Supplementary material for "Robust Density Power Divergence Estimates for Panel Data Models"

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E Integral of the first term of the DPD measure

$$
\begin{aligned}
&\int_y f_\theta^{1+\gamma}(y|x_i)dy \\
&= (2\pi)^{-\frac{T(1+\gamma)}{2}}|\Omega|^{-\frac{1+\gamma}{2}} \int_y \exp\left\{-\frac{1+\gamma}{2}(y-x_i\beta)^T\Omega^{-1}(y-x_i\beta)dy\right\} \\
&= (2\pi)^{-\frac{T\gamma}{2}}|\Omega|^{-\frac{\gamma}{2}}(1+\gamma)^{-\frac{1}{2}}\left[(2\pi)^{-\frac{T}{2}}\left|\frac{\Omega}{1+\gamma}\right|^{-\frac{1}{2}}\int_y \exp\left\{-\frac{1}{2}(y-x_i\beta)^T\left(\frac{\Omega}{1+\gamma}\right)^{-1}(y-x_i\beta)dy\right\}\right] \\
&= (2\pi)^{-\frac{T\gamma}{2}}|\Omega|^{-\frac{\gamma}{2}}(1+\gamma)^{-\frac{1}{2}}.
\end{aligned} \tag{17}
$$

F Vector $\xi^{(i)}$ at Model

From Equations (6) and (15) of the main paper, we get

$$
\begin{aligned}
\xi_\beta^{(i)} &= \int_{y_i} u_\beta(y_i|x_i) f_\theta^{1+\gamma}(y_i|x_i)dy_i \\
&= \int_{y_i}\left[\frac{1}{\sigma_\epsilon^2}\sum_{t=1}^T x_{it}(y_{it}-x_{it}\beta) - \frac{T\bar{x}_i\sigma_\alpha^2}{\sigma_\epsilon^2(\sigma_\epsilon^2+T\sigma_\alpha^2)}\sum_{t=1}^T(y_{it}-x_{it}\beta)\right] f_\theta^{1+\gamma}(y_i|x_i)dy_i \\
&= 0, \quad \text{from (22)}.
\end{aligned}
$$

$$
\begin{aligned}
\xi_{\sigma_\alpha^2}^{(i)} &= \int_{y_i} u_{\sigma_\alpha^2}(y|x_i) f_\theta^{1+\gamma}(y_i|x_i)dy_i \\
&= \int_{y_i}\left[-\frac{T}{2(\sigma_\epsilon^2+T\sigma_\alpha^2)} + \frac{1}{2(\sigma_\epsilon^2+T\sigma_\alpha^2)^2}\left\{\sum_{t=1}^T(y_{it}-x_{it}\beta)\right\}^2\right] f_\theta^{1+\gamma}(y_i|x_i)dy_i \\
&= -\frac{T}{2(\sigma_\epsilon^2+T\sigma_\alpha^2)} \times M(1+\gamma), \text{ using (23)} \\
&\qquad + \frac{1}{2(\sigma_\epsilon^2+T\sigma_\alpha^2)^2} \times MT(\sigma_\epsilon^2+T\sigma_\alpha^2), \text{ using (24)} \\
&= -\frac{MT\gamma}{2(\sigma_\epsilon^2+T\sigma_\alpha^2)}.
\end{aligned}
$$

$$
\begin{aligned}
\xi^{(i)}_{\sigma^2_\epsilon} &= \int_{y_i} u_{\sigma^2_\epsilon}(y|x_i) f_\theta^{1+\gamma}(y_i|x_i) dy_i \\
&= \int_{y_i} \Bigg[-\frac{T\left[\sigma^2_\epsilon + (T-1)\sigma^2_\alpha\right]}{2\sigma^2_\epsilon(\sigma^2_\epsilon + T\sigma^2_\alpha)} + \frac{1}{2\sigma^4_\epsilon} \sum_{t=1}^{T} (y_{it} - x_{it}\beta)^2 \\
&\qquad\qquad - \frac{\sigma^2_\alpha(2\sigma^2_\epsilon + T\sigma^2_\alpha)}{2\sigma^4_\epsilon(\sigma^2_\epsilon + T\sigma^2_\alpha)^2} \left\{ \sum_{t=1}^{T} (y_{it} - x_{it}\beta) \right\}^2 \Bigg] f_\theta^{1+\gamma}(y_i|x_i) dy_i \\
&= -\frac{T\left[\sigma^2_\epsilon + (T-1)\sigma^2_\alpha\right]}{2\sigma^2_\epsilon(\sigma^2_\epsilon + T\sigma^2_\alpha)} \times M(1+\gamma), \text{ using (23)} \\
&\qquad\qquad + \frac{1}{2\sigma^4_\epsilon} \times TM(\sigma^2_\epsilon + \sigma^2_\alpha), \text{ using (20)} \\
&\qquad\qquad - \frac{\sigma^2_\alpha(2\sigma^2_\epsilon + T\sigma^2_\alpha)}{2\sigma^4_\epsilon(\sigma^2_\epsilon + T\sigma^2_\alpha)^2} \times MT(\sigma^2_\epsilon + T\sigma^2_\alpha), \text{ using (24)} \\
&= -\frac{MT\gamma\left[\sigma^2_\epsilon + (T-1)\sigma^2_\alpha\right]}{2\sigma^2_\epsilon(\sigma^2_\epsilon + T\sigma^2_\alpha)}.
\end{aligned}
$$

