Supplement to “A tuning-free efficient test for marginal linear effects in high-dimensional quantile regression”

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Proof of Theorem 1

By the scale and translation invariance properties of $\hat{qcor}(Y, X_k)$, $k = 1, \ldots, p$, we assume without loss of generality that $E(X_1) = \ldots = E(X_p) = 0$ and $\text{var}(X_1) = \ldots = \text{var}(X_p) = 1$. Define the infeasible maximum-type statistic by $\hat{S}^2_T = \max_{1 \leq k \leq p} |\hat{qcor}^2_k(Y, X_k)|$, where $\hat{qcor}^2_k(Y, X_k) = \{\tau(1 - \tau)\}^{-1/2}n^{-1}\sum_{i=1}^n [\psi_{\tau}\{Y_i - \hat{Q}_\tau(Y)\} - \psi_{\tau}\{Y_i - Q_{\tau}(Y)\}]X_{ik}$, for $k = 1, \ldots, p$. By the definitions of $\hat{qcor}(Y, X_k)$ and $\hat{qcor}^2_k(Y, X_k)$, we can decompose $\hat{qcor}(Y, X_k) - \hat{qcor}^2_k(Y, X_k)$ as $\hat{qcor}(Y, X_k) - \hat{qcor}^2_k(Y, X_k) = \sum_{i=1}^7 I_{kl}$, where

$I_{k1} = -\{\tau(1 - \tau)\}^{-1/2}X_kn^{-1}\sum_{i=1}^n [\psi_{\tau}\{Y_i - \hat{Q}_\tau(Y)\} - \psi_{\tau}\{Y_i - Q_{\tau}(Y)\}],$

$I_{k2} = \{\tau(1 - \tau)\}^{-1/2}n^{-1}\sum_{i=1}^n [\psi_{\tau}\{Y_i - \hat{Q}_\tau(Y)\} - \psi_{\tau}\{Y_i - Q_{\tau}(Y)\}]X_{ik},$

$I_{k3} = -\{\tau(1 - \tau)\}^{-1/2}(\hat{\sigma}_k^{-1} - 1)X_kn^{-1}\sum_{i=1}^n \psi_{\tau}\{Y_i - Q_{\tau}(Y)\},$

$I_{k4} = \{\tau(1 - \tau)\}^{-1/2}(\hat{\sigma}_k^{-1} - 1)n^{-1}\sum_{i=1}^n \psi_{\tau}\{Y_i - Q_{\tau}(Y)\}X_{ik},$

$I_{k5} = -\{\tau(1 - \tau)\}^{-1/2}(\hat{\sigma}_k^{-1} - 1)X_kn^{-1}\sum_{i=1}^n [\psi_{\tau}\{Y_i - \hat{Q}_\tau(Y)\} - \psi_{\tau}\{Y_i - Q_{\tau}(Y)\}],$

$I_{k6} = \{\tau(1 - \tau)\}^{-1/2}(\hat{\sigma}_k^{-1} - 1)n^{-1}\sum_{i=1}^n [\psi_{\tau}\{Y_i - \hat{Q}_\tau(Y)\} - \psi_{\tau}\{Y_i - Q_{\tau}(Y)\}]X_{ik},$

$I_{k7} = -\{\tau(1 - \tau)\}^{-1/2}X_kn^{-1}\sum_{i=1}^n \psi_{\tau}\{Y_i - Q_{\tau}(Y)\}$. 

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By the triangle inequality, \(| \widehat{S}_\tau - \widehat{S}_\tau^2 | \leq \sum_{l=1}^{7} \max_{1 \leq k \leq p} | I_{kl} |\). In what follows, we provide non-asymptotic bounds on \(\max_{1 \leq k \leq p} | I_{kl} |, l = 1, \ldots, 7\), under two scenarios of \(X\): (i) \(X\) is strongly bounded; (ii) \(X\) has i.i.d. sub-Gaussian rows. Throughout the proof, the notations \(C\) and \(c\) are generic constants, which may take different values at each appearance.

We first deal with \(\max_{1 \leq k \leq p} | I_{k1} |\). Recalling the definition of \(I_{k1}\), we have

\[
\max_{1 \leq k \leq p} | I_{k1} | = \{(1 - \tau)\}^{-1/2} \left| n^{-1} \sum_{i=1}^{n} \left[ \psi_\tau \{ Y_i - \hat{Q}_\tau(Y) \} - \psi_\tau \{ Y_i - Q_\tau(Y) \} \right] \right| \max_{1 \leq k \leq p} | \bar{X}_k | \\
\leq \{(1 - \tau)\}^{-1/2} n^{-1} \sum_{i=1}^{n} \left| \psi_\tau \{ Y_i - \hat{Q}_\tau(Y) \} - \psi_\tau \{ Y_i - Q_\tau(Y) \} \right| \max_{1 \leq k \leq p} | \bar{X}_k |.
\]

For any given \(\epsilon, \bar{\epsilon} > 0\), it can be easily shown that

\[
P(\max_{1 \leq k \leq p} | I_{k1} | \geq \epsilon \bar{\epsilon}) \leq P \left[ n^{-1} \sum_{i=1}^{n} \left| \psi_\tau \{ Y_i - \hat{Q}_\tau(Y) \} - \psi_\tau \{ Y_i - Q_\tau(Y) \} \right| \geq \{(1 - \tau)\}^{1/2} \epsilon \right] \\
+ P(\max_{1 \leq k \leq p} | \bar{X}_k | \geq \bar{\epsilon}). \tag{0.1}
\]

When \(\epsilon\) is sufficiently small and by Lemma 3, the first term on the right-hand side of \(0.1\) is bounded by \(3 \exp\{ -2c\tau(1 - \tau)n\epsilon^2 \}\). By Lemma 8 of [Chernozhukov et al. (2015)], it is routine to verify that \(E(\max_{1 \leq k \leq p} | \bar{X}_k |) \lesssim \{ \log(p)/n \}^{1/2} + \{ E(\max_{1 \leq i \leq n} \max_{1 \leq k \leq p} X_{ik}^2) \}^{1/2} \{ \log(p)/n \}\). Applying Lemma 5, we have for every
\( t > 0 \) and \( r > 2 \),

\[
\mathbb{P}\{ \max_{1 \leq k \leq p} |X_k| \geq 2E( \max_{1 \leq k \leq p} |X_k|) + t \} \\
\lesssim \exp\{- (nt)^2 / (3n \max_{1 \leq i \leq n} \max_{1 \leq k \leq p} E|X_{ik}|^2)\} + (nt)^{-r} \sum_{i=1}^{n} E( \max_{1 \leq k \leq p} |X_{ik}|^r). \quad (0.2)
\]

