# Supplementary Material for "Test for Conditional Quantile Change in General Conditional Heteroscedastic Time Series Models"

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This supplementary file contains detailed results regarding the simulation studies conducted in Section 4. Specifically, Tables S.1 to S.3 describe the quality of the quantile regression estimators of the proposed CUSUM tests of all six cases introduced in Section 4.1, which report the bias, standard deviation (SD), root mean-squared error (RMSE), and the rejection rate of two coverage tests. Moreover, Tables S.4 to S.9 depict empirical sizes and powers of our proposed test statistics for all six cases. Finally, Tables S.10 to S.11 report the mean and the standard deviation (SD) of the location of the detected change point, where the quantile and the location of the change is given as $\tau \in \{0.9, 0.1\}$ and $n = 1,000$.

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Table S.1: Quality of the quantile regression estimators for Cases 1-3, where $\tau \in \{0.9, 0.1\}$. SD and RMSE respectively are abbreviations for the standard deviation and root mean-squared error.

| | | | $\tau = 0.9$ | | | | $\tau = 0.1$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | $\alpha_1$ | $\alpha_2$ | $\beta$ | $\xi$ | $\alpha_1$ | $\alpha_2$ | $\beta$ | $\xi$ |
| Case 1 | $n = 500$ | Bias | 0.116 | 0.250 | 0.043 | -0.028 | 0.203 | 0.211 | 0.034 | 0.024 |
| | | SD | 1.195 | 0.729 | 0.282 | 0.093 | 1.166 | 0.698 | 0.277 | 0.091 |
| | | RMSE | 1.201 | 0.771 | 0.285 | 0.097 | 1.184 | 0.729 | 0.279 | 0.094 |
| | $n = 1,000$ | Bias | 0.094 | 0.135 | 0.006 | -0.011 | 0.176 | 0.101 | 0.016 | 0.014 |
| | | SD | 0.871 | 0.540 | 0.246 | 0.078 | 0.945 | 0.520 | 0.242 | 0.077 |
| | | RMSE | 0.876 | 0.556 | 0.246 | 0.079 | 0.961 | 0.530 | 0.242 | 0.079 |
| | $n = 1,500$ | Bias | 0.122 | 0.073 | -0.005 | -0.005 | 0.149 | 0.082 | -0.003 | 0.006 |
| | | SD | 0.763 | 0.431 | 0.215 | 0.068 | 0.798 | 0.440 | 0.218 | 0.070 |
| | | RMSE | 0.773 | 0.438 | 0.215 | 0.069 | 0.811 | 0.448 | 0.218 | 0.070 |
| Case 2 | $n = 500$ | Bias | 0.124 | 0.154 | 0.041 | -0.017 | 0.209 | -0.017 | 0.041 | 0.016 |
| | | SD | 1.822 | 2.623 | 0.294 | 0.066 | 1.872 | 2.575 | 0.284 | 0.065 |
| | | RMSE | 1.826 | 2.628 | 0.297 | 0.068 | 1.883 | 2.575 | 0.287 | 0.066 |
| | $n = 1,000$ | Bias | 0.046 | 0.060 | 0.022 | -0.011 | 0.087 | -0.004 | 0.024 | 0.011 |
| | | SD | 1.437 | 2.046 | 0.265 | 0.069 | 1.477 | 2.101 | 0.258 | 0.058 |
| | | RMSE | 1.438 | 2.047 | 0.266 | 0.070 | 1.480 | 2.101 | 0.259 | 0.059 |
| | $n = 1,500$ | Bias | 0.014 | 0.176 | -0.017 | -0.001 | 0.101 | 0.028 | -0.009 | 0.002 |
| | | SD | 1.241 | 1.774 | 0.229 | 0.050 | 1.234 | 1.767 | 0.228 | 0.049 |
| | | RMSE | 1.241 | 1.783 | 0.230 | 0.050 | 1.238 | 1.767 | 0.228 | 0.049 |
| Case 3 | $n = 500$ | Bias | -0.032 | -0.142 | -0.088 | -0.242 | -0.024 | -0.157 | -0.088 | 0.238 |
| | | SD | 0.100 | 0.176 | 0.150 | 0.186 | 0.102 | 0.193 | 0.152 | 0.188 |
| | | RMSE | 0.105 | 0.226 | 0.174 | 0.305 | 0.105 | 0.249 | 0.175 | 0.303 |
| | $n = 1,000$ | Bias | -0.029 | -0.108 | -0.056 | -0.285 | -0.022 | -0.118 | -0.052 | 0.290 |
| | | SD | 0.080 | 0.132 | 0.097 | 0.118 | 0.082 | 0.137 | 0.096 | 0.121 |
| | | RMSE | 0.085 | 0.170 | 0.112 | 0.308 | 0.085 | 0.181 | 0.109 | 0.314 |
| | $n = 1,500$ | Bias | -0.023 | -0.088 | -0.040 | -0.310 | -0.024 | -0.098 | -0.037 | 0.310 |
| | | SD | 0.070 | 0.105 | 0.071 | 0.086 | 0.068 | 0.114 | 0.077 | 0.095 |
| | | RMSE | 0.074 | 0.137 | 0.082 | 0.322 | 0.072 | 0.150 | 0.086 | 0.325 |

Table S.2: Quality of the quantile regression estimators for Cases 4-6, where $\tau \in \{0.9, 0.1\}$.

| | | | $\tau = 0.9$ | | | | | $\tau = 0.1$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $\phi$ | $\psi$ | $\alpha$ | $\beta$ | $\xi$ | $\phi$ | $\psi$ | $\alpha$ | $\beta$ | $\xi$ |
| Case 4 | $n = 500$ | Bias | -0.008 | -0.036 | 0.487 | -0.063 | 0.086 | -0.005 | -0.042 | 0.484 | -0.067 | -0.089 |
| | | SD | 0.158 | 0.158 | 0.733 | 0.226 | 0.090 | 0.160 | 0.162 | 0.753 | 0.217 | 0.089 |
| | | RMSE | 0.159 | 0.162 | 0.880 | 0.234 | 0.125 | 0.160 | 0.167 | 0.895 | 0.227 | 0.126 |
| | $n = 1,000$ | Bias | -0.005 | -0.040 | 0.435 | -0.081 | 0.097 | 0.003 | -0.050 | 0.433 | -0.095 | -0.102 |
| | | SD | 0.112 | 0.114 | 0.568 | 0.179 | 0.071 | 0.115 | 0.118 | 0.551 | 0.174 | 0.069 |
| | | RMSE | 0.112 | 0.121 | 0.715 | 0.196 | 0.120 | 0.115 | 0.129 | 0.701 | 0.198 | 0.124 |
| | $n = 1,500$ | Bias | 0.000 | -0.045 | 0.382 | -0.096 | 0.105 | -0.008 | -0.039 | 0.370 | -0.089 | -0.103 |
| | | SD | 0.094 | 0.097 | 0.456 | 0.152 | 0.060 | 0.094 | 0.095 | 0.449 | 0.161 | 0.064 |
| | | RMSE | 0.094 | 0.107 | 0.595 | 0.180 | 0.121 | 0.094 | 0.103 | 0.582 | 0.184 | 0.121 |
| Case 5 | $n = 500$ | Bias | 0.000 | 0.046 | 0.177 | -0.080 | 0.160 | 0.001 | 0.046 | 0.161 | -0.109 | -0.183 |
| | | SD | 0.080 | 0.090 | 0.266 | 0.204 | 0.138 | 0.079 | 0.094 | 0.269 | 0.181 | 0.129 |
| | | RMSE | 0.080 | 0.101 | 0.320 | 0.220 | 0.211 | 0.079 | 0.105 | 0.313 | 0.211 | 0.224 |
| | $n = 1,000$ | Bias | -0.001 | 0.049 | 0.150 | -0.095 | 0.173 | -0.005 | 0.050 | 0.149 | -0.107 | -0.181 |
| | | SD | 0.055 | 0.065 | 0.206 | 0.171 | 0.113 | 0.056 | 0.068 | 0.201 | 0.162 | 0.110 |
| | | RMSE | 0.055 | 0.081 | 0.255 | 0.195 | 0.207 | 0.056 | 0.084 | 0.250 | 0.194 | 0.212 |
| | $n = 1,500$ | Bias | 0.001 | 0.044 | 0.148 | -0.108 | 0.184 | 0.002 | 0.044 | 0.148 | -0.109 | -0.184 |
| | | SD | 0.046 | 0.053 | 0.165 | 0.142 | 0.095 | 0.044 | 0.053 | 0.167 | 0.142 | 0.094 |
| | | RMSE | 0.046 | 0.069 | 0.222 | 0.179 | 0.207 | 0.044 | 0.069 | 0.223 | 0.179 | 0.207 |
| Case 6 | $n = 500$ | Bias | 0.232 | -0.576 | -0.205 | -0.197 | 1.232 | 0.221 | -0.570 | -0.204 | -0.185 | -1.203 |
| | | SD | 0.680 | 0.675 | 0.054 | 0.161 | 0.497 | 0.641 | 0.636 | 0.054 | 0.158 | 0.467 |
| | | RMSE | 0.719 | 0.888 | 0.212 | 0.255 | 1.328 | 0.678 | 0.855 | 0.211 | 0.243 | 1.291 |
| | $n = 1,000$ | Bias | 0.188 | -0.529 | -0.198 | -0.157 | 1.108 | 0.210 | -0.558 | -0.199 | -0.155 | -1.112 |
| | | SD | 0.669 | 0.665 | 0.042 | 0.107 | 0.340 | 0.619 | 0.614 | 0.043 | 0.102 | 0.327 |
| | | RMSE | 0.694 | 0.850 | 0.202 | 0.191 | 1.159 | 0.654 | 0.830 | 0.204 | 0.186 | 1.159 |
| | $n = 1,500$ | Bias | 0.172 | -0.515 | -0.200 | -0.147 | 1.088 | 0.178 | -0.526 | -0.198 | -0.146 | -1.079 |
| | | SD | 0.658 | 0.655 | 0.035 | 0.076 | 0.249 | 0.590 | 0.585 | 0.035 | 0.075 | 0.242 |
| | | RMSE | 0.680 | 0.833 | 0.203 | 0.165 | 1.116 | 0.616 | 0.787 | 0.201 | 0.164 | 1.105 |

