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Gradual change-point analysis based on Spearman matrices for multivariate time series

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Abstract
It may happen that the behavior of a multivariate time series is such that the underlying joint distribution is gradually moving from one distribution to another between unknown times of change. Under this context of a possible gradual-change, tests of change-point detection in the dependence structure of multivariate series are developed around the associated sequence of Spearman matrices. It is formally established that the proposed test statistics for that purpose are asymptotically marginal-free under a general strong-mixing assumption, and written as functions of integrated Brownian bridges. Consistent estimators of the pair of times of change, as well as of the before-the-change and after-the-change Spearman matrices, are also proposed. A simulation study examines the sampling properties of the introduced tools, and the methodologies are illustrated on a synthetic dataset.

Keywords α-mixing · Copula · Gradual-change model · Integrated Brownian bridge · Marginal-free procedures

1 Introduction

A problem that is often of interest is to examine whether the stochastic behavior of a given phenomenon has changed over time. This research question has generated a large body of the literature. In the context of real-valued processes, one can mention the early works of Page (1955) and Kander and Zacks (1966), where tests are devised to detect a change that may occur at an unspecified time. Several authors, in particular (Pettitt 1979; Carlstein 1988), have looked at the problem from a non-parametric point of view. The use of U-statistics was at the heart of many developments that followed, for example Gombay and Horváth

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(1995) and its generalization to α-mixing by Dehling et al. (2022). The use of the characteristic function has also gotten a lot of attention. See for example Hušková and Meintanis (2006b), Hušková and Meintanis (2006a). Many authors have also looked at the problem in the case of multivariate time series, for example Gombay and Horváth (1999) in the case of observations that are serially independent and (Inoue 2001) under serial dependence.

The above-mentioned works on the so-called change-point problem assume that a change will occur abruptly if it ever happens. This restriction may not be realistic in many situations, especially when it comes to environmental phenomena that are affected by climate change. To relax it, many authors suggested models where a given moment (e.g., mean, variance, covariance) starts to drift at an unknown time point. In that context, parametric approaches assuming a linear drift have been developed (see Bissell 1984; Gan 1992, for instance), as well as extensions to accommodate non-linear drifts, for example by Hušková (1999) and its generalization to non i.i.d. data by Aue and Steinebach (2002). More recently, Vogt and Dette (2015) tackled this problem from a non-parametric point-of-view.

An interesting alternative is to consider a model wherein the stochastic behavior of a process shifts from one distribution to another between two unknown times. This approach was taken by Brodsky and Darkhovsky (1993), where a consistent estimator is proposed for a general vector of parameters. However, this theoretical framework does not allow for the estimation of the unknown times of change nor the estimation of the distributions before and after the change. To overcome these limitations, one will consider a model proposed by Quessy (2019) for the realizations $\mathbf{Y}_1, \ldots, \mathbf{Y}_T$ of a multivariate process in $\mathbb{R}^d$. Specifically, it is assumed that the underlying distribution of this process gradually shifts from F to G between times $K_1 = \lfloor Tk_1 \rfloor$ and $K_2 = \lfloor Tk_2 \rfloor$, where $0 < k_1 < k_2 < 1$, in such a way that

$$\mathbb{P}(\mathbf{Y}_t \leq \mathbf{y}) = \left(1 - w_t^{\mathbf{k}}\right) F(\mathbf{y}) + w_t^{\mathbf{k}}\, G(\mathbf{y}), \quad \mathbf{y} \in \mathbb{R}^d, \tag{1}$$

where for $\beta : \mathbb{R} \to [0,1]$ such that $\beta(x) = 0$ when $x \leq 0$ and $\beta(x) = 1$ when $x \geq 1$,

$$w_t^{\mathbf{k}} = \beta\left(\frac{t - \lfloor Tk_1 \rfloor}{\lfloor Tk_2 \rfloor - \lfloor Tk_1 \rfloor} \right), \quad t \in \{1, \ldots, T\}. \tag{2}$$

In Quessy (2019), it is assumed that $\beta(x) = x$, but other choices are possible. Observe that (1) generalizes the abrupt-change pattern since $w_t^{\mathbf{k}} \to \mathbb{I}(t \leq k_1)$ as $k_2 \downarrow k_1$.

This paper focuses on the dependence structure of multivariate processes managed by the gradual-change model in (1). The starting point is that if F and G have continuous marginals $F_1, \ldots, F_d$ and $G_1, \ldots, G_d$, respectively, then a celebrated theorem of Sklar (1959) ensures that there exist unique copulas $C^F, C^G : [0,1]^d \to [0,1]$ such that $F(\mathbf{y}) = C^F\{F_1(y_1), \ldots, F_d(y_d)\}$ and $G(\mathbf{y}) = C^G\{G_1(y_1), \ldots, G_d(y_d)\}$ for all

$\mathbf{y} = (y_1, \dots, y_d) \in \mathbb{R}^d$. Hence, as long as one is willing to assume that the d univariate time series are stochastically stable in time in the sense that $F_\ell = G_\ell$ for each $\ell \in \{1, \dots, d\}$, the sequence of copulas $C_1, \dots, C_T$ associated to $\mathbf{Y}_1, \dots, \mathbf{Y}_T$ evolves according to the gradual-change model

$$C_t(\mathbf{u}) = \left(1 - w_t^{\mathbf{k}}\right) C^F(\mathbf{u}) + w_t^{\mathbf{k}}\, C^G(\mathbf{u}), \quad \mathbf{u} \in [0,1]^d. \tag{3}$$

A consequence of model (3) is that the associated sequence of Spearman matrices $\Sigma_1^{\mathrm{Sp}}, \dots, \Sigma_T^{\mathrm{Sp}}$ is managed by a similar gradual-change pattern. Specifically, knowing that a Spearman matrix contains the pair-by-pair Spearman's rank correlation coefficient in its off-diagonal elements, it can be shown that

$$\Sigma_t^{\mathrm{Sp}} = \left(1 - w_t^{\mathbf{k}}\right) \Sigma_F^{\mathrm{Sp}} + w_t^{\mathbf{k}}\, \Sigma_G^{\mathrm{Sp}}, \tag{4}$$

where Σ_F^{Sp} and Σ_G^{Sp} are the Spearman matrices of F and G, respectively. This follows readily from the fact that as long as the marginals are continuous, Spearman's measure of association expresses as a linear functional of the copula up to normalizing constants (see (5) in Sect. 2). The model in (4) will be exploited for three purposes:

(1) To develop formal nonparametric tests to check whether a significant gradual change occurred in a multivariate time series under a general α-mixing condition;
(2) To provide a consistent estimator of the pair of times of change $\mathbf{k} = (k_1, k_2)$, where $0 < k_1 < k_2 < 1$;
(3) To provide estimators of the Spearman matrices Σ_F^{Sp} and Σ_G^{Sp} that correspond to the before-the-change and after-the-change dependence structures.

The fact that these tools are based on the well-known Spearman measure of association facilitates the interpretation of the results. Such approaches based on a dependence measure were also taken by Quessy et al. (2013), Dehling et al. (2017) using Kendall's tau and by Wied et al. (2014), Kojadinovic et al. (2016) based on Spearman's rho. However, these methodologies were developed under an abrupt-change scenario. So is the case for the majority of change-point tests for copulas, for example Bücher and Ruppert (2013), Bücher et al. (2014), and Nasri et al. (2022).

The remainder of the paper is organized as follows. Asymptotically distribution-free tests of change-point detection against gradual changes are proposed in Sect. 2 and their performance under several scenarios is investigated in Sect. 3. In Sect. 4, consistent estimators of the location of the change and of the before-the-change and after-the-change Spearman matrices are developed. Their efficiency is investigated with simulations. The introduced tools are illustrated on a synthetic dataset in Sect. 5. The mathematical proofs can be found in Appendix A and the MATLAB code is available at www.uqtr.ca/MyMatlabWebpage.

2 Tests of change-point detection

2.1 A criterion for testing the null hypothesis of no change

The procedures that will be developed in the sequel are based on a formalization by Quessy (2019) of a model proposed by Lombard (1987) for tackling gradual changes in a given moment of a univariate series. To this end, define the standardized scalar product between two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^T$ as

$$\langle \mathbf{a}, \mathbf{b} \rangle = \frac{1}{T} \sum_{t=1}^{T} \left(a_t - \bar{a}\right)\left(b_t - \bar{b}\right) = \frac{1}{T} \sum_{t=1}^{T} \left(a_t - \bar{a}\right) b_t,$$

where $\bar{a}$ and $\bar{b}$ are the means, respectively, of $\mathbf{a}$ and $\mathbf{b}$. Now let $C_{\ell\ell',t}$ be the copula of the ℓ-th and ℓ'-th variables at time $t \in \{1, \ldots, T\}$. From a well-known relationship between the copula of a population and Spearman's measure of association (see Nelsen 2006, for instance), Spearman's rank correlation coefficient between the ℓ-th and ℓ'-th variables at time $t \in \{1, \ldots, T\}$ may be expressed by

$$\rho^{\mathrm{Sp}}_{\ell\ell',t} = 12 \int_{[0,1]^2} C_{\ell\ell',t}(u, v)\, \mathrm{d}u\, \mathrm{d}v - 3. \tag{5}$$

Letting $\mathbf{w}^{\mathbf{k}} = (w^{\mathbf{k}}_1, \ldots, w^{\mathbf{k}}_T)$ be the vector of the weights defined in (2) and defining $\mathscr{S}^{\mathrm{Sp}}_{\ell\ell'} = (\rho^{\mathrm{Sp}}_{\ell\ell',1}, \ldots, \rho^{\mathrm{Sp}}_{\ell\ell',T})$, introduce the symmetric matrix $\Upsilon^{\mathbf{k}} \in \mathbb{R}^{d \times d}$ such that $\Upsilon^{\mathbf{k}}_{11} = \cdots = \Upsilon^{\mathbf{k}}_{dd} = 0$ and for each $\ell < \ell' \in \{1, \ldots, d\}$,

$$\Upsilon^{\mathbf{k}}_{\ell\ell'} = \left\langle \mathbf{w}^{\mathbf{k}}, \mathscr{S}^{\mathrm{Sp}}_{\ell\ell'} \right\rangle = \frac{1}{T} \sum_{t=1}^{T} \left(w^{\mathbf{k}}_t - \bar{w}^{\mathbf{k}}\right) \rho^{\mathrm{Sp}}_{\ell\ell',t}. \tag{6}$$

The following proposition suggests that $\Upsilon^{\mathbf{k}}$ is a reasonable measure of the discrepancy between the before-the-change and after-the-change distributions at the level of their Spearman matrices. This simple result will serve as a criterion for determining the presence or absence of a significant change in the dependence structure of a multivariate time series.

Proposition 1 *If the sequence of copulas $C_1, \ldots, C_T$ follows the gradual-change model in* (3) *with time of change $\mathbf{k}_0 \in \Delta = \{(k_1, k_2) \in (0,1)^2 : k_1 < k_2\}$, then*

$$\Upsilon^{\mathbf{k}} = \left(\Sigma^{\mathrm{Sp}}_G - \Sigma^{\mathrm{Sp}}_F\right) \left\langle \mathbf{w}^{\mathbf{k}}, \mathbf{w}^{\mathbf{k}_0} \right\rangle.$$

The hypothesis of no gradual change under the model in (3) means that $C^F = C^G$. It then entails that the corresponding Spearman matrices are stable in time, i.e., $\Sigma^{\mathrm{Sp}}_G = \Sigma^{\mathrm{Sp}}_F$. The converse implication is, of course, not true in general. In view of Proposition 1, the null and alternative hypotheses of a gradual change-point can be formulated as

$$\mathbb{H}_0^{\text{Cp}} : \sup_{\mathbf{k}\in\Delta} \mathscr{V}(\Upsilon^{\mathbf{k}}) = \mathbf{0}_L \quad \text{against} \quad \mathbb{H}_1^{\text{Cp}} : \sup_{\mathbf{k}\in\Delta} \mathscr{V}(\Upsilon^{\mathbf{k}}) \neq \mathbf{0}_L, \tag{7}$$

where $\Delta = \{(k_1, k_2) \in (0,1)^2 : k_1 < k_2\}$, $L = d(d-1)/2$ and $\mathscr{V}$ is the operator that *vectorizes* symmetric matrices. Specifically, if $\mathscr{T}_d$ is the space of $d \times d$ symmetric matrices and $A \in \mathscr{T}_d$, then $\mathscr{V} : \mathscr{T}_d \mapsto \mathbb{R}^L$ is such that

$$\mathscr{V}(A)_j = A_{\ell\ell'}, \quad \text{with } j = (\ell-1)d + \ell' - \binom{\ell+1}{2}.$$

For example, $\mathscr{V}(A) = A_{12}$ when $d = 2$ and $\mathscr{V}(A) = (A_{12}, A_{13}, A_{23})$ when $d = 3$.

