Supplementary material for “Robust Density Power Divergence Estimates for Panel Data Models”

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E Integral of the first term of the DPD measure

\[
\int_{y_i} f_{\theta}^{1+\gamma}(y|x_i)dy
\]

\[
= (2\pi)^{-\frac{T(1+\gamma)}{2}} |\Omega|^{-\frac{1+\gamma}{2}} \int_{y_i} \exp \left\{ -\frac{1+\gamma}{2} (y_i - x_i \beta)^T \Omega^{-1} (y_i - x_i \beta) \right\} dy_i
\]

\[
= (2\pi)^{-\frac{T}{2}} |\Omega|^{-\frac{1}{2}} (1+\gamma)^{-\frac{1}{2}} \int_{y_i} \left[ \frac{\Omega}{1+\gamma} \right]^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} \left( y_i - x_i \beta \right)^T \left( \frac{\Omega}{1+\gamma} \right)^{-1} (y_i - x_i \beta) \right\} dy_i
\]

\[
= (2\pi)^{-\frac{T}{2}} |\Omega|^{-\frac{1}{2}} (1+\gamma)^{-\frac{1}{2}}.
\]

F Vector $\xi^{(i)}$ at Model

From Equations (6) and (15) of the main paper, we get

\[
\xi^{(i)}_{\beta} = \int_{y_i} u_{\beta}(y_i|x_i) f_{\theta}^{1+\gamma}(y_i|x_i)dy_i
\]

\[
= \int_{y_i} \left[ \frac{1}{\sigma^2} \sum_{t=1}^{T} x_{it} (y_{it} - x_i \beta) - \frac{T \bar{x}_i \sigma^2}{\sigma^2 + T \alpha^2} \sum_{t=1}^{T} (y_{it} - x_i \beta) \right] f_{\theta}^{1+\gamma}(y_i|x_i)dy_i
\]

\[
= 0, \text{ from (22)}.
\]

\[
\xi^{(i)}_{\sigma^2} = \int_{y_i} u_{\sigma^2}(y_i|x_i) f_{\theta}^{1+\gamma}(y_i|x_i)dy_i
\]

\[
= \int_{y_i} \left[ -\frac{T}{2(\sigma^2 + T \alpha^2)} + \frac{1}{2(\sigma^2 + T \alpha^2)^2} \left\{ \sum_{t=1}^{T} (y_{it} - x_i \beta) \right\} \right]^{T} f_{\theta}^{1+\gamma}(y_i|x_i)dy_i
\]

\[
= -\frac{T}{2(\sigma^2 + T \alpha^2)} \times M(1+\gamma), \text{ using (23)}
\]

\[
+ \frac{1}{2(\sigma^2 + T \alpha^2)^2} \times MT(\sigma^2 + T \alpha^2), \text{ using (24)}
\]

\[
= -\frac{MT\gamma}{2(\sigma^2 + T \alpha^2)}.
\]
From Equations (6) and (15) of the main paper, we get

\[
J_{\sigma_{a}^{2}} = \int_{y_{i}} u_{\sigma_{a}^{2}}(y|x_{i}) f_{\theta}^{1+\gamma}(y_{i}|x_{i}) dy_{i}
\]

\[
= \int_{y_{i}} \left[ -\frac{T}{2\sigma_{a}^{2}(\sigma_{a}^{2} + T\sigma_{a}^{2})} + \frac{1}{2\sigma_{a}^{2}} \sum_{t=1}^{T} (y_{it} - x_{it}\beta)^{2} \right] f_{\theta}^{1+\gamma}(y_{i}|x_{i}) dy_{i}
\]

\[
= -\frac{T}{2\sigma_{a}^{2}(\sigma_{a}^{2} + T\sigma_{a}^{2})} \int_{y_{i}} f_{\theta}^{1+\gamma}(y_{i}|x_{i}) dy_{i} + \frac{1}{2\sigma_{a}^{2}} \sum_{t=1}^{T} (y_{it} - x_{it}\beta)^{2} \int_{y_{i}} f_{\theta}^{1+\gamma}(y_{i}|x_{i}) dy_{i}
\]