G Matrix $J^{(i)}$ at Model

From Equations (6) and (15) of the main paper, we get

$$
\begin{aligned}
J^{(i)}_{\sigma^2_\alpha} &= \int_{y_i} u^2_{\sigma^2_\alpha}(y|x_i) f_\theta^{1+\gamma}(y_i|x_i) dy_i \\
&= \int_{y_i} \left[-\frac{T}{2(\sigma^2_\epsilon + T\sigma^2_\alpha)} + \frac{1}{2(\sigma^2_\epsilon + T\sigma^2_\alpha)^2} \left\{ \sum_{t=1}^{T} (y_{it} - x_{it}\beta) \right\}^2 \right]^2 f_\theta^{1+\gamma}(y_i|x_i) dy_i \\
&= \frac{T^2}{4(\sigma^2_\epsilon + T\sigma^2_\alpha)^2} \int_{y_i} f_\theta^{1+\gamma}(y_i|x_i) dy_i - \frac{T}{2(\sigma^2_\epsilon + T\sigma^2_\alpha)^3} \int_{y_i} \left\{ \sum_{t=1}^{T} (y_{it} - x_{it}\beta) \right\}^2 f_\theta^{1+\gamma}(y_i|x_i) dy_i \\
&\qquad + \frac{1}{4(\sigma^2_\epsilon + T\sigma^2_\alpha)^4} \int_{y_i} \left\{ \sum_{t=1}^{T} (y_{it} - x_{it}\beta) \right\}^4 f_\theta^{1+\gamma}(y_i|x_i) dy_i \\
&= M(1+\gamma) \frac{T^2}{4(\sigma^2_\epsilon + T\sigma^2_\alpha)^2}, \text{ using (23)} \\
&\quad - \frac{T}{2(\sigma^2_\epsilon + T\sigma^2_\alpha)^3} \times MT(\sigma^2_\epsilon + T\sigma^2_\alpha), \text{ using (24)} \\
&\quad + \frac{1}{4(\sigma^2_\epsilon + T\sigma^2_\alpha)^4} \times \frac{3MT^2}{(1+\gamma)} (\sigma^2_\epsilon + T\sigma^2_\alpha)^2, \text{ using (25)} \\
&= \frac{MT^2(\gamma^2 + 2)}{4(1+\gamma)(\sigma^2_\epsilon + T\sigma^2_\alpha)^2}.
\end{aligned}
$$