2.2 Sample version of $\Upsilon^{\mathbf{k}}$

Let $\mathbf{Y}_1, \dots, \mathbf{Y}_T$, where $\mathbf{Y}_t = (Y_{t1}, \dots, Y_{td})$, be a realization of a process $(\mathbf{Y}_t)_{t\in\mathbb{Z}}$ that follows the gradual-change model (1) with time of change $\mathbf{k} \in \Delta$. The process $(\mathbf{Y}_t)_{t\in\mathbb{Z}}$ is assumed to satisfy the *strong-mixing* assumption. This setup covers a wide variety of models including the autoregressive-moving-average and autoregressive conditional heteroskedasticity processes. Specifically, let $(\eta_t)_{t\in\mathbb{Z}}$ be a sequence defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and write $\mathscr{F}_a^b$ for the σ-field generated by $(\eta_t)_{a\le t\le b}$, where $a, b \in \mathbb{Z} \cup \{-\infty, +\infty\}$. As defined for instance by Rio (2000), the α-mixing coefficients of $(\eta_t)_{t\in\mathbb{Z}}$ are given by $\alpha_\eta(0) = 1/2$ and for $r \ge 1$,

$$\alpha_\eta(r) = \sup_{t\in\mathbb{Z}} \sup_{(A,B)\in\mathscr{F}_{-\infty}^t \times \mathscr{F}_{t+r}^\infty} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\,\mathbb{P}(B)|.$$

The process $(\eta_t)_{t\in\mathbb{Z}}$ is said to be strongly mixing if $\alpha_\eta(r) \to 0$ as $r \to \infty$.

In the sequel, it is assumed that the d univariate time series are stochastically stable in time with respective marginal distributions $F_1, \dots, F_d$. One can then define $\mathbf{U}_t = (F_1(Y_{t1}), \dots, F_d(Y_{td}))$ for each $t \in \{1, \dots, T\}$, so that the process $(\mathbf{U}_t)_{t\in\mathbb{Z}}$ is managed by a sequence of copulas that evolves according to (3). Usually, $\mathbf{U}_t$ is not observable since the marginal distributions are unknown. It is customary in this case to use $\widehat{\mathbf{U}}_t = (\widehat{U}_{t1}, \dots, \widehat{U}_{td})$ instead, where $\widehat{U}_{t\ell} = \widehat{F}_\ell(Y_{t\ell})$ and $\widehat{F}_\ell$ is the empirical distribution function based on $Y_{1\ell}, \dots, Y_{T\ell}$.

In order to provide a sample version for $\Upsilon^{\mathbf{k}} \in \mathbb{R}^{d\times d}$, consider the estimator of $\rho_{\ell\ell',t}^{\text{Sp}}$ given by $\widehat{\rho}_{\ell\ell',t}^{\text{Sp}} = 12(1 - \widehat{U}_{t\ell})(1 - \widehat{U}_{t\ell'}) - 3$. Letting $\widehat{\mathscr{S}}_{\ell\ell'}^{\text{Sp}} = 12((1 - \widehat{U}_{1\ell})(1 - \widehat{U}_{1\ell'}), \dots, (1 - \widehat{U}_{T\ell})(1 - \widehat{U}_{T\ell'})) - 3$, an estimator of $\Upsilon^{\mathbf{k}}$ is given by $\widehat{\Upsilon}^{\mathbf{k}} \in \mathbb{R}^{d\times d}$ such that $\widehat{\Upsilon}_{11}^{\mathbf{k}} = \cdots = \widehat{\Upsilon}_{dd}^{\mathbf{k}} = 0$ and for $\ell < \ell' \in \{1, \dots, d\}$,

$$\widehat{\Upsilon}_{\ell\ell'}^{\mathbf{k}} = \left\langle \mathbf{w}^{\mathbf{k}}, \widehat{\mathscr{S}}_{\ell\ell'}^{\text{Sp}} \right\rangle = \frac{12}{T} \sum\nolimits_{t=1}^{T} \left(w_t^{\mathbf{k}} - \bar{w}^{\mathbf{k}}\right)\left(1 - \widehat{U}_{t\ell}\right)\left(1 - \widehat{U}_{t\ell'}\right). \tag{8}$$

2.3 Weak convergence of test statistics for change-point detection

It is common that empirical processes for abrupt changes have limits that are distribution-free up to an asymptotic variance that can be easily estimated. As established in Proposition 2 that follows, this feature also occurs for $\widehat{\Upsilon}^{\mathbf{k}}$ under a gradual-change pattern. Before stating the result, let $\mathbb{B}$ be the Brownian bridge, i.e., $\mathbb{B}$ is a Gaussian process on [0, 1] such that $\mathrm{E}\{\mathbb{B}(s)\} = 0$ and $\mathrm{E}\{\mathbb{B}(s)\,\mathbb{B}(s')\} = \min(s, s') - ss'$ for all $s, s' \in [0, 1]$. Then, define for $\mathbf{k} = (k_1, k_2) \in \Delta$ the integrated Brownian bridge as

$$\widetilde{\mathbb{B}}(\mathbf{k}) = \frac{1}{k_2 - k_1} \int_{k_1}^{k_2} \mathbb{B}(s)\,\mathrm{d}s. \tag{9}$$

For $\ell < \ell' \in \{1, \dots, d\}$ and $m < m' \in \{1, \dots, d\}$, let also $\Omega \in \mathbb{R}^{L \times L}$ be such that for $j := (\ell - 1)d + \ell' - \binom{\ell+1}{2}$ and $k := (m - 1)d + m' - \binom{m+1}{2}$,

$$\Omega_{jk} = \sum_{t \in \mathbb{Z}} \mathrm{cov}\left\{(1 - U_{0\ell})(1 - U_{0\ell'}), (1 - U_{tm})(1 - U_{tm'})\right\}.$$

Here and in the sequel, $\rightsquigarrow$ means "converges in distribution to".

Proposition 2 *Suppose that the α-mixing coefficients of $(\mathbf{U}_t)_{t \in \mathbb{Z}}$ are such that $\alpha(r) = O(r^{-4-d(1+\epsilon)})$ for some $\epsilon \in (0, 1/4]$. Then under the null hypothesis $\mathbb{H}_0^{\mathrm{Cp}} : C_F = C_G$,*

$$\frac{\sqrt{T}\,\mathscr{V}\left(\widehat{\Upsilon}^{\mathbf{k}}\right)\Omega^{-1/2}}{12} \rightsquigarrow \left(\widetilde{\mathbb{B}}_1(\mathbf{k}), \dots, \widetilde{\mathbb{B}}_L(\mathbf{k})\right),$$

where $\widetilde{\mathbb{B}}_1(\mathbf{k}), \dots, \widetilde{\mathbb{B}}_L(\mathbf{k})$ are independent integrated Brownian bridges. The convergence takes place in $\ell^\infty(\Delta) \times \cdots \times \ell^\infty(\Delta)$, where $\ell^\infty(\Delta)$ is the space of bounded functions on Δ equipped with the supremum norm.

The formulation of the null and alternative hypotheses in (7) suggests to base tests of change-point detection on supremum statistics of the form

$$S_{T,p} = \sup_{\mathbf{k} \in \Delta} \eta(\mathbf{k}) \left\| \frac{\mathscr{V}\left(\widehat{\Upsilon}^{\mathbf{k}}\right)\Omega^{-1/2}}{12} \right\|_p, \tag{10}$$

where $\|\ \|_p$ is the L_p-norm and $\eta : \Delta \to \mathbb{R}^+$ is a continuous weight function. The consequence of Proposition 2 on the asymptotic behavior of this statistic under the null hypothesis is immediate.

Proposition 3 *Suppose that the* α*-mixing coefficients of* $(\mathbf{U}_t)_{t\in\mathbb{Z}}$ *are such that* $\alpha(r) = O(r^{-4-d(1+\epsilon)})$ *for some* $\epsilon \in (0, 1/4]$*. If* $\widetilde{\mathbb{B}}_1(\mathbf{k}), \ldots, \widetilde{\mathbb{B}}_L(\mathbf{k})$ *are independent integrated Brownian bridges, one has under the null hypothesis* $\mathbb{H}_0^{\mathrm{Cp}} : C_F = C_G$ *that*

$$\sqrt{T}\, S_{T,p} \rightsquigarrow \sup_{\mathbf{k}\in\Delta} \eta(\mathbf{k}) \left\| (\widetilde{\mathbb{B}}_1(\mathbf{k}), \ldots, \widetilde{\mathbb{B}}_L(\mathbf{k})) \right\|_p .$$

2.4 Computation of critical values

The asymptotic results of Proposition 3 will still hold if Ω is replaced by a consistent estimator $\widehat{\Omega} \in \mathbb{R}^{d\times d}$. Such an estimator can be based on the kernel covariance estimator for autocorrelated data proposed by Andrews (1991). Specifically, define $\mathbf{Z}_t = (Z_{t1}, \ldots, Z_{tL})$ with

$$Z_{tj} = \left(1 - \widehat{U}_{t\ell}\right)\left(1 - \widehat{U}_{t\ell'}\right) - \frac{1}{T}\sum_{t=1}^{T} \left(1 - \widehat{U}_{t\ell}\right)\left(1 - \widehat{U}_{t\ell'}\right).$$

Equation (2.5) of Andrews (1991) with $r = 1$ can then be rewritten

$$\widehat{\Omega} = \frac{1}{T-1}\left[\sum_{t=1}^{T} \mathbf{Z}_t^\top \mathbf{Z}_t + \sum_{j=1}^{T-1} \gamma_j \left\{ \sum_{t=j+1}^{T} \left(\mathbf{Z}_t^\top \mathbf{Z}_{t-j} + \mathbf{Z}_{t-j}^\top \mathbf{Z}_t\right)\right\}\right], \tag{11}$$

where $\gamma_1, \ldots, \gamma_{T-1}$ is a sequence of decreasing weights. Whereas the version proposed here is not based on the raw data $\mathbf{U}_1, \ldots, \mathbf{U}_T$ but on their estimated versions $\widehat{\mathbf{U}}_1, \ldots, \widehat{\mathbf{U}}_T$, the result on the consistency of $\widehat{\Omega}$ by Andrews (1991) still holds. Grossly, the difference between Z_{tj} and $Z_{tj}^\star = (1 - U_{t\ell})(1 - U_{t\ell'}) - (1/T)\sum_{t=1}^{T}(1 - U_{t\ell})(1 - U_{t\ell'})$ can be seen to be $o_{\mathbb{P}}(1/\sqrt{T})$. The estimated version of the test statistic $S_{T,p}$ defined in (10) is then

$$\widehat{S}_p = \sup_{\mathbf{k}\in\Delta} \eta(\mathbf{k}) \left\| \frac{\mathcal{V}(\widehat{Y}^{\mathbf{k}})\, \widehat{\Omega}^{-1/2}}{12} \right\|_p .$$

A test of gradual change-point based on $\widehat{S}_p$ consists in rejecting $\mathbb{H}_0^{\mathrm{Cp}}$ at level α when $\sqrt{T}\,\widehat{S}_p > q_p(\alpha)$, where $q_p(\alpha)$ is the α-th upper percentile of the distribution of

Table 1 Estimation, based on 10,000 samples of size $T = 5000$, of the upper percentiles of the limit distribution of $\sqrt{T}\widehat{S}_1$, $\sqrt{T}\widehat{S}_2$, $\sqrt{T}\widehat{S}_\infty$ and $\sqrt{T}\widehat{S}_\infty^{\mathrm{Abr}}$ under $\mathbb{H}_0^{\mathrm{Cp}}$ when $\alpha = .05$

d	$q_1(\alpha)$	$q_2(\alpha)$	$q_\infty(\alpha)$	$q_\infty^{\mathrm{Abr}}(\alpha)$
2	0.5721	0.5721	0.5721	1.3049
3	1.2444	0.8143	0.6912	1.4934
4	2.1289	1.0259	0.7617	1.6028

$$\sup_{\mathbf{k}\in\Delta} \eta(\mathbf{k}) \left\| (\widetilde{\mathbb{B}}_1(\mathbf{k}), \ldots, \widetilde{\mathbb{B}}_L(\mathbf{k})) \right\|_p.$$

Table 1 reports the estimated values of $q_1(\alpha)$, $q_2(\alpha)$ and $q_\infty(\alpha)$ when $\eta(\mathbf{k}) = k_2 - k_1$ obtained from 10,000 replicates of size $T = 5000$ from an i.i.d. Gaussian time series with an identity variance–covariance matrix. The estimated percentiles of the statistic $\sqrt{T}\widehat{S}_\infty^{\rm Abr}$ that is specifically designed to detect abrupt changes has also been included; this statistic is described in Sect. 3.2. In view of Proposition 3, any distribution under the null hypothesis would provide such valid critical values.