In the strongly bounded case, it is straightforward to see that

\[
\mathbb{E}(\max_{1 \leq k \leq p} |X_k|) \lesssim \{\log(p)/n\}^{1/2} \vee K_n \{\log(p)/n\} \land \mathbb{P}\{ \max_{1 \leq k \leq p} |X_k| \geq 2E( \max_{1 \leq k \leq p} |X_k|) + t \} \lesssim \exp(- nt^2 / 3) + n^{1-r} t^{-r} K_n^r. \]

By taking \( t \asymp K_n \{\log(p)/n\}^{1/2} \), it follows from (0.2) that

\[
\mathbb{P}\{ \max_{1 \leq k \leq p} |X_k| \leq CK_n \{\log(p)/n\} \vee CK_n \{\log(p)/n\}^{1/2} \} = 1 - O(p^{-c} + n^{1-r/2}),
\]

for some positive constants \( C, c > 0 \). Let

\[ \tilde{\epsilon} \asymp K_n \{\log(p)/n\} \vee K_n \{\log(p)/n\}^{1/2}, \]

and

\[ \epsilon \asymp \{\log(p)/n\}^{1/2}. \]

Using (0.1), we can easily prove that

\[
\mathbb{P}\{ \max_{1 \leq k \leq p} |I_{k1}| \leq CK_n \{\log(p)/n\}^{3/2} \vee CK_n \{\log(p)/n\} \} = 1 - O(p^{-c} + n^{1-r/2}), \quad (0.3)
\]

for some positive constants \( C, c > 0 \). For the sub-Gaussian case, we define the function

\[ \psi_\beta : [0, \infty) \to [0, \infty) \text{ by } \psi_\beta(x) = \exp(x^\beta) - 1 \text{ for } \beta > 0, \]

and for a real-valued random variable \( \xi \), we define

\[ \|\xi\|_{\psi_\beta} \overset{\text{def}}{=} \inf\{\lambda > 0 : E[\psi_\beta(|\xi| / \lambda)] \leq 1\}. \]
By Problem 2.2.5 and Lemma 2.2.2 in van der Vaart and Wellner (1996), it is not difficult to verify that

\[
E \left( \max_{1 \leq k \leq p} |X_{ik}|^r \right) \leq (\Pi_{l=1}^r l^r) \max_{1 \leq k \leq p} X_{ik}^r \leq (\Pi_{l=1}^r l^r \log^{r/2}(2)) \max_{1 \leq k \leq p} X_{ik}^r,
\]

\[
\lesssim \log^{r/2}(p),
\]

\[
E \left( \max_{1 \leq i \leq n} \max_{1 \leq k \leq p} X_{ik}^2 \right) \leq 4 \max_{1 \leq i \leq n} \max_{1 \leq k \leq p} X_{ik}^2 \leq 4 \log(2) \max_{1 \leq i \leq n} \max_{1 \leq k \leq p} X_{ik}^2 \leq \log(pn),
\]

when \( X \) has i.i.d. sub-Gaussian rows. This, together with (0.2), entails immediately that

\[
E(\max_{1 \leq k \leq p} |X_k|) \lesssim \{\log(p)/n\}^{1/2} \vee \log^{1/2}(pn)\{\log(p)/n\} \text{ and } \mathbb{P}\{\max_{1 \leq k \leq p} |X_k| \geq 2E(\max_{1 \leq k \leq p} |X_k|) + t\} \lesssim \exp(-nt^2/3) + n^{-r}t^{-r} \log^{r/2}(p). \]

This implies by taking \( t \approx \{\log(p)/n\}^{1/2} \) that \( \mathbb{P}\{\max_{1 \leq k \leq p} |X_k| \leq C\{\log(p)/n\} \vee C\log^{1/2}(pn)\{\log(p)/n\}\} = 1 - O(p^{-c} + n^{-1-r/2}) \), for some positive constants \( C,c > 0 \). Let

\[
\tilde{\epsilon} \approx \{\log(p)/n\} \vee \log^{1/2}(pn)\{\log(p)/n\}^{1/2},
\]

and

\[
\epsilon \approx \{\log(p)/n\}^{1/2}.
\]

In the sub-Gaussian case, apply (0.1) to obtain that

\[
\mathbb{P}\{\max_{1 \leq k \leq p} I_{k1} \leq C\{\log(p)/n\}^{3/2} \vee C\log^{1/2}(pn)\{\log(p)/n\}\} = 1 - O(p^{-c} + n^{-1-r/2})(0.4)
\]

for some positive constants \( C,c > 0 \).

\[
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\]
Next we establish the bound for \( \max_{1 \leq k \leq p} |I_{k2}| \). Note that

\[
\left| \sum_{i=1}^{n} \left[ \psi_{\tau}\{Y_i - \hat{Q}_{\tau}(Y)\} - \psi_{\tau}\{Y_i - Q_{\tau}(Y)\} \right] X_{ik} \right|
\]

\[
= |\sum_{i=1}^{n} I\{\hat{Q}_{\tau}(Y) < Y_i \leq Q_{\tau}(Y)\} X_{ik} + \sum_{i=1}^{n} I\{Q_{\tau}(Y) < Y_i \leq \hat{Q}_{\tau}(Y)\} X_{ik}| \]

for \( 1 \leq k \leq p \). Then, for any given \( \epsilon > 0 \),

\[
\mathbb{P}(\max_{1 \leq k \leq p} |I_{k2}| \geq \epsilon)
\]

\[
\leq \mathbb{P}(\max_{1 \leq k \leq p} |n^{-1} \sum_{i=1}^{n} I\{\hat{Q}_{\tau}(Y) < Y_i \leq Q_{\tau}(Y)\} X_{ik} | \geq \{\tau(1 - \tau)\}^{1/2} \epsilon / 2)
\]

\[
+ \mathbb{P}(\max_{1 \leq k \leq p} |n^{-1} \sum_{i=1}^{n} I\{Q_{\tau}(Y) < Y_i \leq \hat{Q}_{\tau}(Y)\} X_{ik} | \geq \{\tau(1 - \tau)\}^{1/2} \epsilon / 2)
\]

\[
= \mathbb{P}(\sup_{u_{k}, k=1, \ldots, p} |n^{-1} \sum_{i=1}^{n} I\{\hat{Q}_{\tau}(Y) < Y_i \leq Q_{\tau}(Y)\} x_{i}^{T} u_{k} | \geq \{\tau(1 - \tau)\}^{1/2} \epsilon / 2)
\]