$$
\begin{aligned}
J_\beta^{(i)} &= \int_{y_i} u_\beta(y_i|x_i) u_\beta^T(y_i|x_i) f_\theta^{1+\gamma}(y_i|x_i) dy_i \\
&= \int_{y_i} \left[\frac{1}{\sigma_\epsilon^2} \sum_{t=1}^T x_{it}(y_{it} - x_{it}\beta) - \frac{T\bar{x}_i\sigma_\alpha^2}{\sigma_\epsilon^2(\sigma_\epsilon^2 + T\sigma_\alpha^2)} \sum_{t=1}^T (y_{it} - x_{it}\beta) \right] \\
&\qquad \left[\frac{1}{\sigma_\epsilon^2} \sum_{t=1}^T x_{it}(y_{it} - x_{it}\beta) - \frac{T\bar{x}_i\sigma_\alpha^2}{\sigma_\epsilon^2(\sigma_\epsilon^2 + T\sigma_\alpha^2)} \sum_{t=1}^T (y_{it} - x_{it}\beta) \right]^T f_\theta^{1+\gamma}(y_i|x_i) dy_i \\
&= \sum_{t=1}^T \left[\frac{1}{\sigma_\epsilon^4} x_{it}x_{it}^T - \frac{2T\sigma_\alpha^2 x_{it}\bar{x}_i^T}{\sigma_\epsilon^4(\sigma_\epsilon^2 + T\sigma_\alpha^2)} + \frac{T^2\sigma_\alpha^4\bar{x}_i\bar{x}_i^T}{\sigma_\epsilon^4(\sigma_\epsilon^2 + T\sigma_\alpha^2)^2} \right] \int_{y_i} (y_{it} - x_{it}\beta)^2 f_\theta^{1+\gamma}(y_i|x_i) dy_i \\
&\qquad + \sum_{t \neq t'} \left[\frac{1}{\sigma_\epsilon^4} x_{it}x_{it'}^T - \frac{2T\sigma_\alpha^2 x_{it}\bar{x}_i^T}{\sigma_\epsilon^4(\sigma_\epsilon^2 + T\sigma_\alpha^2)} + \frac{T^2\sigma_\alpha^4\bar{x}_i\bar{x}_i^T}{\sigma_\epsilon^4(\sigma_\epsilon^2 + T\sigma_\alpha^2)^2} \right] \int_{y_i} (y_{it} - x_{it}\beta)(y_{it'} - x_{it'}\beta) f_\theta^{1+\gamma}(y_i|x_i) dy_i \\
&= M(\sigma_\epsilon^2 + \sigma_\alpha^2) \sum_{t=1}^T \left[\frac{1}{\sigma_\epsilon^4} x_{it}x_{it}^T - \frac{2T\sigma_\alpha^2 x_{it}\bar{x}_i^T}{\sigma_\epsilon^4(\sigma_\epsilon^2 + T\sigma_\alpha^2)} + \frac{T^2\sigma_\alpha^4\bar{x}_i\bar{x}_i^T}{\sigma_\epsilon^4(\sigma_\epsilon^2 + T\sigma_\alpha^2)^2} \right], \text{ using (20)} \\
&\qquad + M\sigma_\alpha^2 \sum_{t \neq t'} \left[\frac{1}{\sigma_\epsilon^4} x_{it}x_{it'}^T - \frac{2T\sigma_\alpha^2 x_{it}\bar{x}_i^T}{\sigma_\epsilon^4(\sigma_\epsilon^2 + T\sigma_\alpha^2)} + \frac{T^2\sigma_\alpha^4\bar{x}_i\bar{x}_i^T}{\sigma_\epsilon^4(\sigma_\epsilon^2 + T\sigma_\alpha^2)^2} \right], \text{ using (21)} \\
&= M\sigma_\alpha^2 \sum_{t,t'=1}^T \left[\frac{1}{\sigma_\epsilon^4} x_{it}x_{it'}^T - \frac{2T\sigma_\alpha^2 x_{it}\bar{x}_i^T}{\sigma_\epsilon^4(\sigma_\epsilon^2 + T\sigma_\alpha^2)} + \frac{T^2\sigma_\alpha^4\bar{x}_i\bar{x}_i^T}{\sigma_\epsilon^4(\sigma_\epsilon^2 + T\sigma_\alpha^2)^2} \right] \\
&\qquad + M\sigma_\epsilon^2 \sum_{t=1}^T \left[\frac{1}{\sigma_\epsilon^4} x_{it}x_{it}^T - \frac{2T\sigma_\alpha^2 x_{it}\bar{x}_i^T}{\sigma_\epsilon^4(\sigma_\epsilon^2 + T\sigma_\alpha^2)} + \frac{T^2\sigma_\alpha^4\bar{x}_i\bar{x}_i^T}{\sigma_\epsilon^4(\sigma_\epsilon^2 + T\sigma_\alpha^2)^2} \right] \\
&= M\sigma_\alpha^2 \left[\frac{T^2}{\sigma_\epsilon^4} \bar{x}_i\bar{x}_i^T - \frac{2T^3\sigma_\alpha^2\bar{x}_i\bar{x}_i^T}{\sigma_\epsilon^4(\sigma_\epsilon^2 + T\sigma_\alpha^2)} + \frac{T^4\sigma_\alpha^4\bar{x}_i\bar{x}_i^T}{\sigma_\epsilon^4(\sigma_\epsilon^2 + T\sigma_\alpha^2)^2} \right], \\
&\qquad + M\sigma_\epsilon^2 \sum_{t=1}^T \frac{1}{\sigma_\epsilon^4} x_{it}x_{it}^T + M\sigma_\epsilon^2 \left[- \frac{2T^2\sigma_\alpha^2\bar{x}_i\bar{x}_i^T}{\sigma_\epsilon^4(\sigma_\epsilon^2 + T\sigma_\alpha^2)} + \frac{T^3\sigma_\alpha^4\bar{x}_i\bar{x}_i^T}{\sigma_\epsilon^4(\sigma_\epsilon^2 + T\sigma_\alpha^2)^2} \right] \\
&= M \left[\frac{\sigma_\epsilon^2}{\sigma_\epsilon^4} \sum_{t=1}^T x_{it}x_{it}^T + \frac{T^2\sigma_\alpha^2}{\sigma_\epsilon^4} \bar{x}_i\bar{x}_i^T + (\sigma_\epsilon^2 + T\sigma_\alpha^2) \left\{ - \frac{2T^2\sigma_\alpha^2\bar{x}_i\bar{x}_i^T}{\sigma_\epsilon^4(\sigma_\epsilon^2 + T\sigma_\alpha^2)} + \frac{T^3\sigma_\alpha^4\bar{x}_i\bar{x}_i^T}{\sigma_\epsilon^4(\sigma_\epsilon^2 + T\sigma_\alpha^2)^2} \right\} \right] \\
&= M\sigma_\epsilon^{-4} \left[\sigma_\epsilon^2 \sum_{t=1}^T x_{it}x_{it}^T + T^2\sigma_\alpha^2\bar{x}_i\bar{x}_i^T - 2T^2\sigma_\alpha^2\bar{x}_i\bar{x}_i^T + \frac{T^3\sigma_\alpha^4\bar{x}_i\bar{x}_i^T}{(\sigma_\epsilon^2 + T\sigma_\alpha^2)} \right] \\
&= M\sigma_\epsilon^{-4} \left[\sigma_\epsilon^2 \sum_{t=1}^T x_{it}x_{it}^T + T^2\sigma_\alpha^2 \left(\frac{T\sigma_\alpha^2}{(\sigma_\epsilon^2 + T\sigma_\alpha^2)} - 1 \right) \bar{x}_i\bar{x}_i^T \right].
\end{aligned}
$$