\[
= -\frac{1}{4(\sigma_{a}^{2} + T\sigma_{a}^{2})^{2}} \int_{y_{i}} f_{\theta}^{1+\gamma}(y_{i}|x_{i}) dy_{i} - \frac{T}{2(\sigma_{a}^{2} + T\sigma_{a}^{2})^{3}} \int_{y_{i}} \left\{ \sum_{t=1}^{T} (y_{it} - x_{it}\beta) \right\}^{2} f_{\theta}^{1+\gamma}(y_{i}|x_{i}) dy_{i}
\]

\[
= M(1 + \gamma) \frac{1}{4(\sigma_{a}^{2} + T\sigma_{a}^{2})^{2}} + \frac{T}{2(\sigma_{a}^{2} + T\sigma_{a}^{2})^{3}} \times MT(\sigma_{a}^{2} + T\sigma_{a}^{2}), \text{ using (23)}
\]

\[
+ \frac{1}{4(\sigma_{a}^{2} + T\sigma_{a}^{2})^{3}} \times \frac{3MT^{2}}{(1 + \gamma)(\sigma_{a}^{2} + T\sigma_{a}^{2})^{2}}, \text{ using (25)}
\]

\[
= MT^{2}(\gamma^{2} + 2) + \frac{3MT^{2}}{(1 + \gamma)(\sigma_{a}^{2} + T\sigma_{a}^{2})^{2}} + \frac{1}{4(1 + \gamma)(\sigma_{a}^{2} + T\sigma_{a}^{2})^{2}}.
\]
\[ J^{(i)} = \int_{y_i} u_\beta(y_i | x_i) w_T(y_i | x_i) f^{1+\gamma}(y_i | x_i) dy_i \]

\[ = \int_{y_i} \left[ \frac{1}{\sigma^2} \sum_{t=1}^T x_{it}(y_{it} - x_{it} \beta) - \frac{T \bar{x}_i \sigma^2}{\sigma^2 + T \sigma^2} \sum_{t=1}^T (y_{it} - x_{it} \beta) \right] \]

\[ + \left[ \frac{1}{\sigma^2} \sum_{t=1}^T x_{it} \beta (y_{it} - x_{it} \beta) - \frac{T \bar{x}_i \sigma^2}{\sigma^2 + T \sigma^2} \sum_{t=1}^T (y_{it} - x_{it} \beta) \right] f^{1+\gamma}(y_i | x_i) dy_i \]

\[ = \sum_{t=1}^T \left[ \frac{1}{\sigma^2} x_{it}^T \bar{x}_i^T - \frac{2T \sigma^2 x_{it} \bar{x}_i^T}{\sigma^2 + T \sigma^2} + \frac{T^2 \sigma^2 \bar{x}_i \bar{x}_i^T}{\sigma^2 + T \sigma^2} \right] \int_{y_i} (y_{it} - x_{it} \beta)^2 f^{1+\gamma}(y_i | x_i) dy_i \]

\[ \text{using (20)} \]

\[ + \sum_{t \neq t'} \left[ \frac{1}{\sigma^2} x_{it}^T \bar{x}_{i'}^T - \frac{2T \sigma^2 x_{it} \bar{x}_{i'}^T}{\sigma^2 + T \sigma^2} + \frac{T^2 \sigma^2 \bar{x}_i \bar{x}_{i'}^T}{\sigma^2 + T \sigma^2} \right] \int_{y_i} (y_{it} - x_{it} \beta)(y_{it'} - x_{it'} \beta)^2 f^{1+\gamma}(y_i | x_i) dy_i \]