\[
+ \mathbb{P}(\sup_{u_{k}, k=1, \ldots, p} |n^{-1} \sum_{i=1}^{n} I\{Q_{\tau}(Y) < Y_i \leq \hat{Q}_{\tau}(Y)\} x_{i}^{T} u_{k} | \geq \{\tau(1 - \tau)\}^{1/2} \epsilon / 2),
\]

where \( u_{k} \) is the \( k \)th column of the \( p \times p \) identity matrix. Let the function class \( \mathcal{F} \) be \( \{I\{Q_{\tau}(Y) < Y \leq \hat{Q}_{\tau}(Y)\} X_{k}, k = 1, \ldots, p\} \). Clearly, \( \mathcal{F} \) has envelope \( \max_{1 \leq k \leq p} |X_{k}| \). Moreover, the function class is VC type in view of Lemma 2.6.18 in [van der Vaart and Wellner (1996)]. Due to Assumption (C4) and Serfling (1980, Theorem 2.3.2), we have \( \sup_{u_{k}, k=1, \ldots, p} |n^{-1} \sum_{i=1}^{n} E[I\{Q_{\tau}(Y) < Y_i \leq \hat{Q}_{\tau}(Y)\} x_{i}^{T} u_{k}] | \leq cn^{-1/2} \sup_{y \in [Q_{\tau}(Y) - \delta_0, Q_{\tau}(Y) + \delta_0]} \max_{1 \leq k \leq p} E(f_{Y|x_{k}}(y)|X_{k}) \). Then, by applying Lemma 4, it is not difficult to obtain that with probability \( 1 - o(1) \),

\[
\max_{1 \leq k \leq p} |I_{k2}| \leq CK_n \{\log(p) / n\}^{3/4} \vee CK_n \{\log(p) / n\}, \quad (0.5)
\]
in the strongly bounded case, and

$$\max_{1 \leq k \leq p} |I_{k2}| \leq C\{\log(p)/n\}^{3/4} \lor C\log^{1/2}(pn)\{\log(p)/n\},$$
(0.6)

in the sub-Gaussian case.

For bounding $\max_{1 \leq k \leq p} |I_{k7}|$, we apply Bernstein’s inequality (van der Vaart and Wellner, 1996, Lemma 2.2.11) and the fact $|\psi_{\tau}\{Y_i - Q_{\tau}(Y)\}| \leq 2$ for $i = 1, \ldots, n$, to yield

$$\mathbb{P}(\max_{1 \leq k \leq p} |I_{k7}| \geq \tilde{c}) \leq \mathbb{P}\left[ \left| n^{-1} \sum_{i=1}^{n} \psi_{\tau}\{Y_i - Q_{\tau}(Y)\} \right| \geq \{\tau(1 - \tau)\}^{1/2}\epsilon \right]$$

$$\quad + \mathbb{P}(\max_{1 \leq k \leq p} |X_k| \geq \tilde{c})$$

$$\leq 2\exp\{-\tau(1 - \tau)n\epsilon^2/8\} + \mathbb{P}(\max_{1 \leq k \leq p} |X_k| \geq \tilde{c}).$$

By using similar arguments to those in the derivation of $\max_{1 \leq k \leq p} |I_{k1}|$, there exist some constants $r > 2$ and $C, c > 0$ such that

$$\mathbb{P}(\max_{1 \leq k \leq p} |I_{k7}| \leq CK_n\{\log(p)/n\}^{3/2} \lor CK_n\{\log(p)/n\}] = 1 - O(p^{-c} + n^{1-r/2}),$$
(0.7)

in the strongly bounded case, and

$$\mathbb{P}(\max_{1 \leq k \leq p} |I_{k7}| \leq C\{\log(p)/n\}^{3/2} \lor C\log^{1/2}(pn)\{\log(p)/n\}] = 1 - O(p^{-c} + n^{1-r/2})(.8)$$

in the sub-Gaussian case.

It remains to bound the probabilities $\mathbb{P}(\max_{1 \leq k \leq p} |I_{kl}| \geq \epsilon), l = 3, 4, 5, 6$. To that end, we need to describe the nonasymptotic bound on $\max_{1 \leq k \leq p} |\widehat{\sigma}_k^2 - 1|$. By the triangle
inequality, for any $\tilde{\epsilon} > 0$, we can obtain that

$$
\mathbb{P}(\max_{1 \leq k \leq p} |\tilde{\sigma}_k^2 - 1| \geq 2\tilde{\epsilon})
\leq \mathbb{P}(\max_{1 \leq k \leq p} \sum_{i=1}^{n} X_{ik}^2/n - 1 | + \max_{1 \leq k \leq p} |X_k| \geq 2\tilde{\epsilon})
\leq \mathbb{P}(\max_{1 \leq k \leq p} \sum_{i=1}^{n} X_{ik}^2/n - 1 | \geq \tilde{\epsilon}) + \mathbb{P}(\max_{1 \leq k \leq p} |X_k| \geq \tilde{\epsilon}.
$$

(0.9)

Invoking Lemma 5, we have for every $t > 0$ and $r > 2$,

$$
\mathbb{P}\{\max_{1 \leq k \leq p} \sum_{i=1}^{n} X_{ik}^2/n - 1 \geq 2E(\max_{1 \leq k \leq p} \sum_{i=1}^{n} X_{ik}^2/n - 1 ) + t\}
\lesssim \exp\{-(nt)^2/(3n \max_{1 \leq k \leq p} E|X_{1k}|^4\} + (nt)^{-r} \sum_{i=1}^{n} E(\max_{1 \leq k \leq p} |X_{ik}^r|).
$$

(0.10)

Obviously, $\max_{1 \leq k \leq p} E(X_{1k}^4) \lesssim \max_{1 \leq k \leq p} E(X_{1k}^2 R_n^2) = K_n^2$ and $n^{-1} \sum_{i=1}^{n} E(\max_{1 \leq k \leq p} |X_{ik}^r|) \lesssim K_n^{2r}$ in the strongly bounded case. When $X$ has i.i.d. sub-Gaussian rows, it is routine to verify that $\max_{1 \leq k \leq p} E|X_{1k}|^4 \lesssim 1$, $E(\max_{1 \leq k \leq p} |X_{ik}|^{2r}) \lesssim \log^r(p)$ and $E(\max_{1 \leq k \leq p} \max_{1 \leq i \leq n} X_{ik}^4) \lesssim \log^2(pn)$. Therefore, the right-hand side of (0.10) has the upper bound $C \exp\{-(nt)^2/(3n K_n^{2r})\} + Cn^{1-r}t^{-r} K_n^{2r}$ in the strongly bounded case, and $C \exp\{-(nt)^2/(3n)\} + Cn^{1-r}t^{-r} \log^r(p)$ in the sub-Gaussian case. Moreover, it follows from Lemma 1 in \cite{Chernozhukov et al. (2015)} that

$$
E(\max_{1 \leq k \leq p} \sum_{i=1}^{n} X_{ik}^2/n - 1 )
\lesssim n^{-1/2} \log(p)^{1/2} \{\max_{1 \leq k \leq p} E(X_{1k}^4)\}^{1/2} + n^{-1} \log(p)\{E(\max_{1 \leq i \leq n} \max_{1 \leq k \leq p} X_{ik}^4)\}^{1/2}.
$$