$$
\begin{aligned}
J_{\sigma_\epsilon^2}^{(i)} &= \int_{y_i} u_{\sigma_\epsilon^2}^2(y|x_i) f_\theta^{1+\gamma}(y_i|x_i)dy_i \\
&= \int_{y_i} \Bigg[- \frac{T\left[\sigma_\epsilon^2 + (T-1)\sigma_\alpha^2\right]}{2\sigma_\epsilon^2(\sigma_\epsilon^2 + T\sigma_\alpha^2)} + \frac{1}{2\sigma_\epsilon^4}\sum_{t=1}^{T}(y_{it} - x_{it}\beta)^2 \\
&\qquad\qquad - \frac{\sigma_\alpha^2(2\sigma_\epsilon^2 + T\sigma_\alpha^2)}{2\sigma_\epsilon^4(\sigma_\epsilon^2 + T\sigma_\alpha^2)^2}\left\{\sum_{t=1}^{T}(y_{it} - x_{it}\beta)\right\}^2\Bigg]^2 f_\theta^{1+\gamma}(y_i|x_i)dy_i \\
&= \frac{T^2\left[\sigma_\epsilon^2 + (T-1)\sigma_\alpha^2\right]^2}{4\sigma_\epsilon^4(\sigma_\epsilon^2 + T\sigma_\alpha^2)^2} \times M(1+\gamma), \text{ using (23)} \\
&\quad + \frac{1}{4\sigma_\epsilon^8} \times M\left[T(T+2)\sigma_\epsilon^4 + 2T(T+2)\sigma_\epsilon^2\sigma_\alpha^2 + 3T^2\sigma_\alpha^4\right], \text{ using (26)} \\
&\quad + \frac{\sigma_\alpha^4(2\sigma_\epsilon^2 + T\sigma_\alpha^2)^2}{4\sigma_\epsilon^8(\sigma_\epsilon^2 + T\sigma_\alpha^2)^4} \times \frac{3MT^2}{(1+\gamma)}(\sigma_\epsilon^2 + T\sigma_\alpha^2)^2, \text{ using (25)} \\
&\quad - \frac{T\left[\sigma_\epsilon^2 + (T-1)\sigma_\alpha^2\right]}{2\sigma_\epsilon^6(\sigma_\epsilon^2 + T\sigma_\alpha^2)} \times TM(\sigma_\epsilon^2 + \sigma_\alpha^2), \text{ using (20)} \\
&\quad + \frac{\sigma_\alpha^2(2\sigma_\epsilon^2 + T\sigma_\alpha^2)T\left[\sigma_\epsilon^2 + (T-1)\sigma_\alpha^2\right]}{2\sigma_\epsilon^6(\sigma_\epsilon^2 + T\sigma_\alpha^2)^3} \times MT(\sigma_\epsilon^2 + T\sigma_\alpha^2), \text{ using (24)} \\
&\quad - \frac{\sigma_\alpha^2(2\sigma_\epsilon^2 + T\sigma_\alpha^2)}{2\sigma_\epsilon^8(\sigma_\epsilon^2 + T\sigma_\alpha^2)^2} \times TM(1+\gamma)\Big[(T+2)\sigma_\epsilon^4 + (T^2+2T+3)\sigma_\epsilon^2\sigma_\alpha^2 \\
&\qquad\qquad + 3(T^2 - T + 1)\sigma_\alpha^4\Big], \text{ using (27)} \\
&= \frac{MT^2(\gamma-1)\left[\sigma_\epsilon^2 + (T-1)\sigma_\alpha^2\right]^2}{4\sigma_\epsilon^4(\sigma_\epsilon^2 + T\sigma_\alpha^2)^2} \\
&\quad + \frac{MT}{4\sigma_\epsilon^8}\left[(T+2)\sigma_\epsilon^4 + 2(T+2)\sigma_\epsilon^2\sigma_\alpha^2 + 3T\sigma_\alpha^4\right] + \frac{3MT^2\sigma_\alpha^4(2\sigma_\epsilon^2 + T\sigma_\alpha^2)^2}{4\sigma_\epsilon^8(1+\gamma)(\sigma_\epsilon^2 + T\sigma_\alpha^2)^2} \\
&\quad - \frac{TM(1+\gamma)\sigma_\alpha^2(2\sigma_\epsilon^2 + T\sigma_\alpha^2)}{2\sigma_\epsilon^8(\sigma_\epsilon^2 + T\sigma_\alpha^2)^2}\Big[(T+2)\sigma_\epsilon^4 + (T^2+2T+3)\sigma_\epsilon^2\sigma_\alpha^2 + 3(T^2 - T + 1)\sigma_\alpha^4\Big].
\end{aligned}
$$

$$
\begin{aligned}
J_{\beta,\,\sigma_\alpha^2}^{(i)} &= \int_{y_i} u_\beta(y_i|x_i) u_{\sigma_\alpha^2}(y_i|x_i) f_\theta^{1+\gamma}(y_i|x_i)dy_i \\
&= \int_{y_i} \left[\frac{1}{\sigma_\epsilon^2}\sum_{t=1}^{T} x_{it}(y_{it} - x_{it}\beta) - \frac{T\bar{x}_i\sigma_\alpha^2}{\sigma_\epsilon^2(\sigma_\epsilon^2 + T\sigma_\alpha^2)}\sum_{t=1}^{T}(y_{it} - x_{it}\beta)\right] \\
&\qquad \times \left[-\frac{T}{2(\sigma_\epsilon^2 + T\sigma_\alpha^2)} + \frac{1}{2(\sigma_\epsilon^2 + T\sigma_\alpha^2)^2}\left\{\sum_{t=1}^{T}(y_{it} - x_i\beta)\right\}^2\right] f_\theta^{1+\gamma}(y_i|x_i)dy_i \\
&= 0 \text{ as all odd moments similar to (22).}
\end{aligned}
$$