\[ \text{using (21)} \]
\[
J_{\sigma_a^2}^{(i)} = \int_{y_i} u_{\sigma_a^2}(y_i|x_i)f_{\theta}^{1+\gamma}(y_i|x_i)dy_i \\
= \int_{y_i} \left[-\frac{T}{2\sigma_a^2} \left(\sigma_a^2 + (T-1)\sigma_a^2\right) + \frac{1}{2\sigma_a^2} \sum_{t=1}^{T} (y_{it} - x_{it}\beta)^2 \right. \\
\left. - \frac{\sigma_a^2}{2\sigma_a^2} \left(\sigma_a^2 + T\sigma_a^2\right) \left\{ \sum_{t=1}^{T} (y_{it} - x_{it}\beta) \right\}^2 \right] f_{\theta}^{1+\gamma}(y_i|x_i)dy_i \\
= \frac{T^2 \left[\sigma_a^2 + (T-1)\sigma_a^2\right]^2}{4\sigma_a^2(\sigma_a^2 + T\sigma_a^2)^2} \times M(1 + \gamma), \text{ using (23)} \\
+ \frac{1}{4\sigma_a^2} \times M \left[T(T+2)\sigma_a^4 + 2T(T+2)\sigma_a^2\sigma_a^2 + 3T^2\sigma_a^4\right], \text{ using (26)} \\
+ \frac{\sigma_a^4(2\sigma_a^2 + T\sigma_a^2)^2}{4\sigma_a^2(\sigma_a^2 + T\sigma_a^2)^4} \times 3MT^2 \left(\sigma_a^2 + T\sigma_a^2\right)^2, \text{ using (25)} \\
- \frac{T}{2\sigma_a^2} \left(\sigma_a^2 + (T-1)\sigma_a^2\right) \times TM(\sigma_a^2 + \sigma_a^2), \text{ using (20)} \\
+ \frac{\sigma_a^2}{2\sigma_a^2} (2\sigma_a^2 + T\sigma_a^2)^2 \times TM(1 + \gamma) \left[(T+2)\sigma_a^4 + (T^2 + 2T + 3)\sigma_a^2\sigma_a^2 \right. \\
\left. + 3(T^2 - T + 1)\sigma_a^4\right], \text{ using (27)} \\
= \frac{MT^2(\gamma - 1) \left[\sigma_a^2 + (T-1)\sigma_a^2\right]^2}{4\sigma_a^2(\sigma_a^2 + T\sigma_a^2)^2} \\
+ \frac{MT}{4\sigma_a^2} \left[(T+2)\sigma_a^4 + 2(T+2)\sigma_a^2\sigma_a^2 + 3T\sigma_a^4\right] + \frac{3MT^2\sigma_a^4(2\sigma_a^2 + T\sigma_a^2)^2}{4\sigma_a^2(1 + \gamma)(\sigma_a^2 + T\sigma_a^2)^2} \\
- \frac{TM(1 + \gamma)\sigma_a^2(2\sigma_a^2 + T\sigma_a^2)^2}{2\sigma_a^2(\sigma_a^2 + T\sigma_a^2)^2} \left[(T+2)\sigma_a^4 + (T^2 + 2T + 3)\sigma_a^2\sigma_a^2 + 3(T^2 - T + 1)\sigma_a^4\right].
\]

\[
J_{\beta, \sigma_a^2}^{(i)} = \int_{y_i} u_{\beta}(y_i|x_i)u_{\sigma_a^2}(y_i|x_i)f_{\theta}^{1+\gamma}(y_i|x_i)dy_i \\
= \int_{y_i} \left[\frac{1}{\sigma_a^2} \sum_{t=1}^{T} x_{it}(y_{it} - x_{it}\beta) - \frac{T x_i\sigma_a^2}{\sigma_a^2(\sigma_a^2 + T\sigma_a^2)} \sum_{t=1}^{T} (y_{it} - x_{it}\beta) \right. \\
\left. \times \left[-\frac{T}{2(\sigma_a^2 + T\sigma_a^2)} + \frac{1}{2(\sigma_a^2 + T\sigma_a^2)^2} \left\{ \sum_{t=1}^{T} (y_{it} - x_{it}\beta) \right\}^2 \right] f_{\theta}^{1+\gamma}(y_i|x_i)dy_i \\
= 0 \text{ as all odd moments similar to (22).}
\]
\[ J_{\beta, \sigma_2}^{(i)} = \int_{y_i} u_\beta(y_i|x_i) u_\sigma_2(y_i|x_i) f^{1+\gamma}_\theta(y_i|x_i) dy_i \]

