Supplement to "A tuning-free efficient test for marginal linear effects in high-dimensional quantile regression"

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Proof of Theorem 1

By the scale and translation invariance properties of $\widehat{\mathrm{qcor}}_\tau(Y, X_k)$, $k = 1, \ldots, p$, we assume without loss of generality that $E(X_1) = \ldots = E(X_p) = 0$ and $\mathrm{var}(X_1) = \ldots = \mathrm{var}(X_p) = 1$. Define the infeasible maximum-type statistic by $\widehat{S}_\tau^\natural = \max_{1\le k\le p} \mid \widehat{\mathrm{qcor}}_\tau^\natural(Y, X_k) \mid$, where $\widehat{\mathrm{qcor}}_\tau^\natural(Y, X_k) = \{\tau(1-\tau)\}^{-1/2} n^{-1} \sum_{i=1}^n \psi_\tau\{Y_i - Q_\tau(Y)\} X_{ik}$, for $k = 1, \ldots, p$. By the definitions of $\widehat{\mathrm{qcor}}_\tau(Y, X_k)$ and $\widehat{\mathrm{qcor}}_\tau^\natural(Y, X_k)$, we can decompose $\widehat{\mathrm{qcor}}_\tau(Y, X_k) - \widehat{\mathrm{qcor}}_\tau^\natural(Y, X_k)$ as $\widehat{\mathrm{qcor}}_\tau(Y, X_k) - \widehat{\mathrm{qcor}}_\tau^\natural(Y, X_k) = \sum_{l=1}^7 I_{kl}$, where

$$
\begin{aligned}
I_{k1} &= -\{\tau(1-\tau)\}^{-1/2} \overline{X}_k n^{-1} \sum_{i=1}^n [\psi_\tau\{Y_i - \widehat{Q}_\tau(Y)\} - \psi_\tau\{Y_i - Q_\tau(Y)\}], \\
I_{k2} &= \{\tau(1-\tau)\}^{-1/2} n^{-1} \sum_{i=1}^n [\psi_\tau\{Y_i - \widehat{Q}_\tau(Y)\} - \psi_\tau\{Y_i - Q_\tau(Y)\}] X_{ik}, \\
I_{k3} &= -\{\tau(1-\tau)\}^{-1/2} (\widehat{\sigma}_k^{-1} - 1) \overline{X}_k n^{-1} \sum_{i=1}^n \psi_\tau\{Y_i - Q_\tau(Y)\}, \\
I_{k4} &= \{\tau(1-\tau)\}^{-1/2} (\widehat{\sigma}_k^{-1} - 1) n^{-1} \sum_{i=1}^n \psi_\tau\{Y_i - Q_\tau(Y)\} X_{ik}, \\
I_{k5} &= -\{\tau(1-\tau)\}^{-1/2} (\widehat{\sigma}_k^{-1} - 1) \overline{X}_k n^{-1} \sum_{i=1}^n [\psi_\tau\{Y_i - \widehat{Q}_\tau(Y)\} - \psi_\tau\{Y_i - Q_\tau(Y)\}], \\
I_{k6} &= \{\tau(1-\tau)\}^{-1/2} (\widehat{\sigma}_k^{-1} - 1) n^{-1} \sum_{i=1}^n [\psi_\tau\{Y_i - \widehat{Q}_\tau(Y)\} - \psi_\tau\{Y_i - Q_\tau(Y)\}] X_{ik}, \\
I_{k7} &= -\{\tau(1-\tau)\}^{-1/2} \overline{X}_k n^{-1} \sum_{i=1}^n \psi_\tau\{Y_i - Q_\tau(Y)\}.
\end{aligned}
$$

By the triangle inequality, $\mid \widehat{S}_\tau - \widehat{S}_\tau^\natural \mid \leq \sum_{l=1}^{7} \max_{1\leq k\leq p} \mid I_{kl} \mid$. In what follows, we provide non-asymptotic bounds on $\max_{1\leq k\leq p} \mid I_{kl} \mid, l = 1, \ldots, 7$, under two scenarios of $\mathbf{X}$: (i) $\mathbf{X}$ is strongly bounded; (ii) $\mathbf{X}$ has i.i.d. sub-Gaussian rows. Throughout the proof, the notations C and c are generic constants, which may take different values at each appearance.

We first deal with $\max_{1\leq k\leq p} \mid I_{k1} \mid$. Recalling the definition of I_{k1}, we have

$$\begin{aligned}
\max_{1\leq k\leq p} \mid I_{k1} \mid &= \{\tau(1-\tau)\}^{-1/2} \left| n^{-1} \sum_{i=1}^{n} [\psi_\tau\{Y_i - \widehat{Q}_\tau(Y)\} - \psi_\tau\{Y_i - Q_\tau(Y)\}] \right| \max_{1\leq k\leq p} \mid \overline{X}_k \mid \\
&\leq \{\tau(1-\tau)\}^{-1/2} n^{-1} \sum_{i=1}^{n} \left| \psi_\tau\{Y_i - \widehat{Q}_\tau(Y)\} - \psi_\tau\{Y_i - Q_\tau(Y)\} \right| \max_{1\leq k\leq p} \mid \overline{X}_k \mid .
\end{aligned}$$

For any given $\epsilon, \widetilde{\epsilon} > 0$, it can be easily shown that

$$\begin{aligned}
\mathbb{P}(\max_{1\leq k\leq p} \mid I_{k1} \mid \geq \epsilon\widetilde{\epsilon}) \leq\ & \mathbb{P}\left[n^{-1} \sum_{i=1}^{n} \left| \psi_\tau\{Y_i - \widehat{Q}_\tau(Y)\} - \psi_\tau\{Y_i - Q_\tau(Y)\} \right| \geq \{\tau(1-\tau)\}^{1/2}\epsilon \right] \\
& + \mathbb{P}(\max_{1\leq k\leq p} \mid \overline{X}_k \mid \geq \widetilde{\epsilon}).
\end{aligned} \tag{0.1}$$

When ϵ is sufficiently small and by Lemma 3, the first term on the right-hand side of (0.1) is bounded by $3\exp\{-2c\tau(1-\tau)n\epsilon^2\}$. By Lemma 8 of Chernozhukov et al. (2015), it is routine to verify that $E(\max_{1\leq k\leq p} \mid \overline{X}_k \mid) \lesssim \{\log(p)/n\}^{1/2} + \{E(\max_{1\leq i\leq n} \max_{1\leq k\leq p} X_{ik}^2)\}^{1/2} \{\log(p)/n\}$. Applying Lemma 5, we have for every

$t > 0$ and $r > 2$,

$$\mathbb{P}\{\max_{1\le k\le p} \mid \overline{X}_k \mid\ge 2E(\max_{1\le k\le p} \mid \overline{X}_k \mid) + t\}$$
$$\lesssim \exp\{-(nt)^2/(3n \max_{1\le i\le n}\max_{1\le k\le p} E|X_{ik}|^2)\} + (nt)^{-r}\sum_{i=1}^{n} E(\max_{1\le k\le p}|X_{ik}|^r). \quad (0.2)$$

