Annals of the Institute of Statistical Mathematics (2023) 75:705–729
https://doi.org/10.1007/s10463-022-00859-x

A copula spectral test for pairwise time reversibility

Shibin Zhang[1]

Received: 4 November 2021 / Revised: 20 October 2022 / Accepted: 22 November 2022 /
Published online: 26 December 2022

Abstract

In this paper, we propose a new frequency domain test for pairwise time reversibility at any specific couple of quantiles of two-dimensional marginal distribution. The proposed test is applicable to a very broad class of time series, regardless of the existence of moments and Markovian properties. By varying the couple of quantiles, the test can detect any violation of pairwise time reversibility. Our approach is based on an estimator of the L^2-distance between the imaginary part of copula spectral density kernel and its value under the null hypothesis. We show that the limiting distribution of the proposed test statistic is normal and investigate the finite sample performance by means of a simulation study. We illustrate the use of the proposed test by applying it to stock price data.

Keywords Copula · Discrete Fourier transform · Time reversibility · Periodogram · Spectral density

1 Introduction

A strictly stationary univariate time series $\{x_t\}$ is called time reversible if all its finite-dimensional distributions are invariant to the reversal of time indices. As is well known, Gaussian ARMA models are examples of time reversibility, in addition to independent identically distributed (iid) sequences. A necessary condition of the time reversibility, which is a weaker but more tangible case, is the pairwise time reversibility, i.e., the bivariate distribution function of the time series satisfies $F_{x_t,x_{t-k}}(a,b) = F_{x_t,x_{t+k}}(a,b)$ for any $(a,b) \in \mathbb{R}^2$ and $k \in \mathbb{N}$. For a Markov process, it is time reversible if and only if it is pairwise time reversible. A stationary Gaussian process is surely pairwise time reversible since its dependence is completely determined by the auto-covariance. Therefore, the rejection of pairwise

✉ Shibin Zhang
zhang_shibin@shnu.edu.cn

[1] Department of Mathematics, Shanghai Normal University, 100 Guilin Rd, Shanghai 200234, China

time reversibility is usually used to capture any deviation of the model from the Gaussianity or linearity. In the field of econometrics, the non-Gaussianity and nonlinearity have already drawn much interest in recent decades; see, e.g., Bollerslev (1986), Tong (1990) and Chen and Kuan (2002).

To test pairwise time reversibility, some pioneer works focus on checking distributional symmetry of $x_t - x_{t-k}$ for each k. For instance, Ramsey and Rothman (1996) proposed a moment-based test (RR test) by checking whether the difference $x_t - x_{t-k}$ has zero third moment. Noticing that the distributional symmetry is equivalent to that the imaginary part of its characteristic function is zero, Chen et al. (2000) developed a test (CCK test) by verifying whether a weighted sine transform of $x_t - x_{t-k}$ is zero mean. Choosing a proper weight function is crucial for implementing this test. Chen et al. (2000) set the weight function as the density of the exponential or half-normal distribution. Chen and Kuan (2002) modified the CCK test to allow it to be capable of detecting the symmetry of conditional distribution. Both Paparoditis and Politis (2002) and Psaradakis (2008) suggested using resampling techniques to test whether the difference $x_t - x_{t-k}$ is zero median. But the former focused only on the processes with Markovian structure.

There also existed several tests that are constructed by the aid of some nonsymmetric measures involving the joint and marginal densities of (x_t, x_{t-k}). Assuming that the joint density of (x_t, x_{t-k}) admits a nonlinear canonical decomposition, Darolles et al. (2004) considered a kernel-based test procedure by exploiting nonlinear canonical correlation analysis. By comparing the joint density with the product of the marginal densities of (x_t, x_{t-k}), Racine and Maasoumi (2007) provided a metric entropy statistic as a robust test for time reversibility.

Frequency domain approaches are another type of choices for testing pairwise time reversibility. According to a spectral decomposition of skewness, Hinich and Rothman (1998) proposed a frequency domain test involving a square summation of the imaginary parts of bispectrum. Recently, Wild et al. (2014) generalized the method of Hinich and Rothman (1998) to a sum-statistic by using the trispectrum, which constitutes a spectral decomposition of kurtosis. These existing frequency domain tests are actually carried out by verifying the symmetries of some third or fourth moments.

A common drawback of the afore-mentioned tests lies that they are only consistent to some specific forms of pairwise time reversibility, i.e., each tested null hypothesis is just a necessary other than sufficient condition of pairwise time reversibility. More recent works of Sharifdoost et al. (2009) and Beare and Seo (2014) essentially tested the actual features of pairwise time reversibility from the joint distributions, but their applicability is limited to the Markov models. For time-domain methods, another drawback is that they contain a lag parameter. In practice, it probably encounters a situation in which the test is significant for some lags but insignificant for others. To avoid drawing contradictory conclusions, Chen (2003) proposed a portmanteau version of the RR or CCK test based on the sum of finite number of pairwise differences of time series with different lags. However, this modified test is only applicable to detecting serial independence against time irreversibility. Finally, many tests, such as the RR and most spectral-domain methods, require the existence of the high-order moments of the time series, which rules out many time series in fields of economics and finance.

Actually, the pairwise time reversibility of a stationary time series can also be depicted by the concept of copula spectral density kernel (CSDK), which was proposed by Dette et al. (2015). The CSDA can depict the pairwise dependence of x_t at each specific couple of quantiles (q_{τ_1}, q_{τ_2}), where q_τ is the τ-quantile of the one-dimensional distribution of x_t, and $0 \leq \tau_1 \leq \tau_2 \leq 1$ are two probability levels. The pairwise time reversibility at (q_{τ_1}, q_{τ_2}) means that $\mathbf{P}(x_t \leq q_{\tau_1}, x_{t-k} \leq q_{\tau_2}) = \mathbf{P}(x_t \leq q_{\tau_1}, x_{t+k} \leq q_{\tau_2})$ for all values of k. Equivalently, a stationary time series is pairwise time reversible if and only if the imaginary part of CSDA vanishes for each quantile pair (cf. Dette et al. 2015, Proposition 2.1). Then, the test of pairwise time reversibility at the (q_{τ_1}, q_{τ_2}) can be designed by checking whether the imaginary part of CSDK vanishes or not. By varying the quantile pairs, the test can provide more specific results on time reversibility than various existing tests. The output of the test may be that there exists the time reversibility at some quantile pairs; otherwise, neither does at some others. It allows the approach to provide more insights on whether a given model fits the real data sufficiently in aspects of time reversibility.

In this paper, by using the CSDK, the test of pairwise time reversibility at a couple of quantiles can be transformed to a nonparametric test about the spectral density. Our idea to construct the test statistic is quite similar to that of Dette et al. (2011a). However, the variance of the statistic of Dette et al. (2011a) is constructed by periodogram, while ours is by smoothed periodogram. We prove that the normalized test statistic follows an asymptotic standard normal distribution, with different normalized constants under the null and alternative hypotheses. For our proposed test, two advantages are manifested. One is that it is applicable to a very broad class of time series, regardless of the existence of moments and Markovian properties. The other lies in that it has the potential to check whether a given model fits the real data sufficiently well in aspects of pairwise time reversibility. More specifically, by varying couple of quantiles, the proposed test can provide more detailed detection of pairwise time irreversibility than any other existing approach, in the meaning that the real data may achieve the pairwise time reversibility at some couples of quantiles but may not at others. These test results provide a good reference in fitting the real data by a model with the accordant pairwise time reversibility.

The remainder of the article is organized as follows. In Sect. 2, we introduce some preliminary concepts and results on CSDK and pairwise time reversibility. Section 3 presents the proposed statistic and its asymptotic results. Section 4 discusses the computational issues of the proposed approach and reports empirical results for examining performance of the test. Section 5 illustrates the applications in analyzing the real data. Section 6 contains our conclusion. All proofs are deferred to the appendix.

2 Preliminaries

Being entirely covariance based, the classical spectral approach for time series analysis is essentially limited to modeling first- and second-order dynamics (Birr et al. 2017). To capture dependence features beyond the second-order, many researchers tried quantile-based spectral analysis tools in recent decades. One of the most attractive quantile-based spectra is the CSDKs, which share the quantile-based favor while allowing to address such important features as pairwise time reversibility and dependence in extremes (Dette et al. 2015). In this section, we first give a brief introduction of the concept of CSDK, with its relation to pairwise time reversibility and then provide the statistics used in constructing the proposed statistic.