$$
\begin{aligned}
J^{(i)}_{\beta,\ \sigma^2_\epsilon} &= \int_{y_i} u_\beta(y_i|x_i) u_{\sigma^2_\epsilon}(y_i|x_i) f_\theta^{1+\gamma}(y_i|x_i) dy_i \\
&= \int_{y_i} \left[\frac{1}{\sigma^2_\epsilon} \sum_{t=1}^{T} x_{it}(y_{it} - x_{it}\beta) - \frac{T\bar{x}_i \sigma^2_\alpha}{\sigma^2_\epsilon(\sigma^2_\epsilon + T\sigma^2_\alpha)} \sum_{t=1}^{T} (y_{it} - x_{it}\beta) \right] \\
&\quad \times \left[- \frac{T\left[\sigma^2_\epsilon + (T-1)\sigma^2_\alpha\right]}{2\sigma^2_\epsilon(\sigma^2_\epsilon + T\sigma^2_\alpha)} + \frac{1}{2\sigma^4_\epsilon} \sum_{t=1}^{T} (y_{it} - x_{it}\beta)^2 \right. \\
&\qquad \left. - \frac{\sigma^2_\alpha(2\sigma^2_\epsilon + T\sigma^2_\alpha)}{2\sigma^4_\epsilon(\sigma^2_\epsilon + T\sigma^2_\alpha)^2} \left\{ \sum_{t=1}^{T} (y_{it} - x_{it}\beta) \right\}^2 \right] f_\theta^{1+\gamma}(y_i|x_i) dy_i \\
&= 0 \text{ as all odd moments similar to (22).}
\end{aligned}
$$

$$
\begin{aligned}
J^{(i)}_{\sigma^2_\alpha,\ \sigma^2_\epsilon} &= \int_{y_i} u_{\sigma^2_\alpha}(y_i|x_i) u_{\sigma^2_\epsilon}(y_i|x_i) f_\theta^{1+\gamma}(y_i|x_i) dy_i \\
&= \int_{y_i} \left[-\frac{T}{2(\sigma^2_\epsilon + T\sigma^2_\alpha)} + \frac{1}{2(\sigma^2_\epsilon + T\sigma^2_\alpha)^2} \left\{ \sum_{t=1}^{T} (y_{it} - x_{it}\beta) \right\}^2 \right] \\
&\quad \times \left[- \frac{T\left[\sigma^2_\epsilon + (T-1)\sigma^2_\alpha\right]}{2\sigma^2_\epsilon(\sigma^2_\epsilon + T\sigma^2_\alpha)} + \frac{1}{2\sigma^4_\epsilon} \sum_{t=1}^{T} (y_{it} - x_{it}\beta)^2 \right. \\
&\qquad \left. - \frac{\sigma^2_\alpha(2\sigma^2_\epsilon + T\sigma^2_\alpha)}{2\sigma^4_\epsilon(\sigma^2_\epsilon + T\sigma^2_\alpha)^2} \left\{ \sum_{t=1}^{T} (y_{it} - x_{it}\beta) \right\}^2 \right] f_\theta^{1+\gamma}(y_i|x_i) dy_i \\
&= \frac{T^2\left[\sigma^2_\epsilon + (T-1)\sigma^2_\alpha\right]}{4\sigma^2_\epsilon(\sigma^2_\epsilon + T\sigma^2_\alpha)^2} \times M(1+\gamma), \text{ using (23)} \\
&\quad + \frac{1}{4\sigma^4_\epsilon(\sigma^2_\epsilon + T\sigma^2_\alpha)^2} \times TM(1+\gamma)\Big[(T+2)\sigma^4_\epsilon + (T^2+2T+3)\sigma^2_\epsilon\sigma^2_\alpha \\
&\qquad + 3(T^2 - T + 1)\sigma^4_\alpha\Big], \text{ using (27)} \\
&\quad - \frac{\sigma^2_\alpha(2\sigma^2_\epsilon + T\sigma^2_\alpha)}{4\sigma^4_\epsilon(\sigma^2_\epsilon + T\sigma^2_\alpha)^4} \times \frac{3MT^2}{(1+\gamma)}(\sigma^2_\epsilon + T\sigma^2_\alpha)^2, \text{ using (25)} \\
&\quad - \frac{T}{4\sigma^4_\epsilon(\sigma^2_\epsilon + T\sigma^2_\alpha)} \times TM(\sigma^2_\epsilon + \sigma^2_\alpha), \text{ using (20)} \\
&\quad + \frac{T[T\sigma^4_\alpha - (T-3)\sigma^2_\alpha\sigma^2_\epsilon - \sigma^4_\epsilon]}{4\sigma^4_\epsilon(\sigma^2_\epsilon + T\sigma^2_\alpha)^3} \times MT(\sigma^2_\epsilon + T\sigma^2_\alpha), \text{ using (24)} \\
&= \frac{TM(1+\gamma)}{4\sigma^4_\epsilon(\sigma^2_\epsilon + T\sigma^2_\alpha)^2}\left[2(T+1)\sigma^4_\epsilon + (2T^2 + T + 3)\sigma^2_\epsilon\sigma^2_\alpha + 3(T^2 - T + 1)\sigma^4_\alpha\right] \\
&\quad - \frac{3MT^2\sigma^2_\alpha(2\sigma^2_\epsilon + T\sigma^2_\alpha)}{4\sigma^4_\epsilon(1+\gamma)(\sigma^2_\epsilon + T\sigma^2_\alpha)^2} - \frac{MT^2\left[(T-1)\sigma^2_\alpha + \sigma^2_\epsilon\right]}{2\sigma^2_\epsilon(\sigma^2_\epsilon + T\sigma^2_\alpha)^2}.
\end{aligned}
$$

