i}$$

(49)

$$= \prod_{i=1}^{n} (c_i a_i)^{\delta_i} (1-a_i)^{d_i - \delta_i} \equiv L_1(\alpha; \beta)$$

where $e_i = d_1 + \cdots + d_n$. For fixed $\beta$ satisfying (AS1), from the 1st and 2nd partial derivatives of $\log L_1(\alpha; \beta)$ with respect to $a_i$'s, we know that the solution of equations

$$\frac{\partial \log L_1}{\partial a_i} = 0, 1 \leq i \leq n,$$

is given by $\hat{a}_i = \frac{\delta_i}{e_i}, \ i = 1, \cdots, n$, which maximizes $L_1(\alpha; \beta)$ with all $0 \leq \hat{a}_i \leq 1$. Thus, (27) follows from the fact that the d.f. $F$ corresponding to $\hat{a}_i$'s is given by $\bar{F}_n(t; \beta) = \prod_{V_i \leq t}(1 - \hat{a}_i)$. □
Proof of (28) and (30)-(31): Notice that we obtain (28) by plugging \( \hat{a}_i \)'s into (49). From notations \( c'_i = \frac{dc_i}{d\beta} \) and \( e'_i = \frac{de_i}{d\beta} \), we have \( \frac{c'_i}{c_i} = \tilde{Z}_i(t_{i,k}) \) and we obtain equation (30) from the following:

\[
\Psi_n(\beta) = n^{-1} \frac{d}{d\beta} \left( \log \ell(\beta) \right) = n^{-1} \frac{d}{d\beta} \left( \sum_{i=1}^{n} \left[ \delta_i (\ln c_i - \ln e_i) + (e_i - \delta_i) \left( \ln(e_i - \delta_i) - \ln e_i \right) \right] \right) \\
= n^{-1} \sum_{i=1}^{n} \left[ \delta_i \left( \frac{c'_i}{c_i} - \frac{e'_i}{e_i} \right) + e'_i \log \left( 1 - \frac{\delta_i}{e_i} \right) + (e_i - \delta_i) \left( \frac{e'_i}{e_i - \delta_i} - \frac{e'_i}{e_i} \right) \right] \\
= n^{-1} \sum_{i=1}^{n} \left[ \delta_i \tilde{Z}_i(t_{i,k}) + e'_i \log \left( 1 - \frac{\delta_i}{e_i} \right) \right] = \sum_{i=1}^{n} \delta_i \left( \tilde{Z}_i(t_{i,k}) + e'_i \log \left( 1 - \frac{1}{e_i} \right) \right).
\]

Using Taylor’s expansion \( \log(1 - x) \approx -x \), term \( \log \left( 1 - \frac{1}{e_i} \right) \) in (30) is approximated by \( -\frac{1}{e_i} \), which gives (31). \( \square \)

Proof of Lemma 4 (a): Using the notations of \( A_{kij} \)'s and \( d_{ij} \)'s in the proof of Lemma 3, we have the following for \( e_i \) in (27) based on \( d_i \)'s given in (26):

\[
e_i = \sum_{q=i}^{n} d_q = \sum_{q=i}^{n} \left( c_{qJ_i} + \sum_{k=1}^{n} \sum_{j=1}^{J_k} d_{kj} I\{V_q \leq t_{kj} < V_{q+1}\} \right) \\
= \sum_{k=i}^{n} c_{kJ_k} + \sum_{k=1}^{n} \sum_{j=1}^{J_k} d_{kj} A_{qkj} I\{q \geq i\} \\
= \sum_{k=i}^{n} c_{kJ_k} + \sum_{k=1}^{n} \sum_{j=1}^{J_k} \sum_{q=1}^{n} d_{kj} A_{qkj} I\{q \geq i\} \\
= \sum_{k=i}^{n} c_{kJ_k} + \sum_{k=1}^{n} \sum_{j=1}^{J_k} d_{kj} I\{V_q \leq t_{kj} < V_{q+1}\} \\
= \sum_{k=i}^{n} c_{kJ_k} + \sum_{k=1}^{n} \sum_{j=1}^{J_k} d_{kj} I\{V_{i} \leq t_{kj}\}.
\]