(0.11)

By arguments similar to those for dealing with (0.10), the right-hand side of (0.11)
has the upper bound $Cn^{-1/2}K_n \log^{1/2}(p) + Cn^{-1}K_n^2 \log(p)$ in the strongly bounded case, and $Cn^{-1/2} \log^{1/2}(p) + Cn^{-1} \log(p) \log(pn)$ in the sub-Gaussian case. Let $t \asymp n^{-1/2}K_n^2 \log^{1/2}(p)$ and $\tilde{\epsilon} \asymp n^{-1}K_n^2 \log(p) \vee n^{-1/2}K_n^2 \log^{1/2}(p)$ in the strongly bounded case, and $t \asymp n^{-1/2} \log(p) \vee n^{-1} \log(p) \log(pn)$ in the sub-Gaussian case. Together, (0.9), (0.10) and (0.11) yield that $P\{ \max_{1 \leq k \leq p} | \hat{\sigma}_k^2 - 1 | \leq Cn^{-1}K_n^2 \log(p) \vee Cn^{-1/2}K_n^2 \log^{1/2}(p) \} = O(p^{-c} + n^{1-r/2})$, in the strongly bounded case, and $P\{ \max_{1 \leq k \leq p} | \hat{\sigma}_k^2 - 1 | \leq Cn^{-1/2} \log(p) \vee Cn^{-1} \log(p) \log(pn) \} = O(p^{-c} + n^{1-r/2})$, in the sub-Gaussian case.

For any given $\epsilon, \tilde{\epsilon} > 0$, it is immediate to see that

\[
P\{ \max_{1 \leq k \leq p} | I_{k3} | \geq \epsilon \tilde{\epsilon} \} \leq P\{ \max_{1 \leq k \leq p} | I_{k7} | \geq \epsilon \} + P\{ \max_{1 \leq k \leq p} | \hat{\sigma}_k^{-1} - 1 | \geq \tilde{\epsilon} \},
\]

\[
P\{ \max_{1 \leq k \leq p} | I_{k5} | \geq \epsilon \tilde{\epsilon} \} \leq P\{ \max_{1 \leq k \leq p} | I_{k1} | \geq \epsilon \} + P\{ \max_{1 \leq k \leq p} | \hat{\sigma}_k^{-1} - 1 | \geq \tilde{\epsilon} \},
\]

\[
P\{ \max_{1 \leq k \leq p} | I_{k6} | \geq \epsilon \tilde{\epsilon} \} \leq P\{ \max_{1 \leq k \leq p} | I_{k2} | \geq \epsilon \} + P\{ \max_{1 \leq k \leq p} | \hat{\sigma}_k^{-1} - 1 | \geq \tilde{\epsilon} \}.
\]

Under Assumption (C2) and combining the nonasymptotic bounds for $\max_{1 \leq k \leq p} | I_{k1} |$, $\max_{1 \leq k \leq p} | I_{k2} |$ and $\max_{1 \leq k \leq p} | I_{k7} |$, we have

\[
P\{ \max_{1 \leq k \leq p} | I_{k3} | \leq Cn^{-5/2}K_n^3 \log^{5/2}(p) \vee Cn^{-2}K_n^3 \log^2(p) \}
\]

\[
\vee Cn^{-3/2}K_n^3 \log^{3/2}(p) \} = 1 - O(p^{-c} + n^{1-r/2}), \quad (0.12)
\]

\[
P\{ \max_{1 \leq k \leq p} | I_{k5} | \leq Cn^{-5/2}K_n^3 \log^{5/2}(p) \vee Cn^{-2}K_n^3 \log^2(p) \}
\]

\[
\vee Cn^{-3/2}K_n^3 \log^{3/2}(p) \} = 1 - O(p^{-c} + n^{1-r/2}), \quad (0.13)
\]

\[
P\{ \max_{1 \leq k \leq p} | I_{k6} | \leq Cn^{-1}K_n^3 \log^{3/2}(p) \} = 1 - O(p^{-c} + n^{1-r/2}), \quad (0.14)
\]
in the strongly bounded case, and

\[ \mathbb{P}\{ \max_{1 \leq k \leq p} | I_{k3} | \leq Cn^{-2} \log^{5/2}(p) \vee Cn^{-5/2} \log^{5/2}(p) \log(pn) \vee Cn^{-3/2} \log^2(p) \log^{1/2}(pn) \} = 1 - O(p^{-c} + n^{1-r/2}), \]  

(0.15)

\[ \mathbb{P}\{ \max_{1 \leq k \leq p} | I_{k5} | \leq Cn^{-2} \log^{5/2}(p) \vee Cn^{-5/2} \log^{5/2}(p) \log(pn) \vee Cn^{-3/2} \log^2(p) \log^{1/2}(pn) \} = 1 - O(p^{-c} + n^{1-r/2}), \]  

(0.16)

\[ \mathbb{P}\{ \max_{1 \leq k \leq p} | I_{k6} | \leq Cn^{-1} \log^{3/2}(p) \log^{1/2}(n) \log^{1/2}(pn) \} = 1 - O(p^{-c} + n^{1-r/2}). \]  