2.1 CSDK and pairwise time reversibility

Let $\{x_t\}_{t\in\mathbb{Z}}$ be a strictly stationary univariate process, and let $F(x)$ be the one-dimensional marginal cumulative distribution function (cdf). For each $(\tau_1, \tau_2) \in [0,1]^2$, the *copula cross-covariance kernel* of lag $k \in \mathbb{Z}$ of $\{x_t\}_{t\in\mathbb{Z}}$ is defined as

$$\gamma_k^U(\tau_1, \tau_2) := \mathrm{Cov}(\mathbb{I}_{(0,\tau_1]}(U_t), \mathbb{I}_{(0,\tau_2]}(U_{t-k})), \tag{1}$$

where $U_t = F(x_t)$, and $\mathbb{I}_A(x)$ is an indicator, which equals 1 if $x \in A$ and 0 otherwise. Thus, the dependence information between x_t and x_{t-k} is fully involved in the dependence between U_t and U_{t-k}, and the latter is implied in a cross-covariance set

$$\gamma_k^U := \{\gamma_k^U(\tau_1, \tau_2) | (\tau_1, \tau_2) \in [0,1]^2\}.$$

For each $(\tau_1, \tau_2) \in [0,1]^2$, we define the *copula spectral density kernel (CSDK)*, which is proposed by Dette et al. (2015), as

$$f_{q_{\tau_1}, q_{\tau_2}}(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_k^U(\tau_1, \tau_2) \mathrm{e}^{-\mathrm{i}k\omega}. \tag{2}$$

The collection of CSDKs for different (τ_1, τ_2) provides a full characterization of the copulas associated with the pairs (x_t, x_{t-k}) and accounts for many important dynamic features of $\{x_t\}$, such as changes in the conditional shape (skewness, kurtosis), time irreversibility, or dependence in the extremes that the traditional second-order spectra cannot capture.

The pairwise time reversibility means that $(x_t, x_{t+k}) \overset{d}{=} (x_t, x_{t-k})$ holds for all $k \in \mathbb{Z}$, where $\overset{d}{=}$ denotes equality in distribution. A stationary time series is pairwise time reversible if and only if for all $(\tau_1, \tau_2) \in [0,1]^2$ and $k \in \mathbb{Z}$, the equality $\gamma_k^U(\tau_1, \tau_2) = \gamma_{-k}^U(\tau_1, \tau_2)$ holds, which is equivalent to $\mathrm{Im} f_{q_{\tau_1}, q_{\tau_2}}(\omega) = 0$ for all $(\tau_1, \tau_2) \in [0,1]^2$ and $\omega \in (0, \pi)$ (Proposition 2.1 of Dette et al. 2015) , where $f_{q_{\tau_1}, q_{\tau_2}}(\cdot)$ is a CSDK defined in (2), and $\mathrm{Im}\, c$ denotes the imaginary part of a complex number c. This paper is devoted to test the hypothesis that $\{x_t\}_{t\in\mathbb{Z}}$ is pairwise time reversible at the couple of quantiles (q_{τ_1}, q_{τ_2}), i.e.,

$$H_0^{(\tau_1,\tau_2)}: \ \mathrm{Im} f_{q_{\tau_1}, q_{\tau_2}}(\omega) = 0 \text{ for all } \omega \in (0, \pi). \tag{3}$$

A non-vanishing imaginary part for $f_{q_{\tau_1}, q_{\tau_2}}(\omega)$ indicates that $\mathbf{P}(x_t \leq q_{\tau_1}, x_{t-k} \leq q_{\tau_2})$ differs from $\mathbf{P}(x_t \leq q_{\tau_1}, x_{t+k} \leq q_{\tau_2})$ for some values of k, which implies that $\{x_t\}$ is time irreversible at the couple of quantiles (q_{τ_1}, q_{τ_2}).

2.2 Rank-based periodogram and its smoothed version

The construction of the proposed test statistic relies on the estimation of CSDK. We use the rank-based copula periodogram (shortly, CR-periodogram) and its smoothed version in estimating the CSDKs (Kley et al. 2016).

To define the CR-periodogram, we first present the discrete Fourier transform (DFT) of clipped time series introduced by Hong (2000). For $\tau \in (0,1)$ and $\omega = \omega_{k,n} = 2\pi k/n$, $k = 1, \dots, n-1$, we define the DFT based on clipped time series by

$$y^{\tau}_{n,c}(\omega) := (\sqrt{2\pi n})^{-1} \sum_{t=1}^{n} \mathbb{I}_{(0,\tau]}(n^{-1}R_t^{(n)})\mathrm{e}^{\mathrm{i}\omega t},$$

where $R_t^{(n)}$ denotes the rank of x_t among $x_1, \dots, x_n$. We extend the definition of $y^{\tau}_{n,c}(\omega)$ to a piecewise constant function on $(0, 2\pi)$ as

$$y^{\tau}_{n,c}(\omega) = \begin{cases} y^{\tau}_{n,c}(\omega_{k,n}), & \text{if } \omega_{k,n} - \pi/n < \omega \le \omega_{k,n} + \pi/n, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

This extension is just for ease of notation, since we use the DFT at frequencies not always coinciding with $\omega_{k,n}$'s in constructing the proposed test statistic.

Following Kley et al. (2016), we consider the cross-periodograms associated with arbitrary couples (τ_1, τ_2) and define the CR-periodogram as

$$I^{\tau_1,\tau_2}_{n,R}(\omega) := \overline{y^{\tau_1}_{n,c}(\omega)} y^{\tau_2}_{n,c}(\omega), \quad \text{for } \omega \in (0, 2\pi),\ (\tau_1, \tau_2) \in [0,1]^2, \tag{5}$$

where $\bar{\cdot}$ denotes the complex conjugate. Consider a smoothed version of $I^{\tau_1,\tau_2}_{n,R}(\omega)$, namely for $(\tau_1, \tau_2) \in [0,1]^2$ and $\omega \in (0, \pi]$, the average of the form

$$\hat{G}_{n,R}(\tau_1, \tau_2; \omega) = \frac{2\pi}{n} \sum_{k=1}^{n-1} W_n(\omega - \omega_{k,n}) I^{\tau_1,\tau_2}_{n,R}(\omega_{k,n}), \tag{6}$$

where $W_n(\cdot)$ denotes a sequence of weighting functions. Under some regularity conditions, $\hat{G}_{n,R}(\tau_1, \tau_2; \omega)$ is a consistent estimator of $f_{q_{\tau_1}, q_{\tau_2}}(\omega)$ (cf. Kley et al. 2016, Theorem 3.5).

In this paper, we employ $\mathrm{Im}\, I^{\tau_1,\tau_2}_{n,R}(\omega)$ in constructing a statistic to estimate the deviation of $\mathrm{Im} f_{q_{\tau_1}, q_{\tau_2}}(\omega)$ from 0, while the variance of the statistic is estimated by using the smoothed CR-periodogram (6).

3 The test statistic and asymptotic results

In this section, we first construct the test statistic, and then prove the asymptotic normality under the null and alternative hypotheses.

3.1 The test statistic

The hypothesis (3) can be re-expressed by the L^2-distance between the imaginary part of CSDK and its value under the null, i.e.,

$$H_0^{(\tau_1,\tau_2)}\ :\ T^{(\tau_1,\tau_2)} := \int_0^{\pi} \left(\mathrm{Im} f_{q_{\tau_1},q_{\tau_2}}(\omega)\right)^2 \mathrm{d}\omega = 0, \tag{7}$$

which is independent of ω. Thus, the test of pairwise time reversibility at a couple of quantiles can be transformed to a nonparametric test about the spectral density.

The quantity $T^{(\tau_1,\tau_2)}$ in (7) can be treated as the L^2-distance between the imaginary part of CSDK and its value under the null hypothesis (3). The $T^{(\tau_1,\tau_2)}$ vanishes if and only if the null hypothesis is satisfied.

To estimate $T^{(\tau_1,\tau_2)}$, for each $M > 1$, we consider the test statistic

$$T_{n,M}^{(\tau_1,\tau_2)} = \frac{\pi}{M} \sum_{m=1}^{M} \mathrm{Im}\, I_{n,R}^{\tau_1,\tau_2}\big(\frac{m}{M}\pi\big)\, \mathrm{Im}\, I_{n,R}^{\tau_1,\tau_2}\big(\frac{m-1}{M}\pi\big), \tag{8}$$

where $I_{n,R}^{\tau_1,\tau_2}(\omega)$ is defined in (5).

Remark 1 Based on the estimator of the integrated deviation from the null, there have been developed a great number of nonparametric approaches for testing hypotheses on spectra; see, e.g., Eichler (2008), Dette and Paparoditis (2009), Dette and Hildebrandt (2012), Jentsch and Pauly (2015), Dette et al. (2011a) and Dette et al. (2011b), to name a few. Among them, to test two specific null hypotheses, Dette et al. (2011a) proposed a method that requires an appropriate summation of the periodogram. The idea of ours in constructing the test statistic is quite similar to that of Dette et al. (2011a).