Table S.3: Rejection rate of the Kupiec's and Christoffersen's coverage tests for all six cases, under $\tau \in \{0.9, 0.1\}$.

| | | Kupiec | | Christoffersen | |
|---|---|---|---|---|---|
| | | $\tau = 0.9$ | $\tau = 0.1$ | $\tau = 0.9$ | $\tau = 0.1$ |
| Case 1 | $n = 500$ | 0 | 0 | 0.007 | 0.007 |
| | $n = 1,000$ | 0 | 0 | 0.002 | 0.009 |
| | $n = 1,500$ | 0 | 0 | 0.004 | 0.007 |
| Case 2 | $n = 500$ | 0 | 0 | 0.012 | 0.008 |
| | $n = 1,000$ | 0 | 0 | 0.010 | 0.005 |
| | $n = 1,500$ | 0 | 0 | 0.008 | 0.006 |
| Case 3 | $n = 500$ | 0 | 0 | 0.024 | 0.006 |
| | $n = 1,000$ | 0 | 0 | 0.025 | 0.010 |
| | $n = 1,500$ | 0 | 0 | 0.018 | 0.012 |
| Case 4 | $n = 500$ | 0 | 0 | 0.003 | 0.007 |
| | $n = 1,000$ | 0 | 0 | 0.008 | 0.012 |
| | $n = 1,500$ | 0 | 0 | 0.011 | 0.004 |
| Case 5 | $n = 500$ | 0 | 0 | 0.008 | 0.011 |
| | $n = 1,000$ | 0 | 0 | 0.011 | 0.011 |
| | $n = 1,500$ | 0 | 0 | 0.011 | 0.011 |
| Case 6 | $n = 500$ | 0 | 0 | 0.034 | 0.014 |
| | $n = 1,000$ | 0 | 0 | 0.037 | 0.036 |
| | $n = 1,500$ | 0 | 0 | 0.04 | 0.029 |

Table S.4: Empirical sizes and powers of the CUSUM test when the underlying model is GJR-GARCH(1,1) with $(\omega, \alpha_1, \alpha_2, \beta) = (0.3, 0.5, 0.1, 0.3)$ (Case 1), under $\tau \in \{0.9, 0.1\}$.

| | | $n = 500$ | | | | | | $n = 1,000$ | | | | | | $n = 1,500$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\hat{T}_n$ | | | | $\tilde{T}_{n,2}$ | $\check{T}_{n,2}$ | $\hat{T}_n$ | | | | $\tilde{T}_{n,2}$ | $\check{T}_{n,2}$ | $\hat{T}_n$ | | | | $\tilde{T}_{n,2}$ | $\check{T}_{n,2}$ |
| | | $\gamma = 0.2$ | $\gamma = 0.5$ | $\gamma = 1$ | $\gamma = 2$ | | | $\gamma = 0.2$ | $\gamma = 0.5$ | $\gamma = 1$ | $\gamma = 2$ | | | $\gamma = 0.2$ | $\gamma = 0.5$ | $\gamma = 1$ | $\gamma = 2$ | | |
| $\tau = 0.9$ | size | 0.044 | 0.041 | 0.050 | 0.042 | 0.044 | 0.042 | 0.047 | 0.047 | 0.044 | 0.049 | 0.053 | 0.050 | 0.058 | 0.059 | 0.068 | 0.051 | 0.049 | 0.049 |
| | $\omega \to 0.5$ | 0.988 | 0.965 | 0.807 | 0.108 | 1.000 | 1.000 | 1.000 | 0.999 | 0.989 | 0.353 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.623 | 1.000 | 1.000 |
| | $\omega \to 0.7$ | 0.998 | 0.984 | 0.824 | 0.052 | 1.000 | 1.000 | 1.000 | 1.000 | 0.996 | 0.173 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.258 | 1.000 | 1.000 |
| | $\omega \to 0.1$ | 1.000 | 1.000 | 1.000 | 0.230 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.548 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.853 | 1.000 | 1.000 |
| | $\omega \to 0.01$ | 1.000 | 1.000 | 0.901 | 0.061 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.208 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.362 | 1.000 | 1.000 |
| | $\alpha_1 \to 2.5$ | 0.324 | 0.324 | 0.227 | 0.046 | 0.778 | 0.776 | 0.608 | 0.625 | 0.556 | 0.095 | 0.976 | 0.977 | 0.771 | 0.808 | 0.742 | 0.143 | 0.999 | 1.000 |
| | $\alpha_2 \to 2.5$ | 0.467 | 0.469 | 0.352 | 0.064 | 0.929 | 0.931 | 0.801 | 0.813 | 0.735 | 0.160 | 1.000 | 0.999 | 0.924 | 0.946 | 0.898 | 0.262 | 1.000 | 1.000 |
| | $\beta \to 0.6$ | 0.771 | 0.675 | 0.435 | 0.107 | 1.000 | 0.999 | 0.974 | 0.953 | 0.792 | 0.246 | 1.000 | 1.000 | 1.000 | 0.990 | 0.941 | 0.437 | 1.000 | 1.000 |
| | $\beta \to 0.7$ | 0.968 | 0.926 | 0.688 | 0.111 | 0.999 | 0.999 | 1.000 | 1.000 | 0.970 | 0.323 | 1.000 | 1.000 | 1.000 | 1.000 | 0.995 | 0.562 | 1.000 | 1.000 |
| | $\eta_t \sim Z_5$ | 0.219 | 0.146 | 0.045 | 0.043 | 0.990 | 0.986 | 0.457 | 0.330 | 0.071 | 0.030 | 1.000 | 1.000 | 0.616 | 0.494 | 0.123 | 0.027 | 1.000 | 1.000 |
| | $\eta_t \sim Z_7$ | 0.301 | 0.176 | 0.025 | 0.052 | 0.988 | 0.983 | 0.573 | 0.402 | 0.047 | 0.041 | 0.998 | 0.998 | 0.709 | 0.574 | 0.089 | 0.037 | 0.999 | 0.999 |
| $\tau = 0.1$ | size | 0.040 | 0.046 | 0.046 | 0.036 | 0.057 | 0.053 | 0.040 | 0.043 | 0.047 | 0.046 | 0.044 | 0.045 | 0.057 | 0.055 | 0.050 | 0.052 | 0.053 | 0.056 |
| | $\omega \to 0.5$ | 0.981 | 0.958 | 0.789 | 0.112 | 1.000 | 1.000 | 1.000 | 1.000 | 0.982 | 0.369 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.632 | 1.000 | 1.000 |
| | $\omega \to 0.7$ | 0.999 | 0.994 | 0.833 | 0.056 | 0.999 | 0.999 | 1.000 | 1.000 | 0.994 | 0.159 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.276 | 1.000 | 1.000 |
| | $\omega \to 0.1$ | 1.000 | 1.000 | 1.000 | 0.259 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.559 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.873 | 1.000 | 1.000 |
| | $\omega \to 0.01$ | 1.000 | 1.000 | 0.893 | 0.074 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.208 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.344 | 1.000 | 1.000 |
| | $\alpha_1 \to 2.5$ | 0.329 | 0.330 | 0.233 | 0.070 | 0.772 | 0.773 | 0.555 | 0.585 | 0.515 | 0.134 | 0.968 | 0.972 | 0.803 | 0.838 | 0.767 | 0.219 | 0.998 | 0.999 |
| | $\alpha_2 \to 2.5$ | 0.496 | 0.501 | 0.360 | 0.041 | 0.935 | 0.926 | 0.817 | 0.829 | 0.715 | 0.115 | 1.000 | 0.999 | 0.941 | 0.954 | 0.906 | 0.181 | 1.000 | 1.000 |
| | $\beta \to 0.6$ | 0.756 | 0.669 | 0.440 | 0.105 | 1.000 | 1.000 | 0.976 | 0.944 | 0.808 | 0.267 | 1.000 | 1.000 | 0.999 | 0.993 | 0.944 | 0.452 | 1.000 | 1.000 |
| | $\beta \to 0.7$ | 0.965 | 0.925 | 0.712 | 0.124 | 1.000 | 1.000 | 1.000 | 0.999 | 0.975 | 0.358 | 1.000 | 1.000 | 1.000 | 1.000 | 0.996 | 0.602 | 1.000 | 1.000 |
| | $\eta_t \sim Z_5$ | 0.235 | 0.165 | 0.045 | 0.050 | 0.989 | 0.986 | 0.430 | 0.335 | 0.090 | 0.039 | 1.000 | 1.000 | 0.613 | 0.475 | 0.123 | 0.033 | 0.999 | 0.998 |
| | $\eta_t \sim Z_7$ | 0.319 | 0.204 | 0.044 | 0.043 | 0.983 | 0.980 | 0.544 | 0.399 | 0.062 | 0.036 | 1.000 | 0.999 | 0.706 | 0.529 | 0.077 | 0.047 | 0.997 | 0.996 |