(0.17)

in the sub-Gaussian case. Under Assumption (C4) and by Lemma 1, we have that for all \( 1 \leq k \leq p, \)

\[ E \left[ n^{-1} \sum_{i=1}^{n} \psi_{\tau} \{ Y_i - Q_{\tau}(Y) \} X_{ik} \right] = 0, \]

under the null hypothesis in (1.2). Using the fact \( | \psi_{\tau} \{ Y_i - Q_{\tau}(Y) \} | \leq 2 \) for \( i = 1, \ldots, n, \) it is routine to show that \( \mathbb{P}\{ \max_{1 \leq k \leq p} | n^{-1} \sum_{i=1}^{n} \psi_{\tau} \{ Y_i - Q_{\tau}(Y) \} X_{ik} \} \leq CK_n \{ \log(p)/n \} \vee CK_n \{ \log(p)/n \}^{1/2} = 1 - O(p^{-c} + n^{1-r/2}) \) in the strongly bounded case, and \( \mathbb{P}\{ \max_{1 \leq k \leq p} | n^{-1} \sum_{i=1}^{n} \psi_{\tau} \{ Y_i - Q_{\tau}(Y) \} X_{ik} \} \leq C \{ \log(p)/n \} \vee C \log^{1/2}(pn) \{ \log(p)/n \}^{1/2} = 1 - O(p^{-c} + n^{1-r/2}) \) in the sub-Gaussian case. Consequently, it follows from the argument similar to that used to bound \( \max_{1 \leq k \leq p} | I_{k6} | \) that

\[ \mathbb{P}\{ \max_{1 \leq k \leq p} | I_{k4} | \leq Cn^{-2} K_n^3 \log^2(p) \vee Cn^{-3/2} K_n^3 \log^{3/2}(p) \vee Cn^{-1} K_n^3 \log(p) \} = 1 - O(p^{-c} + n^{1-r/2}), \]  

(0.18)
in the strongly bounded case, and

\[
\mathbb{P}\left\{ \max_{1 \leq k \leq p} | I_{k4} | \leq C n^{-3/2} \log^2(p) \vee C n^{-2} \log^2(p) \log(pn) \vee C n^{-1} \log^{3/2}(p) \log^{1/2}(pn) \right. \\
\left. \vee C n^{-3/2} \log^{3/2}(p) \log^{3/2}(pn) \right\} = 1 - O(p^{-c} + n^{-r/2}), \quad (0.19)
\]

in the sub-Gaussian case. Combining (0.3), (0.5), (0.7), (0.12), (0.13), (0.14) and (0.18), we obtain that with probability 1 - o(1),

\[
| \hat{S}_r - \hat{S}_r^\tau | \lesssim n^{-3/4} K_n^3 \log^{3/4}(p)
\]

in the strongly bounded case. Combining (0.4), (0.6), (0.8), (0.15), (0.16), (0.17) and (0.19), we obtain that with probability 1 - o(1),

\[
| \hat{S}_r - \hat{S}_r^\tau | \lesssim n^{-2} \log^2(p) \log^{3/2}(pn)
\]

in the sub-Gaussian case. As a result, there exist \( \zeta_1, \zeta_2 > 0 \) such that

\[
\mathbb{P}\left( \left| n^{1/2} \hat{S}_r - \max_{1 \leq k \leq p} \{ \tau(1 - \tau) \}^{-1/2} n^{-1/2} \sum_{i=1}^n \psi_\tau \{ Y_i - Q_\tau(Y) \} X_{ik} \right| \geq \zeta_1 \right) < \zeta_2, \quad (0.20)
\]

where \( \zeta_1 \asymp n^{-1/4} K_n^3 \log^{3/4}(p) \) in the strongly bounded case, and \( \zeta_1 \asymp n^{-3/2} \log^2(p) \log^{3/2}(pn) \) in the sub-Gaussian case and \( \zeta_2 = o(1) \).

Let

\[
Z_{ik} = \{ \tau(1 - \tau) \}^{-1/2} \psi_\tau \{ Y_i - Q_\tau(Y) \} X_{ik},
\]

for \( i = 1, \ldots, n \) and \( k = 1, \ldots, p \). When \( X \) is strongly bounded, we take \( B_n = 2\{ \tau(1 - \tau) \}^{-1/2} K_n \). It is trivial that \( n^{-1} \sum_{i=1}^n E( | Z_{ik} |^{2+l} ) \leq n^{-1} \sum_{i=1}^n E( | X_{ik} |^{2+l} ) B_n^l = B_n^l \) for all \( k = 1, \ldots, p \) and \( l = 1, 2 \), and \( E( \max_{1 \leq k \leq p} | Z_{ik} | / B_n^{q}) \leq E( \max_{1 \leq k \leq p} | X_{ik} | / K_n q) \leq 2 \) for all \( i = 1, \ldots, n \) and \( q \geq 3 \). An application of Chernozhukov et al. (2017, Proposition 2.1) under these conditions leads to

\[
\sup_{t \in \mathbb{R}} | \mathbb{P}( \max_{1 \leq k \leq p} n^{1/2} Z_{ik} \leq \]

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\(t - \mathbb{P}(\max_{1 \leq k \leq p} n^{1/2} G_k \leq t) \leq \{n^{-1} K_n^2 \log^7(pn)\}^{1/6}\), where \(Z_k = n^{-1} \sum_{i=1}^{n} Z_{ik}\) and \(G_k = n^{-1} \sum_{i=1}^{n} G_{ik}\) with \(\{g_i = (G_{i1}, \ldots, G_{ip})\}_{i=1}^{n}\) being a sequence of independent centred Gaussian random vectors such that each \(g_i\) has the same covariance matrix as \(z_i = (Z_{i1}, \ldots, Z_{ip})^T\). Consequently,

\[
\begin{align*}
\sup_{t \in \mathbb{R}^+} \left| \mathbb{P}(\max_{1 \leq k \leq p} n^{1/2} G_k \leq t) - \mathbb{P}(\max_{1 \leq k \leq p} n^{1/2} Z_k \leq t) \right| \\
\leq \sup_{t \in \mathbb{R}^+} \left| \mathbb{P}(\max_{1 \leq k \leq p} n^{1/2} Z_k \leq t) - \mathbb{P}(\max_{1 \leq k \leq p} n^{1/2} G_k \leq t) \right| \\
+ \sup_{t \in \mathbb{R}} \left| \mathbb{P}(\max_{1 \leq k \leq p} n^{1/2} Z_k \leq -t) - \mathbb{P}(\max_{1 \leq k \leq p} n^{1/2} G_k \leq -t) \right| \\
\leq 2 \sup_{t \in \mathbb{R}} \left| \mathbb{P}(\max_{1 \leq k \leq p} n^{1/2} Z_k \leq t) - \mathbb{P}(\max_{1 \leq k \leq p} n^{1/2} G_k \leq t) \right| \leq \left\{n^{-1} K_n^2 \log^7(pn)\right\}^{1/6}.
\end{align*}
\]

(0.21)