3 Performance analysis of the tests of change-point detection

3.1 Serial models under investigation

This section explores the performance of the proposed tests for change-point detection. To this end, data will be simulated from stochastic processes of the form $\mathbf{Y}_t = \mathbf{X}_t + \mathbf{e}_t$, where $(\mathbf{X}_t)_{t\in\mathbb{Z}}$ is a process of independent d-dimensional Gaussian vectors with mean zero and a correlation structure $\Sigma_t^{\mathbf{X}}$ subject to the gradual-change pattern $\Sigma_t^{\mathbf{X}} = (1 - w_t^{\mathbf{k}})\Sigma_F + w_t^{\mathbf{k}}\,\Sigma_G$. The innovation $(\mathbf{e}_t)_{t\in\mathbb{Z}}$ is a stationary process of centered d-variate Gaussian vectors with covariance matrix $\theta\, I_d$ for some $\theta \in \mathbb{R}$. The process $(\mathbf{Y}_t)_{t\in\mathbb{Z}}$ inherits the serial nature of $\mathbf{e}_t$, as $\mathrm{E}(\mathbf{Y}_t^\top \mathbf{Y}_{t-q}) = \mathrm{E}(\mathbf{e}_t^\top \mathbf{e}_{t-q})$ for $q > 0$. Also, since $\mathrm{E}(\mathbf{Y}_t) = \mathbf{0}_d$ and $\mathrm{E}(\mathbf{Y}_t^\top \mathbf{Y}_t) = (1 - w_t^{\mathbf{k}})(\Sigma_F + \theta\, I_d) + w_t^{\mathbf{k}}\,(\Sigma_G + \theta\, I_d)$, the marginal distributions of $(\mathbf{Y}_t)_{t\in\mathbb{Z}}$ are centered Normal with variance $1 + \theta$ and its correlation structure follows the gradual-change pattern

$$\widetilde{\Sigma}_t = (1 - w_t^{\mathbf{k}})\left(\frac{\Sigma_F + \theta\, I_d}{1+\theta}\right) + w_t^{\mathbf{k}}\left(\frac{\Sigma_G + \theta\, I_d}{1+\theta}\right).$$

From the well-known relationship $\rho = 2\sin(\pi\rho^{\rm Sp}/6)$ between the correlation coefficient ρ and Spearman's rank correlation $\rho^{\rm Sp}$ in the bivariate Normal distribution (see Nelsen (2006), for instance), the process $(\mathbf{Y}_t)_{t\in\mathbb{Z}}$ leads to a sequence of Spearman matrices that follows the pattern in (4) with

$$\left(\Sigma_F^{\rm Sp}\right)_{\ell\ell'} = \frac{6}{\pi}\sin^{-1}\left\{\frac{(\Sigma_F)_{\ell\ell'}}{2(1+\theta)}\right\} \quad\text{and}\quad \left(\Sigma_G^{\rm Sp}\right)_{\ell\ell'} = \frac{6}{\pi}\sin^{-1}\left\{\frac{(\Sigma_G)_{\ell\ell'}}{2(1+\theta)}\right\}. \tag{12}$$

In the following investigations, two types of serial dependence will be considered for the process $(\mathbf{e}_t)_{t\in\mathbb{Z}}$. The first is the moving average process of order one $\mathbf{e}_t = (\beta\,\mathbf{Z}_{t-1} + \mathbf{Z}_t)/\sqrt{1+\beta^2}$, where $\mathbf{Z}_1, \mathbf{Z}_2, \ldots,$ are i.i.d. centered d-variate Normal with covariance $\theta\, I_d$. The second model is the autoregressive process of order one $\mathbf{e}_t = \beta\,\mathbf{e}_{t-1} + \sqrt{1-\beta^2}\,\mathbf{Z}_t$. As required, $\mathrm{E}(\mathbf{e}_t^\top \mathbf{e}_t) = \theta\, I_d$ in both cases.

The serial parameters are fixed to $\beta = 1/2$ and $\theta = 1/4$ all over. For simplicity, the correlation matrices Σ_F and Σ_G are taken equicorrelated, i.e., their off-diagonal elements are $(\Sigma_F)_{\ell\ell'} = \rho_F$ and $(\Sigma_G)_{\ell\ell'} = \rho_G$ for all $\ell < \ell' \in \{1, \ldots, d\}$. In order that $(\Sigma_F^{\mathrm{Sp}})_{\ell\ell'} = 1/4$ and $(\Sigma_G^{\mathrm{Sp}})_{\ell\ell'} = 3/4$, one must set $\rho_F = 0.327$ and $\rho_G = 0.957$.

3.2 Test statistics under investigation

The performance of the tests of change-point detection based on the statistics $\widehat{S}_1$, $\widehat{S}_2$ and $\widehat{S}_\infty$ will be investigated when the weight function is set to $\eta(\mathbf{k}) = k_2 - k_1$. Other possibilities exist, such as putting $\eta(\mathbf{k}) = 1$, but simulations not presented indicate that the resulting tests are less efficient in terms of size and power. To serve as a benchmark, a version of $\widehat{S}_\infty$ specific for abrupt changes has been included in the investigation. The latter is denoted by $\widehat{S}_\infty^{\mathrm{Abr}}$ and arises as $k_2 \to k_1 = k \in (0, 1)$; it corresponds in fact to a statistic proposed by Wied et al. (2014). The critical values at level $\alpha = .05$ for these four tests are found in Table 1. The weights in the definition of the estimator of the covariance matrix Ω in (11) are based on the Parzen kernel introduced by Parzen (1962), i.e., $\gamma_j = \gamma(j/\sqrt{T})$ with $\gamma(x) = (1 - 6x^2 + 6|x|^3)\, \mathbb{I}(|x| \leq 1/2) + 2(1 - |x|)^3\, \mathbb{I}(1/2 < |x| \leq 1)$. Finally, it should be noted that the transition function in the definition of the weights in (2) is assumed to be $\beta(x) = x$.

3.3 Size and robustness of the tests

Before exploring the power of a test, it is important to check its ability to keep the nominal level under the null hypothesis. It is also an opportunity to explore at which point the assumption that $\beta(x) = x$ has an influence on the results. Since it seems reasonable to assume strictly increasing transitions, three scenarios have been considered, namely $\beta(x) \in \{x, x^2, x^3\}$. The estimated probabilities of rejection of the null hypothesis under $\mathbb{H}_0^{\mathrm{Cp}}$ for sample sizes $T \in \{350, 700, 1400\}$ when $d \in \{2, 3, 4\}$ are found in Table 2. The sequences have been simulated using the autoregressive process described in Sect. 3.1.

The first thing to observe is the robustness of the tests against a misspecification of the transition function β, i.e., when $\beta(x) \in \{x^2, x^3\}$. When $\beta(x) = x$, the tests keep their 5% nominal level well when $d = 2$ and $d = 3$, but larger sample sizes are needed in the trivariate case. When $d = 4$, the four tests are conservative, even though the probabilities of rejection tend toward the theoretical 5% level as T increases. This should not be surprising, as the number of elements of the Spearman matrix increases quadratically with d. Finally, note that results not reported here show that the behavior of the tests is quite similar under serial dependence managed by the moving-average process.

3.4 Power under serial dependence

The power of the tests based on $\widehat{S}_1$, $\widehat{S}_2$, $\widehat{S}_\infty$ and $\widehat{S}_\infty^{\mathrm{Abr}}$ has been investigated under the autoregressive and moving-average processes when the Spearman matrix gradually moves from Σ_F^{Sp} to Σ_G^{Sp} for three patterns of times of change, namely $\mathbf{k}_1 = (1/3, 1/2)$, $\mathbf{k}_2 = (1/3, 2/3)$ and $\mathbf{k}_3 = (1/2, 1/2 + 1/T)$. The results in Table 3 concern the AR(1) process, whereas those in Table 4 concern the MA(1) process.

Under both serial patterns, one can see that the power increases as the sample size increases for all the considered dimensions and locations of the change. This is expected. Overall, the tests become less powerful as the dimension increases. The

Table 2 Percentages of rejection, as estimated from 2500 replicates, of the tests of gradual change-point detection based on $\widehat{S}_1$, $\widehat{S}_2$, $\widehat{S}_\infty$ and $\widehat{S}_\infty^{\mathrm{Abr}}$ under AR(1) Gaussian times series when $\beta = .5$ and $\sigma = .25$ under $\mathbb{H}_0^{\mathrm{Cp}}$

Spearman matrix	Test statistic	$T = 350$			$T = 700$			$T = 1400$		
		x	x^2	x^3	x	x^2	x^3	x	x^2	x^3
Σ_F^{Sp}	$\widehat{S}_\infty$	5.32	5.20	5.08	4.12	4.76	4.68	5.44	4.84	5.60
	$\widehat{S}_\infty^{\mathrm{Abr}}$	4.40	4.00	4.56	4.20	3.48	3.84	4.48	4.40	5.24
Σ_G^{Sp}	$\widehat{S}_\infty$	5.00	4.80	4.84	5.52	4.32	5.04	4.40	5.16	4.88
	$\widehat{S}_\infty^{\mathrm{Abr}}$	4.40	3.80	3.76	4.40	4.44	4.20	4.56	4.20	4.60
Σ_F^{Sp}	$\widehat{S}_1$	3.32	3.88	4.28	3.88	4.12	4.16	4.32	4.04	4.60
	$\widehat{S}_2$	2.84	4.04	3.44	3.56	3.96	3.76	4.00	3.92	4.16
	$\widehat{S}_\infty$	3.56	3.92	4.32	4.20	4.32	4.32	3.76	4.16	4.32
	$\widehat{S}_\infty^{\mathrm{Abr}}$	2.24	3.00	2.28	2.64	3.64	2.96	3.16	3.76	3.56
Σ_G^{Sp}	$\widehat{S}_1$	3.76	4.04	3.84	3.76	3.84	4.36	3.80	4.48	4.28
	$\widehat{S}_2$	3.60	3.36	3.80	3.20	3.56	4.44	3.92	4.64	4.04
	$\widehat{S}_\infty$	4.48	3.64	4.28	4.00	3.96	4.60	4.64	5.52	5.20
	$\widehat{S}_\infty^{\mathrm{Abr}}$	1.92	2.08	2.68	3.24	2.72	3.96	3.56	3.56	3.88
Σ_F^{Sp}	$\widehat{S}_1$	1.52	2.24	2.36	2.68	2.96	3.40	3.88	3.28	3.52
	$\widehat{S}_2$	1.32	1.48	1.28	2.04	2.12	2.44	3.08	3.20	3.36
	$\widehat{S}_\infty$	2.32	2.68	2.76	2.64	2.56	2.92	3.28	3.52	4.40
	$\widehat{S}_\infty^{\mathrm{Abr}}$	0.96	1.12	1.24	2.08	1.76	2.20	2.92	2.64	2.76
Σ_G^{Sp}	$\widehat{S}_1$	1.64	2.00	2.36	2.48	3.12	2.84	4.40	3.72	4.28
	$\widehat{S}_2$	1.32	1.48	1.04	2.16	2.56	2.72	3.76	3.44	3.76
	$\widehat{S}_\infty$	2.52	2.96	2.76	2.60	3.68	3.24	3.76	3.88	3.84
	$\widehat{S}_\infty^{\mathrm{Abr}}$	0.84	1.20	0.92	1.72	1.60	2.08	3.04	2.92	2.68

Upper panel: $d = 2$; middle panel: $d = 3$; lower panel: $d = 4$

performance of $\widehat{S}_1$, $\widehat{S}_2$ and $\widehat{S}_\infty$ is little influenced by the location of the change. The power of $\widehat{S}_\infty^{\text{Abr}}$ fluctuates more and achieves its best performance under an abrupt change. Among the newly introduced statistics for gradual change-points, $\widehat{S}_1$ and $\widehat{S}_2$ outperform $\widehat{S}_\infty$, especially for large sample sizes. In the case of an abrupt change, there is no price to pay for using a test specifically designed for gradual changes. In fact, the power of the three tests surpasses that of $\widehat{S}_\infty^{\text{abr}}$ when $\mathbf{k} := \mathbf{k}_3$, except when $d = 2$, in which case the four tests perform equivalently. Finally, note that the tests are slightly more powerful under the MA(1) process compared to the AR(1) process.