Table S.5: Empirical sizes and powers of the CUSUM test when the underlying model is GJR-GARCH(1,1) with $(\omega, \alpha_1, \alpha_2, \beta) = (0.2, 0.3, 0.7, 0.3)$ (Case 2), under $\tau \in \{0.9, 0.1\}$.

| | | $n = 500$ | | | | | | $n = 1,000$ | | | | | | $n = 1,500$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\hat{T}_n$ | | | | $\tilde{T}_{n,2}$ | $\check{T}_{n,2}$ | $\hat{T}_n$ | | | | $\tilde{T}_{n,2}$ | $\check{T}_{n,2}$ | $\hat{T}_n$ | | | | $\tilde{T}_{n,2}$ | $\check{T}_{n,2}$ |
| | | $\gamma = 0.2$ | $\gamma = 0.5$ | $\gamma = 1$ | $\gamma = 2$ | | | $\gamma = 0.2$ | $\gamma = 0.5$ | $\gamma = 1$ | $\gamma = 2$ | | | $\gamma = 0.2$ | $\gamma = 0.5$ | $\gamma = 1$ | $\gamma = 2$ | | |
| $\tau = 0.9$ | size | 0.049 | 0.052 | 0.049 | 0.041 | 0.053 | 0.046 | 0.052 | 0.051 | 0.052 | 0.055 | 0.069 | 0.062 | 0.049 | 0.050 | 0.049 | 0.050 | 0.058 | 0.054 |
| | $\omega \to 0.25$ | 0.513 | 0.424 | 0.272 | 0.086 | 0.989 | 0.987 | 0.847 | 0.778 | 0.594 | 0.221 | 1.000 | 1.000 | 0.951 | 0.919 | 0.770 | 0.315 | 1.000 | 1.000 |
| | $\omega \to 0.35$ | 0.998 | 0.991 | 0.911 | 0.157 | 1.000 | 1.000 | 1.000 | 1.000 | 0.998 | 0.490 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.775 | 1.000 | 1.000 |
| | $\omega \to 0.15$ | 0.757 | 0.681 | 0.472 | 0.131 | 1.000 | 1.000 | 0.974 | 0.944 | 0.830 | 0.340 | 1.000 | 1.000 | 0.995 | 0.990 | 0.933 | 0.519 | 1.000 | 1.000 |
| | $\omega \to 0.1$ | 1.000 | 0.999 | 0.991 | 0.255 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.730 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.960 | 1.000 | 1.000 |
| | $\alpha_1 \to 3.5$ | 0.460 | 0.472 | 0.347 | 0.069 | 0.925 | 0.915 | 0.757 | 0.780 | 0.704 | 0.154 | 0.998 | 0.998 | 0.914 | 0.931 | 0.900 | 0.299 | 1.000 | 1.000 |
| | $\alpha_2 \to 3.5$ | 0.295 | 0.299 | 0.242 | 0.063 | 0.793 | 0.793 | 0.591 | 0.627 | 0.560 | 0.182 | 0.981 | 0.977 | 0.778 | 0.814 | 0.768 | 0.296 | 1.000 | 1.000 |
| | $(\alpha_1, \alpha_2) \to (1.5, 2.5)$ | 0.506 | 0.526 | 0.452 | 0.143 | 0.947 | 0.942 | 0.816 | 0.848 | 0.802 | 0.368 | 0.998 | 0.998 | 0.929 | 0.951 | 0.942 | 0.585 | 1.000 | 1.000 |
| | $\beta \to 0.05$ | 0.393 | 0.349 | 0.267 | 0.116 | 0.923 | 0.922 | 0.707 | 0.657 | 0.521 | 0.246 | 1.000 | 1.000 | 0.866 | 0.813 | 0.691 | 0.375 | 1.000 | 1.000 |
| | $\beta \to 0.7$ | 0.970 | 0.910 | 0.689 | 0.125 | 1.000 | 1.000 | 0.999 | 0.998 | 0.958 | 0.346 | 1.000 | 1.000 | 1.000 | 1.000 | 0.994 | 0.573 | 1.000 | 1.000 |
| | $\eta_t \sim Z_5$ | 0.261 | 0.179 | 0.051 | 0.040 | 0.993 | 0.989 | 0.462 | 0.377 | 0.097 | 0.032 | 0.999 | 0.997 | 0.662 | 0.531 | 0.141 | 0.031 | 1.000 | 1.000 |
| | $\eta_t \sim Z_7$ | 0.369 | 0.248 | 0.044 | 0.052 | 0.985 | 0.980 | 0.584 | 0.452 | 0.086 | 0.038 | 0.996 | 0.994 | 0.815 | 0.669 | 0.113 | 0.055 | 0.998 | 0.998 |
| $\tau = 0.1$ | size | 0.048 | 0.049 | 0.043 | 0.031 | 0.068 | 0.065 | 0.051 | 0.053 | 0.053 | 0.041 | 0.066 | 0.060 | 0.042 | 0.046 | 0.055 | 0.041 | 0.048 | 0.045 |
| | $\omega \to 0.25$ | 0.479 | 0.400 | 0.255 | 0.079 | 0.981 | 0.980 | 0.845 | 0.764 | 0.561 | 0.210 | 1.000 | 0.999 | 0.951 | 0.912 | 0.782 | 0.351 | 1.000 | 1.000 |
| | $\omega \to 0.35$ | 0.999 | 0.995 | 0.914 | 0.155 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.485 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.773 | 1.000 | 1.000 |
| | $\omega \to 0.15$ | 0.749 | 0.666 | 0.448 | 0.134 | 1.000 | 1.000 | 0.977 | 0.951 | 0.830 | 0.363 | 1.000 | 1.000 | 0.999 | 0.988 | 0.943 | 0.536 | 1.000 | 1.000 |
| | $\omega \to 0.1$ | 1.000 | 1.000 | 0.992 | 0.287 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.758 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.960 | 1.000 | 1.000 |
| | $\alpha_1 \to 3.5$ | 0.447 | 0.461 | 0.364 | 0.089 | 0.911 | 0.903 | 0.781 | 0.816 | 0.743 | 0.232 | 0.998 | 0.998 | 0.920 | 0.948 | 0.913 | 0.373 | 1.000 | 1.000 |
| | $\alpha_2 \to 3.5$ | 0.365 | 0.361 | 0.287 | 0.058 | 0.803 | 0.800 | 0.597 | 0.624 | 0.574 | 0.118 | 0.979 | 0.981 | 0.795 | 0.823 | 0.776 | 0.263 | 0.998 | 0.998 |
| | $(\alpha_1, \alpha_2) \to (1.5, 2.5)$ | 0.494 | 0.525 | 0.456 | 0.129 | 0.942 | 0.935 | 0.836 | 0.860 | 0.825 | 0.347 | 0.999 | 0.999 | 0.938 | 0.959 | 0.950 | 0.592 | 1.000 | 1.000 |
| | $\beta \to 0.05$ | 0.381 | 0.339 | 0.257 | 0.121 | 0.931 | 0.935 | 0.715 | 0.640 | 0.487 | 0.219 | 0.999 | 0.999 | 0.866 | 0.811 | 0.698 | 0.361 | 1.000 | 1.000 |
| | $\beta \to 0.7$ | 0.958 | 0.908 | 0.662 | 0.130 | 1.000 | 1.000 | 0.999 | 0.998 | 0.971 | 0.345 | 1.000 | 1.000 | 1.000 | 1.000 | 0.995 | 0.573 | 1.000 | 1.000 |
| | $\eta_t \sim Z_5$ | 0.273 | 0.206 | 0.055 | 0.037 | 0.987 | 0.981 | 0.471 | 0.360 | 0.081 | 0.026 | 1.000 | 0.999 | 0.646 | 0.520 | 0.140 | 0.021 | 1.000 | 1.000 |
| | $\eta_t \sim Z_7$ | 0.382 | 0.276 | 0.042 | 0.037 | 0.974 | 0.964 | 0.623 | 0.472 | 0.074 | 0.047 | 0.998 | 0.998 | 0.794 | 0.646 | 0.105 | 0.044 | 0.998 | 0.998 |

Table S.6: Empirical sizes and powers of the CUSUM test when the underlying model is GJR-GARCH(1,1) with $(\omega, \alpha_1, \alpha_2, \beta) = (0.7, 0.1, 0.5, 0.6)$ (Case 3), under $\tau \in \{0.9, 0.1\}$.