In the strongly bounded case, it is straightforward to see that $E(\max_{1\le k\le p} \mid \overline{X}_k \mid) \lesssim \{\log(p)/n\}^{1/2} \vee K_n\{\log(p)/n\}$ and $\mathbb{P}\{\max_{1\le k\le p} \mid \overline{X}_k \mid\ge 2E(\max_{1\le k\le p} \mid \overline{X}_k \mid) + t\} \lesssim \exp(-nt^2/3) + n^{1-r}t^{-r}K_n^r$. By taking $t \asymp K_n\{\log(p)/n\}^{1/2}$, it follows from (0.2) that $\mathbb{P}[\max_{1\le k\le p} \mid \overline{X}_k \mid\le CK_n\{\log(p)/n\} \vee CK_n\{\log(p)/n\}^{1/2}] = 1 - O(p^{-c} + n^{1-r/2})$, for some positive constants $C, c > 0$. Let

$$\widetilde{\epsilon} \asymp K_n\{\log(p)/n\} \vee K_n\{\log(p)/n\}^{1/2},$$

and

$$\epsilon \asymp \{\log(p)/n\}^{1/2}.$$

Using (0.1), we can easily prove that

$$\mathbb{P}[\max_{1\le k\le p} \mid I_{k1} \mid\le CK_n\{\log(p)/n\}^{3/2} \vee CK_n\{\log(p)/n\}] = 1 - O(p^{-c} + n^{1-r/2}), \quad (0.3)$$

for some positive constants $C, c > 0$. For the sub-Gaussian case, we define the function $\psi_\beta : [0,\infty) \to [0,\infty)$ by $\psi_\beta(x) = \exp(x^\beta) - 1$ for $\beta > 0$, and for a real-valued random variable ξ, we define

$$\|\xi\|_{\psi_\beta} \stackrel{\text{def}}{=} \inf\{\lambda > 0 : E[\psi_\beta(\mid \xi \mid /\lambda)] \le 1\}.$$

By Problem 2.2.5 and Lemma 2.2.2 in van der Vaart and Wellner (1996), it is not difficult to verify that

$$
\begin{aligned}
E(\max_{1\le k\le p}|X_{ik}|^r) &\le (\Pi_{l=1}^r l)^r \|\max_{1\le k\le p} X_{ik}\|_{\psi_1}^r \le (\Pi_{l=1}^r l)^r \log^{r/2}(2)\|\max_{1\le k\le p} X_{ik}\|_{\psi_2}^r, \\
&\lesssim \log^{r/2}(p), \\
E\left(\max_{1\le i\le n}\max_{1\le k\le p} X_{ik}^2\right) &\le 4\|\max_{1\le i\le n}\max_{1\le k\le p} X_{ik}\|_{\psi_1}^2 \le 4\log(2)\|\max_{1\le i\le n}\max_{1\le k\le p} X_{ik}\|_{\psi_2}^2 \\
&\lesssim \log(pn)\max_{1\le i\le n}\max_{1\le k\le p}\|X_{ik}\|_{\psi_2}^2 \lesssim \log(pn),
\end{aligned}
$$

when $\mathbf{X}$ has i.i.d. sub-Gaussian rows. This, together with (0.2), entails immediately that $E(\max_{1\le k\le p} \mid \overline{X}_k \mid) \lesssim \{\log(p)/n\}^{1/2} \vee \log^{1/2}(pn)\{\log(p)/n\}$ and $\mathbb{P}\{\max_{1\le k\le p} \mid \overline{X}_k \mid\ge 2E(\max_{1\le k\le p} \mid \overline{X}_k \mid) + t\} \lesssim \exp(-nt^2/3) + n^{1-r}t^{-r}\log^{r/2}(p)$. This implies by taking $t \asymp \{\log(p)/n\}^{1/2}$ that $\mathbb{P}[\max_{1\le k\le p} \mid \overline{X}_k \mid\le C\{\log(p)/n\} \vee C\log^{1/2}(pn)\{\log(p)/n\}^{1/2}] = 1 - O(p^{-c} + n^{1-r/2})$, for some positive constants $C, c > 0$. Let

$$\widetilde{\epsilon} \asymp \{\log(p)/n\} \vee \log^{1/2}(pn)\{\log(p)/n\}^{1/2},$$

and

$$\epsilon \asymp \{\log(p)/n\}^{1/2}.$$

In the sub-Gaussian case, apply (0.1) to obtain that

$$\mathbb{P}[\max_{1\le k\le p} \mid I_{k1} \mid\le C\{\log(p)/n\}^{3/2} \vee C\log^{1/2}(pn)\{\log(p)/n\}] = 1 - O(p^{-c} + n^{1-r/2}) \quad (0.4)$$

for some positive constants $C, c > 0$.

Next we establish the bound for $\max_{1\le k\le p} \mid I_{k2} \mid$. Note that

$$\begin{aligned}
&\mid \sum_{i=1}^{n}[\psi_\tau\{Y_i - \widehat{Q}_\tau(Y)\} - \psi_\tau\{Y_i - Q_\tau(Y)\}]X_{ik} \mid \\
=&\mid \sum_{i=1}^{n} I\{\widehat{Q}_\tau(Y) < Y_i \le Q_\tau(Y)\}X_{ik} + \sum_{i=1}^{n} I\{Q_\tau(Y) < Y_i \le \widehat{Q}_\tau(Y)\}X_{ik} \mid
\end{aligned}$$

for $1 \le k \le p$. Then, for any given $\epsilon > 0$,

$$\begin{aligned}
&\mathbb{P}(\max_{1\le k\le p} \mid I_{k2} \mid \ge \epsilon) \\
\le& \mathbb{P}[\max_{1\le k\le p} \mid n^{-1}\sum_{i=1}^{n} I\{\widehat{Q}_\tau(Y) < Y_i \le Q_\tau(Y)\}X_{ik} \mid \ge \{\tau(1-\tau)\}^{1/2}\epsilon/2] \\
+&\mathbb{P}[\max_{1\le k\le p} \mid n^{-1}\sum_{i=1}^{n} I\{Q_\tau(Y) < Y_i \le \widehat{Q}_\tau(Y)\}X_{ik} \mid \ge \{\tau(1-\tau)\}^{1/2}\epsilon/2] \\
=& \mathbb{P}[\sup_{\mathbf{u}_k, k=1,\ldots,p} \mid n^{-1}\sum_{i=1}^{n} I\{\widehat{Q}_\tau(Y) < Y_i \le Q_\tau(Y)\}\mathbf{x}_i^{\mathrm{T}}\mathbf{u}_k \mid \ge \{\tau(1-\tau)\}^{1/2}\epsilon/2] \\
+&\mathbb{P}[\sup_{\mathbf{u}_k, k=1,\ldots,p} \mid n^{-1}\sum_{i=1}^{n} I\{Q_\tau(Y) < Y_i \le \widehat{Q}_\tau(Y)\}\mathbf{x}_i^{\mathrm{T}}\mathbf{u}_k \mid \ge \{\tau(1-\tau)\}^{1/2}\epsilon/2],
\end{aligned}$$