\[ = \int_{y_i} \left[ \frac{1}{\sigma_i^2} \sum_{t=1}^T x_{it}(y_{it} - x_{it}\beta) - \frac{T x_i^2 \sigma_0^2}{\sigma_i^2 (\sigma_i^2 + T \sigma_0^2)} \sum_{t=1}^T (y_{it} - x_{it}\beta) \right] \]

\[ \times \left[ \frac{T [\sigma_i^2 + (T - 1)\sigma_0^2]}{2\sigma_i^2 (\sigma_i^2 + T \sigma_0^2)} + \frac{1}{2\sigma_i^2} \sum_{t=1}^T (y_{it} - x_{it}\beta)^2 \right] \]

\[ - \frac{\sigma_0^2 (2\sigma_i^2 + T \sigma_0^2)}{2\sigma_i^2 (\sigma_i^2 + T \sigma_0^2)^2} \left\{ \sum_{t=1}^T (y_{it} - x_{it}\beta) \right\}^2 \] \[ f^{1+\gamma}_\theta(y_i|x_i) dy_i \]

\[ = 0 \text{ as all odd moments similar to (22)}. \]

\[ J_{\sigma_2, \sigma_2}^{(i)} = \int_{y_i} u_{\sigma_2}(y_i|x_i) u_{\sigma_2}(y_i|x_i) f^{1+\gamma}_\theta(y_i|x_i) dy_i \]

\[ = \int_{y_i} \left[ -\frac{T}{2(\sigma_i^2 + T \sigma_0^2)} + \frac{1}{2(\sigma_i^2 + T \sigma_0^2)^2} \left\{ \sum_{t=1}^T (y_{it} - x_{it}\beta) \right\}^2 \right] \]

\[ \times \left[ \frac{T [\sigma_i^2 + (T - 1)\sigma_0^2]}{2\sigma_i^2 (\sigma_i^2 + T \sigma_0^2)} + \frac{1}{2\sigma_i^2} \sum_{t=1}^T (y_{it} - x_{it}\beta)^2 \right] \]

\[ - \frac{\sigma_0^2 (2\sigma_i^2 + T \sigma_0^2)}{2\sigma_i^2 (\sigma_i^2 + T \sigma_0^2)^2} \left\{ \sum_{t=1}^T (y_{it} - x_{it}\beta) \right\}^2 \] \[ f^{1+\gamma}_\theta(y_i|x_i) dy_i \]

\[ = \frac{T^2 [\sigma_i^2 + (T - 1)\sigma_0^2]}{4\sigma_i^2 (\sigma_i^2 + T \sigma_0^2)^2} \times M(1 + \gamma), \text{ using (23)} \]

\[ + \frac{1}{4\sigma_i^4 (\sigma_i^2 + T \sigma_0^2)^2} \times TM(1 + \gamma) \left( T + 2 \sigma_i^2 + (T^2 + 2T + 3)\sigma_i^2 \sigma_0^2 \right) \]

\[ + 3(T^2 - T + 1)\sigma_i^2 \right], \text{ using (27)} \]

\[ - \frac{\sigma_0^2 (2\sigma_i^2 + T \sigma_0^2)}{4\sigma_i^4 (\sigma_i^2 + T \sigma_0^2)^2} \times 3MT^2 \]

\[ (1 + \gamma) (\sigma_i^2 + T \sigma_0^2)^2, \text{ using (25)} \]

\[ - \frac{T}{4\sigma_i^4 (\sigma_i^2 + T \sigma_0^2)^2} \times TM(\sigma_i^2 + \sigma_0^2), \text{ using (20)} \]