3.5 Power under serial independence

The results in Table 5 were obtained from serially independent observations. The tests have been performed under the assumption of serial independence, i.e., the estimation of the limit covariance function simplifies to the sample covariance of $\widehat{\mathbf{U}}_1, \dots, \widehat{\mathbf{U}}_T$. The same scenarios as those in Tables 3 and 4 have been considered when $\beta = \sigma = 0$, so that $\rho_F = 2\sin\{(\pi(1/4)/6\} = 0.261$ and $\rho_G = 2\sin\{(\pi(3/4)/6\} = 0.765$ in order that $\rho_F^{\text{Sp}} = 1/4$ and $\rho_G^{\text{Sp}} = 3/4$.

When looking at Table 5, the first thing that catches the eye is that the tests are better at keeping their nominal level compared to their behavior under serial dependence, especially when $d \in \{3, 4\}$. This suggests that the tests are not keeping their nominal level well due to the estimation of Ω in the serial case. If one looks at the power of the tests, conclusions similar to those drawn from Tables 3, 4 can

Table 3 Percentages of rejection, as estimated from 2500 replicates, of the tests of gradual change-point detection based on $\widehat{S}_1$, $\widehat{S}_2$, $\widehat{S}_\infty$ and $\widehat{S}_\infty^{\text{Abr}}$ under AR(1) Gaussian times series when $\beta = .5$ and $\sigma = .25$

Test	$T = 350$			$T = 700$			$T = 1400$		
statistic	$\mathbf{k}_1$	$\mathbf{k}_2$	$\mathbf{k}_3$	$\mathbf{k}_1$	$\mathbf{k}_2$	$\mathbf{k}_3$	$\mathbf{k}_1$	$\mathbf{k}_2$	$\mathbf{k}_3$
$\widehat{S}_\infty$	17.80	19.64	19.60	33.16	33.92	34.16	58.48	58.68	61.80
$\widehat{S}_\infty^{\text{Abr}}$	14.96	16.08	19.04	30.76	29.40	34.96	57.72	57.04	65.04
$\widehat{S}_1$	10.48	11.36	11.16	21.24	23.40	24.36	45.48	47.76	48.96
$\widehat{S}_2$	10.16	11.08	10.76	20.56	21.08	23.64	43.04	45.92	47.00
$\widehat{S}_\infty$	9.96	9.88	10.40	17.32	16.68	19.12	34.08	35.76	36.24
$\widehat{S}_\infty^{\text{Abr}}$	6.20	6.68	7.24	13.80	13.72	15.92	30.08	31.56	34.52
$\widehat{S}_1$	4.40	3.92	5.20	13.56	13.12	13.56	32.80	33.52	36.20
$\widehat{S}_2$	4.48	3.68	4.36	12.44	11.96	12.96	30.60	30.72	34.28
$\widehat{S}_\infty$	5.96	5.04	5.76	9.00	10.28	10.84	19.04	18.04	19.12
$\widehat{S}_\infty^{\text{Abr}}$	3.40	2.92	3.60	6.56	7.56	8.40	16.60	14.92	18.68

Upper panel: $d = 2$; middle panel: $d = 3$; lower panel: $d = 4$

Table 4 Percentages of rejection, as estimated from 2500 replicates, of the tests of gradual change-point detection based on $\widehat{S}_1$, $\widehat{S}_2$, $\widehat{S}_\infty$ and $\widehat{S}_\infty^{\mathrm{Abr}}$ under MA(1) Gaussian times series when $\beta = .5$ and $\sigma = .25$

Test	$T = 350$			$T = 700$			$T = 1400$		
statistic	$\mathbf{k}_1$	$\mathbf{k}_2$	$\mathbf{k}_3$	$\mathbf{k}_1$	$\mathbf{k}_2$	$\mathbf{k}_3$	$\mathbf{k}_1$	$\mathbf{k}_2$	$\mathbf{k}_3$
$\widehat{S}_\infty$	19.36	18.88	22.08	34.88	37.24	39.08	61.84	63.24	65.52
$\widehat{S}_\infty^{\mathrm{Abr}}$	16.76	16.72	20.44	32.56	34.72	38.76	62.08	61.04	69.92
$\widehat{S}_1$	10.96	11.48	13.08	21.92	22.52	24.56	47.36	48.60	54.36
$\widehat{S}_2$	10.16	10.92	12.72	22.16	21.44	23.40	45.72	45.72	50.48
$\widehat{S}_\infty$	10.72	10.04	11.72	18.76	18.16	19.68	36.44	34.48	39.36
$\widehat{S}_\infty^{\mathrm{Abr}}$	6.36	6.88	8.76	15.08	14.88	16.56	31.16	29.20	39.00
$\widehat{S}_1$	5.52	5.40	5.28	14.08	13.44	15.92	33.96	37.00	38.60
$\widehat{S}_2$	4.84	4.00	4.68	13.44	12.04	14.44	31.68	32.80	36.00
$\widehat{S}_\infty$	7.08	6.00	5.40	10.72	8.80	10.80	18.28	18.12	20.16
$\widehat{S}_\infty^{\mathrm{Abr}}$	3.72	4.04	4.44	8.24	7.24	9.40	17.56	15.36	19.36

Upper panel: $d = 2$; middle panel: $d = 3$; lower panel: $d = 4$

be drawn, except that their values here under serial independence are much higher, especially when $d \in \{3, 4\}$. Although it is not exorbitant, there is a price to be paid for considering the very general alpha-mixing setup.

Table 5 Percentages of rejection, as estimated from 2500 replicates, of the tests of gradual change-point detection based on $\widehat{S}_1$, $\widehat{S}_2$, $\widehat{S}_\infty$ and $\widehat{S}_\infty^{\mathrm{Abr}}$ under serially independent time series

Test	$T = 350$				$T = 700$				$T = 1400$			
stat	$\mathbb{H}_0^{\mathrm{Cp}}$	$\mathbf{k}_1$	$\mathbf{k}_2$	$\mathbf{k}_3$	$\mathbb{H}_0^{\mathrm{Cp}}$	$\mathbf{k}_1$	$\mathbf{k}_2$	$\mathbf{k}_3$	$\mathbb{H}_0^{\mathrm{Cp}}$	$\mathbf{k}_1$	$\mathbf{k}_2$	$\mathbf{k}_3$
$\widehat{S}_\infty$	5.04	23.20	21.68	23.60	4.72	39.84	41.72	45.08	4.84	68.60	69.72	72.16
$\widehat{S}_\infty^{\mathrm{Abr}}$	4.80	21.64	20.08	24.68	4.60	40.24	40.40	47.52	4.48	69.24	68.72	76.76
$\widehat{S}_1$	5.28	16.32	15.40	17.20	4.84	29.08	29.72	34.32	4.56	59.36	59.04	63.00
$\widehat{S}_2$	4.96	17.16	15.72	17.76	5.00	30.04	29.84	34.76	4.28	58.68	57.64	61.76
$\widehat{S}_\infty$	5.28	16.56	15.24	16.64	6.08	26.96	27.12	29.68	4.32	50.32	48.36	51.64
$\widehat{S}_\infty^{\mathrm{Abr}}$	4.56	14.84	13.56	15.52	5.04	24.16	25.16	29.60	4.56	47.84	45.88	53.48
$\widehat{S}_1$	5.40	13.36	13.16	14.76	4.52	25.92	25.32	25.80	6.20	50.36	48.52	55.72
$\widehat{S}_2$	5.40	14.92	13.84	15.64	4.64	26.64	26.08	26.84	5.80	50.68	47.96	54.24
$\widehat{S}_\infty$	5.04	15.00	12.16	13.64	5.68	21.28	19.44	20.12	5.60	37.16	31.76	36.88
$\widehat{S}_\infty^{\mathrm{Abr}}$	4.00	13.88	12.00	14.08	4.76	21.84	19.08	21.72	4.68	37.64	32.96	41.76

Upper panel: $d = 2$; middle panel: $d = 3$; lower panel: $d = 4$

4 Parameter estimation under gradual changes

4.1 Times of the change

After a formal test has established that a significant change has occurred in the sequence of Spearman matrices, it is of interest to identify the moment when this change occurred. In other words, the goal is to estimate the real time of change $\mathbf{k}_0 \in \Delta$ under the assumption that the sequence of copulas $C_1, \dots, C_T$ follows model (3). From Proposition (1) and an application of the Cauchy–Schwarz inequality,

$$\begin{aligned}\left\| \mathscr{V}\left(\Upsilon^{\mathbf{k}}\right) \right\|_p &= \left\| \mathscr{V}\left(\Sigma_G^{\mathrm{Sp}} - \Sigma_G^{\mathrm{Sp}} \right) \right\|_p \left| \langle \mathbf{w}^{\mathbf{k}}, \mathbf{w}^{\mathbf{k}_0} \rangle \right| \\ &\leq \left\| \mathscr{V}\left(\Sigma_G^{\mathrm{Sp}} - \Sigma_G^{\mathrm{Sp}} \right) \right\|_p \left\langle \mathbf{w}^{\mathbf{k}}, \mathbf{w}^{\mathbf{k}} \right\rangle^{1/2} \left\langle \mathbf{w}^{\mathbf{k}_0}, \mathbf{w}^{\mathbf{k}_0} \right\rangle^{1/2},\end{aligned}$$

with equality if and only if $\mathbf{k} = \mathbf{k}_0$. As a consequence, assuming that $\mathbf{k} \in \Delta^\star = \{(k_1, k_2) \in (0,1)^2 : k_1 < k_2, \langle \mathbf{w}^{\mathbf{k}}, \mathbf{w}^{\mathbf{k}} \rangle > 0\}$,

$$\frac{\left\| \mathscr{V}\left(\Upsilon^{\mathbf{k}}\right) \right\|_p}{\langle \mathbf{w}^{\mathbf{k}}, \mathbf{w}^{\mathbf{k}} \rangle^{1/2}} \leq \left\| \mathscr{V}\left(\Sigma_G^{\mathrm{Sp}} - \Sigma_G^{\mathrm{Sp}} \right) \right\|_p \langle \mathbf{w}^{\mathbf{k}_0}, \mathbf{w}^{\mathbf{k}_0} \rangle^{1/2}.$$

As long as $\Sigma_F^{\mathrm{Sp}} \neq \Sigma_G^{\mathrm{Sp}}$, it then follows that

$$\operatorname*{argsup}_{\mathbf{k} \in \Delta^\star} \frac{\| \mathscr{V}(\Upsilon^{\mathbf{k}}) \|_p}{\langle \mathbf{w}^{\mathbf{k}}, \mathbf{w}^{\mathbf{k}} \rangle^{1/2}} = \mathbf{k}_0.$$

For $\widehat{\Upsilon}^{\mathbf{k}}$ defined in (8), an estimator of $\mathbf{k}_0 \in \Delta$ based on the L_p norm is then

$$\widehat{\mathbf{k}}_{0,p} = \operatorname*{argsup}_{\mathbf{k} \in \Delta^\star} \frac{\| \mathscr{V}(\widehat{\Upsilon}^{\mathbf{k}}) \|_p}{\langle \mathbf{w}^{\mathbf{k}}, \mathbf{w}^{\mathbf{k}} \rangle^{1/2}}.$$

As established next, $\widehat{\mathbf{k}}_{0,p}$ is a consistent estimator for $\mathbf{k}_0$.