where $\mathbf{u}_k$ is the kth column of the $p \times p$ identity matrix. Let the function class $\mathcal{F}$ be $\{I\{Q_\tau(Y) < Y \le \widehat{Q}_\tau(Y)\}X_k, k = 1, \ldots, p\}$. Clearly, $\mathcal{F}$ has envelope $\max_{1\le k\le p} \mid X_k \mid$. Moreover, the function class is VC type in view of Lemma 2.6.18 in van der Vaart and Wellner (1996). Due to Assumption (C4) and Serfling (1980, Theorem 2.3.2), we have $\sup_{\mathbf{u}_k, k=1,\ldots,p} \mid n^{-1}\sum_{i=1}^{n} E[I\{Q_\tau(Y) < Y_i \le \widehat{Q}_\tau(Y)\}\mathbf{x}_i^{\mathrm{T}}\mathbf{u}_k] \mid \le cn^{-1/2}\sup_{y\in[Q_\tau(Y)-\delta_0, Q_\tau(Y)+\delta_0]} \max_{1\le k\le p} E(f_{Y|X_k}(y)|X_k|)$. Then, by applying Lemma 4, it is not difficult to obtain that with probability $1 - o(1)$,

$$\max_{1\le k\le p} \mid I_{k2} \mid \le CK_n\{\log(p)/n\}^{3/4} \vee CK_n\{\log(p)/n\}, \tag{0.5}$$

in the strongly bounded case, and

$$\max_{1\le k\le p} \mid I_{k2} \mid\le C\{\log(p)/n\}^{3/4} \vee C\log^{1/2}(pn)\{\log(p)/n\}, \tag{0.6}$$

in the sub-Gaussian case.

For bounding $\max_{1\le k\le p} \mid I_{k7} \mid$, we apply Bernstein's inequality (van der Vaart and Wellner, 1996, Lemma 2.2.11) and the fact $\mid \psi_\tau\{Y_i - Q_\tau(Y)\} \mid\le 2$ for $i = 1, \ldots, n$, to yield

$$\begin{aligned}
\mathbb{P}(\max_{1\le k\le p} \mid I_{k7} \mid\ge \epsilon\widetilde{\epsilon}) &\le \mathbb{P}\left[\left|n^{-1}\sum_{i=1}^{n}\psi_\tau\{Y_i - Q_\tau(Y)\}\right| \ge \{\tau(1-\tau)\}^{1/2}\epsilon\right] \\
&\quad +\mathbb{P}(\max_{1\le k\le p} \mid \overline{X}_k \mid\ge \widetilde{\epsilon}) \\
&\le 2\exp\{-\tau(1-\tau)n\epsilon^2/8\} + \mathbb{P}(\max_{1\le k\le p} \mid \overline{X}_k \mid\ge \widetilde{\epsilon}).
\end{aligned}$$

By using similar arguments to those in the derivation of $\max_{1\le k\le p} \mid I_{k1} \mid$, there exist some constants $r > 2$ and $C, c > 0$ such that

$$\mathbb{P}[\max_{1\le k\le p} \mid I_{k7} \mid\le CK_n\{\log(p)/n\}^{3/2} \vee CK_n\{\log(p)/n\}] = 1 - O(p^{-c} + n^{1-r/2}), \tag{0.7}$$

in the strongly bounded case, and

$$\mathbb{P}[\max_{1\le k\le p} \mid I_{k7} \mid\le C\{\log(p)/n\}^{3/2} \vee C\log^{1/2}(pn)\{\log(p)/n\}] = 1 - O(p^{-c} + n^{1-r/2}) \tag{0.8}$$

in the sub-Gaussian case.

It remains to bound the probabilities $\mathbb{P}(\max_{1\le k\le p} \mid I_{kl} \mid\ge \epsilon), l = 3, 4, 5, 6$. To that end, we need to describe the nonasymptotic bound on $\max\limits_{1\le k\le p} \mid \widehat{\sigma}_k^2 - 1 \mid$. By the triangle inequality, for any $\widetilde{\widetilde{\epsilon}} > 0$, we can obtain that

$$
\begin{aligned}
&\mathbb{P}(\max_{1\leq k\leq p} \mid \widehat{\sigma}_k^2 - 1 \mid \geq 2\widetilde{\widetilde{\epsilon}}) \\
&\leq \mathbb{P}(\max_{1\leq k\leq p} \mid \sum_{i=1}^{n} X_{ik}^2/n - 1 \mid + \max_{1\leq k\leq p} \mid \overline{X}_k \mid^2 \geq 2\widetilde{\widetilde{\epsilon}}) \\
&\leq \mathbb{P}(\max_{1\leq k\leq p} \mid \sum_{i=1}^{n} X_{ik}^2/n - 1 \mid \geq \widetilde{\widetilde{\epsilon}}) + \mathbb{P}(\max_{1\leq k\leq p} \mid \overline{X}_k \mid^2 \geq \widetilde{\widetilde{\epsilon}}). \qquad (0.9)
\end{aligned}
$$

Invoking Lemma 5, we have for every $t > 0$ and $r > 2$,

$$
\begin{aligned}
&\mathbb{P}\{\max_{1\leq k\leq p} \mid \sum_{i=1}^{n} X_{ik}^2/n - 1 \mid \geq 2E(\max_{1\leq k\leq p} \mid \sum_{i=1}^{n} X_{ik}^2/n - 1 \mid) + t\} \\
&\lesssim \exp\{-(nt)^2/(3n \max_{1\leq k\leq p} E|X_{1k}|^4)\} + (nt)^{-r} \sum_{i=1}^{n} E(\max_{1\leq k\leq p} |X_{ik}^2|^r). \qquad (0.10)
\end{aligned}
$$

Obviously, $\max_{1\leq k\leq p} E(X_{1k}^4) \lesssim \max_{1\leq k\leq p} E(X_{1k}^2 K_n^2) = K_n^2$ and $n^{-1} \sum_{i=1}^{n} E(\max_{1\leq k\leq p} |X_{ik}^2|^r) \lesssim K_n^{2r}$ in the strongly bounded case. When $\mathbf{X}$ has i.i.d. sub-Gaussian rows, it is routine to verify that $\max_{1\leq k\leq p} E|X_{1k}|^4 \lesssim 1$, $E(\max_{1\leq k\leq p} |X_{ik}|^{2r}) \lesssim \log^r(p)$ and $E\left(\max_{1\leq i\leq n} \max_{1\leq k\leq p} X_{ik}^4\right)$ $\lesssim \log^2(pn)$. Therefore, the right-hand side of (0.10) has the upper bound $C\exp\{-(nt)^2/(3nK_n^2)\} + Cn^{1-r}t^{-r}K_n^{2r}$ in the strongly bounded case, and $C\exp\{-(nt)^2/(3n)\} + Cn^{1-r}t^{-r}\log^r(p)$ in the sub-Gaussian case. Moreover, it follows from Lemma 1 in Chernozhukov et al. (2015) that

$$
\begin{aligned}
&E(\max_{1\leq k\leq p} \mid \sum_{i=1}^{n} X_{ik}^2/n - 1 \mid) \\
&\lesssim n^{-1/2}\log(p)^{1/2}\{\max_{1\leq k\leq p} E(X_{1k}^4)\}^{1/2} + n^{-1}\log(p)\{E(\max_{1\leq i\leq n}\max_{1\leq k\leq p} X_{ik}^4)\}^{1/2}. \qquad (0.11)
\end{aligned}
$$