H Integrals for $J^{(i)}$

$$\begin{aligned}
\int_{y_i} & (y_{it}-x_{it}\beta)(y_{it'}-x_{it'}\beta)f_\theta^{1+\gamma}(y_i|x_i)dy_i \\
&= \int_{z_i} z_{it}z_{it'} f_\theta^{1+\gamma}(z_i|0)dz_i, \text{ where } f_\theta(z_i|0) \text{ is } N_T(0,\Omega) \\
&= (2\pi)^{-\frac{T\gamma}{2}}|\Omega|^{-\frac{\gamma}{2}} \int_{z_i} z_{it}z_{it'}(2\pi)^{-\frac{T}{2}}|\Omega|^{-\frac{1}{2}} \exp\left\{-\frac{1+\gamma}{2}z_i'\Omega^{-1}z_i\right\}dz_i \\
&= (2\pi)^{-\frac{T\gamma}{2}}|\Omega|^{-\frac{\gamma}{2}}(1+\gamma)^{-\frac{T}{2}} \int_{z_i} z_{it}z_{it'}(2\pi)^{-\frac{T}{2}}\left|\frac{\Omega}{1+\gamma}\right|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}z_i'\left(\frac{\Omega}{1+\gamma}\right)^{-1}z_i\right\}dz_i \\
&= (2\pi)^{-\frac{T\gamma}{2}}|\Omega|^{-\frac{\gamma}{2}}(1+\gamma)^{-\frac{T+2}{2}}\Omega_{tt'} \\
&= M\Omega_{tt'},
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
M &= (2\pi)^{-\frac{T\gamma}{2}}|\Omega|^{-\frac{\gamma}{2}}(1+\gamma)^{-\frac{T+2}{2}} \\
&= (2\pi)^{-\frac{T\gamma}{2}}(1+\gamma)^{-\frac{T+2}{2}}\left\{\sigma_\epsilon^{2(T-1)}(\sigma_\epsilon^2+T\sigma_\alpha^2)\right\}^{-\frac{\gamma}{2}}, \text{ using (5) of the main paper} \\
&= (2\pi)^{-\frac{T\gamma}{2}}(1+\gamma)^{-\frac{T+2}{2}}\sigma_\epsilon^{-\gamma(T-1)}(\sigma_\epsilon^2+T\sigma_\alpha^2)^{-\frac{\gamma}{2}}.
\end{aligned} \tag{19}$$

For $t = t'$, combining (3) of the main paper and (18), we get

$$\int_{y_i}(y_{it}-x_{it}\beta)^2 f_\theta^{1+\gamma}(y_i|x_i)dy_i = M(\sigma_\epsilon^2+\sigma_\alpha^2). \tag{20}$$

For $t \neq t'$, combining (3) of the main paper and (18), we get

$$\int_{y_i}(y_{it}-x_{it}\beta)(y_{it'}-x_{it'}\beta) f_\theta^{1+\gamma}(y_i|x_i)dy_i = M\sigma_\alpha^2. \tag{21}$$

For two integer r and s, where $(r+s)$ is an odd number, we have

$$\begin{aligned}
\int_{y_i} & (y_{it}-x_{it}\beta)^r(y_{it}-x_{it}\beta)^s f_\theta^{1+\gamma}(y_i|x_i)dy_i \\
&= \int_{z_i} z_{it}^r z_{it}^s f_\theta^{1+\gamma}(z_i|0)dz_i, \text{ where } f_\theta(z_i|0) \text{ is } N_T(0,\Omega) \\
&= 0, \text{ using (31).}
\end{aligned} \tag{22}$$

Now

$$\begin{aligned}
\int_{y_i} f_\theta^{1+\gamma}(y_i|x_i)dy_i &= \int_{z_i} f_\theta^{1+\gamma}(z_i|0)dz_i, \text{ where } f_\theta(z_i|0) \text{ is } N_T(0,\Omega) \\
&= (2\pi)^{-\frac{T\gamma}{2}}|\Omega|^{-\frac{\gamma}{2}}\int_{z_i}(2\pi)^{-\frac{T}{2}}|\Omega|^{-\frac{1}{2}}\exp\left\{-\frac{1+\gamma}{2}z_i'\Omega^{-1}z_i\right\}dz_i \\
&= (2\pi)^{-\frac{T\gamma}{2}}|\Omega|^{-\frac{\gamma}{2}}(1+\gamma)^{-\frac{T}{2}}\int_{z_i}(2\pi)^{-\frac{T}{2}}\left|\frac{\Omega}{1+\gamma}\right|^{-\frac{1}{2}}\exp\left\{-\frac{1}{2}z_i'\left(\frac{\Omega}{1+\gamma}\right)^{-1}z_i\right\}dz_i \\
&= (2\pi)^{-\frac{T\gamma}{2}}|\Omega|^{-\frac{\gamma}{2}}(1+\gamma)^{-\frac{T}{2}} \\
&= M(1+\gamma), \text{ using (19).}
\end{aligned} \tag{23}$$