3.2 Asymptotic results

The rigorous derivation of the asymptotic properties of $T_{n,M}^{(\tau_1,\tau_2)}$ requires the assumption on the dependence structure of $\{x_t\}_{t\in\mathbb{Z}}$:

(M) The process $\{x_t\}_{t\in\mathbb{Z}}$ is strictly stationary and exponentially β-mixing, that is,

$$\beta(n) := \sup\{\mathbf{P}(B|A) - \mathbf{P}(B)\ :\ A \in \sigma(x_k;k \le 0),\, B \in \sigma(x_k;k \ge n),\, \mathbf{P}(A) > 0\} \le K\eta^n,$$

holds for each $n \in \mathbb{N}$, some $K < \infty$ and $\eta \in (0, 1)$, where for any $\mathbb{K} \subset \mathbb{Z}$, $\sigma(x_k;k \in \mathbb{K})$ denotes the σ-algebra generated by $\{x_k;k \in \mathbb{K}\}$.

The class of β-mixing processes is well studied, which contains a wide range of linear and nonlinear processes. Under mild additional conditions, many familiar models satisfy assumption (M), such as ARMA, ARCH, GARCH and so on (e.g. (Mokkadem, 1988; Carrasco and Chen, 2002; Fryzlewicz and Rao, 2011)); for more examples, we refer to Dette et al. (2015).

Let Re c denote the real part of a complex number c. To obtain the asymptotic distribution of $T_{n,M}^{(\tau_1,\tau_2)}$, M should go to infinity with an appropriate rate. Precisely, the increasing rate of M should satisfy:

Condition 1 As $n \to \infty$, $M = O(n^{1/2-\delta})$, where δ can be taken any constant satisfying $0 < \delta < 1/2$. For ease of notation, we omit the dependence of M on n.

Theorem 1 *Suppose assumption* (*M*) *holds*. *If Condition* 1 *is satisfied*, *then for each* $\tau_1, \tau_2 \in [0, 1]$,

$$\sqrt{M}\big(T_{n,M}^{(\tau_1,\tau_2)} - T^{(\tau_1,\tau_2)}\big) \rightsquigarrow N(0, V^{(\tau_1,\tau_2)}), \tag{9}$$

where "$\rightsquigarrow$" *denotes convergence in distribution*, *and*

$$\begin{aligned} V^{(\tau_1,\tau_2)} =& \frac{\pi}{4}\int_0^{\pi} \big\{f_{q_{\tau_1},q_{\tau_1}}(\omega)f_{q_{\tau_2},q_{\tau_2}}(\omega) + (\mathrm{Im}f_{q_{\tau_1},q_{\tau_2}}(\omega))^2 - (\mathrm{Re}f_{q_{\tau_1},q_{\tau_2}}(\omega))^2\big\}^2 \,\mathrm{d}\omega \\ & + 2\pi\int_0^{\pi} (\mathrm{Im}f_{q_{\tau_1},q_{\tau_2}}(\omega))^2\big\{f_{q_{\tau_1},q_{\tau_1}}(\omega)f_{q_{\tau_2},q_{\tau_2}}(\omega) + (\mathrm{Im}f_{q_{\tau_1},q_{\tau_2}}(\omega))^2 \\ & - (\mathrm{Re}f_{q_{\tau_1},q_{\tau_2}}(\omega))^2\big\} \,\mathrm{d}\omega. \end{aligned} \tag{10}$$

Remark 2 If the null hypothesis in (3) is satisfied, then

$$\sqrt{M}T_{n,M}^{(\tau_1,\tau_2)} \rightsquigarrow N(0, V_0^{(\tau_1,\tau_2)}),$$

where the asymptotic variance $V_0^{(\tau_1,\tau_2)}$ has a simple expression

$$V_0^{(\tau_1,\tau_2)} = \frac{\pi}{4}\int_0^{\pi} \big\{f_{q_{\tau_1},q_{\tau_1}}(\omega)f_{q_{\tau_2},q_{\tau_2}}(\omega) - (\mathrm{Re}f_{q_{\tau_1},q_{\tau_2}}(\omega))^2\big\}^2 \,\mathrm{d}\omega. \tag{11}$$

Remark 3 In the process of analyzing the asymptotic behavior of $T_{n,M}^{(\tau_1,\tau_2)}$, we approximate it by an unobservable quantity

$$\widetilde{T}_{n,M}^{(\tau_1,\tau_2)} = \frac{\pi}{M}\sum_{m=1}^{M} \mathrm{Im}\, I_{n,U}^{\tau_1,\tau_2}\big(\frac{m}{M}\pi\big)\,\mathrm{Im}\, I_{n,U}^{\tau_1,\tau_2}\big(\frac{m-1}{M}\pi\big), \tag{12}$$

where

$$I_{n,U}^{\tau_1,\tau_2}(\omega) := \overline{y_{n,U}^{\tau_1}(\omega)}y_{n,U}^{\tau_2}(\omega), \quad \omega \in (0, 2\pi),\ (\tau_1,\tau_2) \in [0,1]^2, \tag{13}$$

and $y_{n,U}^{\tau_1}(\omega)$ is a piecewise constant function extended from the DFT

$$y_{n,U}^{\tau}(\omega_{k,n}) = (\sqrt{2\pi n})^{-1}\sum_{t=1}^{n} \mathbb{I}_{(0,\tau]}(U_t)\mathrm{e}^{\mathrm{i}\omega_{k,n}t} \tag{14}$$

as in (4). Note that,

$$\sqrt{M}\big(T_{n,M}^{(\tau_1,\tau_2)} - T^{(\tau_1,\tau_2)}\big) = \sqrt{M}\big(T_{n,M}^{(\tau_1,\tau_2)} - \widetilde{T}_{n,M}^{(\tau_1,\tau_2)}\big) + \sqrt{M}\big(\widetilde{T}_{n,M}^{(\tau_1,\tau_2)} - T^{(\tau_1,\tau_2)}\big).$$

The imposed assumption (M) guarantees that the asymptotic normality $\sqrt{M}\big(\widetilde{T}_{n,M}^{(\tau_1,\tau_2)} - T^{(\tau_1,\tau_2)}\big) \rightsquigarrow N(0, V^{(\tau_1,\tau_2)})$ and the equality $\sqrt{M}(T_{n,M}^{(\tau_1,\tau_2)} - \widetilde{T}_{n,M}^{(\tau_1,\tau_2)}) = o_P(1)$ hold, under Condition 1.

Recall that under the null, the asymptotic variance $V^{(\tau_1,\tau_2)}$ in (10) simplifies to $V_0^{(\tau_1,\tau_2)}$ in (11). Consequently, if $\hat{V}_0^{(\tau_1,\tau_2)}$ is a consistent estimator of $V_0^{(\tau_1,\tau_2)}$, we define a normalized version of the test statistic by

$$\mathcal{Z}_{n,M}^{(\tau_1,\tau_2)} = \sqrt{M}T_{n,M}^{(\tau_1,\tau_2)} \Big/ \sqrt{\hat{V}_0^{(\tau_1,\tau_2)}}. \tag{15}$$

It follows from Theorem 1 that a test with asymptotic level of significance α is obtained by rejecting the null hypothesis if

$$\mathcal{Z}_{n,M}^{(\tau_1,\tau_2)} \geq z_{1-\alpha}, \tag{16}$$

where $z_{1-\alpha}$ denotes the $1-\alpha$ quantile of the standard normal distribution.

We employ the smoothed CR-periodogram (6) to construct the consistent estimator of $V_0^{(\tau_1,\tau_2)}$. That is, we define $\hat{V}_0^{(\tau_1,\tau_2)}$ by the same expression as that of $V_0^{(\tau_1,\tau_2)}$ defined in Remark 2, but with $f_{q_{\tau_1},q_{\tau_2}}(\omega)$ in (11) replaced by $\hat{G}_{n,R}(\tau_1,\tau_2;\omega)$ defined in (6). In the sequel, we consider the test statistic (15), with $\hat{V}_0^{(\tau_1,\tau_2)}$ defined in this way.

In order to establish the asymptotic distribution of $\mathcal{Z}_{n,M}^{(\tau_1,\tau_2)}$, we require the weights W_n in (6) to satisfy the following assumption, which is quite standard in classical time series analysis (see, e.g., Brillinger 2001, pp. 147).

(W) The weight function W is real-valued and even, with support $[-\pi,\pi]$; moreover, it has bounded variation and satisfies $\int_{-\pi}^{\pi} W(u)\,\mathrm{d}u = 1$.