| | | $n=500$ | | | | | | $n=1,000$ | | | | | | $n=1,500$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\hat{T}_n$ | | | | $\tilde{T}_{n,2}$ | $\check{T}_{n,2}$ | $\hat{T}_n$ | | | | $\tilde{T}_{n,2}$ | $\check{T}_{n,2}$ | $\hat{T}_n$ | | | | $\tilde{T}_{n,2}$ | $\check{T}_{n,2}$ |
| | | $\gamma=0.2$ | $\gamma=0.5$ | $\gamma=1$ | $\gamma=2$ | | | $\gamma=0.2$ | $\gamma=0.5$ | $\gamma=1$ | $\gamma=2$ | | | $\gamma=0.2$ | $\gamma=0.5$ | $\gamma=1$ | $\gamma=2$ | | |
| $\tau=0.9$ | size | 0.050 | 0.054 | 0.049 | 0.033 | 0.091 | 0.089 | 0.058 | 0.061 | 0.051 | 0.044 | 0.081 | 0.077 | 0.063 | 0.066 | 0.051 | 0.032 | 0.076 | 0.081 |
| | $\omega\to1$ | 0.723 | 0.564 | 0.229 | 0.063 | 1.000 | 1.000 | 0.963 | 0.899 | 0.532 | 0.061 | 1.000 | 1.000 | 0.997 | 0.986 | 0.749 | 0.094 | 1.000 | 1.000 |
| | $\omega\to1.5$ | 0.962 | 0.719 | 0.117 | 0.133 | 0.999 | 0.999 | 0.999 | 0.920 | 0.195 | 0.131 | 1.000 | 1.000 | 1.000 | 0.978 | 0.275 | 0.138 | 1.000 | 1.000 |
| | $\omega\to0.5$ | 0.760 | 0.652 | 0.372 | 0.068 | 1.000 | 1.000 | 0.967 | 0.918 | 0.730 | 0.136 | 1.000 | 1.000 | 0.997 | 0.988 | 0.862 | 0.189 | 1.000 | 1.000 |
| | $\omega\to0.1$ | 1.000 | 1.000 | 0.379 | 0.039 | 1.000 | 1.000 | 1.000 | 1.000 | 0.873 | 0.041 | 1.000 | 1.000 | 1.000 | 1.000 | 0.983 | 0.049 | 1.000 | 1.000 |
| | $\alpha_1\to0.6$ | 0.576 | 0.443 | 0.142 | 0.056 | 0.976 | 0.973 | 0.871 | 0.807 | 0.350 | 0.053 | 0.999 | 0.999 | 0.969 | 0.942 | 0.581 | 0.053 | 1.000 | 1.000 |
| | $\alpha_2\to1$ | 0.432 | 0.354 | 0.155 | 0.048 | 0.921 | 0.924 | 0.695 | 0.632 | 0.325 | 0.077 | 0.996 | 0.996 | 0.888 | 0.850 | 0.517 | 0.081 | 0.999 | 0.999 |
| | $\beta\to0.1$ | 0.943 | 0.850 | 0.481 | 0.056 | 1.000 | 1.000 | 0.999 | 0.994 | 0.837 | 0.121 | 1.000 | 1.000 | 1.000 | 0.999 | 0.970 | 0.180 | 1.000 | 1.000 |
| | $\beta\to0.75$ | 0.707 | 0.522 | 0.196 | 0.050 | 1.000 | 1.000 | 0.942 | 0.848 | 0.452 | 0.063 | 1.000 | 1.000 | 0.995 | 0.976 | 0.639 | 0.080 | 1.000 | 1.000 |
| | $\eta_t\sim Z_5$ | 0.505 | 0.230 | 0.059 | 0.121 | 0.994 | 0.984 | 0.697 | 0.406 | 0.087 | 0.142 | 0.997 | 0.997 | 0.863 | 0.578 | 0.094 | 0.164 | 0.998 | 0.998 |
| | $\eta_t\sim Z_7$ | 0.577 | 0.191 | 0.076 | 0.176 | 0.971 | 0.961 | 0.780 | 0.269 | 0.090 | 0.213 | 0.988 | 0.985 | 0.875 | 0.359 | 0.085 | 0.181 | 0.985 | 0.985 |
| $\tau=0.1$ | size | 0.063 | 0.055 | 0.047 | 0.020 | 0.100 | 0.099 | 0.049 | 0.046 | 0.054 | 0.031 | 0.089 | 0.085 | 0.061 | 0.053 | 0.047 | 0.025 | 0.067 | 0.069 |
| | $\omega\to1$ | 0.745 | 0.537 | 0.168 | 0.022 | 1.000 | 0.999 | 0.969 | 0.897 | 0.477 | 0.020 | 1.000 | 1.000 | 0.997 | 0.986 | 0.755 | 0.026 | 1.000 | 1.000 |
| | $\omega\to1.5$ | 0.979 | 0.723 | 0.048 | 0.189 | 0.999 | 0.999 | 0.999 | 0.946 | 0.099 | 0.199 | 1.000 | 1.000 | 1.000 | 0.983 | 0.181 | 0.198 | 1.000 | 1.000 |
| | $\omega\to0.5$ | 0.793 | 0.675 | 0.372 | 0.052 | 1.000 | 1.000 | 0.973 | 0.921 | 0.725 | 0.118 | 1.000 | 1.000 | 0.999 | 0.992 | 0.896 | 0.188 | 1.000 | 1.000 |
| | $\omega\to0.1$ | 1.000 | 0.999 | 0.370 | 0.016 | 1.000 | 1.000 | 1.000 | 1.000 | 0.871 | 0.025 | 1.000 | 1.000 | 1.000 | 1.000 | 0.980 | 0.040 | 1.000 | 1.000 |
| | $\alpha_1\to0.6$ | 0.586 | 0.444 | 0.143 | 0.049 | 0.984 | 0.983 | 0.885 | 0.793 | 0.358 | 0.049 | 1.000 | 1.000 | 0.975 | 0.945 | 0.579 | 0.057 | 1.000 | 1.000 |
| | $\alpha_2\to1$ | 0.453 | 0.350 | 0.102 | 0.044 | 0.916 | 0.913 | 0.731 | 0.663 | 0.282 | 0.036 | 0.997 | 0.998 | 0.887 | 0.851 | 0.487 | 0.035 | 1.000 | 1.000 |
| | $\beta\to0.1$ | 0.958 | 0.876 | 0.462 | 0.037 | 1.000 | 1.000 | 0.999 | 0.991 | 0.861 | 0.075 | 1.000 | 1.000 | 1.000 | 0.999 | 0.972 | 0.149 | 1.000 | 1.000 |
| | $\beta\to0.75$ | 0.720 | 0.531 | 0.136 | 0.039 | 1.000 | 1.000 | 0.958 | 0.862 | 0.366 | 0.033 | 1.000 | 1.000 | 0.994 | 0.966 | 0.620 | 0.029 | 1.000 | 1.000 |
| | $\eta_t\sim Z_5$ | 0.473 | 0.198 | 0.049 | 0.162 | 0.989 | 0.986 | 0.725 | 0.402 | 0.050 | 0.184 | 1.000 | 1.000 | 0.856 | 0.546 | 0.064 | 0.220 | 0.998 | 0.997 |
| | $\eta_t\sim Z_7$ | 0.617 | 0.160 | 0.100 | 0.234 | 0.967 | 0.962 | 0.808 | 0.256 | 0.109 | 0.228 | 0.983 | 0.983 | 0.912 | 0.340 | 0.117 | 0.261 | 0.987 | 0.986 |

Table S.7: Empirical sizes and powers of the CUSUM test when the underlying model is ARMA(1,1)-GARCH(1,1) with $(\phi, \psi, \omega, \alpha, \beta) = (0.2, 0.2, 0.3, 0.3, 0.3)$ (Case 4), under $\tau \in \{0.9, 0.1\}$.