\[ + \frac{T [\sigma_i^4 - (T - 3)\sigma_0^2 \sigma_i^2 - \sigma_0^4]}{4\sigma_i^4 (\sigma_i^2 + T \sigma_0^2)^3} \times MT(\sigma_i^2 + T \sigma_0^2), \text{ using (24)} \]

\[ = \frac{TM(1 + \gamma)}{4\sigma_i^2 (\sigma_i^2 + T \sigma_0^2)^2} \left[ 2(T + 1)\sigma_i^4 + (2T^2 + T + 3)\sigma_i^2 \sigma_0^2 + 3(T^2 - T + 1)\sigma_0^4 \right] \]

\[ - \frac{3MT^2 \sigma_0^2 (2\sigma_i^2 + T \sigma_0^2)}{4\sigma_i^4 (1 + \gamma)(\sigma_i^2 + T \sigma_0^2)^2} - \frac{MT^2 [T - (1)\sigma_0^2 + \sigma_i^2]}{2\sigma_i^2 (\sigma_i^2 + T \sigma_0^2)^2}. \]
H Integrals for $J^{(i)}$

\[
\int_{y_i} \left( y_{it} - x_{it} \right)(y_{it} - x_{it}) f_{\theta}^{1+\gamma}(y_i | \gamma) dy_i = \int_{z_i} z_{it} z_{it'} f_{\theta}^{1+\gamma}(z_i | 0) dz_i, \text{ where } f_{\theta}(z_i | 0) \text{ is } N_T(0, \Omega)
\]

\[
= (2\pi)^{-\frac{3}{2}} |\Omega|^{-\frac{1}{2}} \int_{z_i} z_{it} z_{it'} (2\pi)^{-\frac{1}{2}} |\Omega|^{-\frac{1}{2}} \exp \left\{ -\frac{1 + \gamma}{2} z_i^2 \Omega^{-1} z_i \right\} dz_i
\]

\[
= (2\pi)^{-\frac{3}{2}} |\Omega|^{-\frac{1}{2}} (1 + \gamma)^{-\frac{1}{2}} \int_{z_i} z_{it} z_{it'} (2\pi)^{-\frac{1}{2}} \left\{ \Omega^{-\frac{1}{2}} \right\} \exp \left\{ -\frac{1}{2} z_i^2 \left( \frac{\Omega}{1 + \gamma} \right)^{-1} z_i \right\} dz_i
\]

\[
= (2\pi)^{-\frac{3}{2}} |\Omega|^{-\frac{1}{2}} (1 + \gamma)^{-\frac{1}{2}} \Omega_{it'} = M \Omega_{it'},
\]

where

\[
M = (2\pi)^{-\frac{3}{2}} |\Omega|^{-\frac{1}{2}} (1 + \gamma)^{-\frac{1}{2}} \Omega_{it'}
\]

\[
= (2\pi)^{-\frac{3}{2}} (1 + \gamma)^{-\frac{1}{2}} \left\{ \sigma_{\theta}^2 \left( \frac{\Omega}{1 + \gamma} \right)^{-1} \right\}^{-\frac{1}{2}}, \text{ using (5) of the main paper}
\]

\[
= (2\pi)^{-\frac{3}{2}} (1 + \gamma)^{-\frac{1}{2}} \sigma_{\theta}^{-\frac{1}{2}} \left( \frac{\Omega}{1 + \gamma} \right)^{-1} \left( \frac{\Omega}{1 + \gamma} \right)^{-\frac{1}{2}}.
\]

For $t = t'$, combining (3) of the main paper and (18), we get

\[
\int_{y_i} \left( y_{it} - x_{it} \right)(y_{it} - x_{it}) f_{\theta}^{1+\gamma}(y_i | \gamma) dy_i = M \sigma_{\theta}^2 + \sigma_{\alpha}^2.
\]

For $t \neq t'$, combining (3) of the main paper and (18), we get

\[
\int_{y_i} \left( y_{it} - x_{it} \right)(y_{it} - x_{it}) f_{\theta}^{1+\gamma}(y_i | \gamma) dy_i = M \sigma_{\alpha}^2.
\]

For two integer $r$ and $s$, where $(r + s)$ is an odd number, we have

\[
\int_{y_i} \left( y_{it} - x_{it} \right)(y_{it} - x_{it}) f_{\theta}^{1+\gamma}(y_i | \gamma) dy_i = 0,
\]

using (31).