Denoting by $b_n > 0$, $n = 1, 2, \cdots$, a sequence of scaling parameters such that $b_n \to 0$ and $nb_n \to \infty$ as n goes to infinity, define

$$W_n(u) := \sum_{j=-\infty}^{\infty} b_n^{-1} W(b_n^{-1}[u + 2\pi j]).$$

Then, the asymptotic result of $\mathcal{Z}_{n,M}^{(\tau_1,\tau_2)}$ is stated as follows.

Theorem 2 *Suppose assumptions* (*M*) *and* (*W*) *hold. Assume that* x_0 *has a continuous distribution function F and that there exist constants* $\kappa > 0$ *and* $k \in \mathbb{N}$, *such that*

$$h_n = o(n^{-1/(2k+1)}) \quad \text{and} \quad b_n n^{1-\kappa} \to \infty.$$

Then, *under Condition* 1, *if the null hypothesis* (3) *holds and* $\tau_1, \tau_2 \in [0,1]$, *then the test statistic* $\mathcal{Z}_{n,M}^{(\tau_1,\tau_2)}$ *converges in distribution to the standard normal distribution.*

Remark 4 One key merit for the statistic of Dette et al. (2011a) is that the variance is estimated by a statistic that is independent of any tuning parameters. But the consistency of the variance estimator relies on that the time series $\{x_t\}$ is a linear process.

However, for our proposed statistic, since the process $\{\mathbb{I}_{(0,\tau]}(U_t),\ t \in \mathbb{Z}\}$ is nonlinear, the consistent estimator of $V_0^{(\tau_1,\tau_2)}$ that is independent of tuning parameters is unavailable.

4 Practical issues and numerical results

In this section, we first give some remarks on choosing tuning parameters, and then report some numerical results to evaluate the finite sample performance of the proposed test.

4.1 Some remarks on choosing tuning parameters

When applying the proposed statistic (15) in practice, there are three tuning parameters, the weight function W, the scaling parameter b_n and the integer M. They should be pre-specified. In this section, we discuss their choices.

1. Choice of W. The weight function recommended is the Epanechnikov kernel, i.e., $W(x) = \frac{3}{4\pi}(1-(\frac{x}{\pi})^2)\mathbb{I}_{[-\pi,\pi]}(x)$. It is the optimal kernel of order (0,2), in the meaning that it minimizes both the asymptotic variance and the asymptotic integrated mean square error for a given scaling parameter (Gasser et al. 1985).

2. Choice of b_n. It is well known that the choice of b_n is much more crucial than the choice of W in estimating a density function (e.g. Silverman 1986). However, the choice of b_n exerts the effect on an integral of spectral estimates in $\mathcal{Z}_{n,M}^{(\tau_1,\tau_2)}$. The b_n first appears in the estimate of $f_{q_{\tau_1},q_{\tau_2}}(\omega)$. Then, their functions are integrated with respect to ω. The integral operation results in that the estimate $\hat{V}_0^{(\tau_1,\tau_2)}$ is not very sensible to the choice of b_n. As an example to show the insensibility, we consider an $AR(1)$ model of the form $x_t = -0.3x_{t-1} + \varepsilon_t$ (Model A), where ε_t, $t = 1, 2, \ldots$, are independent $N(0,\ 1)$-distributed random variables. For each sample size $n = 125, 250, 500, 1000$ and each pair of levels, with values in $\{0.10, 0.50, 0.90\}$, we compute the ratios of $\hat{V}_0^{(\tau_1,\tau_2)}$ under two different choices of b_n, $n^{-1/5}$ and $n^{-1/3}$, based on 500 Monte Carlo replications. Presented in Fig. 1 are the boxplots for the ratios of $\hat{V}_0^{(\tau_1,\tau_2)}$. It can be evidenced from Fig. 1 that the b_n are different, but the medians of ratios are very close to 1 for each sample size n and each pre-specified level pair (τ_1, τ_2). Moreover, as n goes to infinity, the ratio of $\hat{V}_0^{(\tau_1,\tau_2)}$ approaches 1.

3. Choice of M. On one hand, M should be taken to satisfy $M \le [n/2]$ since the CR-periodogram is defined as a piecewise constant function and its value changes only at $\omega_{k,n} = 2\pi k/n,\ k = 1, \ldots, [n/2]$. On the other hand, to guarantee that the standard normal distribution can provide a good approximation to the null distribution of the test statistic $\mathcal{Z}_{n,M}^{(\tau_1,\tau_2)}$, M should satisfy Condition 1. Combining these two hands, it is natural to choose M with the form of $M = \min\{[n/2], [Cn^{1/2-\delta}]\}$, where $C > 0$ and $\delta \in (0, 1/2)$ are two constants. By our empirical results, if $n \le 2500$ and $M = [n/2]$, the proposed test (16) performs well. For $n > 2500$, we can set $M = [Cn^{1/3}]$ with $C \in [80, 100]$ for security. Therefore, we recommend using $M = \min\{[n/2], [Cn^{1/3}]\}$ in practice, where $C \in [80, 100]$ is a pre-specified constant.

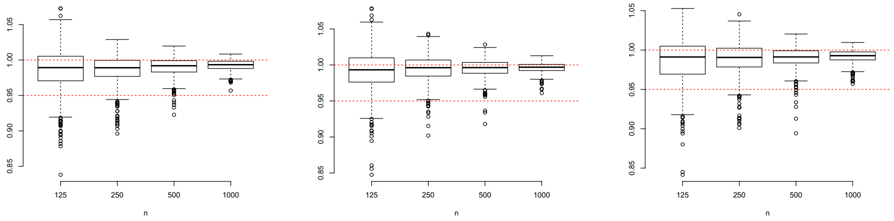

Fig. 1 Boxplots for the ratios of $\hat{V}_0^{(\tau_1,\tau_2)}$ under two different choice of b_n. Left: $(\tau_1, \tau_2) = (0.10, 0.50)$; Middle: $(\tau_1, \tau_2) = (0.10, 0.90)$; Right: $(\tau_1, \tau_2) = (0.50, 0.90)$

4.2 Numerical examples

In this section, we report some numerical evidence pertaining to evaluate the finite sample performance of the proposed test (16) when applied to test the hypothesis (3). All results in the considered examples are based on 1000 Monte Carlo replications.

In all the simulation examples, we set the nominal size $\alpha = 5\%$. In computing the values of test statistic, we choose the weight function W to the Epanechnikov kernel, and set $b_n = n^{-1/5}$ and $M = [n/2]$. All these settings are in accordance with the remarks in Sect. 4.1. We consider three different choices of (τ_1, τ_2), (0.10,0.50) (0.10,0.90) and (0.50,0.90). Note that, under the finite sample circumstance, the $y^{\tau}_{n,c}(\omega)$ approaches a constant, as τ tends to zero or one. It means that the proposed test may fail to work if τ_1 is taken too small or τ_2 too large. In practice, we recommend choosing the smallest value of τ_1 to 0.05 or 0.10, and the largest value of τ_2 to 0.90 or 0.95.

To illustrate the performance of test (16) under the null, we consider three models that behave the pairwise time reversibility at three considered quantile pairs. The models are given as

Model A ($AR(1)$ model.) $x_t = -0.3x_{t-1} + \varepsilon_t, t = 1, \dots, n$.

Model B ($ARMA(1, 1)$ model.) $x_t = -0.8x_{t-1} + 1.25\varepsilon_{t-1} + \varepsilon_t, t = 1, \dots, n$.

Model C (independent $t(1)$ model.) $x_t = \varepsilon_t, t = 1, \dots, n$.

In Models A and B, ε_t, $t = 1, 2, \dots$, are independent $N(0, 1)$-distributed random variables, while in Model C, they are independent $t(1)$-distributed random variables, where $t(\nu)$ denotes the Student t distribution with ν degrees of freedom. The time reversibility of Models A, B and C is obvious since they are series of either Gaussianity or independence.

To show the performance of test (16) for the models of time irreversibility, we consider four models. The first two, Models A$'$ and B$'$, admit, respectively, the same expressions as Models A and B, but ε_t, $t = 1, 2, \dots$, are independent $t(\nu)$-distributed random variables. The last two are given as:

Model D ($ARCH(1)$ model.) $x_t = (1/1.9 + 0.9x^2_{t-1})^{1/2}\varepsilon_t$, $t = 1, \dots, n$, where ε_t, $t = 1, 2, \dots$, are independent $t(1)$-distributed random variables.