| | | $n = 500$ | | | | | | $n = 1,000$ | | | | | | $n = 1,500$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\hat{T}_n$ | | | | $\tilde{T}_n$ | $\check{T}_n$ | $\hat{T}_n$ | | | | $\tilde{T}_n$ | $\check{T}_n$ | $\hat{T}_n$ | | | | $\tilde{T}_n$ | $\check{T}_n$ |
| | | $\gamma = 0.2$ | $\gamma = 0.5$ | $\gamma = 1$ | $\gamma = 2$ | | | $\gamma = 0.2$ | $\gamma = 0.5$ | $\gamma = 1$ | $\gamma = 2$ | | | $\gamma = 0.2$ | $\gamma = 0.5$ | $\gamma = 1$ | $\gamma = 2$ | | |
| $\tau = 0.9$ | size | 0.040 | 0.044 | 0.048 | 0.033 | 0.030 | 0.035 | 0.045 | 0.050 | 0.052 | 0.033 | 0.032 | 0.034 | 0.056 | 0.051 | 0.057 | 0.032 | 0.034 | 0.038 |
| | $\phi \to 0.8$ | 0.091 | 0.138 | 0.202 | 0.270 | 0.998 | 0.998 | 0.103 | 0.177 | 0.264 | 0.331 | 1.000 | 1.000 | 0.121 | 0.199 | 0.321 | 0.364 | 1.000 | 1.000 |
| | $\phi \to -0.8$ | 0.028 | 0.025 | 0.020 | 0.026 | 0.975 | 0.975 | 0.034 | 0.023 | 0.035 | 0.036 | 0.996 | 0.996 | 0.052 | 0.046 | 0.039 | 0.027 | 1.000 | 1.000 |
| | $\omega \to 0.5$ | 0.965 | 0.892 | 0.521 | 0.043 | 1.000 | 1.000 | 1.000 | 0.997 | 0.910 | 0.073 | 1.000 | 1.000 | 1.000 | 1.000 | 0.997 | 0.170 | 1.000 | 1.000 |
| | $\omega \to 0.7$ | 0.995 | 0.966 | 0.436 | 0.019 | 1.000 | 1.000 | 1.000 | 1.000 | 0.915 | 0.021 | 1.000 | 1.000 | 1.000 | 1.000 | 0.991 | 0.026 | 1.000 | 1.000 |
| | $\omega \to 0.1$ | 0.997 | 0.998 | 0.978 | 0.083 | 1.000 | 1.000 | 1.000 | 0.999 | 1.000 | 0.195 | 1.000 | 1.000 | 0.997 | 1.000 | 1.000 | 0.386 | 1.000 | 1.000 |
| | $(\alpha, \beta) \to (0.1, 0.8)$ | 0.854 | 0.567 | 0.126 | 0.040 | 0.999 | 0.998 | 0.984 | 0.908 | 0.363 | 0.048 | 1.000 | 1.000 | 0.993 | 0.976 | 0.608 | 0.060 | 1.000 | 1.000 |
| | $(\alpha, \beta) \to (0.3, 0.6)$ | 0.950 | 0.838 | 0.396 | 0.030 | 1.000 | 1.000 | 0.999 | 0.992 | 0.836 | 0.079 | 1.000 | 1.000 | 1.000 | 1.000 | 0.976 | 0.156 | 1.000 | 1.000 |
| | $\eta_t \sim Z_{4.5}$ | 0.266 | 0.183 | 0.032 | 0.037 | 0.966 | 0.943 | 0.546 | 0.412 | 0.066 | 0.040 | 0.993 | 0.992 | 0.714 | 0.615 | 0.103 | 0.049 | 0.999 | 0.998 |
| | $\eta_t \sim Z_{1,3}$ | 0.544 | 0.437 | 0.100 | 0.012 | 0.956 | 0.932 | 0.891 | 0.817 | 0.298 | 0.013 | 1.000 | 0.998 | 0.976 | 0.949 | 0.573 | 0.015 | 0.999 | 0.999 |
| $\tau = 0.1$ | size | 0.041 | 0.043 | 0.034 | 0.020 | 0.037 | 0.041 | 0.051 | 0.050 | 0.050 | 0.034 | 0.043 | 0.048 | 0.039 | 0.043 | 0.037 | 0.033 | 0.037 | 0.037 |
| | $\phi \to 0.8$ | 0.089 | 0.139 | 0.194 | 0.248 | 0.990 | 0.990 | 0.112 | 0.183 | 0.270 | 0.322 | 1.000 | 1.000 | 0.114 | 0.204 | 0.309 | 0.351 | 1.000 | 1.000 |
| | $\phi \to -0.8$ | 0.029 | 0.023 | 0.015 | 0.035 | 0.962 | 0.962 | 0.023 | 0.031 | 0.025 | 0.037 | 0.995 | 0.995 | 0.040 | 0.036 | 0.035 | 0.030 | 1.000 | 1.000 |
| | $\omega \to 0.5$ | 0.951 | 0.881 | 0.529 | 0.035 | 1.000 | 1.000 | 1.000 | 0.998 | 0.912 | 0.081 | 1.000 | 1.000 | 1.000 | 1.000 | 0.988 | 0.160 | 1.000 | 1.000 |
| | $\omega \to 0.7$ | 0.996 | 0.967 | 0.456 | 0.018 | 1.000 | 1.000 | 1.000 | 1.000 | 0.882 | 0.021 | 1.000 | 1.000 | 1.000 | 1.000 | 0.993 | 0.029 | 1.000 | 1.000 |
| | $\omega \to 0.1$ | 0.994 | 0.988 | 0.973 | 0.093 | 1.000 | 1.000 | 0.996 | 0.998 | 0.997 | 0.217 | 1.000 | 1.000 | 0.998 | 0.998 | 0.998 | 0.378 | 1.000 | 1.000 |
| | $(\alpha, \beta) \to (0.1, 0.8)$ | 0.864 | 0.585 | 0.153 | 0.055 | 0.999 | 0.998 | 0.976 | 0.905 | 0.409 | 0.049 | 1.000 | 1.000 | 0.995 | 0.978 | 0.584 | 0.050 | 1.000 | 1.000 |
| | $(\alpha, \beta) \to (0.3, 0.6)$ | 0.953 | 0.840 | 0.420 | 0.036 | 1.000 | 1.000 | 0.999 | 0.996 | 0.864 | 0.078 | 1.000 | 1.000 | 1.000 | 1.000 | 0.973 | 0.139 | 1.000 | 1.000 |
| | $\eta_t \sim Z_{4.5}$ | 0.273 | 0.181 | 0.041 | 0.081 | 0.962 | 0.928 | 0.521 | 0.375 | 0.053 | 0.105 | 0.988 | 0.984 | 0.757 | 0.623 | 0.094 | 0.077 | 0.994 | 0.994 |
| | $\eta_t \sim Z_{-1,3}$ | 0.572 | 0.456 | 0.103 | 0.009 | 0.966 | 0.940 | 0.882 | 0.806 | 0.343 | 0.009 | 0.999 | 0.997 | 0.969 | 0.933 | 0.562 | 0.015 | 1.000 | 1.000 |

Table S.8: Empirical sizes and powers of the CUSUM test when the underlying model is ARMA(1,1)-GARCH(1,1) with $(\phi, \psi, \omega, \alpha, \beta) = (-0.5, -0.2, 0.5, 0.2, 0.3)$ (Case 5), under $\tau \in \{0.9, 0.1\}$.