Let \(\tilde{c}_{\tau, \alpha} = \inf\{t \in R^+ : \mathbb{P}(\max_{1 \leq j \leq p} n^{1/2} G_j \leq t) \geq 1 - \alpha\}\) and note \(c_{\tau, \alpha} = \inf\{t \in \mathbb{R}^+ : \mathbb{P}(n^{1/2} S_\tau \leq t \mid Y_i, x_i, i = 1, \ldots, n) \geq 1 - \alpha\}\). Using the similar arguments in the proof of Lemma 3.2 in [Chernozhukov et al. (2013)] we have that for every \(v > 0\),

\[
\mathbb{P}(\Delta > v) \geq \mathbb{P}\{c_{\tau, \alpha} \geq \tilde{c}_{\tau, \alpha + \pi(v)}\} \vee \mathbb{P}\{\tilde{c}_{\tau, \alpha} \geq c_{\tau, \alpha + \pi(v)}\},
\]

(0.22)

where \(\pi(v) \asymp v^{1/3} (1 \vee \log(p/v))^{2/3}\) and

\[
\Delta = \max_{1 \leq k, l \leq p} \left| n^{-1} \sum_{i=1}^{n} \{Z_{ik} Z_{il} - E(Z_{ik} Z_{il})\} \right|.
\]

By the triangle inequality,

\[
\begin{align*}
\left| \mathbb{P}\left(\Psi_{\bar{S}_\tau, \alpha} = 1\right) - \alpha \right| & \leq \left| \mathbb{P}\left(n^{1/2} \bar{S}_\tau > c_{\tau, \alpha}\right) - \mathbb{P}\left(|\sqrt{n} Z_k| > c_{\tau, \alpha}\right) \right| \\
& \quad + \left| \mathbb{P}\left(|\sqrt{n} Z_k| > c_{\tau, \alpha}\right) - \alpha \right|.
\end{align*}
\]
Apply the inequality $|I(a < c) - I(b < c)| \leq I(|a - c| < |a - b|)$ to show that

$$\left| \mathbb{P}\left( n^{1/2} \hat{S}_\tau > c_{r,\alpha} \right) - \mathbb{P}(|\sqrt{n} \bar{Z}_k > c_{r,\alpha}|) \right|$$

$$\leq \mathbb{P}( |n^{1/2} \hat{S}_\tau - \sqrt{n} \bar{Z}_k | > |\sqrt{n} \bar{Z}_k - c_{r,\alpha}| )$$

$$= \mathbb{P}( |n^{1/2} \hat{S}_\tau - \sqrt{n} \bar{Z}_k | > |\sqrt{n} \bar{Z}_k - c_{r,\alpha}| ) + \mathbb{P}( |n^{1/2} \hat{S}_\tau - \sqrt{n} \bar{Z}_k | < |\sqrt{n} \bar{Z}_k - c_{r,\alpha}| )$$

$$\leq \mathbb{P}( |n^{1/2} \hat{S}_\tau - \sqrt{n} \bar{Z}_k | \geq \xi_1 ) + \mathbb{P}( |\xi_1 > |\sqrt{n} \bar{Z}_k - c_{r,\alpha}| | - \mathbb{P}( |\xi_1 > |\sqrt{n} \bar{G}_k - c_{r,\alpha}| | )$$

$$\lesssim \zeta_2 + \left\{ n^{-1} K_n^2 \log^7(pn) \right\}^{1/6} + \mathbb{P}( |\xi_1 > |\sqrt{n} \bar{G}_k - c_{r,\alpha}| | )$$

$$\lesssim \zeta_2 + \left\{ n^{-1} K_n^2 \log^7(pn) \right\}^{1/6} + \zeta_1 \{ 1 \lor \log(p/\zeta_1) \}^{1/2},$$

where the third inequality follows from (0.20) and (0.21) and the last inequality holds due to the anti-concentration inequality in Chernozhukov et al. (2015). Further, apply (0.21), (0.22) and the triangle inequality to obtain

$$\left| \mathbb{P}( |\sqrt{n} \bar{Z}_k | > c_{r,\alpha} | - \alpha \right|$$

$$\lesssim \mathbb{P}( |\sqrt{n} \bar{G}_k | > c_{r,\alpha} | - \{ \alpha + \pi(v) \} | + \pi(v) + \left\{ n^{-1} K_n^2 \log^7(pn) \right\}^{1/6}$$

$$\lesssim \mathbb{P}( \Delta > v ) + \pi(v) + \left\{ n^{-1} K_n^2 \log^7(pn) \right\}^{1/6}.$$

By the maximal inequality in Lemma E.1 of Chernozhukov et al. (2017) and the boundness of the function $\psi_r(\cdot)$, it is routine to verify that $\mathbb{P}\{ \Delta \leq C n^{-1} K_n^2 \log(p) \lor C n^{-1/2} K_n^2 \log^{1/2}(p) \} = 1 - O(p^{-c} + n^{1-r/2})$, for some positive constants $c > 0, r > 2$. Therefore, in the strongly bounded case and choosing $v \asymp n^{-1} K_n^2 \log(p) \lor n^{-1/2} K_n^2 \log^{1/2}(p)$,
we obtain

\[
\left| \mathbb{P}\left( \Psi_{\hat{S}, \alpha} = 1 \right) - \alpha \right| \lesssim v^{1/3} \{1 \vee \log(p/v)\}^{2/3} + \zeta_2 + \{n^{-1}K_n^2 \log^7(pn)\}^{1/6} \\
+ \zeta_1 \{1 \vee \log(p/\zeta_1)\}^{1/2} + p^{-c} + n^{-r/2},
\]

(0.23)

for some constants \( c > 0, r > 2 \). Under the assumption \( K_n^2 \{\log(pn)\}^7/n \lesssim n^{-c_1} \) with some constant \( c_1 > 0 \), we deduce the desired conclusion in the strongly bounded case.