Proposition 4 *Let the sequence of copulas $C_1, \dots, C_T$ follow the gradual-change model in (3) with time of change $\mathbf{k}_0 \in \Delta$ in such a way that $\Sigma_F^{\mathrm{Sp}} \neq \Sigma_G^{\mathrm{Sp}}$. Then as long as the α-mixing coefficients of $(\mathbf{U}_t)_{t \in \mathbb{Z}}$ are such that $\alpha(r) = O(r^{-4-d(1+\epsilon)})$ for some $\epsilon \in (0, 1/4]$, $\widehat{\mathbf{k}}_{0,p}$ converges in probability to $\mathbf{k}_0$.*

Table 6 reports the estimated bias and mean-squared errors of $\widehat{\mathbf{k}}_{0,1}$ under serially independent bivariate observations. In that case, $\widehat{\mathbf{k}}_{0,1} = \widehat{\mathbf{k}}_{0,2} = \widehat{\mathbf{k}}_{0,\infty}$. The data are generated such that Spearman's measure shifts from $\rho_F^{\mathrm{Sp}} = 1/4$ to $\rho_G^{\mathrm{Sp}} = 3/4$ under four locations for the change-points. As anticipated, the accuracy of the estimator increases with increasing sample size. When $k_{01} = 1/6$, the estimator has a larger bias, even when $T = 1400$. This suggests that changes close to the boundary are more difficult to estimate. In general, it can nevertheless be asserted that the proposed estimator is reliable.

4.2 Before-the-change and after-the-change Spearman matrices

The goal here is to derive estimators of the Spearman matrices under the assumption of a sequence of copulas that follows model (3). Whereas $\Upsilon^{\mathbf{k}}$ involves the two matrices of interest, as stated in Proposition 1, another measure involving Σ_F^{Sp} and Σ_G^{Sp} is needed to derive such estimators. To this end, let $\bar{\mathscr{S}} \in \mathbb{R}^{d\times d}$ be the symmetric matrix such that $\bar{\mathscr{S}}_{\ell\ell'} = (\rho^{\mathrm{Sp}}_{\ell\ell',1}, \ldots, \rho^{\mathrm{Sp}}_{\ell\ell',T})/T$ for each $\ell < \ell' \in \{1, \ldots, d\}$ and for $\mathbf{k} \in \Delta^\star$, let

$$\Psi^{\mathbf{k}} = 2\left\langle \mathbf{w}^{\mathbf{k}}, \mathbf{w}^{\mathbf{k}} \right\rangle \bar{\mathscr{S}} + \left(1 - 2\,\bar{w}^{\mathbf{k}}\right)\Upsilon^{\mathbf{k}}.$$

Under model (3) with time of change $\mathbf{k}_0$, it can be shown that

$$\Psi^{\mathbf{k}} = 2\left\langle \mathbf{w}^{\mathbf{k}}, \mathbf{w}^{\mathbf{k}} \right\rangle \left\{ \left(1 - \bar{w}^{\mathbf{k}_0}\right)\Sigma_F^{\mathrm{Sp}} + \bar{w}^{\mathbf{k}_0}\Sigma_G^{\mathrm{Sp}} \right\} + \left(1 - 2\,\bar{w}^{\mathbf{k}}\right)\left\langle \mathbf{w}^{\mathbf{k}}, \mathbf{w}^{\mathbf{k}_0} \right\rangle \left(\Sigma_G^{\mathrm{Sp}} - \Sigma_F^{\mathrm{Sp}}\right).$$

If one assumes that $\mathbf{k} = \mathbf{k}_0$, the above formula as well as the expression of $\Upsilon^{\mathbf{k}}$ in Proposition 1 reduce respectively to

Table 6 Estimation, based on 2500 replicates, of the bias and mean-squared errors of $\widehat{\mathbf{k}}_0 = (\widehat{k}_{01}, \widehat{k}_{02})$

Sample size	$\mathbf{k}_0$	Bias		Mean-squared error	
		$\widehat{k}_{01}$	$\widehat{k}_{02}$	$\widehat{k}_{01}$	$\widehat{k}_{02}$
	[1/3, 1/2]	0.069	0.077	0.043	0.041
$T = 350$	[1/3, 2/3]	0.101	− 0.064	0.048	0.039
	[1/6, 1/2]	0.228	0.075	0.097	0.046
	[1/2, 1/2]	− 0.048	0.082	0.032	0.035
	[1/3, 1/2]	0.045	0.068	0.035	0.038
$T = 700$	[1/3, 2/3]	0.097	− 0.061	0.042	0.035
	[1/6, 1/2]	0.186	0.049	0.079	0.042
	[1/2, 1/2]	− 0.050	0.075	0.024	0.027
	[1/3, 1/2]	0.006	0.050	0.028	0.031
$T = 1400$	[1/3, 2/3]	0.063	− 0.039	0.035	0.030
	[1/6, 1/2]	0.126	0.036	0.055	0.042
	[1/2, 1/2]	− 0.052	0.071	0.019	0.022
	[1/3, 1/2]	− 0.024	0.039	0.016	0.019
$T = 5000$	[1/3, 2/3]	0.027	− 0.017	0.021	0.021
	[1/6, 1/2]	0.023	0.022	0.020	0.027
	[1/2, 1/2]	− 0.040	0.048	0.008	0.009

$$\Psi^{\mathbf{k}_0} = \left\langle \mathbf{w}^{\mathbf{k}_0}, \mathbf{w}^{\mathbf{k}_0} \right\rangle \left(\Sigma_F^{\mathrm{Sp}} + \Sigma_G^{\mathrm{Sp}} \right) \quad \text{and} \quad \Upsilon^{\mathbf{k}_0} = \left\langle \mathbf{w}^{\mathbf{k}_0}, \mathbf{w}^{\mathbf{k}_0} \right\rangle \left(\Sigma_G^{\mathrm{Sp}} - \Sigma_F^{\mathrm{Sp}} \right).$$

As a consequence, one can write

$$\Sigma_F^{\mathrm{Sp}} = \frac{\Psi^{\mathbf{k}_0} - \Upsilon^{\mathbf{k}_0}}{2 \langle \mathbf{w}^{\mathbf{k}_0}, \mathbf{w}^{\mathbf{k}_0} \rangle} \quad \text{and} \quad \Sigma_G^{\mathrm{Sp}} = \frac{\Psi^{\mathbf{k}_0} + \Upsilon^{\mathbf{k}_0}}{2 \left\langle \mathbf{w}^{\mathbf{k}_0}, \mathbf{w}^{\mathbf{k}_0} \right\rangle}.$$

The fact that $\Psi^{\mathbf{k}_0}$ and $\Upsilon^{\mathbf{k}_0}$ can easily be estimated from the data will provide estimators of Σ_F^{Sp} and Σ_G^{Sp}. Specifically, letting

$$\widehat{\overline{\mathscr{S}}}_{\ell\ell'} = \frac{12}{T} \sum_{t=1}^{T} \left(1 - \widehat{U}_{t\ell} \right) \left(1 - \widehat{U}_{t\ell'} \right) - 3,$$

an estimator of $\Psi^{\mathbf{k}_0}$ is given by $\widehat{\Psi}^{\mathbf{k}_0} = 2 \langle \mathbf{w}^{\mathbf{k}_0}, \mathbf{w}^{\mathbf{k}_0} \rangle \, \widehat{\overline{\mathscr{S}}} + (1 - 2\, \bar{w}^{\mathbf{k}})\, \widehat{\Upsilon}^{\mathbf{k}_0}$. The proposed plug-in estimators of Σ_F^{Sp} and Σ_G^{Sp} are thus respectively

$$\widehat{\Sigma}_F^{\mathrm{Sp}} = \frac{\widehat{\Psi}^{\mathbf{k}_0} - \widehat{\Upsilon}^{\mathbf{k}_0}}{2 \left\langle \mathbf{w}^{\mathbf{k}_0}, \mathbf{w}^{\mathbf{k}_0} \right\rangle} = \widehat{\overline{\mathscr{S}}} - \frac{\bar{w}^{\mathbf{k}_0}}{\left\langle \mathbf{w}^{\mathbf{k}_0}, \mathbf{w}^{\mathbf{k}_0} \right\rangle} \widehat{\Upsilon}^{\mathbf{k}_0}$$

$$\text{and} \quad \widehat{\Sigma}_G^{\mathrm{Sp}} = \frac{\widehat{\Psi}^{\mathbf{k}_0} + \widehat{\Upsilon}^{\mathbf{k}_0}}{2 \left\langle \mathbf{w}^{\mathbf{k}_0}, \mathbf{w}^{\mathbf{k}_0} \right\rangle} = \widehat{\overline{\mathscr{S}}} + \frac{\left(1 - \bar{w}^{\mathbf{k}_0}\right)}{\left\langle \mathbf{w}^{\mathbf{k}_0}, \mathbf{w}^{\mathbf{k}_0} \right\rangle} \widehat{\Upsilon}^{\mathbf{k}_0}.$$

It can be shown that these estimators are asymptotically unbiased if $\mathbf{k}_0$ is set to the true time of the change. In practice, $\mathbf{k}_0$ is unknown and may be replaced by the consistent estimator $\widehat{\mathbf{k}}_{0,p}$ encountered in Sect. 4.1.

The accuracy of this estimator has been investigated in the bivariate case both when $\mathbf{k}_0$ is known and when $\mathbf{k}_0$ is estimated. The two versions are referred to $(\widehat{\rho}_F^{\mathrm{Sp}}, \widehat{\rho}_G^{\mathrm{Sp}})$ and $(\widehat{\rho}_F^{\mathrm{Sp},\star}, \widehat{\rho}_G^{\mathrm{Sp},\star})$, respectively. The results are found in Table 7 in the case of serially independent observations, such that $\rho_F^{\mathrm{Sp}} = 1/4$ and $\rho_G^{\mathrm{Sp}} = 3/4$. When $\mathbf{k}_0$ is known, the estimator has very small bias and mean-squared errors. The version where $\mathbf{k}_0$ is replaced by $\widehat{\mathbf{k}}_{0,1}$ is less accurate, showing that there is a significant price to pay for not knowing the real time of change. Nevertheless, in absolute terms, this estimator is still very accurate under all the scenarios investigated.

5 Illustration on synthetic data

A 3-dimensional time series $\mathbf{Y}_1, \ldots, \mathbf{Y}_T$ of length $T = 1250$ has been generated following one of the patterns described in Sect. 3.1 and can be found in Fig. 1. The actual parameters of this simulation will be revealed after its analysis.