| | | $n = 500$ | | | | | | $n = 1,000$ | | | | | | $n = 1,500$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\hat{T}_n$ | | | | $\tilde{T}_n$ | $\check{T}_n$ | $\hat{T}_n$ | | | | $\tilde{T}_n$ | $\check{T}_n$ | $\hat{T}_n$ | | | | $\tilde{T}_n$ | $\check{T}_n$ |
| | | $\gamma = 0.2$ | $\gamma = 0.5$ | $\gamma = 1$ | $\gamma = 2$ | | | $\gamma = 0.2$ | $\gamma = 0.5$ | $\gamma = 1$ | $\gamma = 2$ | | | $\gamma = 0.2$ | $\gamma = 0.5$ | $\gamma = 1$ | $\gamma = 2$ | | |
| $\tau = 0.9$ | size | 0.042 | 0.038 | 0.033 | 0.028 | 0.042 | 0.042 | 0.056 | 0.052 | 0.043 | 0.036 | 0.054 | 0.051 | 0.055 | 0.063 | 0.054 | 0.033 | 0.050 | 0.043 |
| | $\phi \to 0.9$ | 0.173 | 0.266 | 0.347 | 0.366 | 0.983 | 0.985 | 0.361 | 0.441 | 0.495 | 0.478 | 1.000 | 1.000 | 0.509 | 0.595 | 0.621 | 0.572 | 1.000 | 1.000 |
| | $\phi \to -0.95$ | 0.082 | 0.108 | 0.134 | 0.134 | 0.992 | 0.992 | 0.106 | 0.167 | 0.190 | 0.151 | 1.000 | 1.000 | 0.179 | 0.248 | 0.263 | 0.187 | 1.000 | 1.000 |
| | $\psi \to -0.9$ | 0.040 | 0.041 | 0.047 | 0.076 | 0.987 | 0.986 | 0.047 | 0.043 | 0.066 | 0.119 | 1.000 | 1.000 | 0.047 | 0.054 | 0.060 | 0.129 | 1.000 | 1.000 |
| | $\omega \to 0.7$ | 0.755 | 0.640 | 0.358 | 0.048 | 1.000 | 1.000 | 0.976 | 0.949 | 0.753 | 0.143 | 1.000 | 1.000 | 1.000 | 0.995 | 0.917 | 0.243 | 1.000 | 1.000 |
| | $\omega \to 1$ | 0.991 | 0.947 | 0.512 | 0.037 | 1.000 | 1.000 | 1.000 | 1.000 | 0.903 | 0.048 | 1.000 | 1.000 | 1.000 | 1.000 | 0.992 | 0.066 | 1.000 | 1.000 |
| | $\omega \to 0.3$ | 0.976 | 0.943 | 0.704 | 0.090 | 1.000 | 1.000 | 1.000 | 1.000 | 0.960 | 0.242 | 1.000 | 1.000 | 1.000 | 1.000 | 0.998 | 0.403 | 1.000 | 1.000 |
| | $\alpha \to 0.65$ | 0.514 | 0.484 | 0.254 | 0.047 | 0.919 | 0.899 | 0.847 | 0.841 | 0.620 | 0.054 | 1.000 | 1.000 | 0.957 | 0.964 | 0.856 | 0.065 | 1.000 | 1.000 |
| | $\beta \to 0.7$ | 0.964 | 0.854 | 0.354 | 0.032 | 1.000 | 1.000 | 0.999 | 0.985 | 0.765 | 0.060 | 1.000 | 1.000 | 1.000 | 0.997 | 0.902 | 0.081 | 1.000 | 1.000 |
| | $\eta_t \sim Z_{4.5}$ | 0.296 | 0.192 | 0.050 | 0.070 | 0.942 | 0.907 | 0.565 | 0.444 | 0.085 | 0.083 | 0.993 | 0.990 | 0.732 | 0.641 | 0.152 | 0.087 | 0.997 | 0.996 |
| | $\eta_t \sim Z_{1,3}$ | 0.468 | 0.356 | 0.123 | 0.052 | 0.927 | 0.893 | 0.800 | 0.723 | 0.312 | 0.036 | 0.999 | 0.998 | 0.934 | 0.891 | 0.547 | 0.043 | 1.000 | 1.000 |
| $\tau = 0.1$ | size | 0.046 | 0.048 | 0.048 | 0.038 | 0.049 | 0.046 | 0.058 | 0.054 | 0.052 | 0.042 | 0.058 | 0.054 | 0.044 | 0.044 | 0.042 | 0.032 | 0.045 | 0.043 |
| | $\phi \to 0.9$ | 0.183 | 0.248 | 0.336 | 0.358 | 0.986 | 0.987 | 0.388 | 0.467 | 0.517 | 0.482 | 0.999 | 0.999 | 0.501 | 0.591 | 0.613 | 0.547 | 1.000 | 1.000 |
| | $\phi \to -0.95$ | 0.077 | 0.125 | 0.159 | 0.170 | 0.991 | 0.989 | 0.122 | 0.170 | 0.201 | 0.175 | 1.000 | 1.000 | 0.188 | 0.259 | 0.262 | 0.180 | 1.000 | 1.000 |
| | $\psi \to -0.9$ | 0.053 | 0.045 | 0.048 | 0.071 | 0.971 | 0.971 | 0.046 | 0.047 | 0.077 | 0.111 | 1.000 | 1.000 | 0.045 | 0.041 | 0.071 | 0.113 | 1.000 | 1.000 |
| | $\omega \to 0.7$ | 0.776 | 0.659 | 0.379 | 0.059 | 1.000 | 1.000 | 0.979 | 0.941 | 0.735 | 0.118 | 1.000 | 1.000 | 0.999 | 0.994 | 0.918 | 0.192 | 1.000 | 1.000 |
| | $\omega \to 1$ | 0.991 | 0.950 | 0.519 | 0.036 | 0.999 | 0.999 | 1.000 | 1.000 | 0.906 | 0.053 | 1.000 | 1.000 | 1.000 | 1.000 | 0.988 | 0.058 | 1.000 | 1.000 |
| | $\omega \to 0.3$ | 0.973 | 0.912 | 0.665 | 0.111 | 1.000 | 1.000 | 1.000 | 0.999 | 0.950 | 0.245 | 1.000 | 1.000 | 1.000 | 1.000 | 0.999 | 0.360 | 1.000 | 1.000 |
| | $\alpha \to 0.65$ | 0.532 | 0.505 | 0.273 | 0.045 | 0.934 | 0.913 | 0.869 | 0.863 | 0.633 | 0.059 | 0.996 | 0.997 | 0.975 | 0.976 | 0.851 | 0.062 | 1.000 | 1.000 |
| | $\beta \to 0.7$ | 0.969 | 0.881 | 0.374 | 0.030 | 1.000 | 1.000 | 0.998 | 0.987 | 0.772 | 0.060 | 1.000 | 1.000 | 1.000 | 0.997 | 0.914 | 0.086 | 1.000 | 1.000 |
| | $\eta_t \sim Z_{4.5}$ | 0.291 | 0.209 | 0.056 | 0.087 | 0.955 | 0.932 | 0.553 | 0.438 | 0.094 | 0.089 | 0.992 | 0.988 | 0.736 | 0.637 | 0.153 | 0.089 | 0.992 | 0.990 |
| | $\eta_t \sim Z_{-1,3}$ | 0.503 | 0.373 | 0.109 | 0.035 | 0.949 | 0.921 | 0.803 | 0.715 | 0.312 | 0.043 | 1.000 | 0.996 | 0.929 | 0.878 | 0.486 | 0.049 | 0.999 | 0.999 |

Table S.9: Empirical sizes and powers of the CUSUM test when the underlying model is ARMA(1,1)-GARCH(1,1) with $(\phi, \psi, \omega, \alpha, \beta) = (-0.3, 0.6, 0.6, 0.2, 0.7)$ (Case 6), under $\tau \in \{0.9, 0.1\}$.

| | | $n=500$ | | | | | | $n=1,000$ | | | | | | $n=1,500$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\hat{T}_n$ | | | | $\tilde{T}_n$ | $\check{T}_n$ | $\hat{T}_n$ | | | | $\tilde{T}_n$ | $\check{T}_n$ | $\hat{T}_n$ | | | | $\tilde{T}_n$ | $\check{T}_n$ |
| | | $\gamma=0.2$ | $\gamma=0.5$ | $\gamma=1$ | $\gamma=2$ | | | $\gamma=0.2$ | $\gamma=0.5$ | $\gamma=1$ | $\gamma=2$ | | | $\gamma=0.2$ | $\gamma=0.5$ | $\gamma=1$ | $\gamma=2$ | | |
| $\tau=0.9$ | size | 0.057 | 0.048 | 0.043 | 0.025 | 0.106 | 0.106 | 0.050 | 0.055 | 0.041 | 0.022 | 0.072 | 0.076 | 0.045 | 0.043 | 0.037 | 0.024 | 0.071 | 0.071 |
| | $\phi \to 0.9$ | 0.134 | 0.220 | 0.317 | 0.324 | 0.989 | 0.989 | 0.165 | 0.282 | 0.367 | 0.359 | 1.000 | 1.000 | 0.196 | 0.382 | 0.497 | 0.454 | 1.000 | 1.000 |
| | $\psi \to -0.95$ | 0.055 | 0.046 | 0.047 | 0.057 | 0.997 | 0.997 | 0.050 | 0.044 | 0.059 | 0.068 | 1.000 | 1.000 | 0.069 | 0.074 | 0.081 | 0.093 | 1.000 | 1.000 |
| | $\omega \to 1$ | 0.645 | 0.340 | 0.052 | 0.091 | 0.998 | 0.998 | 0.922 | 0.721 | 0.085 | 0.071 | 1.000 | 1.000 | 0.981 | 0.906 | 0.176 | 0.085 | 1.000 | 1.000 |
| | $\omega \to 1.5$ | 0.789 | 0.145 | 0.074 | 0.287 | 0.998 | 0.998 | 0.951 | 0.325 | 0.079 | 0.321 | 0.998 | 0.998 | 0.992 | 0.494 | 0.077 | 0.336 | 0.997 | 0.997 |
| | $\omega \to 0.1$ | 0.981 | 0.927 | 0.187 | 0.067 | 1.000 | 1.000 | 0.995 | 0.976 | 0.382 | 0.073 | 1.000 | 1.000 | 0.998 | 0.995 | 0.599 | 0.054 | 1.000 | 1.000 |
| | $\beta \to 0.1$ | 0.910 | 0.766 | 0.196 | 0.043 | 1.000 | 1.000 | 0.961 | 0.914 | 0.402 | 0.051 | 1.000 | 1.000 | 0.989 | 0.957 | 0.612 | 0.062 | 1.000 | 1.000 |
| | $(\alpha,\beta) \to (0.5,0.4)$ | 0.546 | 0.229 | 0.054 | 0.039 | 0.991 | 0.990 | 0.830 | 0.387 | 0.057 | 0.038 | 1.000 | 1.000 | 0.902 | 0.577 | 0.081 | 0.054 | 1.000 | 1.000 |
| | $(\alpha,\beta) \to (0.7,0.2)$ | 0.719 | 0.291 | 0.058 | 0.037 | 0.997 | 0.997 | 0.926 | 0.493 | 0.059 | 0.046 | 0.998 | 0.997 | 0.980 | 0.660 | 0.059 | 0.042 | 1.000 | 1.000 |
| | $\eta_t \sim Z_{4.5}$ | 0.515 | 0.133 | 0.101 | 0.230 | 0.929 | 0.904 | 0.732 | 0.254 | 0.084 | 0.253 | 0.977 | 0.973 | 0.892 | 0.392 | 0.084 | 0.245 | 0.987 | 0.983 |
| | $\eta_t \sim Z_{1,3}$ | 0.503 | 0.234 | 0.037 | 0.162 | 0.946 | 0.913 | 0.791 | 0.512 | 0.033 | 0.161 | 0.998 | 0.997 | 0.913 | 0.730 | 0.049 | 0.177 | 1.000 | 1.000 |
| $\tau=0.1$ | size | 0.056 | 0.051 | 0.040 | 0.016 | 0.092 | 0.094 | 0.066 | 0.056 | 0.045 | 0.024 | 0.076 | 0.075 | 0.057 | 0.052 | 0.037 | 0.017 | 0.061 | 0.059 |
| | $\phi \to 0.9$ | 0.109 | 0.195 | 0.285 | 0.301 | 0.991 | 0.991 | 0.135 | 0.271 | 0.403 | 0.389 | 1.000 | 1.000 | 0.195 | 0.361 | 0.468 | 0.435 | 1.000 | 1.000 |
| | $\psi \to -0.95$ | 0.049 | 0.047 | 0.048 | 0.045 | 0.998 | 0.998 | 0.050 | 0.045 | 0.049 | 0.071 | 1.000 | 1.000 | 0.044 | 0.056 | 0.079 | 0.090 | 1.000 | 1.000 |
| | $\omega \to 1$ | 0.645 | 0.350 | 0.052 | 0.071 | 1.000 | 1.000 | 0.925 | 0.725 | 0.098 | 0.092 | 1.000 | 1.000 | 0.976 | 0.911 | 0.176 | 0.082 | 1.000 | 1.000 |
| | $\omega \to 1.5$ | 0.740 | 0.129 | 0.076 | 0.312 | 1.000 | 1.000 | 0.962 | 0.300 | 0.084 | 0.325 | 1.000 | 1.000 | 0.994 | 0.483 | 0.077 | 0.337 | 1.000 | 1.000 |
| | $\omega \to 0.1$ | 0.987 | 0.947 | 0.192 | 0.055 | 0.999 | 1.000 | 0.994 | 0.984 | 0.363 | 0.066 | 0.999 | 0.999 | 0.997 | 0.991 | 0.564 | 0.078 | 1.000 | 1.000 |
| | $\beta \to 0.1$ | 0.912 | 0.764 | 0.209 | 0.048 | 0.999 | 1.000 | 0.971 | 0.921 | 0.387 | 0.057 | 1.000 | 1.000 | 0.986 | 0.964 | 0.593 | 0.049 | 1.000 | 1.000 |
| | $(\alpha,\beta) \to (0.5,0.4)$ | 0.544 | 0.232 | 0.050 | 0.021 | 0.992 | 0.989 | 0.834 | 0.401 | 0.057 | 0.030 | 0.999 | 0.999 | 0.949 | 0.577 | 0.076 | 0.038 | 1.000 | 1.000 |
| | $(\alpha,\beta) \to (0.7,0.2)$ | 0.743 | 0.316 | 0.045 | 0.028 | 0.991 | 0.991 | 0.929 | 0.499 | 0.052 | 0.039 | 0.999 | 0.999 | 0.974 | 0.678 | 0.063 | 0.048 | 0.999 | 1.000 |
| | $\eta_t \sim Z_{4.5}$ | 0.502 | 0.129 | 0.086 | 0.242 | 0.956 | 0.926 | 0.758 | 0.255 | 0.096 | 0.255 | 0.991 | 0.986 | 0.854 | 0.360 | 0.083 | 0.243 | 0.986 | 0.985 |
| | $\eta_t \sim Z_{-1,3}$ | 0.485 | 0.218 | 0.040 | 0.151 | 0.943 | 0.918 | 0.794 | 0.489 | 0.055 | 0.167 | 0.997 | 0.997 | 0.928 | 0.724 | 0.058 | 0.164 | 0.998 | 0.998 |