Model E ($QAR(1)$ model.) $x_t = 0.1\Phi^{-1}(\upsilon_t) + 1.9(\upsilon_t - 0.5)x_{t-1}$ (cf.Koenker and Xiao 2006), $t = 1, \dots, n$, where $\{\upsilon_t\}$ is a sequence of i.i.d. uniformly distributed random variables on [0,1], and Φ denotes the cdf of the $N(0, 1)$.

For each sample size $n = 125, 250, 500, 1000, 2000$ and each considered model, we calculated the rejection rate of our proposed test (16) from simulated data. Presented in Table 1 are empirical rejection probabilities for Models A, B and C, while those for Models A′, B′, D and E are presented in Table 2.

Results in Table 1 show that each empirical type I error rate is very close to the pre-specified nominal value. We conclude from Table 2 as follows. First, for most of considered models, the performance of test (16) improves significantly with increasing n. Take Model A′ ($\nu = 1$) as an example. From the nonparametric estimates of the CSDKs (Dette et al. 2015), it displays a time irreversible impact of extreme values on the central ones, but does not between the symmetric lower and upper extreme values. These properties are also evidenced from the simulation study of test (16). The empirical rejection ratios at level pairs (0.10,0.50) and (0.50,0.90) increase with increasing n, but those at the pair (0.10,0.90) approach the nominal level of significance. Second, by the aid of test (16), the detailed time irreversibility of a time series at any couple of quantiles can be detected. For Model A′ ($\nu = 1$), it achieves the time irreversible impact of extreme values on the central ones, but the time irreversibility impact between symmetric lower and upper extreme values (i.e., $\mathbf{P}(x_t \le q_{0.1}, x_{t-k} \le q_{0.9}) \ne \mathbf{P}(x_t \le q_{0.1}, x_{t+k} \le q_{0.9})$ for all k) for Model D. For Models B′ ($\nu = 1$) and D, the time irreversibility is shown at all considered couples of quantiles. The time irreversibility of Model D is also consistent to the Bayesian estimates of the CSDKs (Zhang 2019).

The simulation results for Models A′ ($\nu = 5$) and B′ ($\nu = 5$) show that the rejection rates are close to the nominal one, $\alpha = 0.05$. Although the time irreversibility of Models A′ ($\nu = 5$) and B′ ($\nu = 5$) is known, it is nearly not detected by the proposed test. Even so, it can not be concluded that the capability of the proposed test to detect time irreversibility is unbelievable. Indeed the performance of the test is poor for these two models. But it is due to that these models achieve the time reversibility approximately at quantile pairs $(q_{0.1}, q_{0.5})$, $(q_{0.1}, q_{0.9})$ and $(q_{0.5}, q_{0.9})$, though not exactly. Taking

Table 1 Rejection probabilities of the test (16) from simulated data of Models A, B and C

Model	(τ_1, τ_2)	n				
		125	250	500	1000	2000
A	(0.10,0.50)	0.051	0.047	0.059	0.051	0.056
	(0.10,0.90)	0.065	0.052	0.055	0.050	0.051
	(0.50,0.90)	0.046	0.039	0.042	0.045	0.042
B	(0.10,0.50)	0.036	0.050	0.049	0.036	0.042
	(0.10,0.90)	0.049	0.055	0.049	0.054	0.056
	(0.50,0.90)	0.054	0.040	0.057	0.051	0.045
C	(0.10,0.50)	0.057	0.058	0.060	0.050	0.044
	(0.10,0.90)	0.046	0.054	0.064	0.047	0.040
	(0.50,0.90)	0.047	0.054	0.061	0.058	0.055

Table 2 Rejection probabilities of the test (16) from simulated data of Models A′, B′, D and E

Model	(τ_1, τ_2)	n				
		125	250	500	1000	2000
A′($\nu = 1$)	(0.10,0.50)	0.086	0.118	0.104	0.174	0.236
	(0.10,0.90)	0.100	0.087	0.059	0.058	0.057
	(0.50,0.90)	0.078	0.100	0.139	0.187	0.238
A′($\nu = 5$)	(0.10,0.50)	0.037	0.058	0.043	0.063	0.071
	(0.10,0.90)	0.059	0.067	0.049	0.057	0.071
	(0.50,0.90)	0.048	0.046	0.037	0.054	0.057
B′($\nu = 1$)	(0.10,0.50)	0.122	0.138	0.170	0.200	0.237
	(0.10,0.90)	0.274	0.204	0.158	0.118	0.111
	(0.50,0.90)	0.112	0.147	0.187	0.206	0.251
B′($\nu = 5$)	(0.10,0.50)	0.053	0.046	0.057	0.058	0.050
	(0.10,0.90)	0.058	0.047	0.066	0.061	0.072
	(0.50,0.90)	0.044	0.048	0.044	0.054	0.051
D	(0.10,0.50)	0.056	0.052	0.049	0.081	0.106
	(0.10,0.90)	0.269	0.329	0.376	0.431	0.461
	(0.50,0.90)	0.041	0.053	0.072	0.076	0.107
E	(0.10,0.50)	0.125	0.196	0.273	0.406	0.610
	(0.10,0.90)	0.234	0.315	0.482	0.693	0.924
	(0.50,0.90)	0.139	0.170	0.249	0.371	0.598

Model A′ as an example, even for the case of $\nu = 1$, the CSDK $\gamma_k^U(\tau_1, \tau_2)$ is nearly symmetric about $k = 0$ if $(\tau_1, \tau_2) = (0.1, 0.5)$, (0.1,0.9) or (0.5,0.9), not to mention the case of $\nu = 5$; see Fig. 2, where the plots of $\gamma_k^U(0.1, 0.5)$, $\gamma_k^U(0.1, 0.9)$ and $\gamma_k^U(0.5, 0.9)$ for Model A′ ($\nu = 1$) are presented from the left-hand to right-hand sides.

The axial symmetry of $\gamma_k^U(\tau_1, \tau_2)$ about $k = 0$ implies the pairwise time reversibility of $\{x_t\}_{t\in\mathbb{Z}}$ at quantile pair (q_{τ_1}, q_{τ_2}).

In Section S2.1 of the supplement, we also present empirical rejection probabilities for all considered models by using Daniell kernel, $W(x) = \frac{1}{2\pi}\mathbb{I}_{[-\pi,\pi]}(x)$, as the weight function. As is expected, the empirical rejection probabilities are insensitive to the choice of weight function. In Section S2.2 of the supplement, for comparison of the power, we present empirical rejection probabilities of the four tests that target the time irreversibility at all τ' null, the RR test of Ramsey and Rothman (1996), the CCK test of Chen et al. (2000), the PP test of Paparoditis and Politis (2002) and the BS test of Beare and Seo (2014), for the same models that are currently considered. The comparison shows that empirical power of the proposed test (16) is reasonably high.

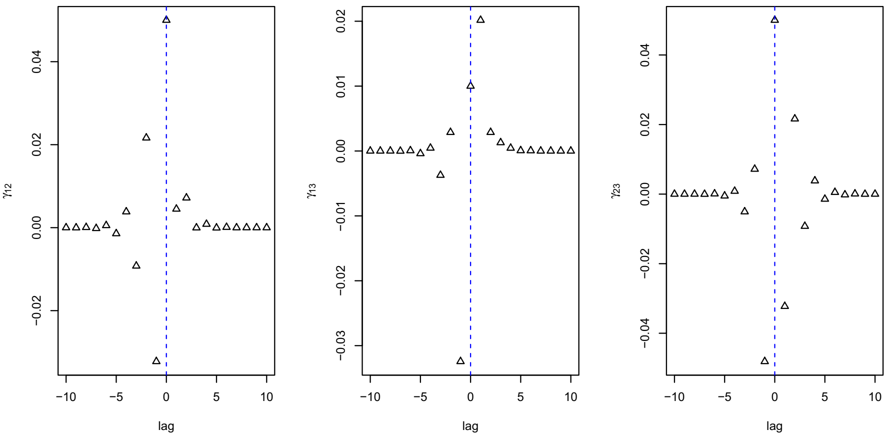

Fig. 2 Plots of $\gamma_k^U(0.1, 0.5)$, $\gamma_k^U(0.1, 0.9)$ and $\gamma_k^U(0.5, 0.9)$ for Model A′ ($\nu = 1$)

5 An empirical example

In this section, we apply the proposed test to analyze the time reversibility of daily return for various stock market indices. The data are taken from Yahoo!Fiance and contain four market indices from January 1, 2010 through December 31, 2018. The indices are the Dow Jones Industrial Average (DJIA), National Association of Securities Dealers Automated Quotations Composite (NASDAQ), Standard and Poor's 500 (S &P500) and Russell 2000 (RS2000). Each series contains $n = 2263$ daily close prices. We let $x_t = 100(\log P_t - \log P_{t-1})$, which denotes the daily return of the index P_t.