The theoretical developments in this paper have been done under the assumption that the univariate time series are stable in time. This can be tested against the hypothesis of a gradual distributional change by the procedure of Quessy (2019). The latter has been performed on the three time series and found no evidence that significant changes occurred. One can then work with

Table 7 Estimation, based on 2500 replicates, of the bias and mean-squared errors of two estimators of Spearman's measure of association before and after a gradual change in the dependence structure of serially independent observations

Sample size	$\mathbf{k}_0$	Bias $\widehat{\rho}_F^{\mathrm{Sp}}$	$\widehat{\rho}_F^{\mathrm{Sp},\star}$	$\widehat{\rho}_G^{\mathrm{Sp}}$	$\widehat{\rho}_G^{\mathrm{Sp},\star}$	Mean-squared error $\widehat{\rho}_F^{\mathrm{Sp}}$	$\widehat{\rho}_F^{\mathrm{Sp},\star}$	$\widehat{\rho}_G^{\mathrm{Sp}}$	$\widehat{\rho}_G^{\mathrm{Sp},\star}$
	[1/3, 1/2]	– 0.012	– 0.089	– 0.024	0.098	0.047	0.116	0.023	0.149
$T = 350$	[1/3, 2/3]	– 0.016	– 0.090	– 0.020	0.098	0.041	0.095	0.038	0.155
	[1/6, 1/2]	– 0.008	– 0.022	– 0.022	0.069	0.076	0.142	0.020	0.153
	[1/2, 1/2]	– 0.013	– 0.099	– 0.026	0.101	0.031	0.097	0.029	0.144
	[1/3, 1/2]	– 0.004	– 0.070	– 0.013	0.073	0.023	0.052	0.012	0.048
$T = 700$	[1/3, 2/3]	– 0.012	– 0.079	– 0.004	0.080	0.021	0.050	0.019	0.071
	[1/6, 1/2]	– 0.007	– 0.027	– 0.010	0.073	0.039	0.077	0.010	0.061
	[1/2, 1/2]	– 0.010	– 0.082	– 0.007	0.080	0.016	0.037	0.014	0.046
	[1/3, 1/2]	– 0.005	– 0.055	– 0.004	0.056	0.013	0.027	0.006	0.018
$T = 1400$	[1/3, 2/3]	– 0.004	– 0.048	– 0.006	0.051	0.010	0.020	0.009	0.026
	[1/6, 1/2]	– 0.004	– 0.026	– 0.005	0.058	0.019	0.039	0.005	0.025
	[1/2, 1/2]	– 0.006	– 0.051	– 0.004	0.053	0.008	0.016	0.007	0.017

the uniformized series $\widehat{\mathbf{U}}_t = (\widehat{F}_1(Y_{t1}), \widehat{F}_2(Y_{t2}), \widehat{F}_3(Y_{t3}))$ for each $t \in \{1, \dots, 1250\}$, where $\widehat{F}_1$, $\widehat{F}_2$ and $\widehat{F}_3$ are the empirical marginal distributions.

The three tests for gradual changes investigated in Sect. 3 have been performed on $\widehat{\mathbf{U}}_1, \dots, \widehat{\mathbf{U}}_{1250}$. One obtained $\widehat{S}_1 = 1.752$ and $\widehat{S}_2 = 1.072$ and $\widehat{S}_\infty = 0.871$. All these values exceed the critical values in Table 1, so that the three tests rejected the null hypothesis $\mathbb{H}_0^{\mathrm{Cp}}$ at the 5% level. The estimations of the location of this change are

$$\widehat{\mathbf{k}}_{0,1} = [.419, .773], \quad \widehat{\mathbf{k}}_{0,2} = [.417, .774] \quad \text{and} \quad \widehat{\mathbf{k}}_{0,\infty} = [.394, .754].$$

The estimation of the before-the-change and after-the-change Spearman matrices considered a gradual change point at [.42, .77], which yielded

$$\widehat{\Sigma}_F^{\mathrm{Sp}} = \begin{pmatrix} 1 & .232 & -.150 \\ .232 & 1 & -.403 \\ -.150 & -.403 & 1 \end{pmatrix} \text{ and } \widehat{\Sigma}_G^{\mathrm{Sp}} = \begin{pmatrix} 1 & .909 & .501 \\ .909 & 1 & .375 \\ .501 & .375 & 1 \end{pmatrix}.$$

In reality, a trivariate autoregressive time series with $\beta = .15$ and $\sigma = .2$ has been simulated with marginal distributions set respectively to $\mathbb{N}(27, 4)$, $\mathrm{Exp}(12)$ and $3 \times \mathrm{Beta}(4, 2)$. The sequence of Spearman matrices has been taken to evolve between times $k_1 = .35$ and $k_2 = .7$ from

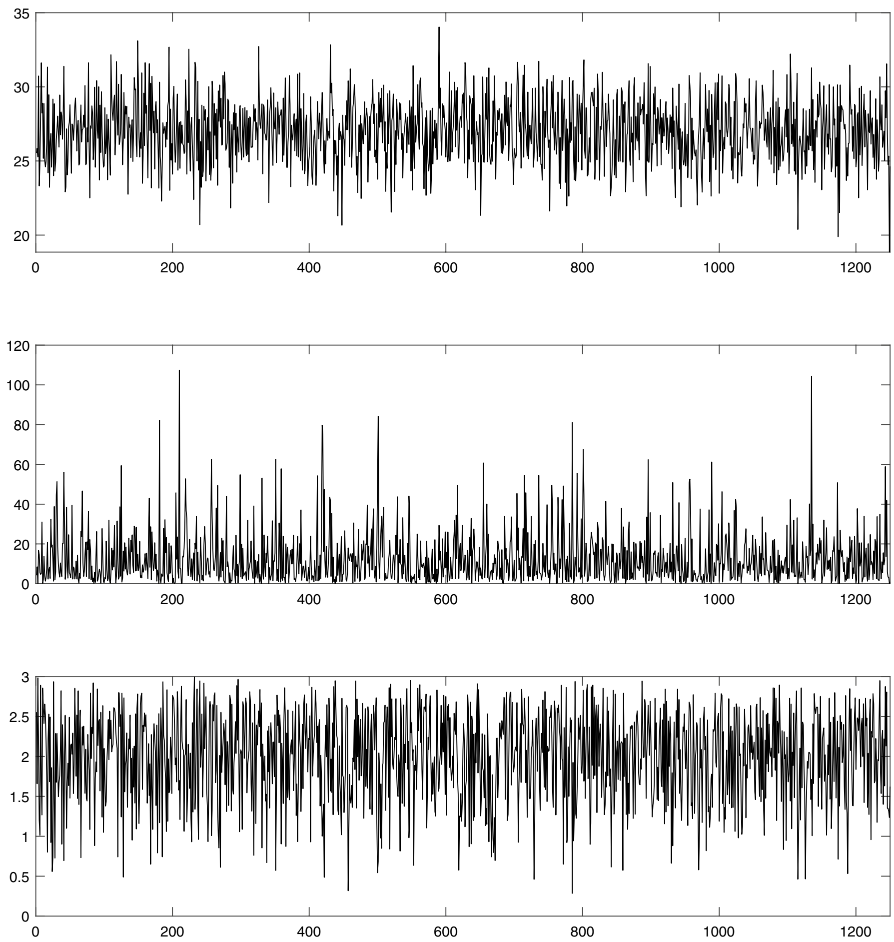

Fig. 1 Three time series of the synthetic data set analyzed in Sect. 5

$$\Sigma_F^{\mathrm{Sp}} = \begin{pmatrix} 1 & .2 & -.1 \\ .2 & 1 & -.4 \\ -.1 & -.4 & 1 \end{pmatrix} \quad \text{to} \quad \Sigma_G^{\mathrm{Sp}} = \begin{pmatrix} 1 & .8 & .4 \\ .8 & 1 & .3 \\ .4 & .3 & 1 \end{pmatrix}.$$

6 Conclusion

In this paper, a set of tools has been developed for performing statistical inference in an extension of the usual abrupt-change model. This so-called gradual change-model allows for a more realistic consideration of changes that may occur in reality, especially in natural phenomena. The introduced methodologies are valid under a general

α-mixing type of serial dependence. The starting point of this work was to consider changes that may occur at the level of the sequence of Spearman matrices induced by a multivariate time series. Doing so allows for interesting interpretations in terms of the well-known Spearman's rank correlation. In particular, asymptotically distribution-free tests of change-point detection have been proposed. In addition, estimators of the time of change, as well as of the before-the-change and after-the-change Spearman matrices, have been derived.

In future works, it would be interesting to propose versions of the tests of gradual change-point whose decision rules would be based on the serial multiplier bootstrap, in line with works by Bücher and Ruppert (2013). Another avenue worth exploring is to succeed to incorporate margins subject to possible gradual changes, which would provide a nice generalization to the procedures by Kojadinovic et al. (2016) for abrupt changes based on Spearman's measure. In the longer term, a generalization of the model to multiple changes would be a valuable contribution.

A Proofs of the main results

A.1: Proof of Proposition 1

From the definition of $\mathscr{S}^{\mathrm{Sp}}_{\ell\ell'} = (\rho^{\mathrm{Sp}}_{\ell\ell',1}, \ldots, \rho^{\mathrm{Sp}}_{\ell\ell',T})$ and in view of (4), a direct computation yields

$$
\begin{aligned}
\Upsilon^{\mathbf{k}}_{\ell\ell'} = \langle \mathbf{w}^{\mathbf{k}}, \mathscr{S}^{\mathrm{Sp}}_{\ell\ell'} \rangle &= \frac{1}{T} \sum\nolimits_{t=1}^{T} \left(w^{\mathbf{k}}_t - \bar{w}^{\mathbf{k}} \right) \rho^{\mathrm{Sp}}_{\ell\ell',t} \\
&= \frac{1}{T} \sum\nolimits_{t=1}^{T} \left(w^{\mathbf{k}}_t - \bar{w}^{\mathbf{k}} \right) \left\{ \left(1 - w^{\mathbf{k}_0}_t \right) \rho^{\mathrm{Sp}}_{F_{\ell\ell'}} + w^{\mathbf{k}_0}_t \rho^{\mathrm{Sp}}_{G_{\ell\ell'}} \right\} \\
&= \frac{1}{T} \sum\nolimits_{t=1}^{T} \left(w^{\mathbf{k}}_t - \bar{w}^{\mathbf{k}} \right) \left\{ \rho^{\mathrm{Sp}}_{F_{\ell\ell'}} + w^{\mathbf{k}_0}_t \left(\rho^{\mathrm{Sp}}_{G_{\ell\ell'}} - \rho^{\mathrm{Sp}}_{F_{\ell\ell'}} \right) \right\} \\
&= \left(\rho^{\mathrm{Sp}}_{G_{\ell\ell'}} - \rho^{\mathrm{Sp}}_{F_{\ell\ell'}} \right) \frac{1}{T} \sum\nolimits_{t=1}^{T} \left(w^{\mathbf{k}}_t - \bar{w}^{\mathbf{k}} \right) w^{\mathbf{k}_0}_t \\
&= \left(\rho^{\mathrm{Sp}}_{G_{\ell\ell'}} - \rho^{\mathrm{Sp}}_{F_{\ell\ell'}} \right) \left\langle \mathbf{w}^{\mathbf{k}}, \mathbf{w}^{\mathbf{k}_0} \right\rangle.
\end{aligned}
$$

It readily follows that $\Upsilon^{\mathbf{k}} = (\Sigma^{\mathrm{Sp}}_G - \Sigma^{\mathrm{Sp}}_F) \, \langle \mathbf{w}^{\mathbf{k}}, \mathbf{w}^{\mathbf{k}_0} \rangle$. □

A.2: Proof of Proposition 2

First recall that for any $\ell < \ell' \in \{1, \ldots, d\}$, one can associate $j \in \{1, \ldots, L\}$ via the rule $j = (\ell - 1)d + \ell' - \binom{\ell+1}{2}$. Letting $\widetilde{\mathbf{u}}_j = (\mathbf{1}_{\ell-1}, u_\ell, \mathbf{1}_{\ell'-\ell-1}, u_{\ell'}, \mathbf{1}_{d-\ell'})$, one can write

$$\begin{aligned}\sqrt{T}\,\widehat{Y}^{\mathbf{k}}_{\ell\ell'} = \sqrt{T}\,\widehat{Y}^{\mathbf{k}}_{j} &= \frac{12}{\sqrt{T}}\sum\nolimits_{t=1}^{T}\left(\mathbf{w}^{\mathbf{k}}_t - \bar{w}^{\mathbf{k}}\right)(1-\widehat{U}_{t\ell})(1-\widehat{U}_{t\ell'})\\ &= \frac{12}{\sqrt{T}}\sum\nolimits_{t=1}^{T}\left(w^{\mathbf{k}}_t - \bar{w}^{\mathbf{k}}\right)\int_{[0,1]^2}\mathbb{I}\left(\widehat{U}_{t\ell}\le u_\ell, \widehat{U}_{t\ell'}\le u_{\ell'}\right)\mathrm{d}u_\ell\mathrm{d}u_{\ell'}\\ &= 12\int_{[0,1]^2}\left\{\frac{1}{\sqrt{T}}\sum\nolimits_{t=1}^{T}\left(w^{\mathbf{k}}_t - \bar{w}^{\mathbf{k}}\right)\mathbb{I}\left(\widehat{\mathbf{U}}_t\le\widetilde{\mathbf{u}}_j\right)\right\}\mathrm{d}u_\ell\mathrm{d}u_{\ell'}\\ &= 12\int_{[0,1]^2}\widehat{\mathbb{L}}^{\mathbf{k}}(\widetilde{\mathbf{u}}_j)\,\mathrm{d}u_\ell\mathrm{d}u_{\ell'},\end{aligned}\tag{13}$$

where $\widehat{\mathbb{L}}^{\mathbf{k}}$ is the empirical process defined for $\mathbf{u}\in[0,1]^d$ by

$$\widehat{\mathbb{L}}^{\mathbf{k}}(\mathbf{u}) = \frac{1}{\sqrt{T}}\sum\nolimits_{t=1}^{T}\left(w^{\mathbf{k}}_t - \bar{w}^{\mathbf{k}}\right)\mathbb{I}\left(\widehat{\mathbf{U}}_t\le\mathbf{u}\right).$$