So,

$$
\begin{aligned}
&\int_{y_i} \left\{ \sum_{t=1}^{T} (y_{it} - x_{it}\beta) \right\}^2 f_\theta^{1+\gamma}(y_i|x_i) dy_i \\
&\quad = \int_{z_i} \left\{ \sum_{t=1}^{T} z_{it} \right\}^2 f_\theta^{1+\gamma}(z_i|0) dz_i, \text{ where } f_\theta(z_i|0) \text{ is } N_T(0, \Omega) \\
&\quad = (2\pi)^{-\frac{T\gamma}{2}} |\Omega|^{-\frac{\gamma}{2}} \int_{z_i} \left\{ \sum_{t=1}^{T} z_{it} \right\}^2 (2\pi)^{-\frac{T}{2}} |\Omega|^{-\frac{1}{2}} \exp\left\{ -\frac{1+\gamma}{2} z_i' \Omega^{-1} z_i \right\} dz_i \\
&\quad = (2\pi)^{-\frac{T\gamma}{2}} |\Omega|^{-\frac{\gamma}{2}} (1+\gamma)^{-\frac{T}{2}} \int_{z_i} \left\{ \sum_{t=1}^{T} z_{it} \right\}^2 (2\pi)^{-\frac{T}{2}} \left| \frac{\Omega}{1+\gamma} \right|^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} z_i' \left(\frac{\Omega}{1+\gamma} \right)^{-1} z_i \right\} dz_i \\
&\quad = M(1+\gamma) E\left(\left[\sum_{t=1}^{T} s_{it} \right]^2 \right), \text{ using (19)} \\
&\quad = MT(\sigma_\epsilon^2 + T\sigma_\alpha^2), \text{ using (28)}.
\end{aligned} \tag{24}
$$

Similarly,

$$
\begin{aligned}
\int_{y_i} \left\{ \sum_{t=1}^{T} (y_{it} - x_{it}\beta) \right\}^4 f_\theta^{1+\gamma}(y_i|x_i) dy_i &= M(1+\gamma) E\left(\left[\sum_{t=1}^{T} s_{it} \right]^4 \right) \\
&= \frac{3MT^2}{(1+\gamma)} (\sigma_\epsilon^2 + T\sigma_\alpha^2)^2, \text{ using (29)},
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
&\int_{y_i} \left\{ \sum_{t=1}^{T} (y_{it} - x_{it}\beta)^2 \right\}^2 f_\theta^{1+\gamma}(y_i|x_i) dy_i \\
&\quad = \int_{y_i} \left\{ \sum_{t=1}^{T} z_{it}^2 \right\}^2 f_\theta^{1+\gamma}(z_i|0) dz_i, \text{ where } f_\theta(z_i|0) \text{ is } N_T(0, \Omega) \\
&\quad = ME\left(\left[\sum_{t=1}^{T} s_{it}^2 \right]^2 \right) \\
&\quad = M(T(T+2)\sigma_\epsilon^4 + 2T(T+2)\sigma_\epsilon^2\sigma_\alpha^2 + 3T^2\sigma_\alpha^4),
\end{aligned} \tag{26}
$$

and

$$
\begin{aligned}
&\int_{y_i} (y_{it} - x_{it}\beta)^2 \left\{ \sum_{t'=1}^{T} (y_{it'} - x_{it'}\beta) \right\}^2 f_\theta^{1+\gamma}(y_i|x_i) dy_i \\
&\quad = M(1+\gamma) E\left(s_{it'}^2 \left[\sum_{t=1}^{T} s_{it} \right]^2 \right) \\
&\quad = M(1+\gamma) \Big\{ (T+2)\sigma_\epsilon^4 + (T^2 + 2T + 3)\sigma_\epsilon^2\sigma_\alpha^2 \\
&\qquad\quad + 3(T^2 - T + 1)\sigma_\alpha^4 \Big\}, \text{ using (30)}.
\end{aligned} \tag{27}
$$

I Expectations for Integrals

Suppose $s_i \sim N_T\left(0, \frac{\Omega}{1+\gamma}\right)$, then

$$\begin{aligned} V\left(\sum_{t=1}^{T} s_{it}\right) &= \sum_{t=1}^{T} V(s_{it}) + \sum_{t\neq t'} cov(s_{it}s_{it'}) \\ &= \frac{1}{1+\gamma}\sum_{t=1}^{T} \Omega_{tt} + \frac{1}{1+\gamma}\sum_{t\neq t'} \Omega_{tt'} \\ &= \frac{1}{1+\gamma}\sum_{t=1}^{T}(\sigma_\epsilon^2 + \sigma_\alpha^2) + \frac{1}{1+\gamma}\sum_{t\neq t'}\sigma_\alpha^2 \\ &= \frac{T}{1+\gamma}(\sigma_\epsilon^2 + T\sigma_\alpha^2). \end{aligned}$$