By arguments similar to those for dealing with (0.10), the right-hand side of (0.11) has the upper bound $Cn^{-1/2}K_n \log^{1/2}(p) + Cn^{-1}K_n^2 \log(p)$ in the strongly bounded case, and $Cn^{-1/2}\log^{1/2}(p) + Cn^{-1}\log(p)\log(pn)$ in the sub-Gaussian case. Let $t \asymp n^{-1/2}K_n^2 \log^{1/2}(p)$ and $\widetilde{\widetilde{\epsilon}} \asymp n^{-1}K_n^2\log(p) \vee n^{-1/2}K_n^2\log^{1/2}(p)$ in the strongly bounded case, and $t \asymp n^{-1/2}\log(p)$ and $\widetilde{\widetilde{\epsilon}} \asymp n^{-1/2}\log(p) \vee n^{-1}\log(p)\log(pn)$ in the sub-Gaussian case. Together, (0.9), (0.10) and (0.11) yield that $\mathbb{P}\{\max_{1\le k\le p} \mid \widehat{\sigma}_k^2 - 1 \mid \le Cn^{-1}K_n^2\log(p) \vee Cn^{-1/2}K_n^2\log^{1/2}(p)\} = 1 - O(p^{-c} + n^{1-r/2})$, in the strongly bounded case, and $\mathbb{P}\{\max_{1\le k\le p} \mid \widehat{\sigma}_k^2 - 1 \mid \le Cn^{-1/2}\log(p) \vee Cn^{-1}\log(p)\log(pn)\} = 1 - O(p^{-c} + n^{1-r/2})$, in the sub-Gaussian case.

For any given $\epsilon, \widetilde{\epsilon} > 0$, it is immediate to see that

$$\begin{aligned}
\mathbb{P}(\max_{1\le k\le p} \mid I_{k3} \mid \ge \epsilon\widetilde{\epsilon}) &\le \mathbb{P}\left(\max_{1\le k\le p} \mid I_{k7} \mid \ge \epsilon\right) + \mathbb{P}(\max_{1\le k\le p} \mid \widehat{\sigma}_k^{-1} - 1 \mid \ge \widetilde{\epsilon}),\\
\mathbb{P}(\max_{1\le k\le p} \mid I_{k5} \mid \ge \epsilon\widetilde{\epsilon}) &\le \mathbb{P}\left(\max_{1\le k\le p} \mid I_{k1} \mid \ge \epsilon\right) + \mathbb{P}(\max_{1\le k\le p} \mid \widehat{\sigma}_k^{-1} - 1 \mid \ge \widetilde{\epsilon}),\\
\mathbb{P}(\max_{1\le k\le p} \mid I_{k6} \mid \ge \epsilon\widetilde{\epsilon}) &\le \mathbb{P}\left(\max_{1\le k\le p} \mid I_{k2} \mid \ge \epsilon\right) + \mathbb{P}(\max_{1\le k\le p} \mid \widehat{\sigma}_k^{-1} - 1 \mid \ge \widetilde{\epsilon}).
\end{aligned}$$

Under Assumption (C2) and combining the nonasymptotic bounds for $\max_{1\le k\le p} \mid I_{k1} \mid$, $\max_{1\le k\le p} \mid I_{k2} \mid$ and $\max_{1\le k\le p} \mid I_{k7} \mid$, we have

$$\mathbb{P}\{\max_{1\le k\le p} \mid I_{k3} \mid \le Cn^{-5/2}K_n^3\log^{5/2}(p) \vee Cn^{-2}K_n^3\log^2(p) \vee Cn^{-3/2}K_n^3\log^{3/2}(p)\} = 1 - O(p^{-c} + n^{1-r/2}), \quad (0.12)$$

$$\mathbb{P}\{\max_{1\le k\le p} \mid I_{k5} \mid \le Cn^{-5/2}K_n^3\log^{5/2}(p) \vee Cn^{-2}K_n^3\log^2(p) \vee Cn^{-3/2}K_n^3\log^{3/2}(p)\} = 1 - O(p^{-c} + n^{1-r/2}), \quad (0.13)$$

$$\mathbb{P}\{\max_{1\le k\le p} \mid I_{k6} \mid \le Cn^{-1}K_n^3\log^{3/2}(p)\} = 1 - O(p^{-c} + n^{1-r/2}), \quad (0.14)$$

in the strongly bounded case, and

$$\mathbb{P}\{\max_{1\le k\le p} \mid I_{k3} \mid\le Cn^{-2}\log^{5/2}(p)\vee Cn^{-5/2}\log^{5/2}(p)\log(pn)\vee Cn^{-3/2}\log^{2}(p)\log^{1/2}(pn)$$
$$\vee Cn^{-2}\log^{2}(p)\log^{3/2}(pn)\} = 1-O(p^{-c}+n^{1-r/2}), \tag{0.15}$$

$$\mathbb{P}\{\max_{1\le k\le p} \mid I_{k5} \mid\le Cn^{-2}\log^{5/2}(p)\vee Cn^{-5/2}\log^{5/2}(p)\log(pn)\vee Cn^{-3/2}\log^{2}(p)\log^{1/2}(pn)$$
$$\vee Cn^{-2}\log^{2}(p)\log^{3/2}(pn)\} = 1-O(p^{-c}+n^{1-r/2}), \tag{0.16}$$

$$\mathbb{P}\{\max_{1\le k\le p} \mid I_{k6} \mid\le Cn^{-1}\log^{3/2}(p)\log^{1/2}(n)\log^{1/2}(pn)$$
$$\vee Cn^{-3/2}\log^{3/2}(p)\log^{1/2}(n)\log^{3/2}(pn)\} = 1-O(p^{-c}+n^{1-r/2}), \tag{0.17}$$

in the sub-Gaussian case. Under Assumption (C4) and by Lemma 1, we have that for all $1\le k\le p$,

$$E\left[n^{-1}\sum_{i=1}^{n}\psi_\tau\{Y_i-Q_\tau(Y)\}X_{ik}\right]=0,$$

under the null hypothesis in (1.2). Using the fact $\mid \psi_\tau\{Y_i-Q_\tau(Y)\} \mid\le 2$ for $i=1,\dots,n$, it is routine to show that $\mathbb{P}[\max_{1\le k\le p} \mid n^{-1}\sum_{i=1}^{n}\psi_\tau\{Y_i-Q_\tau(Y)\}X_{ik} \mid\le CK_n\{\log(p)/n\}\vee CK_n\{\log(p)/n\}^{1/2}] = 1-O(p^{-c}+n^{1-r/2})$ in the strongly bounded case, and $\mathbb{P}[\max_{1\le k\le p} \mid n^{-1}\sum_{i=1}^{n}\psi_\tau\{Y_i-Q_\tau(Y)\}X_{ik} \mid\le C\{\log(p)/n\}\vee C\log^{1/2}(pn)\{\log(p)/n\}^{1/2}] = 1-O(p^{-c}+n^{1-r/2})$ in the sub-Gaussian case. Consequently, it follows from the argument similar to that used to bound $\max_{1\le k\le p}\mid I_{k6}\mid$ that