It will now be shown that $\widehat{\mathbb{L}}^{\mathbf{k}}$ is asymptotically equivalent to

$$\widetilde{\mathbb{L}}^{\mathbf{k}}(\mathbf{u}) = \frac{1}{\sqrt{T}}\sum\nolimits_{t=1}^{T}\left(w^{\mathbf{k}}_t - \bar{w}^{\mathbf{k}}\right)\mathbb{I}\left(\mathbf{U}_t\le\mathbf{u}\right).$$

Letting $\widehat{\mathbf{F}}^{-1}(\mathbf{u}) = (\widehat{F}_1^{-1}(u_1),\ldots,\widehat{F}_d^{-1}(u_d))$, one can write

$$\sup_{\mathbf{u}\in[0,1]^d}\left|\widehat{\mathbb{L}}^{\mathbf{k}}(\mathbf{u}) - \widetilde{\mathbb{L}}^{\mathbf{k}}(\mathbf{u})\right| \le \sup_{\mathbf{u}\in[0,1]^d}\left|\widehat{\mathbb{L}}^{\mathbf{k}}(\mathbf{u}) - \widetilde{\mathbb{L}}^{\mathbf{k}}(\widehat{\mathbf{F}}^{-1}(\mathbf{u}))\right| + \sup_{\mathbf{u}\in[0,1]^d}\left|\widetilde{\mathbb{L}}^{\mathbf{k}}(\widehat{\mathbf{F}}^{-1}(\mathbf{u})) - \widetilde{\mathbb{L}}^{\mathbf{k}}(\mathbf{u})\right|.\tag{14}$$

Under the assumption that the α-mixing coefficients of $(\mathbf{U}_t)_{t\in\mathbb{Z}}$ are such that $\alpha(r) = O(r^{-4-d(1+\epsilon)})$ for some $\epsilon\in(0,1/4]$, one can invoke Proposition 1 of Quessy (2019) and conclude that $\widetilde{\mathbb{L}}^{\mathbf{k}}$ converges weakly in the space $\Delta\times\ell^\infty([0,1]^d)$ to a centered Gaussian process $\mathbb{L}^{\mathbf{k}}$ such that for $(\mathbf{k},\mathbf{u}),(\mathbf{k}',\mathbf{u}')\in\Delta\times[0,1]^d$,

$$\mathrm{E}\left\{\mathbb{L}^{\mathbf{k}}(\mathbf{u})\,\mathbb{L}^{\mathbf{k}'}(\mathbf{u}')\right\} = \Lambda(\mathbf{k},\mathbf{k}')\sum\nolimits_{t\in\mathbb{Z}}\mathrm{cov}\left\{\mathbb{I}(\mathbf{U}_0\le\mathbf{u}),\mathbb{I}(\mathbf{U}_t\le\mathbf{u}')\right\},$$

where for $\mathbf{k}=(k_1,k_2)$ and $\mathbf{k}'=(k_1',k_2')$,

$$\Lambda(\mathbf{k},\mathbf{k}') = \int_{k_1}^{k_2}\int_{k_1'}^{k_2'}\left\{\frac{\min(s,s')-ss'}{(k_2-k_1)(k_2'-k_1')}\right\}\mathrm{d}s\,\mathrm{d}s'.$$

Since $\sqrt{T}(\widehat{\mathbf{F}}^{-1}(\mathbf{u})-\mathbf{u})$ converges weakly and $\widetilde{\mathbb{L}}^{\mathbf{k}}$ is asymptotically continuous, the second expression on the right of inequality (14) converges in probability to zero. To deal with the first expression on the right of inequality (14), note that

$$\widehat{\mathbb{L}}^{\mathbf{k}}(\mathbf{u}) - \widetilde{\mathbb{L}}^{\mathbf{k}}(\widehat{\mathbf{F}}^{-1}(\mathbf{u})) = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left(w_t^{\mathbf{k}} - \bar{w}^{\mathbf{k}}\right) \left\{ \mathbb{I}\left(\widehat{\mathbf{F}}(\mathbf{U}_t) \leq \mathbf{u}\right) - \mathbb{I}\left(\mathbf{U}_t \leq \widehat{\mathbf{F}}^{-1}(\mathbf{u})\right) \right\}.$$

Invoking Lemma 1 in Quessy (2019) that establishes that

$$\sum_{t=1}^{T} \left(w_t^{\mathbf{k}} - \bar{w}^{\mathbf{k}}\right) a_t = \frac{1}{K_2 - K_1} \sum_{j=K_1}^{K_2-1} \left(\sum_{t=1}^{j} a_t - \frac{j}{T} \sum_{t=1}^{T} a_t \right),$$

where $K_1 = \lfloor Tk_1 \rfloor$ and $K_2 = \lfloor Tk_2 \rfloor$, one has

$$\begin{aligned}
\left|\widehat{\mathbb{L}}^{\mathbf{k}}(\mathbf{u}) - \widetilde{\mathbb{L}}^{\mathbf{k}}(\widehat{\mathbf{F}}^{-1}(\mathbf{u}))\right| =& \frac{1}{\sqrt{T}(K_2 - K_1)} \left| \sum_{j=K_1}^{K_2-1} \left[\sum_{t=1}^{j} \left\{ \mathbb{I}\left(\widehat{\mathbf{F}}(\mathbf{U}_t) \leq \mathbf{u}\right) - \mathbb{I}\left(\mathbf{U}_t \leq \widehat{\mathbf{F}}^{-1}(\mathbf{u})\right) \right\} \right.\right. \\
& \left.\left. - \frac{j}{T} \sum_{t=1}^{T} \left\{ \mathbb{I}\left(\widehat{\mathbf{F}}(\mathbf{U}_t) \leq \mathbf{u}\right) - \mathbb{I}\left(\mathbf{U}_t \leq \widehat{\mathbf{F}}^{-1}(\mathbf{u})\right) \right\} \right] \right| \\
\leq & \frac{1}{\sqrt{T}(K_2 - K_1)} \sum_{j=K_1}^{K_2-1} \left| \sum_{t=1}^{j} \left\{ \mathbb{I}\left(\widehat{\mathbf{F}}(\mathbf{U}_t) \leq \mathbf{u}\right) - \mathbb{I}\left(\mathbf{U}_t \leq \widehat{\mathbf{F}}^{-1}(\mathbf{u})\right) \right\} \right| \\
& + \frac{1}{\sqrt{T}(K_2 - K_1)} \sum_{j=K_1}^{K_2-1} \frac{j}{T} \left| \sum_{t=1}^{T} \left\{ \mathbb{I}\left(\widehat{\mathbf{F}}(\mathbf{U}_t) \leq \mathbf{u}\right) - \mathbb{I}\left(\mathbf{U}_t \leq \widehat{\mathbf{F}}^{-1}(\mathbf{u})\right) \right\} \right|.
\end{aligned}$$

Now proceeding as in Fermanian et al. (2004), define the grid $G_T = \{(i_1, ..., i_d)/T : i_1, \ldots, i_d \in \{1, \ldots, T\}\}$ and observe that $\mathbb{I}(\widehat{\mathbf{F}}(\mathbf{U}_t) \leq \mathbf{u}) = \mathbb{I}(\mathbf{U}_t \leq \widehat{\mathbf{F}}^{-1}(\mathbf{u}))$ when $\mathbf{u} \in G_T$. From the fact that

$$\begin{aligned}
\left|\mathbb{I}\left(\mathbf{U}_t \leq \widehat{\mathbf{F}}^{-1}(\mathbf{s})\right) - \mathbb{I}\left(\mathbf{U}_t \leq \widehat{\mathbf{F}}^{-1}(\mathbf{s} - \mathbf{1}/T)\right)\right| =& \left| \sum_{\ell=1}^{d} \prod_{\ell'=1}^{\ell-1} \mathbb{I}\left(U_{t\ell'} \leq \widehat{F}_{\ell'}^{-1}(s_{\ell'})\right) \prod_{\ell'=\ell+1}^{d} \mathbb{I}\left(U_{t\ell'} \leq \widehat{F}_{\ell'}^{-1}(s_{\ell'} - 1/T)\right) \right. \\
& \left. \times \left\{ \mathbb{I}\left(U_{t\ell} \leq \widehat{F}_{\ell}^{-1}(s_\ell)\right) - \mathbb{I}\left(U_{t\ell} \leq \widehat{F}_{\ell}^{-1}(s_\ell - 1/T)\right) \right\} \right| \\
\leq & \sum_{\ell=1}^{d} \left| \mathbb{I}\left(U_{t\ell} \leq \widehat{F}_{\ell}^{-1}(s_\ell)\right) - \mathbb{I}\left(U_{t\ell} \leq \widehat{F}_{\ell}^{-1}(s_\ell - 1/T)\right) \right|,
\end{aligned}$$

one can write

$$\begin{aligned}
\sup_{\mathbf{u}\in[0,1]^d} \left| \sum_{t=1}^{j} \left\{ \mathbb{I}\left(\widehat{\mathbf{F}}(\mathbf{U}_t) \leq \mathbf{u}\right) - \mathbb{I}\left(\mathbf{U}_t \leq \widehat{\mathbf{F}}^{-1}(\mathbf{u})\right) \right\} \right| \leq & \max_{\mathbf{s}\in G_T} \left| \sum_{t=1}^{j} \left\{ \mathbb{I}\left(\mathbf{U}_t \leq \widehat{\mathbf{F}}^{-1}(\mathbf{s})\right) - \mathbb{I}\left(\mathbf{U}_t \leq \widehat{\mathbf{F}}^{-1}(\mathbf{s} - \mathbf{1}/T)\right) \right\} \right| \\
\leq & \max_{\mathbf{s}\in G_T} \sum_{t=1}^{j} \left| \mathbb{I}\left(\mathbf{U}_t \leq \widehat{\mathbf{F}}^{-1}(\mathbf{s})\right) - \mathbb{I}\left(\mathbf{U}_t \leq \widehat{\mathbf{F}}^{-1}(\mathbf{s} - \mathbf{1}/T)\right) \right| \\
\leq & d.
\end{aligned}$$

It follows that

$$\begin{aligned}
\sup_{\mathbf{u}\in[0,1]^d} \left|\widehat{\mathbb{L}}^{\mathbf{k}}(\mathbf{u}) - \widetilde{\mathbb{L}}^{\mathbf{k}}(\widehat{\mathbf{F}}^{-1}(\mathbf{u}))\right| \leq & \frac{d}{\sqrt{T}(K_2 - K_1)} \sum_{j=K_1}^{K_2-1} \left(1 + \frac{j}{T}\right) \\
= & \frac{d}{\sqrt{T}} \left(1 + \frac{K_1 + K_2 - 1}{2T}\right) \approx \frac{d}{\sqrt{T}} \left(1 + \frac{k_1 + k_2}{2}\right).
\end{aligned}$$

This shows that as $T \to \infty$

$$\sup_{\mathbf{u}\in[0,1]^d} \left|\widehat{\mathbb{L}}^{\mathbf{k}}(\mathbf{u}) - \widetilde{\mathbb{L}}^{\mathbf{k}}(\mathbf{u})\right| \xrightarrow{\mathbb{P}} 0.$$