So, $\sum_{t=1}^{T} s_{it} \sim N(0, \frac{T}{1+\gamma}(\sigma_\epsilon^2 + T\sigma_\alpha^2))$. Therefore

$$E\left(\left[\sum_{t=1}^{T} s_{it}\right]^2\right) = \frac{T}{1+\gamma}(\sigma_\epsilon^2 + T\sigma_\alpha^2), \tag{28}$$

$$E\left(\left[\sum_{t=1}^{T} s_{it}\right]^4\right) = \frac{3T^2}{(1+\gamma)^2}(\sigma_\epsilon^2 + T\sigma_\alpha^2)^2, \tag{29}$$

and

$$\begin{aligned} E\left(\left[\sum_{t=1}^{T} s_{it}^2\right]^2\right) &= E\left(\sum_{t=1}^{T} s_{it}^4 + \sum_{t\neq t'} s_{it}^2 s_{it'}^2\right) \\ &= \sum_{t=1}^{T} 3\Omega_{tt}^2 + \sum_{t\neq t'}(\Omega_{tt}\Omega_{t't'} + 2\Omega_{tt'}^2) \\ &= 3T\Omega_{tt}^2 + T(T-1)(\Omega_{tt}^2 + 2\Omega_{tt'}^2) \\ &= T(T+2)(\sigma_\epsilon^2 + \sigma_\alpha^2)^2 + 2T(T-1)\Omega_{tt'}^2 \\ &= T(T+2)\sigma_\epsilon^4 + 2T(T+2)\sigma_\epsilon^2\sigma_\alpha^2 + 3T^2\sigma_\alpha^4. \end{aligned}$$

For $t' = 1, 2, \cdots, T$, we have

$$
\begin{aligned}
E\left(s_{it'}^2 \left[\sum_{t=1}^{T} s_{it}\right]^2\right) &= E\left(s_{it'}^4 + s_{it'}^2 \sum_{t \neq t'} s_{it}^2 + s_{it'}^3 \sum_{t \neq t'} s_{it} + s_{it'}^2 \sum_{t \neq t' \neq t''} s_{it} s_{it''}\right) \\
&= 3\Omega_{t't'}^2 + \left(\Omega_{t't'} \sum_{t \neq t'} \Omega_{tt} + 2 \sum_{t \neq t'} \Omega_{tt'}^2\right) + 3\Omega_{t't'} \sum_{t \neq t'} \Omega_{tt'} \\
&\quad + \left(\Omega_{t't'} \sum_{t \neq t' \neq t''} \Omega_{tt''} + 2 \sum_{t \neq t' \neq t''} \Omega_{tt'} \Omega_{t't''}\right) \qquad (30) \\
&= 3\Omega_{t't'}^2 + (T-1)\Omega_{t't'}^2 + 2(T-1)\Omega_{tt'}^2 + 3(T-1)\Omega_{t't'}\Omega_{tt'} \\
&\qquad + (T-1)(T-2)(\Omega_{t't'}\Omega_{tt'} + 2\Omega_{tt'}^2) \text{ for } t \neq t' \\
&= (T+2)(\sigma_\epsilon^2 + \sigma_\alpha^2)^2 + 2(T-1)\sigma_\alpha^4 + 3(T-1)\sigma_\alpha^2(\sigma_\epsilon^2 + \sigma_\alpha^2) \\
&\qquad + (T-1)(T-2)(\sigma_\epsilon^2\sigma_\alpha^2 + \sigma_\alpha^4 + 2\sigma_\alpha^4), \text{ using (3) of the main paper} \\
&= (T+2)\sigma_\epsilon^4 + (T^2 + 2T + 3)\sigma_\epsilon^2\sigma_\alpha^2 + 3(T^2 - T + 1)\sigma_\alpha^4.
\end{aligned}
$$

Similarly,

$$
E\left(s_{it'} \left[\sum_{t=1}^{T} s_{it}\right]^2\right) = E\left(s_{it'}^3 + s_{it'}^2 \sum_{t \neq t'} s_{it} + s_{it'} \sum_{t \neq t' \neq t''} s_{it} s_{it''}\right) = 0. \qquad (31)
$$

J Oman Weather Stations

Oman weather dataset consists observations from 55 stations across Oman over the period January 2018 to December 2018. The list of weather stations is given in Table 4.

Table 4 Stations for the monthly Oman weather data.

Station	Station	Station	Station	Station
Adam	Diba	Madha	Nizwa	Shalim
Al Amrat	Fahud Airport	Mahdah	Qairoon Hairiti	Shinas
Al Jazir	Haima	Majis	Qalhat	Sohar Airport
Al Khaboura	Ibra	Marmul Airport	Qarn alam	Sunaynah
Al Mudhaibi	Ibri	Masirah	Qurayyat	Suwaiq
Al Qabil	Ibri New	Mina Salalah	RasAlHaad	Taqah
Al-Buraymi	Izki	Mina Sultan Qaboos	Sadah	Thamrayt
Bidiyah	Joba	Mirbat	Saham	Wadi Bani Khalid
Buhla	Khasab Airport	Muqshin	Saiq Airport	Yalooni
Bukha	Khasab Port	Muscat City	Salalah Airport	Yalooni Airport
Dhank (Qumaira)	Liwa	Muscat International Airport	Samail	Yanqul