Notice that from data (11) and assumption (24), we know that for each \( k < i \), we always have \( t_{kj} \leq V_k < V_i \) for any \( j = 1, \ldots, J_k \). Thus, we can simplify and express above...
equation as below:

\[ e_i = \sum_{k=i}^{n} \left( c_{k,J_k} + \sum_{j=1}^{J_k} d_{kj} I\{V_i \leq t_{kj}\} \right) = \sum_{k=i}^{n} B_{ik}, \quad (50) \]

where \( B_{ik} = c_{k,J_k} + \sum_{j=1}^{J_k} d_{kj} I\{V_i \leq t_{kj}\} \). Obviously, if \( \mathcal{E}_{ik} = \emptyset \), we have \( B_{ik} = c_{k,J_k} \). If \( \mathcal{E}_{ik} \neq \emptyset \), for \( J_{ik} \) given in the statement of Lemma 4 we have

\[ B_{ik} = c_{k,J_k} + \sum_{j=J_{ik}+1}^{J_k} d_{kj} = c_{k,J_k} + \sum_{j=J_{ik}+1}^{J_k} (c_{k,J_{ik}-1} - c_{kJ}) = c_{k,J_{ik}} \]

which gives

\[ B_{ik} \overset{(23)}{=} e^{\beta \tilde{Z}_k(t_{kJ_k})} I\{\mathcal{E}_{ik} = \emptyset\} + e^{\beta \tilde{Z}_k(t_{kJ_{ik}})} I\{\mathcal{E}_{ik} \neq \emptyset\}. \quad \square \quad (51) \]

**Proof of Lemma 4 (b):** For \( e_i'' = \frac{d e_i'}{d \beta} \), we obtain derivative of \( \psi_n(\beta) \) in (31):

\[ \psi_n'(\beta) = n^{-1} \sum_{i=1}^{n} \delta_i \left( \frac{(e_i')^2 - e_i' e_i''}{e_i^2} \right). \quad (52) \]

Let \( C_{ik} = \tilde{Z}_k(t_{kJ_k}) I\{\mathcal{E}_{ik} = \emptyset\} + \tilde{Z}_k(t_{kJ_{ik}}) I\{\mathcal{E}_{ik} \neq \emptyset\} \), then from (50)-(51) we have

\[ e_i' = \sum_{k=i}^{n} C_{ik} B_{ik} \quad \text{and} \quad e_i'' = \sum_{k=i}^{n} C_{ik}^2 B_{ik}. \]

From Cauchy-Schwarz inequality, we know

\[ (e_i'')^2 = \left( \sum_{k=i}^{n} C_{ik} B_{ik} \right)^2 = \left( \sum_{k=i}^{n} \sqrt{B_{ik}} \left( C_{ik} \sqrt{B_{ik}} \right) \right)^2 \leq \left( \sum_{k=i}^{n} B_{ik} \right) \left( \sum_{k=i}^{n} C_{ik}^2 B_{ik} \right) = e_i' e_i'' \]

which implies \( \psi_n'(\beta) \leq 0 \) in (52). \( \square \)


**Data-Based Choice of \( M \):** In our simulation studies, we only consider the cases with all nonnegative \( \tilde{Z}_i(t_{ij}) \)'s, thus from the proof of Lemma 4 we know that \( e'_i \geq 0 \) and \( E'_i \geq 0 \) always. Applying Taylor’s expansion to (30) and (37), we have the following:

\[
\begin{align*}
\Psi_n(\beta) &= \psi_n(\beta) - n^{-1} \sum_{i=1}^{n} \frac{\delta_i e'_i}{2(1 - \xi_i)^2 e_i^2} \\
\Psi_{M,n}(\beta) &= \psi_n(\beta) - n^{-1} \sum_{i=1}^{n} \frac{\delta_i E'_i}{2(1 - \zeta_i)^2 E_i^2}
\end{align*}
\]

where \( \xi_i \) is between 0 and \( \frac{1}{e_i} \), and \( \zeta_i \) is between 0 and \( \frac{1}{E_i} \), which implies

\[
\Psi_n(\beta) < \psi_n(\beta) \quad \text{and} \quad \Psi_{M,n}(\beta) < \psi_n(\beta), \quad \forall \beta. \quad (53)
\]