$$\mathbb{P}\{\max_{1\le k\le p} \mid I_{k4} \mid\le Cn^{-2}K_n^3\log^{2}(p)\vee Cn^{-3/2}K_n^3\log^{3/2}(p)$$
$$\vee Cn^{-1}K_n^3\log(p)\} = 1-O(p^{-c}+n^{1-r/2}), \tag{0.18}$$

in the strongly bounded case, and

$$\mathbb{P}\{\max_{1\le k\le p} \mid I_{k4} \mid\le Cn^{-3/2}\log^2(p) \vee Cn^{-2}\log^2(p)\log(pn) \vee Cn^{-1}\log^{3/2}(p)\log^{1/2}(pn)$$
$$\vee Cn^{-3/2}\log^{3/2}(p)\log^{3/2}(pn)\} = 1 - O(p^{-c} + n^{1-r/2}), \qquad (0.19)$$

in the sub-Gaussian case. Combining (0.3), (0.5), (0.7), (0.12), (0.13), (0.14) and (0.18), we obtain that with probability $1-o(1)$, $\mid \widehat{S}_\tau - \widehat{S}_\tau^\natural \mid\lesssim n^{-3/4}K_n^3\log^{3/4}(p)$ in the strongly bounded case. Combining (0.4), (0.6), (0.8), (0.15), (0.16), (0.17) and (0.19), we obtain that with probability $1-o(1)$, $\mid \widehat{S}_\tau - \widehat{S}_\tau^\natural \mid\lesssim n^{-2}\log^2(p)\log^{3/2}(pn)$ in the sub-Gaussian case. As a result, there exist $\zeta_1, \zeta_2 > 0$ such that

$$\mathbb{P}\left(\left| n^{1/2}\widehat{S}_\tau - \max_{1\le k\le p} \mid \{\tau(1-\tau)\}^{-1/2} n^{-1/2}\sum_{i=1}^n \psi_\tau\{Y_i - Q_\tau(Y)\}X_{ik} \mid \right| \ge \zeta_1\right) < \zeta_2, \qquad (0.20)$$

where $\zeta_1 \asymp n^{-1/4}K_n^3\log^{3/4}(p)$ in the strongly bounded case, and $\zeta_1 \asymp n^{-3/2}\log^2(p)\log^{3/2}(pn)$ in the sub-Gaussian case and $\zeta_2 = o(1)$.

Let

$$Z_{ik} = \{\tau(1-\tau)\}^{-1/2}\psi_\tau\{Y_i - Q_\tau(Y)\}X_{ik},$$

for $i = 1, \ldots, n$ and $k = 1, \ldots, p$. When $\mathbf{X}$ is strongly bounded, we take $B_n = 2\{\tau(1-\tau)\}^{-1/2}K_n$. It is trivial that $n^{-1}\sum_{i=1}^n E(\mid Z_{ik} \mid^{2+l}) \le n^{-1}\sum_{i=1}^n E(\mid X_{ik} \mid^2)B_n^l = B_n^l$ for all $k = 1, \ldots, p$ and $l = 1, 2$, and $E\{(\max_{1\le k\le p} \mid Z_{ik} \mid /B_n)^q\} \le E\{(\max_{1\le k\le p} \mid X_{ik} \mid /K_n)^q\} \le 2$ for all $i = 1, \ldots, n$ and $q \ge 3$. An application of Chernozhukov et al. (2017, Proposition 2.1) under these conditions leads to $\sup_{t\in\mathbb{R}} \mid \mathbb{P}(\max_{1\le k\le p} n^{1/2}\ \overline{Z}_k \le$

$t) - \mathbb{P}(\max_{1\le k\le p} n^{1/2}\ \overline{G}_k \le t)\ |\lesssim \{n^{-1}K_n^2 \log^7(pn)\}^{1/6}$, where $\overline{Z}_k = n^{-1}\sum_{i=1}^n Z_{ik}$ and $\overline{G}_k = n^{-1}\sum_{i=1}^n G_{ik}$ with $\{\mathbf{g}_i = (G_{i1},\dots,G_{ip})\}_{i=1}^n$ being a sequence of independent centred Gaussian random vectors such that each $\mathbf{g}_i$ has the same covariance matrix as $\mathbf{z}_i = (Z_{i1},\dots,Z_{ip})^{\mathrm{T}}$. Consequently,

$$
\begin{aligned}
&\sup_{t\in\mathbb{R}^+} |\ \mathbb{P}(\max_{1\le k\le p} |\ \sqrt{n}\ \overline{Z}_k\ |\le t) - \mathbb{P}(\max_{1\le k\le p} |\ \sqrt{n}\ \overline{G}_k\ |\le t)\ | \\
&\le \sup_{t\in\mathbb{R}^+} |\ \mathbb{P}(\max_{1\le k\le p} n^{1/2}\ \overline{Z}_k \le t) - \mathbb{P}(\max_{1\le k\le p} n^{1/2}\ \overline{G}_k \le t)\ | \\
&\qquad + \sup_{t\in\mathbb{R}^+} |\ \mathbb{P}(\max_{1\le k\le p} n^{1/2}\ \overline{Z}_k \le -t) - \mathbb{P}(\max_{1\le k\le p} n^{1/2}\ \overline{G}_k \le -t)\ | \\
&\le 2\sup_{t\in\mathbb{R}} |\ \mathbb{P}(\max_{1\le k\le p} n^{1/2}\ \overline{Z}_k \le t) - \mathbb{P}(\max_{1\le k\le p} n^{1/2}\ \overline{G}_k \le t)\ |\lesssim \{n^{-1}K_n^2\log^7(pn)\}^{1/6}.
\end{aligned}
\tag{0.21}
$$

Let $\widetilde{c}_{\tau,\alpha} = \inf\{t \in R^+ : \mathbb{P}(\max_{1\le j\le p} |n^{1/2}\overline{G}_j| \le t) \ge 1-\alpha\}$ and note $c_{\tau,\alpha} = \inf\{t \in \mathbb{R}^+ : \mathbb{P}(n^{1/2}\widehat{S}_\tau \le t \mid Y_i, \mathbf{x}_i, i = 1,\dots,n) \ge 1-\alpha\}$. Using the similar arguments in the proof of Lemma 3.2 in Chernozhukov et al. (2013) we have that for every $v > 0$,

$$
\mathbb{P}(\Delta > v) \ge \mathbb{P}\{c_{\tau,\alpha} \ge \widetilde{c}_{\tau,\alpha+\pi(v)}\} \vee \mathbb{P}\{\widetilde{c}_{\tau,\alpha} \ge c_{\tau,\alpha+\pi(v)}\}, \tag{0.22}
$$

where $\pi(v) \asymp v^{1/3}\{1 \vee \log(p/v)\}^{2/3}$ and

$$
\Delta = \max_{1\le k,l\le p} \left| n^{-1}\sum_{i=1}^n \{Z_{ik}Z_{il} - E(Z_{ik}Z_{il})\} \right|.
$$