On the other hand, when \( X \) has i.i.d. sub-Gaussian rows and by Lemma 2.2.2 in van der Vaart and Wellner [1996], we have \( \|X_{ik}\|_{\psi_1} \leq \log^{1/2}(2) \max_{1 \leq i \leq n} \max_{1 \leq k \leq p} \|X_{ik}\|_{\psi_2} < \infty \) and \( E(X_{ik}^{2+l}) \leq (\Pi_{m=1}^{2+l} m)^{2+l} \max_{1 \leq i \leq n} \max_{1 \leq k \leq p} \|X_{ik}\|_{\psi_{1+l}}^{2+l} < \infty \) for all \( i = 1, \ldots, n \), \( k = 1, \ldots, p \) and \( l = 1, 2 \). Thus, there exists a large enough constant \( C > 0 \) such that
\[\sum_{i=1}^{n} E(|Z_{ik}|^{2+l}) \leq \{\tau(1 - \tau)/2\}^{-1-l/2} n^{-1} \sum_{i=1}^{n} E(|X_{ik}|^{2+l}) \leq C^l \]
for all \( k = 1, \ldots, p \) and \( l = 1, 2 \), and \( E\{\exp(|Z_{ik}|^{1/C})\} \leq 2\{\tau(1 - \tau)/2\}^{-1/2} \|X_{ik}\|_{\psi_1}/C \leq 2\{\tau(1 - \tau)/2\}^{-1/2} \max_{1 \leq i \leq n} \max_{1 \leq k \leq p} \|X_{ik}\|_{\psi_1}/C \leq 2 \)

for all \( i = 1, \ldots, n \) and \( q \geq 3 \). Together with Chernozhukov et al. [2017] Proposition 2.1, this implies that \( \sup_{t \in \mathbb{R}} \left| \mathbb{P}(\max_{1 \leq k \leq p} Z_k \leq t) - \mathbb{P}(\max_{1 \leq k \leq p} \mathcal{C}_k \leq t) \right| \lesssim \{n^{-1} \log^7(pn)\}^{1/6} \) in the sub-Gaussian case. Taking \( v \asymp n^{-1/2} \log(p) \vee n^{-1} \log(p) \log(pn) \) and employing arguments similar to those for dealing with (0.23), we have

\[
\left| \mathbb{P}\left( \Psi_{\hat{S}, \alpha} = 1 \right) - \alpha \right| \lesssim v^{1/3} \{1 \vee \log(p/v)\}^{2/3} + \zeta_2 + \{n^{-1} \log^7(pn)\}^{1/6} \\
+ \zeta_1 \{1 \vee \log(p/\zeta_1)\}^{1/2} + p^{-c} + n^{-r/2},
\]

for some constants \( c > 0, r > 2 \). Under the assumption \( \{\log(pn)\}^7/n \lesssim n^{-c_1} \) with some constant \( c_1 > 0 \), it is immediate to deduce the desired conclusion in the sub-
Gaussian case.

Proof of Theorem 2

Without loss of generality, we set $\sigma_{11} = \ldots = \sigma_{pp} = 1$. Define $\tilde{S}_{\tau} = \max_{1 \leq k \leq p} | \tilde{q}_{cov}(Y, X_k) - \tilde{q}_{cor}(Y, X_k) |$. Under the assumptions in Theorem 1, it is routine to show that $\mathbb{P} \left( \left| n^{1/2} \tilde{S}_{\tau} - \max_{1 \leq k \leq p} \left| n^{-1/2} \sum_{i=1}^{n} Z_{ik} \right| \geq \zeta_1 \right| \right) < \zeta_2$ for $\zeta_1 \{ 1 \lor \log(p/\zeta_1) \}^{1/2} - \log(p) \lor \log(\log(p)) \lor \log(1/p) = o(1)$ and $\zeta_2 = o(1)$, where $Z_{ik} = \tau \{ (1 - \tau) - \psi_{\tau} \{ Y_i - Q_{\tau}(Y) \} X_{ik} \}$ for $i = 1, \ldots, n$ and $k = 1, \ldots, p$. In another word, the distribution of $n^{1/2} \tilde{S}_{\tau}$ can be approximated by $\max_{1 \leq k \leq p} | G_k |$, where $(G_1, \ldots, G_p)^T$ is the centered Gaussian random vector with mean zero and covariance matrix $\Theta = E[\psi_{\tau}^2 \{ Y - Q_{\tau}(Y) \}\{ x - E(x) \}\{ x - E(x) \}]^T \in \mathbb{R}^{p \times p}$. Since $\lambda_{\max}(\Theta) = \sup_{\beta \in \mathbb{R}^p} \beta^T \Theta \beta / \| \beta \|^2 = \sup_{\beta \in \mathbb{R}^p} E[\psi_{\tau}^2 \{ Y - Q_{\tau}(Y) \}\| \beta \|^2 / \| \beta \|^2 \leq \{ \tau \lor (1 - \tau) \}^2 \sup_{\beta \in \mathbb{R}^p} E[\| \beta \|^2 / \| \beta \|^2 = \{ \tau \lor (1 - \tau) \}^2 \lambda_{\max}(\Sigma)]$, we conclude that under Assumption (C5), by Lemma 6 of Cai et al. (2014), we have for any $x \in \mathbb{R}$ and as $p \to \infty$, $\mathbb{P} \left( \max_{1 \leq k \leq p} | G_k | - 2 \log(p) + \log(\log(p)) \leq x \right) \to F(x) = \exp\{ -\pi^{1/2} \exp(-x/2) \}$. It implies that

$$\mathbb{P} \left( n^{2} \tilde{S}_{\tau}^2 \leq 2 \log(p) - \log(\log(p)) \right) \to 1.$$  \hspace{1cm} (0.24)

The bootstrap consistency result implies that

$$c_{\tau,\alpha}^2 - 2 \log(p) + \log(\log(p)) - q_\alpha = o_P(1),$$

where $q_\alpha$ is the 100(1-$\alpha$)th quantile of $F(x)$. Consider any $k \in \{1, \ldots, p\}$ such that $| q_{cov}(Y, X_k) / \sigma_{kk}^{1/2} | \geq (e_0 + 2^{1/2}) \{ \tau (1 - \tau) \log(p) / n \}^{1/2}$. Using the inequality $2a_1a_2 \leq \ldots$
\[ \delta^{-1}a_1^2 + \delta a_2^2 \text{ for any } \delta > 0, \text{ we have} \]

\[
q_{\text{cor}}^2(Y, X_k) \leq (1 + \delta^{-1}) \left| \widehat{q_{\text{cor}}}(Y, X_k) - q_{\text{cor}}(Y, X_k) \right|^2 \\
+ (1 + \delta)\widehat{q_{\text{cor}}^2}(Y, X_k),
\]

where \( n \) \( | \widehat{q_{\text{cor}}}(Y, X_k) - q_{\text{cor}}(Y, X_k) |^2 /\{\tau(1 - \tau)\hat{\sigma}_{kk}\} = o_P\{\log(p)\} \) as \( k \) is fixed and \( p \) grows. From the proof of Theorem 1, we know the difference between \( n \) \( q_{\text{cor}}^2(Y, X_k) /\{\tau(1 - \tau)\hat{\sigma}_{kk}\} \) and \( n \) \( q_{\text{cor}}^2(Y, X_k) /\{\tau(1 - \tau)\sigma_{kk}\} \) is asymptotically negligible. Thus by (0.25) and the fact that \( \theta \tau \in V_{\tau}(\epsilon_0 + 2^{1/2}) \), we have,

\[
\max_{1 \leq k \leq p} n \left| \widehat{q_{\text{cor}}}(Y, X_k) \right|^2 /\{\tau(1 - \tau)\hat{\sigma}_{kk}\} \\
\geq (1 + \delta)^{-1}[(\epsilon_0 + 2^{1/2})^2 \log(p) - o_P\{\log(p)\}].
\]