We consider testing the null (3) by the test (16). The levels of (τ_1, τ_2) are selected from $\{0.10, 0.50, 0.90\}$. The weight function W, the scaling parameter b_n and the integer M are set the same as in Sect. 4. For the purpose of comparison, we also consider the CCR test (Chen et al. 2000) with exponential weight function $g(\omega) = \exp(-\omega)\mathbb{I}_{(0,\infty)}(\omega)$. Thus, the CCR test statistic is given by

$$\mathcal{C}_{n,k} = \sqrt{n-k}\,\bar{\psi}_{\exp,k}/\hat{\sigma}_{\exp,k}, \tag{17}$$

with $\bar{\psi}_{\exp,k} = \frac{1}{n-k}\sum_{t=k+1}^{n} \psi_{\exp}(x_t - x_{t-k})$ and

$$\hat{\sigma}^2_{\exp,k} = \frac{1}{n-k}\sum_{t=k+1}^{n}(\psi_{\exp}(x_t - x_{t-k}) - \bar{\psi}_{\exp,k})^2$$
$$+ \frac{2}{n-k}\sum_{v=1}^{n-k-1}\kappa(v)\sum_{t=v+k+1}^{n}(\psi_{\exp}(x_t - x_{t-k}) - \bar{\psi}_{\exp,k})(\psi_{\exp}(x_{t-v} - x_{t-v-k}) - \bar{\psi}_{\exp,k}),$$

where $\psi_{\exp}(\omega) = \omega/(1+\omega^2)$, and

$$\kappa(v) = \left(1 - \frac{v}{n-k}\right)[1 - 0.5(n-k)^{-1/3}]^{v} + \frac{v}{n-k}[1 - 0.5(n-k)^{-1/3}]^{n-k-v}$$

is a kernel function ensuring that $\hat{\sigma}^2_{\exp,k}$ is nonnegative. Since both $\mathcal{Z}^{(\tau_1,\tau_2)}_{n,M}$ and $\mathcal{C}_{n,k}$ are asymptotically standard normal under the null and independent of any model moment restriction, the CCR test

$$\mathcal{C}_{n,k} \geq z_{1-\alpha} \tag{18}$$

serves a good reference to evaluate the proposed test (16). For the CCR test (18), we consider four choices of lag parameters $k = 1, 2, 3, 4$. The observed values of test statistics $\mathcal{Z}^{(\tau_1,\tau_2)}_{n,M}$ and $\mathcal{C}_{n,k}$ and the p values are summarized in Table 3.

As shown in Table 3, each considered return series is time irreversible, since there exists at least one value of k such that the p value of test (18) is very close to zero. However, given $\alpha = 0.05$ level of significance, we can obtain more detailed conclusions from test (16) than those from test (18). On one hand, the time irreversible impact between symmetric extreme values is very strong for each return series of considered market indices, since each p value of test (16) is zero for the case of $(\tau_1, \tau_2) = (0.10, 0.90)$. On the other hand, the time irreversible impact of extreme values on the central ones is different among different return series. For the returns of DJIA, NASDAQ and S &P500, the time irreversible impact of lower and upper extreme values on the central ones is clear, since p values of test (16) are less than 0.05 for the cases of $(\tau_1, \tau_2) = (0.10, 0.50)$ and (0.50,0.90). However, for the returns of RS2000, they appear the time reversible impact of the lower extreme values but do not of the upper extreme values, on the central ones. These test results can be used in selecting an appropriate model for the real data. If there are several alternative models that are used to fit the real data, in practice, one can select the model that is consistent with the pairwise time reversibility of the data at pre-specified quantile pairs.

Table 3 Observed values of test statistics (15) and (17) for each daily return of market indices, with corresponding p values

Index		(τ_1, τ_2) in $\mathcal{Z}^{(\tau_1,\tau_2)}_{n,M}$			k in $\mathcal{C}_{n,k}$			
		(0.10,0.50)	(0.10,0.90)	(0.50, 0.90)	1	2	3	4
DJIA	statistic	2.220	3.777	1.725	3.351	2.053	2.102	4.389
	pvalue	0.013	0.000	0.042	0.001	0.020	0.018	0.000
NASDAQ	statistic	2.052	3.955	2.204	2.951	3.006	2.106	5.576
	pvalue	0.020	0.000	0.014	0.002	0.001	0.018	0.000
S &P500	statistic	3.305	4.786	2.889	3.803	2.270	2.395	4.857
	pvalue	0.000	0.000	0.002	0.000	0.012	0.008	0.000
RS2000	statistic	0.847	4.426	2.739	1.616	1.413	2.417	3.906
	pvalue	0.199	0.000	0.003	0.053	0.079	0.008	0.000

6 Conclusion

In this paper, we propose a nonparametric test approach to detect the pairwise time irreversibility at any couple of quantile levels. The test statistic is independent of the lag parameter, and without any moment restriction. Moreover, the implementation of the test is easy in practice since the asymptotic distribution of the test statistic under the null and alternative hypotheses is normal.

Supplementary Information The online version contains supplementary material available at https://doi.org/10.1007/s10463-022-00859-x.

Acknowledgements I would like to thank the Associate Editor and anonymous reviewers for their constructive comments that resulted in an improved version of the paper. This work was supported by the National Natural Science Foundation of China (Grant Numbers: 11671416, 11971116), the Natural Science Foundation of Shanghai (grant number: 20JC1413800) and the research project of Shanghai Normal University (Grant Number: SK202239).

Appendix Details for the proofs in Sect. 3

This appendix contains the detailed proofs of Theorems 1 and 2. For ease of notation, we use C for any generic positive constant.

Main lemmas used in the proofs

Recall that the rth-order joint cumulant $\text{cum}(\zeta_1, \dots, \zeta_r)$ of the random vector $(\zeta_1, \dots, \zeta_r)$ is defined as

$$\text{cum}(\zeta_1, \dots, \zeta_r) := \sum_{\{\nu_1,\dots,\nu_p\}} (-1)^{p-1}(p-1)!\big(\mathbf{E}\prod_{j\in\nu_1}\zeta_j\big)\cdots\big(\mathbf{E}\prod_{j\in\nu_p}\zeta_j\big),$$

with summation extending over all partitions $\{\nu_1, \dots, \nu_p\}$, $p = 1, \dots, r$, of $\{1, \dots, r\}$ (cf. Brillinger 2001, pp. 19). If $\zeta_1 = \cdots = \zeta_r = \zeta$, we use $\text{cum}_r(\zeta)$ to denote $\text{cum}(\zeta_1, \dots, \zeta_r)$.

Lemma 1 *If assumption (M) holds, then the following assumption (C) also holds.*

(C) There exist constants $\rho \in (0,1)$ and $K < \infty$ such that, for arbitrary intervals $A_1, \dots, A_p \subset \mathbb{R}$ and arbitrary $t_1, \dots, t_p \in \mathbb{Z}$,

$$\left|\text{cum}\big(\mathbb{I}_{A_1}(x_{t_1}), \dots, \mathbb{I}_{A_p}(x_{t_p})\big)\right| \le K\rho^{\max_{i,j}|t_i-t_j|}. \tag{19}$$

Proof Let $\alpha(n) := \sup\{\mathbf{P}(AB) - \mathbf{P}(A)\mathbf{P}(B) : A \in \sigma(x_k; k \le 0), B \in \sigma(x_k; k \ge n)\}$. According to Bradley (2005), we have $\alpha(n) \le \frac{1}{2}\beta(n)$. Then, by applying Proposition 3.1 of Kley et al. (2016), we obtain the lemma. □

For $\omega \in (0, \pi]$ and $(\tau_1, \tau_2) \in [0,1]^2$, we denote

$$\mathbf{y}_{n,U}^{\tau_1,\tau_2}(\omega) = \left(\mathrm{Re}\, y_{n,U}^{\tau_1}(\omega), \mathrm{Im}\, y_{n,U}^{\tau_1}(\omega), \mathrm{Re}\, y_{n,U}^{\tau_2}(\omega), \mathrm{Im}\, y_{n,U}^{\tau_2}(\omega)\right)^T,$$

where $y_{n,U}^{\tau}(\omega)$ is defined in (14).

Then, according to (1.6) of Kley et al. (2015), we have the following lemma.