For fixed \( \beta \), we differentiate \( \Psi_{M,n}(\beta) \) with respective to \( M \):

\[
\frac{\partial \Psi_{M,n}(\beta)}{\partial M} = n^{-1} \sum_{i=1}^{n} \delta_i E'_i \left( \log \left( 1 - \frac{1}{E_i} \right) + \frac{1}{E_i} \right) = n^{-1} \sum_{i=1}^{n} \delta_i g \left( \frac{1}{E_i} \right) > 0
\]

where \( g(x) = \log(1 - x) + \frac{x}{1-x}, \ 0 < x < 1 \) satisfies

\[
g'(x) = \frac{-1}{1-x} + \frac{1}{(1-x)^2} = \frac{x}{(1-x)^2} > 0 \quad \Rightarrow \quad g(x) > g(0) = 0.
\]

Thus, we know that for any fixed \( \beta \), \( \Psi_{M,n}(\beta) \) is increasing in \( M \).

Based on above facts, we have the following steps for data-based choice of \( M \), which starts after finding \( \tilde{\beta}_n \) from equation \( \psi_n(\beta) = 0 \).

**Steps to Choose \( M \):**

**Step 1.** Compute \( e_1, \cdots, e_n \) at \( \beta = (\tilde{\beta}_n - \rho) \), then order them into \( e_{(1)} \leq e_{(2)} \leq \cdots \leq e_{(n)} \).

If \( e_{(1)} \geq 1 \), no need to choose \( M \) and go to Step 2; if \( e_{(1)} < 1 \), go to Step 3.
Step 2. Search $\hat{\beta}_n$ from equation $\Psi_n(\beta) = 0$: increase $\rho > 0$ until $\Psi_n(\hat{\beta}_n - \rho) > 0$, then search $\hat{\beta}_n$ in interval $(\hat{\beta}_n - \rho, \hat{\beta}_n)$, because (53) implies $\Psi_n(\hat{\beta}_n) < \psi_n(\hat{\beta}_n) = 0$.

Step 3. Find $e^* = \min_{1 \leq i \leq n}\{e(i) \mid e(i) \geq 10^{-7}\}$ and compute $M_0 = -\log e^*$, then search $\hat{\beta}_n$ from equation $\Psi_{M_0, n}(\beta) = 0$: increase $\rho > 0$ until $\Psi_{M_0, n}(\hat{\beta}_n - \rho) > 0$, then search $\hat{\beta}_n$ in interval $(\hat{\beta}_n - \rho, \hat{\beta}_n)$, because (53) implies $\Psi_{M_0, n}(\hat{\beta}_n) < \psi_n(\hat{\beta}_n) = 0$.

Step 4. If encounter some $E_i \leq 1$ in Step 3, let $\Psi_{M_0, n}(\hat{\beta}_0 n) = 0$ by dropping at most two terms involving $E_i \leq 1$. Then, use $M = M_0 + \gamma$ to search $\hat{\beta}_n$ from equation $\Psi_{M, n}(\beta) = 0$ in interval $(\hat{\beta}_0 n, \hat{\beta}_n)$, where $\gamma > 0$ and $\rho > 0$ need to be appropriately chosen based on important facts aforementioned regarding $\Psi_{M, n}(\beta) = 0$.

Note: Above choice of $M_0$ is to achieve $E_i \geq 1$ in order to avoid computing floating errors, which is difficult to control in intensive Monte Carlo simulation studies. But such choice of $M_0$ is conservative and can still have occasional $E_i \leq 1$, thus adjustment to use $M = M_0 + \gamma$ is for the purpose of not too big $M$ which can eliminate cases $E_i \leq 1$. □

Proof of (44)-(46): For $\tilde{\delta}_k$ and $\hat{c}_k$ given in (45), we have

\[
\prod_{k=1}^{N} \prod_{V_i = U_k} \left[ c_i \; dF(V_i) \right]^{\delta_i} \left[ \tilde{F}(V_i) \right]^{c_i - \delta_i} = \prod_{k=1}^{N} \left( \prod_{V_i = U_k} e^{\beta \tilde{Z}_i(t_{ij} \delta_i)} \left[ dF(V_k) \right]^{\delta_k} \left[ \tilde{F}(V_k) \right] \sum_{i=1}^{n} c_i I\{V_i = U_k\} - \delta_k \right)^{\hat{c}_k} dF(V_k) \left[ \tilde{F}(V_k) \right]^{\sum_{i=1}^{n} c_i I\{V_i = U_k\} - \delta_k}.
\]