By the triangle inequality,

$$
\begin{aligned}
\left|\mathbb{P}\left(\Psi_{\widehat{S}_\tau,\alpha} = 1\right) - \alpha\right| \ &\le\ \left|\mathbb{P}\left(n^{1/2}\widehat{S}_\tau > c_{\tau,\alpha}\right) - \mathbb{P}\left(|\ \sqrt{n}\ \overline{Z}_k\ | > c_{\tau,\alpha}\right)\right| \\
&\quad + \left|\mathbb{P}\left(|\ \sqrt{n}\ \overline{Z}_k\ | > c_{\tau,\alpha}\right) - \alpha\right|.
\end{aligned}
$$

Apply the inequality $| I(a < c) - I(b < c) | \leq I(| b - c | < | a - b |)$ to show that

$$
\begin{aligned}
&\left| \mathbb{P}\left(n^{1/2}\widehat{S}_\tau > c_{\tau,\alpha}\right) - \mathbb{P}\left(| \sqrt{n}\ \overline{Z}_k | > c_{\tau,\alpha}\right)\right| \\
&\leq \mathbb{P}(| n^{1/2}\widehat{S}_\tau - \sqrt{n}\ \overline{Z}_k | > | \sqrt{n}\ \overline{Z}_k - c_{\tau,\alpha} |) \\
&= \mathbb{P}(| n^{1/2}\widehat{S}_\tau - \sqrt{n}\ \overline{Z}_k | > | \sqrt{n}\ \overline{Z}_k - c_{\tau,\alpha} |, | n^{1/2}\widehat{S}_\tau - \sqrt{n}\ \overline{Z}_k | \geq \xi_1) \\
&\qquad + \mathbb{P}(| n^{1/2}\widehat{S}_\tau - \sqrt{n}\ \overline{Z}_k | > | \sqrt{n}\ \overline{Z}_k - c_{\tau,\alpha} |, | n^{1/2}\widehat{S}_\tau - \sqrt{n}\ \overline{Z}_k | < \xi_1) \\
&\leq \mathbb{P}(| n^{1/2}\widehat{S}_\tau - \sqrt{n}\ \overline{Z}_k | \geq \xi_1) + | \mathbb{P}(\xi_1 > | \sqrt{n}\ \overline{Z}_k - c_{\tau,\alpha} |) - \mathbb{P}(\xi_1 > | \sqrt{n}\ \overline{G}_k - c_{\tau,\alpha} |) | \\
&\qquad + \mathbb{P}(\xi_1 > | \sqrt{n}\ \overline{G}_k - c_{\tau,\alpha} |) \\
&\lesssim \zeta_2 + \{n^{-1}K_n^2 \log^7(pn)\}^{1/6} + \mathbb{P}(\xi_1 > | \sqrt{n}\ \overline{G}_k - c_{\tau,\alpha} |) \\
&\lesssim \zeta_2 + \{n^{-1}K_n^2 \log^7(pn)\}^{1/6} + \zeta_1\{1 \vee \log(p/\zeta_1)\}^{1/2},
\end{aligned}
$$

where the third inequality follows from (0.20) and (0.21) and the last inequality holds due to the anti-concentration inequality in Chernozhukov et al. (2015). Further, apply (0.21), (0.22) and the triangle inequality to obtain

$$
\begin{aligned}
&\left| \mathbb{P}\left(| \sqrt{n}\ \overline{Z}_k | > c_{\tau,\alpha}\right) - \alpha\right| \\
&\lesssim \left| \mathbb{P}\left(| \sqrt{n}\ \overline{G}_k | > c_{\tau,\alpha}\right) - \{\alpha + \pi(v)\}\right| + \pi(v) + \{n^{-1}K_n^2 \log^7(pn)\}^{1/6} \\
&\lesssim \mathbb{P}(\Delta > v) + \pi(v) + \{n^{-1}K_n^2 \log^7(pn)\}^{1/6}.
\end{aligned}
$$

By the maximal inequality in Lemma E.1 of Chernozhukov et al. (2017) and the boundness of the function $\psi_\tau(\cdot)$, it is routine to verify that $\mathbb{P}\{\Delta \leq Cn^{-1}K_n^2 \log(p) \vee Cn^{-1/2}K_n^2 \log^{1/2}(p)\} = 1 - O(p^{-c} + n^{1-r/2})$, for some positive constants $c > 0, r > 2$. Therefore, in the strongly bounded case and choosing $v \asymp n^{-1}K_n^2 \log(p) \vee n^{-1/2}K_n^2 \log^{1/2}(p)$, we obtain

$$\left|\mathbb{P}\left(\Psi_{\widehat{S}_\tau,\alpha}=1\right)-\alpha\right| \lesssim v^{1/3}\{1\vee\log(p/v)\}^{2/3}+\zeta_2+\{n^{-1}K_n^2\log^7(pn)\}^{1/6}$$
$$+\zeta_1\{1\vee\log(p/\zeta_1)\}^{1/2}+p^{-c}+n^{1-r/2}, \tag{0.23}$$

for some constants $c>0, r>2$. Under the assumption $K_n^2\{\log(pn)\}^7/n\lesssim n^{-c_1}$ with some constant $c_1>0$, we deduce the desired conclusion in the strongly bounded case.

On the other hand, when $\mathbf{X}$ has i.i.d. sub-Gaussian rows and by Lemma 2.2.2 in van der Vaart and Wellner (1996), we have $\|X_{ik}\|_{\psi_1}\le\log^{1/2}(2)\max_{1\le i\le n}\max_{1\le k\le p}\|X_{ik}\|_{\psi_2}$ $<\infty$ and $E(X_{ik}^{2+l})\le(\Pi_{m=1}^{2+l}m)^{2+l}\max_{1\le i\le n}\max_{1\le k\le p}\|X_{ik}\|_{\psi_1}^{2+l}<\infty$ for all $i=1,\ldots,n$, $k=1,\ldots,p$ and $l=1,2$. Thus, there exists a large enough constant $C>0$ such that $n^{-1}\sum_{i=1}^{n}E(|\ Z_{ik}\ |^{2+l})\le\{\tau(1-\tau)/2\}^{-1-l/2}n^{-1}\sum_{i=1}^{n}E(|\ X_{ik}\ |^{2+l})\le C^l$ for all $k=1,\ldots,p$ and $l=1,2$, and $E\{\exp(|\ Z_{ik}\ |\ /C)\}\le2\{\tau(1-\tau)/2\}^{-1/2}\|X_{ik}\|_{\psi_1}/C\le 2\{\tau(1-\tau)/2\}^{-1/2}\max_{1\le i\le n}\max_{1\le k\le p}\|X_{ik}\|_{\psi_1}/C\le2$ for all $i=1,\ldots,n$ and $q\ge3$. Together with Chernozhukov et al. (2017, Proposition 2.1), this implies that $\sup_{t\in\mathbb{R}}|\ \mathbb{P}(\max_{1\le k\le p}n^{1/2}\ \overline{Z}_k\le t)-\mathbb{P}(\max_{1\le k\le p}n^{1/2}\ \overline{G}_k\le t)\ |\lesssim\{n^{-1}\log^7(pn)\}^{1/6}$ in the sub-Gaussian case. Taking $v\asymp n^{-1/2}\log(p)\vee n^{-1}\log(p)\log(pn)$ and employing arguments similar to those for dealing with (0.23), we have