Lemma 2 *If $\{x_t\}_{t\in\mathbb{Z}}$ is strictly stationary and satisfies assumption (C), then for every $\omega \in (0,\pi]$, we have*

$$\mathbf{y}_{n,U}^{\tau_1,\tau_2}(\omega) \rightsquigarrow \mathbb{Y}^{\tau_1,\tau_2}(\omega),$$

as n goes to infinity, where $\mathbb{Y}^{\tau_1,\tau_2}(\omega) = (\mathbb{C}^{\tau_1}(\omega), \mathbb{D}^{\tau_1}(\omega), \mathbb{C}^{\tau_2}(\omega), \mathbb{D}^{\tau_2}(\omega))^T$ follows a four-dimensional zero-mean Gaussian distribution with covariance matrix $\frac{1}{2}\begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}$, with

$$\Sigma_{ij} = \begin{cases} \begin{pmatrix} f_{q_{\tau_i},q_{\tau_i}}(\omega) & 0 \\ 0 & f_{q_{\tau_i},q_{\tau_i}}(\omega)\end{pmatrix}, & \text{if } i=j, \\ \begin{pmatrix} \mathrm{Re} f_{q_{\tau_i},q_{\tau_j}}(\omega) & \mathrm{Im} f_{q_{\tau_i},q_{\tau_j}}(\omega) \\ -\mathrm{Im} f_{q_{\tau_i},q_{\tau_j}}(\omega) & \mathrm{Re} f_{q_{\tau_i},q_{\tau_j}}(\omega)\end{pmatrix}, & \text{if } i\neq j. \end{cases} \tag{20}$$

Moreover, $\mathbf{y}_{n,U}^{\tau_1,\tau_2}(\omega)$ is asymptotically independent for distinct ω's.

Let $\mathrm{Im}\, I^{\tau_1,\tau_2}(\omega) = \mathbb{C}^{\tau_1}(\omega)\mathbb{D}^{\tau_2}(\omega) - \mathbb{D}^{\tau_1}(\omega)\mathbb{C}^{\tau_2}(\omega)$. Then, we have:

Lemma 3 *If $\{x_t\}_{t\in\mathbb{Z}}$ is strictly stationary and satisfies assumption (C), then $I^{\tau_1,\tau_2}(\omega)$s are independent among distinct ω's; moreover, for every $\omega \in (0,\pi]$ and $\tau_1,\tau_2 \in [0,1]$, it holds*

$$\mathbf{E}\left[\mathrm{Im}\, I^{\tau_1,\tau_2}(\omega)\right] = \mathrm{Im} f_{q_{\tau_1},q_{\tau_2}}(\omega) \tag{21}$$

and

$$\begin{aligned}\mathbf{E}\left[\left(\mathrm{Im}\, I^{\tau_1,\tau_2}(\omega)\right)^2\right] =& \frac{3}{2}\left(\mathrm{Im} f_{q_{\tau_1},q_{\tau_2}}(\omega)\right)^2 + \frac{1}{2} f_{q_{\tau_1},q_{\tau_1}}(\omega) f_{q_{\tau_2},q_{\tau_2}}(\omega) \\ & - \frac{1}{2}\left(\mathrm{Re} f_{q_{\tau_1},q_{\tau_2}}(\omega)\right)^2 =: A_{\tau_1,\tau_2}(\omega).\end{aligned} \tag{22}$$

Proof To prove (21) and (22), from Lemma 2, it suffices to verify

$$\mathbf{E}\left[\mathbb{C}^{\tau_1}(\omega)\mathbb{D}^{\tau_2}(\omega) - \mathbb{D}^{\tau_1}(\omega)\mathbb{C}^{\tau_2}(\omega)\right] = \mathrm{Im} f_{q_{\tau_1},q_{\tau_2}}(\omega) \tag{23}$$

and

$$\mathbf{E}\left[\left(\mathbb{C}^{\tau_1}(\omega)\mathbb{D}^{\tau_2}(\omega) - \mathbb{D}^{\tau_1}(\omega)\mathbb{C}^{\tau_2}(\omega)\right)^2\right] = A_{\tau_1,\tau_2}(\omega). \tag{24}$$

From (20), the equality (23) holds obviously. By using Lemma 2.2 of Nagao (1973) and (20), some straightforward calculations yield the equality (24). □

For $p \geq 2$, $k_1, \ldots, k_{p-1} \in \mathbb{Z}$ and the quantile levels $\tau_1, \ldots, \tau_p \in [0, 1]$, consider the *copula cumulant kernel of order p*

$$\gamma^U_{k_1,\ldots,k_{p-1}}(\tau_1, \ldots, \tau_p) := \mathrm{cum}\big(\mathbb{I}_{(0,\tau_1]}(U_0), \mathbb{I}_{(0,\tau_1]}(U_{k_1}), \ldots, \mathbb{I}_{(0,\tau_p]}(U_{k_{p-1}})\big),$$

where $U_t = F(x_t)$. Note that, under assumption (C), the following quantity, which we call *copula spectral density kernel of order p*,

$$f_{q_{\tau_1},\ldots,q_{\tau_p}}(\omega_1, \ldots, \omega_{p-1}) := \frac{1}{(2\pi)^{p-1}} \sum_{k_1,\ldots,k_{p-1}=\infty}^{\infty} \gamma^U_{k_1,\ldots,k_{p-1}}(\tau_1, \ldots, \tau_p)\, \mathrm{e}^{-\mathrm{i}(k_1\omega_1+\cdots+k_{p-1}\omega_{p-1})}$$

exists for all $p \geq 2$ (pp. 1 of(Kley et al. 2015)).

Let

$$\begin{aligned}\varepsilon(\tau_1, \ldots, \tau_p, \omega_1, \ldots, \omega_p) :=&\mathrm{cum}\big(y^{\tau_1}_{n,U}(\omega_1), \ldots, y^{\tau_p}_{n,U}(\omega_p)\big)\\ &- (2\pi)^{p/2-1} n^{-p/2} \Delta_n\Big(\sum_{i=1}^{p} \omega_i\Big) f_{q_{\tau_1},\ldots,q_{\tau_p}}(\omega_1, \ldots, \omega_{p-1}),\end{aligned}$$

where $\Delta_n(\omega) := \sum_{t=1}^{n} \mathrm{e}^{-\mathrm{i}\omega t}$.

By Theorem 1.3 of Kley et al. (2015), we obtain the following lemma directly.

Lemma 4 *If $\{x_t\}_{t\in\mathbb{Z}}$ is strictly stationary and satisfies assumption (C), then*

$$\sup_{\tau_1,\ldots,\tau_p\in[0,1]} \sup_{\omega_1,\ldots,\omega_p\in(0,\pi]} \big|\varepsilon(\tau_1, \ldots, \tau_p, \omega_1, \ldots, \omega_p)\big| = O(n^{-p/2}). \tag{25}$$

Since $|\Delta_n(\omega)| \leq n$ holds, and $f_{q_{\tau_1},\ldots,q_{\tau_p}}(\omega_1, \ldots, \omega_{p-1})$ is bounded above uniformly for $(\omega_1, \ldots, \omega_{p-1}) \in (0, \pi]^{p-1}$ (pp. 1 of Kley et al. 2015), we obtain:

Corollary 1 *If $\{x_t\}_{t\in\mathbb{Z}}$ is strictly stationary and satisfies assumption (C), then*

$$\sup_{\tau_1,\ldots,\tau_p\in[0,1]} \sup_{\omega_1,\ldots,\omega_p\in(0,\pi]} \big|\mathrm{cum}\big(y^{\tau_1}_{n,U}(\omega_1), \ldots, y^{\tau_p}_{n,U}(\omega_p)\big)\big| = O(n^{1-p/2}). \tag{26}$$

Lemma 5 *If $\{x_t\}_{t\in\mathbb{Z}}$ is strictly stationary and satisfies assumption (C), then*

$$\mathbf{E}\big[\mathrm{Im}\, I^{\tau_1,\tau_2}_{n,U}(\omega)\big] = \mathrm{Im} f_{q_{\tau_1},q_{\tau_2}}(\omega) + O(\frac{1}{n}) \tag{27}$$

and

$$\mathbf{E}\big[\mathrm{Im}\, I^{\tau_1,\tau_2}_{n,U}(\omega_1)\, \mathrm{Im}\, I^{\tau_1,\tau_2}_{n,U}(\omega_2)\big] = \mathrm{Im} f_{q_{\tau_1},q_{\tau_2}}(\omega_1)\, \mathrm{Im} f_{q_{\tau_1},q_{\tau_2}}(\omega_2) + O(\frac{1}{\sqrt{n}}) \tag{28}$$

hold uniformly for all $\tau_1, \tau_2 \in [0, 1]$ and for all $\omega, \omega_1, \omega_2 \in (0, \pi]$. More generally, for each $p \in \mathbb{N}$ and $k_1, \ldots, k_p \in \mathbb{N}$,

$$\mathbf{E}\big[\big(\mathrm{Im}\, I^{\tau_1,\tau_2}_{n,U}(\omega_1)\big)^{k_1} \cdots \big(\mathrm{Im}\, I^{\tau_1,\tau_2}_{n,U}(\omega_p)\big)^{k_p}\big] = \mathbf{E}\big[\big(\mathrm{Im}\, I^{\tau_1,\tau_2}(\omega_1)\big)^{k_1} \cdots \big(\mathrm{Im}\, I^{\tau_1,\tau_2}(\omega_p)\big)^{k_p}\big] + O(\frac{1}{\sqrt{n}}) \tag{29}$$

hold uniformly for all $\tau_1, \tau_2 \in [0,1]$ *and all* $\omega_1, \dots, \omega_2 \in (0,\pi]$.