Using the notations of $A_{kl}$’s and $d_{ij}$’s in the proof of Lemma 3, for $F$ given by $F(t) = \sum_{k=1}^{N} p_k I\{U_k \leq t\}$, we have the following for the last component in the 1st
equation of (44):

\[
\prod_{k=1}^{N} \prod_{i=1}^{n} \prod_{j=1}^{J_i} \left[ \bar{F}(t_{ij}) \right]^{d_{ij}} = \prod_{k=1}^{N} \prod_{i=1}^{n} \prod_{j=1}^{J_i} \left[ \bar{F}(t_{ij}) \right]^{d_{ij}I\{V_i=U_k\}}
\]

\[
= \prod_{k=1}^{N} \prod_{i=1}^{n} \prod_{j=1}^{J_i} \prod_{q=1}^{N} \left[ \bar{F}(U_q) \right]^{A_{qij}d_{ij}I\{V_i=U_k\}}
\]

\[
= \prod_{q=1}^{N} \left[ \bar{F}(U_q) \right]^{\sum_{k=1}^{N} \sum_{i=1}^{n} \sum_{j=1}^{J_i} A_{qij}d_{ij}I\{V_i=U_k\}} = \prod_{k=1}^{N} \left[ \bar{F}(U_k) \right]^{A_k}
\]

where

\[
A_k = \sum_{q=1}^{N} \sum_{i=1}^{n} \sum_{j=1}^{J_i} A_{qij}d_{ij}I\{V_i=U_k\} = \sum_{i=1}^{n} \sum_{j=1}^{J_i} \left( \sum_{q=1}^{N} I\{V_i=U_q\} \right) A_{qij}d_{ij}
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{J_i} A_{kij}d_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{J_i} (c_{ij-1} - c_{ij}) I\{U_k \leq t_{ij} < U_{k+1}\}.
\]

Thus, from (54) and for \(\tilde{d}_k\) given in (45) we have

\[
\sum_{i=1}^{n} c_i I\{V_i = U_k\} + \tilde{A}_k = \sum_{i=1}^{n} \left( c_{i,J_i} I\{V_i = U_k\} + \sum_{j=1}^{J_i} (c_{i,j-1} - c_{ij}) I\{U_k \leq t_{ij} < U_{k+1}\} \right) = \tilde{d}_k,
\]

which gives (44)-(45) by putting (54) and (55) together.

From the derivation of (49) in the proof of Theorem 1, we know that for any fixed \(\beta\) satisfying \(\tilde{d}_k \geq \tilde{\delta}_k, k = 1, \cdots, N\), likelihood function in (44) is maximized by:

\[
1 - \hat{F}_n(t; \beta) = \prod_{U_k \leq t} \left( 1 - \frac{\tilde{\delta}_k}{\tilde{e}_k} \right),
\]

where \(\tilde{e}_k = \tilde{d}_k + \cdots + \tilde{d}_N, k = 1, \cdots, N\); in turn, by similar derivation to that of (28)
the profile likelihood function for $\beta_0$ is given by:

$$\ell(\beta) = L(\beta, \hat{F}_n(\cdot ; \beta)) = \prod_{k=1}^N \left( \tilde{c}_k \tilde{\delta}_k \frac{\tilde{\delta}_k}{\tilde{e}_k} \right) \left( 1 - \frac{\tilde{\delta}_k}{\tilde{e}_k} \right) \tilde{e}_k - \tilde{\delta}_k.$$

From differentiation, we obtain the profile estimating function: $\Psi_n(\beta) = n^{-1} \frac{d}{d\beta} \left( \log \ell(\beta) \right)$, which, after computation and algebraic simplification, is given by the 1st equation in (46) with the given $\tilde{Z}_k$.

From Taylor’s expansion, we use linear approximation $\log(1 - x) \approx -x$ on term $\log \left( 1 - \frac{\tilde{\delta}_k}{\tilde{e}_k} \right)$ in $\Psi_n(\beta)$ given by the 1st equation in (46), then we obtain approximated profile estimating function $\psi_n(\beta)$, which is given by the 2nd equation in (46).

Finally, using the “shifting” method on the likelihood function given in (44) which is described in Section 3.2, we follow the derivation of equation (37) step by step, then with some algebraic work, we obtain generalized profile estimating function $\Psi_{M,n}(\beta)$, which is given by the 3rd equation in (46). □