$$\left|\mathbb{P}\left(\Psi_{\widehat{S}_\tau,\alpha}=1\right)-\alpha\right| \lesssim v^{1/3}\{1\vee\log(p/v)\}^{2/3}+\zeta_2+\{n^{-1}\log^7(pn)\}^{1/6}$$
$$+\zeta_1\{1\vee\log(p/\zeta_1)\}^{1/2}+p^{-c}+n^{1-r/2},$$

for some constants $c>0, r>2$. Under the assumption $\{\log(pn)\}^7/n\lesssim n^{-c_1}$ with some constant $c_1>0$, it is immediate to deduce the desired conclusion in the sub-

Gaussian case.

□

Proof of Theorem 2

Without loss of generality, we set $\sigma_{11} = \ldots = \sigma_{pp} = 1$. Define $\widetilde{S}_\tau = \max_{1\le k\le p} \mid \widehat{\text{qcor}}_\tau(Y, X_k) - \text{qcor}_\tau(Y, X_k) \mid$. Under the assumptions in Theorem 1, it is routine to show that $\mathbb{P}\left(\left| n^{1/2}\widetilde{S}_\tau - \max_{1\le k\le p} \mid n^{-1/2}\sum_{i=1}^{n} Z_{ik} \mid \right| \ge \zeta_1\right) < \zeta_2$ for $\zeta_1\{1 \vee \log(p/\zeta_1)\}^{1/2} = o(1)$ and $\zeta_2 = o(1)$, where $Z_{ik} = \{\tau(1-\tau)\}^{-1/2}\psi_\tau\{Y_i - Q_\tau(Y)\}X_{ik}$ for $i = 1, \ldots, n$ and $k = 1, \ldots, p$. In another word, the distribution of $n^{1/2}\widetilde{S}_\tau$ can be approximated by $\max_{1\le k\le p} \mid G_k \mid$, where $(G_1, \ldots, G_p)^{\mathrm{T}}$ is the centered Gaussian random vector with mean zero and covariance matrix $\boldsymbol{\Theta} = E[\psi_\tau^2\{Y - Q_\tau(Y)\}\{\mathbf{x} - E(\mathbf{x})\}\{\mathbf{x} - E(\mathbf{x})\}^{\mathrm{T}}] \in \mathbb{R}^{p\times p}$. Since $\lambda_{\max}(\boldsymbol{\Theta}) = \sup_{\boldsymbol{\beta}\in\mathbb{R}^p} \boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\Theta}\boldsymbol{\beta}/\|\boldsymbol{\beta}\|^2 = \sup_{\boldsymbol{\beta}\in\mathbb{R}^p} E[\psi_\tau^2\{Y - Q_\tau(Y)\}\|\boldsymbol{\beta}^{\mathrm{T}}\{\mathbf{x} - E(\mathbf{x})\}\|^2]/\|\boldsymbol{\beta}\|^2 \le \{\tau \vee (1-\tau)\}^2 \sup_{\boldsymbol{\beta}\in\mathbb{R}^p} E(\|\boldsymbol{\beta}^{\mathrm{T}}\{\mathbf{x} - E(\mathbf{x})\}\|^2)/\|\boldsymbol{\beta}\|^2 = \{\tau \vee (1-\tau)\}^2\lambda_{\max}(\boldsymbol{\Sigma})$, we conclude that under Assumption (C5), by Lemma 6 of Cai et al. (2014), we have for any $x \in \mathbb{R}$ and as $p \to \infty$, $\mathbb{P}[\max_{1\le k\le p} \mid G_k \mid -2\log(p) + \log\{\log(p)\} \le x] \to F(x) = \exp\{-\pi^{-1/2}\exp(-x/2)\}$. It implies that

$$\mathbb{P}\left[n\widetilde{S}_\tau^2 \le 2\log(p) - \log\{\log(p)\}/2\right] \to 1. \tag{0.24}$$

The bootstrap consistency result implies that

$$c_{\tau,\alpha}^2 - 2\log(p) + \log\{\log(p)\} - q_\alpha = o_P(1),$$

where q_α is the 100(1-α)th quantile of $F(x)$. Consider any $k \in \{1, \ldots, p\}$ such that $\mid \text{qcov}_\tau(Y, X_k)/\sigma_{kk}^{1/2} \mid \ge (\epsilon_0 + 2^{1/2})\{\tau(1-\tau)\log(p)/n\}^{1/2}$. Using the inequality $2a_1a_2 \le$

$\delta^{-1}a_1^2 + \delta a_2^2$ for any $\delta > 0$, we have

$$\begin{aligned} \text{qcor}_\tau^2(Y, X_k) &\leq (1+\delta^{-1}) \mid \widehat{\text{qcor}}_\tau(Y, X_k) - \text{qcor}_\tau(Y, X_k) \mid^2 \\ &\quad +(1+\delta)\widehat{\text{qcor}}_\tau^2(Y, X_k), \end{aligned} \tag{0.25}$$

where $n \mid \widehat{\text{qcor}}_\tau(Y, X_k) - \text{qcor}_\tau(Y, X_k) \mid^2 / \{\tau(1-\tau)\widehat{\sigma}_{kk}\} = o_P\{\log(p)\}$ as k is fixed and p grows. From the proof of Theorem 1, we know the difference between $n\ \text{qcor}_\tau^2(Y, X_k)/\{\tau(1-\tau)\widehat{\sigma}_{kk}\}$ and $n\ \text{qcor}_\tau^2(Y, X_k)/\{\tau(1-\tau)\sigma_{kk}\}$ is asymptotically negligible. Thus by (0.25) and the fact that $\boldsymbol{\theta}_\tau \in \mathcal{V}_\tau(\epsilon_0 + 2^{1/2})$, we have,

$$\begin{aligned} &\max_{1 \leq k \leq p} n \mid \widehat{\text{qcor}}_\tau(Y, X_k) \mid^2 / \{\tau(1-\tau)\widehat{\sigma}_{kk}\} \\ &\geq (1+\delta)^{-1}[(\epsilon_0 + 2^{1/2})^2 \log(p) - o_P\{\log(p)\}]. \end{aligned} \tag{0.26}$$

The conclusion thus follows from (0.24), (0.25) and (0.26) provided that δ is small enough.