Table S.10: Mean and standard deviation (SD) of the location of the detected change point when the underlying model is GJR-GARCH(1,1) (Cases 1-3) with $\tau \in \{0.9, 0.1\}$ and $n = 1,000$. Notice that the true location of the change point is at $t = 1,001$.

| | | $\tau = 0.9$ | | | | | | $\tau = 0.1$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\hat{T}_n$ | | $\tilde{T}_n$ | | $\check{T}_n$ | | $\hat{T}_n$ | | $\tilde{T}_n$ | | $\check{T}_n$ | |
| | | mean | SD | mean | SD | mean | SD | mean | SD | mean | SD | mean | SD |
| Case 1 | $\omega \to 0.5$ | 1021.1 | 54.7 | 1005.3 | 12.7 | 1005.3 | 12.7 | 1019.1 | 51.4 | 1005.3 | 12.7 | 1005.3 | 12.7 |
| | $\omega \to 0.7$ | 1015.9 | 40.5 | 1002.5 | 5.7 | 1002.5 | 5.7 | 1017.3 | 37.7 | 1002.4 | 6.0 | 1002.4 | 6.0 |
| | $\omega \to 0.1$ | 989.0 | 16.8 | 996.2 | 5.1 | 996.2 | 5.1 | 988.1 | 18.0 | 996.5 | 4.9 | 996.5 | 4.9 |
| | $\omega \to 0.01$ | 984.2 | 24.9 | 997.5 | 3.7 | 997.5 | 3.7 | 984.7 | 22.5 | 997.6 | 3.2 | 997.6 | 3.2 |
| | $\alpha_1 \to 2.5$ | 1080.3 | 186.2 | 1041.0 | 108.1 | 1040.9 | 108.5 | 1040.0 | 169.6 | 1049.3 | 121.5 | 1049.0 | 121.6 |
| | $\alpha_2 \to 2.5$ | 1041.4 | 149.8 | 1038.2 | 84.5 | 1038.2 | 84.5 | 1058.2 | 145.1 | 1034.7 | 84.1 | 1034.7 | 84.1 |
| | $\beta \to 0.6$ | 1029.4 | 107.5 | 1015.2 | 34.4 | 1015.2 | 34.4 | 1030.1 | 99.9 | 1014.1 | 31.5 | 1014.1 | 31.5 |
| | $\beta \to 0.7$ | 1022.1 | 57.2 | 1009.9 | 16.5 | 1009.9 | 16.5 | 1022.0 | 58.4 | 1010.5 | 17.9 | 1010.5 | 17.9 |
| | $\eta_t \sim Z_5$ | 1045.0 | 182.5 | 1050.9 | 63.6 | 1050.9 | 63.6 | 1045.5 | 167.8 | 1052.3 | 67.0 | 1052.3 | 67.0 |
| | $\eta_t \sim Z_7$ | 1045.7 | 165.3 | 1051.1 | 62.5 | 1051.1 | 62.5 | 1045.4 | 168.3 | 1052.2 | 67.3 | 1051.8 | 66.4 |
| Case 2 | $\omega \to 0.25$ | 1008.2 | 345.8 | 972.2 | 337.5 | 982.3 | 331.8 | 1022.6 | 316.1 | 972.9 | 341.7 | 956.9 | 346.6 |
| | $\omega \to 0.35$ | 1035.4 | 147.3 | 1014.7 | 57.3 | 1014.7 | 57.3 | 1031.6 | 148.9 | 1013.5 | 64.0 | 1013.5 | 64.0 |
| | $\omega \to 0.15$ | 1019.7 | 43.8 | 1006.2 | 12.2 | 1006.2 | 12.2 | 1017.3 | 43.1 | 1006.1 | 11.8 | 1006.1 | 11.8 |
| | $\omega \to 0.1$ | 971.6 | 112.2 | 988.7 | 34.3 | 988.7 | 34.3 | 964.3 | 102.0 | 988.1 | 33.5 | 988.1 | 33.5 |
| | $\alpha_1 \to 3.5$ | 982.9 | 30.4 | 994.6 | 9.2 | 994.6 | 9.2 | 982.8 | 32.4 | 994.6 | 8.2 | 994.6 | 8.2 |
| | $\alpha_2 \to 3.5$ | 1077.6 | 160.8 | 1038.5 | 88.9 | 1038.5 | 88.9 | 1031.2 | 157.0 | 1039.0 | 97.8 | 1039.0 | 97.8 |
| | $(\alpha_1, \alpha_2) \to (1.5, 2.5)$ | 1040.5 | 169.6 | 1052.1 | 119.1 | 1052.4 | 119.0 | 1071.6 | 170.1 | 1042.8 | 106.7 | 1042.7 | 106.6 |
| | $\beta \to 0.05$ | 1053.2 | 148.7 | 1041.0 | 85.9 | 1041.0 | 85.9 | 1057.6 | 152.4 | 1040.6 | 84.8 | 1040.6 | 84.8 |
| | $\beta \to 0.7$ | 969.0 | 156.0 | 983.2 | 74.3 | 983.2 | 74.3 | 961.0 | 160.0 | 975.1 | 77.1 | 975.1 | 77.1 |
| | $\eta_t \sim Z_5$ | 1023.9 | 63.7 | 1010.2 | 20.5 | 1010.2 | 20.5 | 1024.5 | 67.1 | 1010.2 | 20.1 | 1010.2 | 20.1 |
| | $\eta_t \sim Z_7$ | 1055.2 | 154.7 | 1054.1 | 73.1 | 1053.3 | 70.5 | 1052.5 | 157.5 | 1055.6 | 73.0 | 1055.6 | 73.0 |
| Case 3 | $\omega \to 1$ | 1032.5 | 113.0 | 1010.0 | 34.3 | 1010.0 | 34.3 | 1028.8 | 113.3 | 1009.2 | 32.4 | 1009.2 | 32.4 |
| | $\omega \to 1.5$ | 1033.9 | 72.2 | 1003.3 | 10.1 | 1003.3 | 10.1 | 1026.6 | 60.3 | 1003.4 | 11.7 | 1003.4 | 11.7 |
| | $\omega \to 0.5$ | 970.1 | 109.0 | 989.5 | 32.8 | 989.5 | 32.8 | 968.2 | 103.4 | 989.2 | 32.5 | 989.2 | 32.5 |
| | $\omega \to 0.1$ | 984.1 | 23.8 | 997.4 | 3.3 | 997.4 | 3.3 | 985.0 | 21.3 | 997.3 | 3.3 | 997.3 | 3.3 |
| | $\alpha_1 \to 0.6$ | 1077.1 | 146.3 | 1041.1 | 81.2 | 1041.1 | 81.2 | 1052.8 | 149.2 | 1040.0 | 81.3 | 1040.0 | 81.3 |
| | $\alpha_2 \to 1$ | 1055.8 | 156.9 | 1050.0 | 105.1 | 1050.0 | 105.1 | 1088.0 | 179.5 | 1051.9 | 107.4 | 1051.9 | 107.4 |
| | $\beta \to 0.1$ | 970.0 | 78.3 | 991.3 | 16.5 | 991.3 | 16.5 | 967.1 | 74.3 | 991.3 | 16.3 | 991.3 | 16.3 |
| | $\beta \to 0.75$ | 1038.3 | 120.2 | 1017.9 | 40.0 | 1017.9 | 40.0 | 1046.3 | 119.6 | 1018.9 | 40.2 | 1018.9 | 40.2 |
| | $\eta_t \sim Z_5$ | 1071.1 | 150.3 | 1061.9 | 81.6 | 1061.4 | 81.3 | 1082.9 | 167.4 | 1059.6 | 74.7 | 1059.3 | 72.7 |
| | $\eta_t \sim Z_7$ | 1102.6 | 160.9 | 1073.3 | 104.0 | 1087.4 | 127.0 | 1103.2 | 166.3 | 1074.2 | 103.1 | 1083.0 | 116.9 |