The conclusion thus follows from (0.24), (0.25) and (0.26) provided that \( \delta \) is small enough.

\[ \square \]

**Proof of Lemma 2**

Recall that the random variable \( C \) is independent of \((Y, X)\). It then follows by the law of iterated expectations that \( \{\tau(1 - \tau)\sigma_k^2\}^{1/2}q_{\text{cor}}(Y, X_k) = E[\psi_{\tau}\{Y - Q_{\tau}(Y)\}\{X_k - E(X_k)\}] \text{ and } E[\{\delta/G(Y^*)\} \{\rho_{\tau}(Y^* - \alpha - \theta X_k) - \rho_{\tau}(Y^*)\}] = E\{\rho_{\tau}(Y - \alpha - \theta X_k) - \rho_{\tau}(Y)\}. \)

Lemma 2 then follows immediately from Lemma 1.

\[ \square \]

**Proof of Theorem 3**

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Write $\hat{T}_{\tau}^p = \max_{1 \leq k \leq p} |\hat{c}\text{qcor}_\tau(Y, X_k)|$, where

$$\hat{c}\text{qcor}_\tau^2(Y, X_k) = \{\tau(1 - \tau)\}^{-1/2}n^{-1}\sum_{i=1}^{n} [\tau - w_{\tau}(F)I\{Y_i^* \leq Q_\tau(Y)](X_{ik} - \bar{X}_k),$$

for $k = 1, \ldots, p$, and

$$w_{\tau}(F) = \begin{cases} 1 & \text{if } \Delta_i = 1 \text{ or } F(C_i) > \tau, \\ \frac{\tau - F(C_i)}{1 - F(C_i)} & \text{if } \Delta_i = 0 \text{ and } F(C_i) \leq \tau. \end{cases}$$

Then we can decompose $\hat{c}\text{qcor}_\tau(Y, X_k) - \hat{c}\text{qcor}_\tau^2(Y, X_k)$ as $\hat{c}\text{pcor}_\tau(Y, X_k) - \hat{c}\text{pcor}_\tau^2(Y, X_k) = \sum_{l=1}^{7} J_{kl}$, where

$$\begin{align*}
J_{k1} &= -\{\tau(1 - \tau)\}^{-1/2}\bar{X}_k n^{-1}\sum_{i=1}^{n} [w_{\tau}(F)I\{Y_i^* \leq \hat{Q}_\tau(Y)] - w_{\tau}(\hat{F})I\{Y_i^* \leq \hat{Q}_\tau(Y)]\}, \\
J_{k2} &= \{\tau(1 - \tau)\}^{-1/2}n^{-1}\sum_{i=1}^{n} [w_{\tau}(F)I\{Y_i^* \leq Q_\tau(Y)] - w_{\tau}(\hat{F})I\{Y_i^* \leq \hat{Q}_\tau(Y)]\}X_{ik}, \\
J_{k3} &= -\{\tau(1 - \tau)\}^{-1/2}(\hat{\sigma}_k^{-1} - 1)\bar{X}_k n^{-1}\sum_{i=1}^{n} [\tau - w_{\tau}(F)I\{Y_i^* \leq Q_\tau(Y)]], \\
J_{k4} &= \{\tau(1 - \tau)\}^{-1/2}(\hat{\sigma}_k^{-1} - 1)n^{-1}\sum_{i=1}^{n} [\tau - w_{\tau}(F)I\{Y_i^* \leq Q_\tau(Y)]\}X_{ik}, \\
J_{k5} &= -\{\tau(1 - \tau)\}^{-1/2}(\hat{\sigma}_k^{-1} - 1)\bar{X}_k n^{-1}\sum_{i=1}^{n} [w_{\tau}(F)I\{Y_i^* \leq \hat{Q}_\tau(Y)]]}.
\end{align*}$$
Using (A.2) in Wang and Wang (2009), we have

\[
-w_{iτ}(\hat{F})I\{Y_i^* ≤ \hat{Q}_τ(Y)\} ≤ K_i1 + K_i2 + K_i3,
\]

where

\[
K_i1 = |I\{C_i > \hat{Q}_τ(Y), Y_i ≤ \hat{Q}_τ(Y)\} - I\{C_i > Q_τ(Y), Y_i ≤ Q_τ(Y)\}|,
\]

\[
K_i2 = |I\{C_i ≤ \hat{Q}_τ(Y), Y_i ≤ C_i\} - I\{C_i ≤ Q_τ(Y), Y_i ≤ C_i\}|,
\]

\[
K_i3 = \left| I\{C_i ≤ Q_τ(Y), Y_i > C_i\} \left[1 - \frac{1 - τ}{1 - F(C_i)}I\{F(C_i) < τ\}\right] - I\{C_i ≤ Q_τ(Y), Y_i > C_i\} \left[1 - \frac{1 - τ}{1 - F(C_i)}I\{F(C_i) < τ\}\right] \right|.
\]

From He et al. (2013) Lemma 8.4 and the Hoeffding’s inequality, there exist \(ε_0 > 0\)
and $c > 0$ such that for any $\epsilon \in (0, \epsilon_0)$,

$$
\mathbb{P}\left[ n^{-1} \sum_{i=1}^{n} \left| w_{i\tau}(F)I\{Y_i^* \leq Q_\tau(Y)\} - w_{i\tau}(\hat{F})I\{Y_i^* \leq \hat{Q}_\tau(Y)\} \right| > \epsilon \right]
\leq \mathbb{P}\left( n^{-1} \sum_{i=1}^{n} K_{i1} > \epsilon/3 \right) + \mathbb{P}\left( n^{-1} \sum_{i=1}^{n} K_{i2} > \epsilon/3 \right) + \mathbb{P}\left( n^{-1} \sum_{i=1}^{n} K_{i3} > \epsilon/3 \right)
\lesssim \exp(-cn\epsilon^2).
$$

The rest of the proof is analogous to the last part of Theorem 1. We omit the details for brevity.

\[\blacksquare\]

**REFERENCE**