□

Proof of Lemma 2

Recall that the random variable C is independent of $(Y, \mathbf{x})$. It then follows by the law of iterated expectations that $\{\tau(1-\tau)\sigma_k^2\}^{1/2}\text{cqcor}_\tau(Y, X_k) = E[\psi_\tau\{Y - Q_\tau(Y)\}\{X_k - E(X_k)\}]$ and $E[\{\delta/G(Y^*)\}\{\rho_\tau(Y^* - \alpha - \theta X_k) - \rho_\tau(Y^*)\}] = E\{\rho_\tau(Y - \alpha - \theta X_k) - \rho_\tau(Y)\}$. Lemma 2 then follows immediately from Lemma 1. □

Proof of Theorem 3

Write $\widehat{T}_\tau^\natural = \max_{1\le k\le p} \mid \widehat{\text{cqcor}}_\tau^\natural(Y, X_k) \mid$, where

$$\widehat{\text{cqcor}}_\tau^\natural(Y, X_k) = \{\tau(1-\tau)\}^{-1/2} n^{-1} \sum_{i=1}^n [\tau - w_{i\tau}(F) I\{Y_i^* \le Q_\tau(Y)\}](X_{ik} - \overline{X}_k),$$

for $k = 1, \ldots, p$, and

$$w_{i\tau}(F) = \begin{cases} 1 & \text{if } \Delta_i = 1 \text{ or } F(C_i) > \tau \ , \\ \frac{\tau - F(C_i)}{1-F(C_i)} & \text{if } \Delta_i = 0 \text{ and } F(C_i) \le \tau \ . \end{cases}$$

Then we can decompose $\widehat{\text{cqcor}}_\tau(Y, X_k) - \widehat{\text{cqcor}}_\tau^\natural(Y, X_k)$ as $\widehat{\text{qpcor}}_\tau(Y, X_k) - \widehat{\text{qpcor}}_\tau^\natural(Y, X_k) = \sum_{l=1}^{7} J_{kl}$, where

$$\begin{aligned}
J_{k1} &= -\{\tau(1-\tau)\}^{-1/2} \overline{X}_k n^{-1} \sum_{i=1}^n [w_{i\tau}(F) I\{Y_i^* \le Q_\tau(Y)\} \\
&\qquad - w_{i\tau}(\widehat{F}) I\{Y_i^* \le \widehat{Q}_\tau(Y)\}], \\
J_{k2} &= \{\tau(1-\tau)\}^{-1/2} n^{-1} \sum_{i=1}^n [w_{i\tau}(F) I\{Y_i^* \le Q_\tau(Y)\} \\
&\qquad - w_{i\tau}(\widehat{F}) I\{Y_i^* \le \widehat{Q}_\tau(Y)\}] X_{ik}, \\
J_{k3} &= -\{\tau(1-\tau)\}^{-1/2} (\widehat{\sigma}_k^{-1} - 1) \overline{X}_k n^{-1} \sum_{i=1}^n [\tau - w_{i\tau}(F) I\{Y_i^* \le Q_\tau(Y)\}], \\
J_{k4} &= \{\tau(1-\tau)\}^{-1/2} (\widehat{\sigma}_k^{-1} - 1) n^{-1} \sum_{i=1}^n [\tau - w_{i\tau}(F) I\{Y_i^* \le Q_\tau(Y)\}] X_{ik}, \\
J_{k5} &= -\{\tau(1-\tau)\}^{-1/2} (\widehat{\sigma}_k^{-1} - 1) \overline{X}_k n^{-1} \sum_{i=1}^n [w_{i\tau}(F) I\{Y_i^* \le Q_\tau(Y)\}
\end{aligned}$$

$$
\begin{aligned}
& \qquad -w_{i\tau}(\widehat{F})I\{Y_i^* \le \widehat{Q}_\tau(Y)\}], \\
J_{k6} &= \{\tau(1-\tau)\}^{-1/2}(\widehat{\sigma}_k^{-1}-1)n^{-1}\sum_{i=1}^n [w_{i\tau}(F)I\{Y_i^* \le Q_\tau(Y)\} \\
& \qquad -w_{i\tau}(\widehat{F})I\{Y_i^* \le \widehat{Q}_\tau(Y)\}]X_{ik}, \\
J_{k7} &= -\{\tau(1-\tau)\}^{-1/2}\overline{X}_k n^{-1}\sum_{i=1}^n [\tau - w_{i\tau}(F)I\{Y_i^* \le Q_\tau(Y)\}].
\end{aligned}
$$

Using (A.2) in Wang and Wang (2009), we have

$$
\begin{aligned}
w_\tau(F)I\{Y^* \le Q_\tau(Y)\} &= I\{C > Q_\tau(Y), Y \le Q_\tau(Y)\} + I\{C \le Q_\tau(Y), Y \le C\} \\
& \quad + I\{C \le Q_\tau(Y), Y > C\}\left[1 - \frac{1-\tau}{1-F(C)} I\{F(C) < \tau\}\right].
\end{aligned}
$$

Consequently,

$$
\left| w_{i\tau}(\widehat{F})I\{Y_i^* \le \widehat{Q}_\tau(Y)\} - w_{i\tau}(F)I\{Y_i^* \le Q_\tau(Y)\} \right| \le K_{i1} + K_{i2} + K_{i3},
$$

where

$$
\begin{aligned}
K_{i1} &= |I\{C_i > \widehat{Q}_\tau(Y), Y_i \le \widehat{Q}_\tau(Y)\} - I\{C_i > Q_\tau(Y), Y_i \le Q_\tau(Y)\}|, \\
K_{i2} &= |I\{C_i \le \widehat{Q}_\tau(Y), Y_i \le C_i\} - I\{C_i \le Q_\tau(Y), Y_i \le C_i\}|, \\
K_{i3} &= \Bigg| I\{C_i \le Q_\tau(Y), Y_i > C_i\}\left[1 - \frac{1-\tau}{1-F(C_i)} I\{F(C_i) < \tau\}\right] \\
& \quad - I\{C_i \le Q_\tau(Y), Y_i > C_i\}\left[1 - \frac{1-\tau}{1-F(C_i)} I\{F(C_i) < \tau\}\right] \Bigg|.
\end{aligned}
$$

From He et al. (2013, Lemma 8.4) and the Hoeffding's inequality, there exist $\epsilon_0 > 0$

and $c > 0$ such that for any $\epsilon \in (0, \epsilon_0)$,

$$\begin{aligned}
&\mathbb{P}\left[n^{-1}\sum_{i=1}^{n}\left|w_{i\tau}(F)I\{Y_i^* \le Q_\tau(Y)\} - w_{i\tau}(\widehat{F})I\{Y^* \le \widehat{Q}_\tau(Y)\}\right| > \epsilon\right] \\
&\le \mathbb{P}\left(n^{-1}\sum_{i=1}^{n} K_{i1} > \epsilon/3\right) + \mathbb{P}\left(n^{-1}\sum_{i=1}^{n} K_{i2} > \epsilon/3\right) + \mathbb{P}\left(n^{-1}\sum_{i=1}^{n} K_{i3} > \epsilon/3\right) \\
&\lesssim \exp(-cn\epsilon^2).
\end{aligned}$$

The rest of the proof is analogous to the last part of Theorem 1. We omit the details for brevity.

□

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