As a consequence, referring back to (13),

$$\sqrt{T}\,\widehat{Y}^{\mathbf{k}}_{\ell\ell'} = \sqrt{T}\,\widehat{Y}^{\mathbf{k}}_{j} = 12\int_{[0,1]^2} \widetilde{\mathbb{L}}^{\mathbf{k}}(\widetilde{\mathbf{u}}_j)\,\mathrm{d}u_\ell \mathrm{d}u_{\ell'} + o_{\mathbb{P}}(1).$$

One can then write

$$\frac{\sqrt{T}\,\mathcal{V}\left(\widehat{Y}^{\mathbf{k}}\right)}{12} = \left(\int_{[0,1]^2} \widetilde{\mathbb{L}}^{\mathbf{k}}(\widetilde{\mathbf{u}}_1)\,\mathrm{d}u_1\mathrm{d}u_2, \dots, \int_{[0,1]^2} \widetilde{\mathbb{L}}^{\mathbf{k}}(\widetilde{\mathbf{u}}_L)\,\mathrm{d}u_{d-1}\mathrm{d}u_d\right) + o_{\mathbb{P}}(1).$$

An application of the Continuous mapping Theorem then ensures that $\sqrt{T}\,\mathcal{V}(\widehat{Y}^{\mathbf{k}})/12$ converges weakly to a vector of processes of the form $\mathbb{V}(\mathbf{k}) = \left(\mathbb{V}_1(\mathbf{k}), \dots, \mathbb{V}_L(\mathbf{k})\right)$, where

$$\mathbb{V}_j(\mathbf{k}) = \int_{[0,1]^2} \mathbb{L}^{\mathbf{k}}(\widetilde{\mathbf{u}}_j)\,\mathrm{d}u_\ell\,\mathrm{d}u_{\ell'}, \quad j \in \{1, \dots, L\}.$$

Therefore, $\mathbb{V}$ is a vector of centered Gaussian processes such that for $j' = (m-1)d + m' - \binom{m+1}{2}$ associated to $m < m' \in \{1, \dots, d\}$, its covariance structure is managed for $\mathbf{k}, \mathbf{k}' \in \Delta$ by

Hence, $\mathrm{E}\{\mathbb{V}(\mathbf{k})^\top \mathbb{V}(\mathbf{k}')\} = \Lambda(\mathbf{k}, \mathbf{k}')\,\Omega$ and

$$\begin{aligned}
\mathrm{E}\left\{\mathbb{V}_j(\mathbf{k})\,\mathbb{V}_{j'}(\mathbf{k}')\right\} &= \int_{[0,1]^4} \mathrm{E}\left\{\mathbb{L}^{\mathbf{k}}(\widetilde{\mathbf{u}}_j)\,\mathbb{L}^{\mathbf{k}'}(\widetilde{\mathbf{u}}_{j'})\right\}\mathrm{d}u_\ell\,\mathrm{d}u_{\ell'}\,\mathrm{d}u_m\,\mathrm{d}u_{m'} \\
&= \Lambda(\mathbf{k}, \mathbf{k}') \int_{[0,1]^4} \sum_{t\in\mathbb{Z}} \mathrm{cov}\left\{\mathbb{I}(\mathbf{U}_0 \le \widetilde{\mathbf{u}}_j), \mathbb{I}(\mathbf{U}_t \le \widetilde{\mathbf{u}}_{j'})\right\}\mathrm{d}u_\ell\,\mathrm{d}u_{\ell'}\,\mathrm{d}u_m\,\mathrm{d}u_{m'} \\
&= \Lambda(\mathbf{k}, \mathbf{k}') \sum_{t\in\mathbb{Z}} \mathrm{cov}\left\{\int_{[0,1]^2} \mathbb{I}(\mathbf{U}_0 \le \widetilde{\mathbf{u}}_j)\right. \\
&\qquad \left.\mathrm{d}u_\ell\,\mathrm{d}u_{\ell'}, \int_{[0,1]^2} \mathbb{I}(\mathbf{U}_t \le \widetilde{\mathbf{u}}_{j'})\,\mathrm{d}u_m\,\mathrm{d}u_{m'}\right\} \\
&= \Lambda(\mathbf{k}, \mathbf{k}') \sum_{t\in\mathbb{Z}} \mathrm{cov}\left\{(1-U_{0\ell})(1-U_{0\ell'}), (1-U_{tm})(1-U_{tm'})\right\} \\
&= \Lambda(\mathbf{k}, \mathbf{k}')\,\Omega_{jj'}.
\end{aligned}$$

$$\mathrm{E}\Big\{ \big(\mathbb{V}(\mathbf{k})\,\Omega^{-1/2}\big)^{\top} \big(\mathbb{V}(\mathbf{k}')\,\Omega^{-1/2}\big) \Big\} = \Omega^{-1/2}\,\mathrm{E}\big\{\mathbb{V}(\mathbf{k})^{\top}\mathbb{V}(\mathbf{k}')\big\}\Omega^{-1/2} = \Lambda(\mathbf{k},\mathbf{k}')\,I_L.$$

Since the covariance function of the integrated Brownian bridge is $\mathrm{E}\{\widetilde{\mathbb{B}}(\mathbf{k})\,\widetilde{\mathbb{B}}(\mathbf{k}')\} = \Lambda(\mathbf{k},\mathbf{k}')$, the covariance structure $\Lambda(\mathbf{k},\mathbf{k}')\,I_L$ is the same as that of a vector $(\widetilde{\mathbb{B}}_1(\mathbf{k}),\ldots,\widetilde{\mathbb{B}}_L(\mathbf{k}))$ of independent integrated Brownian bridges. It follows that $\sqrt{T}\,\mathscr{V}(\widehat{Y}^{\mathbf{k}})\,\Omega^{-1/2}/12$ converges weakly to $(\widetilde{\mathbb{B}}_1(\mathbf{k}),\ldots,\widetilde{\mathbb{B}}_L(\mathbf{k}))$. □

A.3: Proof of Proposition 3

From the conclusion of Proposition 2 and the Continuous mapping Theorem,

$$\sqrt{T}\,S_{T,p} = \sup_{\mathbf{k}\in\Delta} \eta(\mathbf{k}) \left\| \frac{\sqrt{T}\,\mathscr{V}(\widehat{Y}^{\mathbf{k}})\,\Omega^{-1/2}}{12} \right\|_p \rightsquigarrow \sup_{\mathbf{k}\in\Delta} \eta(\mathbf{k}) \left\| \left(\widetilde{\mathbb{B}}_1(\mathbf{k}),\ldots,\widetilde{\mathbb{B}}_L(\mathbf{k})\right) \right\|_p,$$

which completes the proof. □

A.4: Proof of Proposition 4

The first step of the proof consists to write

$$\begin{aligned}
\widehat{Y}^{\mathbf{k}}_{\ell\ell'} =& 12 \int_{[0,1]^2} \left[\frac{1}{T} \sum_{t=1}^{T} \left(w_t^{\mathbf{k}} - \bar{w}^{\mathbf{k}}\right) \Big\{ \mathbb{I}(\widehat{\mathbf{U}}_t \le \widetilde{\mathbf{u}}_j) - \mathrm{E}\big(\mathbb{I}(\mathbf{U}_t \le \mathbf{u})\big) + \mathrm{E}\big(\mathbb{I}(\mathbf{U}_t \le \mathbf{u})\big) \Big\} \right] \mathrm{d}u_\ell \mathrm{d}u_{\ell'} \\
=& \frac{12}{\sqrt{T}} \int_{[0,1]^2} \widehat{\mathbb{L}}^{\mathbf{k}}(\widetilde{\mathbf{u}}_j)\,\mathrm{d}u_\ell \mathrm{d}u_{\ell'} + 12\,\langle \mathbf{w}^{\mathbf{k}}, \mathbf{w}^{\mathbf{k}_0} \rangle \big\{ (\Sigma_G)_{\ell\ell'} - (\Sigma_F)_{\ell\ell'} \big\},
\end{aligned}$$

where $\widehat{\mathbb{L}}^{\mathbf{k}}$ is the empirical process defined for $\mathbf{u} \in [0,1]^d$ by

$$\widehat{\mathbb{L}}^{\mathbf{k}}(\mathbf{u}) = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left(w_t^{\mathbf{k}} - \bar{w}^{\mathbf{k}}\right) \Big\{ \mathbb{I}(\widehat{\mathbf{U}}_t \le \mathbf{u}) - \mathrm{E}\big(\mathbb{I}(\mathbf{U}_t \le \mathbf{u})\big) \Big\}.$$

By arguments similar as those in the proof of Proposition 2, $\widehat{\mathbb{L}}^{\mathbf{k}}$ is asymptotically equivalent to

$$\widetilde{\mathbb{L}}^{\mathbf{k}}(\mathbf{u}) = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left(w_t^{\mathbf{k}} - \bar{w}^{\mathbf{k}}\right) \big\{ \mathbb{I}(\mathbf{U}_t \le \mathbf{u}) - \mathrm{E}\big(\mathbb{I}(\mathbf{U}_t \le \mathbf{u})\big) \big\}.$$

It is easy to see that $\widetilde{\mathbb{L}}^{\mathbf{k}}$ converges weakly to a non-degenerate centered Gaussian process whose covariance structure is characterized by the finite-dimensional structure of $\widetilde{\mathbb{L}}^{\mathbf{k}}$. As a consequence,

$$\begin{aligned}
\mathscr{V}(\widehat{Y}^{\mathbf{k}}) =& \frac{12}{\sqrt{T}} \left(\int_{[0,1]^2} \widetilde{\mathbb{L}}^{\mathbf{k}}(\widetilde{\mathbf{u}}_1)\,\mathrm{d}u_1 \mathrm{d}u_2, \ldots, \int_{[0,1]^2} \widetilde{\mathbb{L}}^{\mathbf{k}}(\widetilde{\mathbf{u}}_L)\,\mathrm{d}u_{d-1} \mathrm{d}u_d \right) \\
&+ 12 \left\langle \mathbf{w}^{\mathbf{k}}, \mathbf{w}^{\mathbf{k}_0} \right\rangle \left\{ \mathscr{V}\!\left(\Sigma_G^{\mathrm{Sp}} - \Sigma_F^{\mathrm{Sp}} \right) \right\} + o_{\mathbb{P}}(1/\sqrt{T}).
\end{aligned}$$

Since $\Sigma_F^{\mathrm{Sp}} \neq \Sigma_G^{\mathrm{Sp}}$, one has in probability that

$$\left\|\mathcal{V}(\widehat{Y}^{\mathbf{k}})\right\| \longrightarrow 12\,\Lambda(\mathbf{k},\mathbf{k}_0)\left\|\mathcal{V}\left(\Sigma_G^{\mathrm{Sp}} - \Sigma_F^{\mathrm{Sp}}\right)\right\| > 0,$$

where $\Lambda(\mathbf{k},\mathbf{k}_0) = \lim_{T\to\infty}\langle \mathbf{w}^{\mathbf{k}}, \mathbf{w}^{\mathbf{k}_0}\rangle$. Invoking the argmax continuous mapping Theorem (see Theorem 3.2.2 of van der Vaart and Wellner 1996), one can conclude that in probability,

$$\widehat{\mathbf{k}}_0 = \operatorname*{argsup}_{\mathbf{k}\in\Delta^\star} \frac{\|\mathcal{V}(\widehat{Y}^{\mathbf{k}})\|}{\langle \mathbf{w}^{\mathbf{k}}, \mathbf{w}^{\mathbf{k}}\rangle^{1/2}} \longrightarrow \left\{12\left\|\mathcal{V}\left(\Sigma_G^{\mathrm{Sp}} - \Sigma_F^{\mathrm{Sp}}\right)\right\|\right\}\operatorname*{argsup}_{\kappa\in\Delta^\star} \frac{\left|\Lambda(\mathbf{k},\mathbf{k}_0)\right|}{\{\Lambda(\mathbf{k},\mathbf{k})\}^{1/2}} = \mathbf{k}_0.$$

□

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