Now

\[
\int_{y_i} f_{\theta}^{1+\gamma}(y_i | \gamma) dy_i = \int_{z_i} f_{\theta}^{1+\gamma}(z_i | 0) dz_i, \text{ where } f_{\theta}(z_i | 0) \text{ is } N_T(0, \Omega)
\]

\[
= (2\pi)^{-\frac{3}{2}} |\Omega|^{-\frac{1}{2}} \int_{z_i} (2\pi)^{-\frac{1}{2}} |\Omega|^{-\frac{1}{2}} \exp \left\{ -\frac{1 + \gamma}{2} z_i^2 \Omega^{-1} z_i \right\} dz_i
\]

\[
= (2\pi)^{-\frac{3}{2}} |\Omega|^{-\frac{1}{2}} (1 + \gamma)^{-\frac{1}{2}} \int_{z_i} (2\pi)^{-\frac{1}{2}} \left\{ \Omega^{-\frac{1}{2}} \right\} \exp \left\{ -\frac{1}{2} z_i^2 \left( \frac{\Omega}{1 + \gamma} \right)^{-1} z_i \right\} dz_i
\]

\[
= (2\pi)^{-\frac{3}{2}} |\Omega|^{-\frac{1}{2}} (1 + \gamma)^{-\frac{1}{2}} \Omega = M (1 + \gamma), \text{ using (19)}.
\]
So,

\[
\int_{y_i} \left\{ \sum_{t=1}^{T} (y_{it} - x_{it}\beta) \right\}^2 f_\theta^{1+\gamma}(y_{i}|x_i) dy_i \\
= \int_{z_i} \left\{ \sum_{t=1}^{T} z_{it} \right\}^2 f_\theta^{1+\gamma}(z_{i}|0) dz_i, \text{ where } f_\theta(z_{i}|0) \text{ is } N_T(0, \Omega)
\]

\[
= (2\pi)^{-\frac{T}{2}} |\Omega|^{-\frac{T}{2}} \int_{z_i} \left\{ \sum_{t=1}^{T} z_{it} \right\}^2 (2\pi)^{-\frac{T}{2}} |\Omega|^{-\frac{T}{2}} \exp \left\{ -\frac{1}{2} z_{i}'\Omega^{-1} z_i \right\} dz_i
\]

\[
= (2\pi)^{-\frac{T}{2}} |\Omega|^{-\frac{T}{2}} (1+\gamma)^{-\frac{T}{2}} \int_{z_i} \left\{ \sum_{t=1}^{T} z_{it} \right\}^2 (2\pi)^{-\frac{T}{2}} |\Omega|^{-\frac{T}{2}} \exp \left\{ -\frac{1}{2} z_{i}'\left(\Omega + \frac{1}{1+\gamma}\right)^{-1} z_i \right\} dz_i
\]

\[
= M(1+\gamma)E \left( \sum_{t=1}^{T} s_{it} \right)^2 \text{, using (19)}
\]

\[
= MT(\sigma_z^2 + T\sigma_\alpha^2), \text{ using (28)}.
\]

Similarly,

\[
\int_{y_i} \left\{ \sum_{t=1}^{T} (y_{it} - x_{it}\beta) \right\}^4 f_\theta^{1+\gamma}(y_{i}|x_i) dy_i = M(1+\gamma)E \left( \sum_{t=1}^{T} s_{it} \right)^4
\]

\[
= 3MT^2(1+\gamma) (\sigma_z^2 + T\sigma_\alpha^2)^2, \text{ using (29)},
\]

\[
\int_{y_i} \left\{ \sum_{t=1}^{T} (y_{it} - x_{it}\beta)^2 \right\}^2 f_\theta^{1+\gamma}(y_{i}|x_i) dy_i
\]