Table S.11: Mean and standard deviation (SD) of the location of the detected change point when the underlying model is ARMA(1,1)-GARCH(1,1) (Cases 4-6) with $\tau \in \{0.9, 0.1\}$ and $n = 1,000$.

| | | $\tau = 0.9$ | | | | | | $\tau = 0.1$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\hat{T}_n$ | | $\tilde{T}_n$ | | $\check{T}_n$ | | $\hat{T}_n$ | | $\tilde{T}_n$ | | $\check{T}_n$ | |
| | | mean | SD | mean | SD | mean | SD | mean | SD | mean | SD | mean | SD |
| Case 4 | $\phi \to 0.8$ | 1257.4 | 239.4 | 1016.1 | 25.4 | 1016.2 | 25.3 | 1257.2 | 244.6 | 1016.1 | 24.2 | 1016.1 | 24.2 |
| | $\phi \to -0.8$ | 1101.9 | 267.1 | 1016.7 | 26.2 | 1016.8 | 26.3 | 1072.1 | 226.1 | 1016.2 | 25.6 | 1016.4 | 25.7 |
| | $\omega \to 0.5$ | 805.0 | 236.3 | 993.3 | 62.8 | 993.3 | 62.7 | 793.9 | 268.7 | 989.2 | 66.5 | 989.3 | 67.8 |
| | $\omega \to 0.7$ | 1060.5 | 312.7 | 1002.0 | 31.3 | 1002.0 | 31.4 | 1137.5 | 323.8 | 1001.6 | 32.0 | 1001.4 | 31.9 |
| | $\omega \to 0.1$ | 1025.2 | 71.1 | 1005.4 | 15.5 | 1005.4 | 15.5 | 1019.1 | 67.5 | 1004.4 | 14.3 | 1004.3 | 14.3 |
| | $(\alpha, \beta) \to (0.1, 0.8)$ | 1021.6 | 46.6 | 1002.0 | 5.7 | 1002.0 | 5.7 | 1018.3 | 43.4 | 1002.2 | 6.1 | 1002.2 | 6.1 |
| | $(\alpha, \beta) \to (0.3, 0.6)$ | 985.8 | 22.6 | 996.7 | 4.8 | 996.7 | 4.8 | 987.7 | 21.0 | 996.6 | 4.9 | 996.5 | 4.9 |
| | $\eta_t \sim Z_{4.5}$ | 1034.5 | 175.1 | 1062.9 | 82.5 | 1063.5 | 83.0 | 1048.3 | 177.6 | 1058.9 | 82.5 | 1057.8 | 77.4 |
| | $\eta_t \sim Z_{\pm 1,3}$ | 1028.5 | 131.1 | 1052.8 | 76.0 | 1052.7 | 75.3 | 1033.9 | 138.4 | 1056.9 | 76.2 | 1057.9 | 78.7 |
| Case 5 | $\phi \to 0.9$ | 1191.5 | 228.3 | 1012.5 | 32.8 | 1012.8 | 32.4 | 1182.7 | 220.4 | 1015.1 | 38.6 | 1015.8 | 39.2 |
| | $\phi \to -0.7$ | 1244.5 | 221.5 | 1046.3 | 71.4 | 1046.3 | 71.3 | 1243.3 | 201.0 | 1042.5 | 59.6 | 1042.6 | 59.5 |
| | $\psi \to 0.8$ | 1025.5 | 215.1 | 1011.3 | 55.3 | 1011.4 | 55.4 | 997.8 | 251.8 | 1005.9 | 51.4 | 1006.2 | 51.8 |
| | $\omega \to 1$ | 1031.1 | 110.4 | 1008.7 | 29.6 | 1008.7 | 29.7 | 1024.0 | 100.3 | 1009.1 | 30.9 | 1009.0 | 30.8 |
| | $\omega \to 0.1$ | 1020.9 | 55.5 | 1002.7 | 9.5 | 1002.8 | 9.6 | 1019.4 | 54.5 | 1002.7 | 9.4 | 1002.7 | 9.5 |
| | $\omega \to 0.05$ | 976.0 | 58.2 | 992.5 | 16.1 | 992.6 | 15.8 | 974.7 | 65.3 | 993.2 | 13.6 | 993.2 | 13.6 |
| | $(\alpha, \beta) \to (0.1, 0.8)$ | 1048.8 | 146.7 | 1025.0 | 66.3 | 1025.3 | 66.5 | 1043.7 | 147.3 | 1027.2 | 67.4 | 1027.5 | 68.0 |
| | $(\alpha, \beta) \to (0.3, 0.6)$ | 1021.1 | 59.7 | 1002.1 | 9.5 | 1002.0 | 9.5 | 1021.6 | 59.1 | 1001.9 | 6.3 | 1001.9 | 6.3 |
| | $\eta_t \sim Z_{4.5}$ | 1053.2 | 168.8 | 1062.5 | 85.3 | 1062.7 | 86.3 | 1053.5 | 169.8 | 1060.5 | 78.4 | 1062.1 | 83.1 |
| | $\eta_t \sim Z_{\pm 1,3}$ | 1036.2 | 136.1 | 1056.3 | 78.5 | 1057.1 | 79.2 | 1031.3 | 139.4 | 1053.3 | 75.3 | 1053.7 | 76.2 |
| Case 6 | $\phi \to 0.9$ | 1260.1 | 229.8 | 1022.1 | 40.8 | 1022.2 | 40.8 | 1259.5 | 226.9 | 1023.5 | 36.3 | 1023.5 | 36.3 |
| | $\psi \to -0.95$ | 964.0 | 296.3 | 1004.2 | 30.8 | 1004.2 | 30.8 | 859.3 | 302.5 | 1003.2 | 20.3 | 1003.1 | 20.3 |
| | $\omega \to 1$ | 1026.3 | 114.9 | 1000.8 | 26.7 | 1000.9 | 27.4 | 1034.5 | 107.4 | 999.9 | 34.3 | 999.8 | 34.2 |
| | $\omega \to 1.5$ | 1045.0 | 102.3 | 1000.6 | 29.7 | 1000.5 | 30.0 | 1056.7 | 115.9 | 1001.6 | 33.1 | 1001.5 | 33.0 |
| | $\omega \to 0.1$ | 975.3 | 48.4 | 994.3 | 22.2 | 994.6 | 18.0 | 978.6 | 42.5 | 992.5 | 26.3 | 992.5 | 26.3 |
| | $\beta \to 0.1$ | 963.0 | 82.7 | 989.5 | 28.3 | 989.3 | 28.6 | 962.5 | 81.4 | 991.0 | 24.6 | 991.5 | 23.9 |
| | $(\alpha, \beta) \to (0.5, 0.4)$ | 982.0 | 135.7 | 1001.6 | 62.4 | 1001.8 | 61.1 | 979.7 | 137.3 | 998.9 | 44.2 | 999.2 | 43.9 |
| | $(\alpha, \beta) \to (0.7, 0.2)$ | 975.0 | 117.4 | 1001.6 | 47.3 | 1001.8 | 43.9 | 979.5 | 111.7 | 1000.3 | 34.7 | 1000.4 | 31.0 |
| | $\eta_t \sim Z_{4.5}$ | 1074.3 | 149.3 | 1078.8 | 120.1 | 1081.0 | 124.6 | 1062.6 | 147.8 | 1085.1 | 131.3 | 1085.9 | 133.4 |
| | $\eta_t \sim Z_{\pm 1,3}$ | 1034.3 | 124.3 | 1056.6 | 96.8 | 1057.3 | 97.3 | 1042.1 | 142.9 | 1061.5 | 95.2 | 1062.9 | 97.3 |