\[
= \int_{y_i} \left\{ \sum_{t=1}^{T} z_{it}^2 \right\}^2 f_\theta^{1+\gamma}(z_{i}|0) dz_i, \text{ where } f_\theta(z_{i}|0) \text{ is } N_T(0, \Omega)
\]

\[
= ME \left( \sum_{t=1}^{T} s_{it}^2 \right)^2
\]

\[
= M(T(T+2)\sigma_z^4 + 2T(T+2)\sigma_z^2\sigma_\alpha^2 + 3T^2\sigma_\alpha^4),
\]

and

\[
\int_{y_i} (y_{it} - x_{it}\beta)^2 \left\{ \sum_{t'=1}^{T} (y_{it'} - x_{it'}\beta) \right\}^2 f_\theta^{1+\gamma}(y_{i}|x_i) dy_i \]

\[
= M(1+\gamma)E \left( s_{it}^2 \left[ \sum_{t=1}^{T} s_{it} \right]^2 \right)
\]

\[
= M(1+\gamma) \left\{ (T+2)\sigma_z^4 + (T^2 + 2T + 3)\sigma_z^2\sigma_\alpha^2 + 3(T^2 - T + 1)\sigma_\alpha^4 \right\}, \text{ using (30)}.
\]
I Expectations for Integrals

Suppose $s_i \sim N_T(0, \frac{\sigma^2}{1 + \gamma})$, then

$$V \left( \sum_{t=1}^{T} s_{it} \right) = \sum_{t=1}^{T} V(s_{it}) + \sum_{t \neq t'} \text{cov}(s_{it}s_{it'})$$

$$= \frac{1}{1 + \gamma} \sum_{t=1}^{T} \Omega_{tt} + \frac{1}{1 + \gamma} \sum_{t \neq t'} \Omega_{tt'}$$

$$= \frac{1}{1 + \gamma} \sum_{t=1}^{T} (\sigma^2 + \sigma^2_\alpha) + \frac{1}{1 + \gamma} \sum_{t \neq t'} \sigma^2_\alpha$$

$$= \frac{T}{1 + \gamma} (\sigma^2 + T\sigma^2_\alpha).$$

So, $\sum_{t=1}^{T} s_{it} \sim N(0, \frac{T}{1 + \gamma}(\sigma^2 + T\sigma^2_\alpha))$. Therefore

$$E \left( \left[ \sum_{t=1}^{T} s_{it} \right]^2 \right) = \frac{T}{1 + \gamma} (\sigma^2 + T\sigma^2_\alpha), \quad (28)$$

$$E \left( \left[ \sum_{t=1}^{T} s_{it} \right]^4 \right) = \frac{3T^2}{(1 + \gamma)^2} (\sigma^2 + T\sigma^2_\alpha)^2, \quad (29)$$

and

$$E \left( \left[ \sum_{t=1}^{T} s_{it}^2 \right]^2 \right) = E \left( \sum_{t=1}^{T} s_{it}^4 + \sum_{t \neq t'} s_{it}^2 s_{it'}^2 \right)$$

$$= \sum_{t=1}^{T} 3\Omega^2_{tt} + \sum_{t \neq t'} (\Omega_{tt} \Omega_{tt'} + 2\Omega^2_{tt'})$$

$$= 3T\Omega^2_{tt} + T(T-1)(\Omega^2_{tt} + 2\Omega^2_{tt'})$$

$$= T(T + 2)(\sigma^2 + \sigma^2_\alpha)^2 + 2T(T-1)\Omega^2_{tt'}$$

$$= T(T + 2)\sigma^4 + 2T(T + 2)\sigma^2_\alpha^2 + 3T^2\sigma^4_\alpha.$$
For \( t' = 1, 2, \ldots, T \), we have

\[
E \left( s_{it'}^2 \sum_{t=1}^{T} s_{it} \right)^2 = E \left( s_{it'}^4 + s_{it'}^2 \sum_{t \neq t'}^{T} s_{it}^2 + s_{it'}^2 \sum_{t \neq t', t''} s_{it} s_{it''} \right)
\]

\[
= 3\Omega_{t'}^2 + \left( \Omega_{t'}^2 \sum_{t \neq t'} \Omega_{tt'} + 2 \sum_{t \neq t'} \Omega_{tt'}^2 \right) + 3\Omega_{tt'} \sum_{t \neq t'} \Omega_{tt'}
\]

\[
+ \left( \Omega_{tt'} \sum_{t \neq t', t''} \Omega_{tt'} + 2 \sum_{t \neq t', t''} \Omega_{tt'} \Omega_{tt''} \right)
\]

\( = 3\Omega_{t'}^2 + (T - 1)\Omega_{t'}^2 + 2(T - 1)\Omega_{tt'}^2 + 3(T - 1)\Omega_{tt'} \Omega_{tt'}
\]

\[
+ (T - 1)(T - 2)(\Omega_{tt'} \Omega_{tt'} + 2\Omega_{tt'}^2) \text{ for } t \neq t'
\]

\[
= (T + 2)(\sigma^2_t + \sigma^2_\alpha)^2 + 2(T - 1)\sigma^2_\alpha + 3(T - 1)\sigma^2_\alpha (\sigma^2_t + \sigma^2_\alpha)
\]

\[
+ (T - 1)(T - 2)(\sigma^2_\alpha \sigma^2_\alpha + \sigma^4_\alpha + 2\sigma^4_\alpha), \text{ using (3) of the main paper}
\]

\[
= (T + 2)\sigma^4_t + (T^2 + 2T + 3)\sigma^2_t \sigma^2_\alpha + 3(T^2 - T + 1)\sigma^4_\alpha.
\]

Similarly,

\[
E \left( s_{it'} \sum_{t=1}^{T} s_{it} \right)^2 = E \left( s_{it'}^3 + s_{it'}^2 \sum_{t \neq t'} s_{it} + s_{it'} \sum_{t \neq t', t''} s_{it} s_{it''} \right) = 0.
\]

J Oman Weather Stations

Oman weather dataset consists observations from 55 stations across Oman over the period January 2018 to December 2018. The list of weather stations is given in Table 4.

<table>
<thead>
<tr>
<th>Station</th>
<th>Station</th>
<th>Station</th>
<th>Station</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adam</td>
<td>Diba</td>
<td>Madha</td>
<td>Nizwa</td>
</tr>
<tr>
<td>Al Amrat</td>
<td>Fahud Airport</td>
<td>Mahdah</td>
<td>Qairoon</td>
</tr>
<tr>
<td>Al Jazir</td>
<td>Haima</td>
<td>Majis</td>
<td>Qalhat</td>
</tr>
<tr>
<td>Al Khaboura</td>
<td>Ibra</td>
<td>Marmul Airport</td>
<td>Qarn alam</td>
</tr>
<tr>
<td>Al Mudhaibi</td>
<td>Ibi</td>
<td>Masirah</td>
<td>Qurayyat</td>
</tr>
<tr>
<td>Al Qahil</td>
<td>Ibi New</td>
<td>Mina Salalah</td>
<td>RasAlHaad</td>
</tr>
<tr>
<td>Al-Buraymi</td>
<td>Izki</td>
<td>Mina Sultan</td>
<td>Sadah</td>
</tr>
<tr>
<td>Bidiyah</td>
<td>Joba</td>
<td>Mirbat</td>
<td>Saham</td>
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<tr>
<td>Buhla</td>
<td>Khasab Airport</td>
<td>Muqshin</td>
<td>Saq Airport</td>
</tr>
<tr>
<td>Bukha</td>
<td>Khasab Port</td>
<td>Muscat City</td>
<td>Salalah Airport</td>
</tr>
<tr>
<td>Dhank (Qumaira)</td>
<td>Liwa</td>
<td>Muscat International Airport</td>
<td>Samail</td>
</tr>
</tbody>
</table>