Proof Note that,

$$\operatorname{Im} I_{n,U}^{\tau_1,\tau_2}(\omega) = \frac{I_{n,U}^{\tau_1,\tau_2}(\omega) - I_{n,U}^{\tau_1,\tau_2}(-\omega)}{2\,\mathrm{i}} = \frac{y_{n,U}^{\tau_1}(-\omega)y_{n,U}^{\tau_2}(\omega) - y_{n,U}^{\tau_1}(\omega)y_{n,U}^{\tau_2}(-\omega)}{2\,\mathrm{i}}. \tag{30}$$

First expressing moments of the form $\mathbf{E}[y_{n,U}^{\tau_1}(\omega_1)\cdots y_{n,U}^{\tau_p}(\omega_p)]$ in terms of cumulants, then using (25) for $p = 2$ and (26) for $p \geq 3$ recursively, we can obtain (27)–(29).

To illustrate, we only prove (27).

Expressing $\mathbf{E}[y_{n,U}^{\tau_1}(-\omega)y_{n,U}^{\tau_2}(\omega)]$ in terms of cumulants, we obtain

$$\mathbf{E}[y_{n,U}^{\tau_1}(-\omega)y_{n,U}^{\tau_2}(\omega)] = \operatorname{cum}(y_{n,U}^{\tau_1}(-\omega), y_{n,U}^{\tau_2}(\omega)) + \mathbf{E}[y_{n,U}^{\tau_1}(-\omega)]\mathbf{E}[y_{n,U}^{\tau_2}(\omega)]. \tag{31}$$

According to (25), we have

$$\operatorname{cum}(y_{n,U}^{\tau_1}(-\omega), y_{n,U}^{\tau_2}(\omega)) = f_{q_{\tau_1},q_{\tau_2}}(\omega) + \varepsilon(\tau_1,\tau_2,-\omega,\omega) = f_{q_{\tau_1},q_{\tau_2}}(\omega) + O(\frac{1}{n}), \tag{32}$$

where $O(\frac{1}{n})$ holds uniformly for all $\tau_1, \tau_2 \in [0,1]$ and all $\omega \in (0,\pi]$. Combining (31) with (32), we obtain

$$\mathbf{E}[y_{n,U}^{\tau_1}(-\omega)y_{n,U}^{\tau_2}(\omega)] = f_{q_{\tau_1},q_{\tau_2}}(\omega) + \mathbf{E}[y_{n,U}^{\tau_1}(-\omega)]\mathbf{E}[y_{n,U}^{\tau_2}(\omega)] + O(\frac{1}{n}). \tag{33}$$

With arguments similar to prove (33), we have

$$\mathbf{E}[y_{n,U}^{\tau_1}(\omega)y_{n,U}^{\tau_2}(-\omega)] = f_{q_{\tau_1},q_{\tau_2}}(-\omega) + \mathbf{E}[y_{n,U}^{\tau_1}(\omega)]\mathbf{E}[y_{n,U}^{\tau_2}(-\omega)] + O(\frac{1}{n}). \tag{34}$$

Noting that $\mathbf{E}[y_{n,U}^{\tau}(\omega)] = (\sqrt{2\pi n})^{-1}\tau\sum_{t=1}^{n} \mathrm{e}^{\mathrm{i}\omega t}$, we find

$$\mathbf{E}[y_{n,U}^{\tau_1}(-\omega)]\mathbf{E}[y_{n,U}^{\tau_2}(\omega)] = \mathbf{E}[y_{n,U}^{\tau_1}(\omega)]\mathbf{E}[y_{n,U}^{\tau_2}(-\omega)]. \tag{35}$$

Combining (30) and (33)–(35) yields

$$\mathbf{E}\left[\operatorname{Im} I_{n,U}^{\tau_1,\tau_2}(\omega)\right] = \frac{f_{q_{\tau_1},q_{\tau_2}}(\omega) - f_{q_{\tau_1},q_{\tau_2}}(-\omega)}{2\,\mathrm{i}} + O(\frac{1}{n}) = \operatorname{Im} f_{q_{\tau_1},q_{\tau_2}}(\omega) + O(\frac{1}{n}).$$

□

Lemma 6 *If assumption* (*M*) *is satisfied, then we have for each* $\tau \in [0,1]$,

$$\sup_{\omega\in(0,2\pi]} |y_{n,R}^{\tau}(\omega) - y_{n,U}^{\tau}(\omega)| = o_P(n^{-1/4+\delta/2}) \tag{36}$$

holds, where δ *can be taken to any constant satisfying* $0 < \delta < 1/2$, *as in Condition* 1.

Proof If assumption (M) is satisfied, then for every $0 < \delta < 1/2$, there exists a $\theta > 1/\delta - 1$ such that $\beta(n) = O(n^{-\theta})$. It follows from (A.13) of Dette et al. (2015) that

$$\sup_{\omega \in (0,2\pi]} |y^{\tau}_{n,R}(\omega) - y^{\tau}_{n,U}(\omega)| = n^{-1/4}(n^{\frac{1}{1+\theta}} \log n)^{1/2} \log n \, O_P(1)$$
$$= n^{-1/4+\delta/2} \frac{(\log n)^{3/2}}{n^{\delta/2-1/2(1+\theta)}} O_P(1) = n^{-1/4+\delta/2} \, o_P(1),$$

since $\delta > 1/(1+\theta)$ holds. This proves the lemma. □

Lemma 7 *If assumption (M) is satisfied, then*

$$\sup_{\omega \in (0,2\pi]} |I^{\tau_1,\tau_2}_{n,U}(\omega)| = O_P(n^{1/k}), \tag{37}$$

$$\sup_{\omega \in (0,2\pi]} |I^{\tau_1,\tau_2}_{n,R}(\omega)| = O_P(n^{1/k}) \tag{38}$$

and

$$\sup_{\omega \in (0,2\pi]} |I^{\tau_1,\tau_2}_{n,R}(\omega) - I^{\tau_1,\tau_2}_{n,U}(\omega)| = o_P(n^{-1/4+\delta/2}) \tag{39}$$

hold for any $k \in \mathbb{N}$ and any $\delta \in (0, 1/2)$.

Proof By Lemma 1, assumption (M) implies assumption (C). Since $y^{\tau}_{n,U}(\omega)$ is defined as a piecewise constant function extended from (14), by Lemma A.6 of Kley et al. (2016), we have $\sup_{\omega \in (0,2\pi]} |y^{\tau}_{n,U}(\omega)| = O_P(n^{1/(2k)})$ for any $k \in \mathbb{N}$. Since the inequality $\sup_{\omega \in (0,2\pi]} |I^{\tau_1,\tau_2}_{n,U}(\omega)| \le \big(\sup_{\omega \in (0,2\pi]} |y^{\tau}_{n,U}(\omega)|\big)^2$ holds, we obtain (37).

By applying the triangular inequality, we have

$$|I^{\tau_1,\tau_2}_{n,R}(\omega) - I^{\tau_1,\tau_2}_{n,U}(\omega)| = |y^{\tau_1}_{n,R}(\omega)||y^{\tau_2}_{n,R}(\omega) - y^{\tau_2}_{n,U}(\omega)| + |y^{\tau_2}_{n,U}(\omega)||y^{\tau_1}_{n,R}(\omega) - y^{\tau_1}_{n,U}(\omega)|.$$

For any $\delta \in (0, 1/2)$, choosing any $\delta_0 \in (0, \delta)$, taking $k = [1/(\delta - \delta_0)]$ and applying (36), we obtain

$$\sup_{\omega \in (0,2\pi]} |y^{\tau_1}_{n,R}(\omega)||y^{\tau_2}_{n,R}(\omega) - y^{\tau_2}_{n,U}(\omega)| \le \sup_{\omega \in (0,2\pi]} |y^{\tau_1}_{n,R}(\omega)| \sup_{\omega \in (0,2\pi]} |y^{\tau_2}_{n,R}(\omega) - y^{\tau_2}_{n,U}(\omega)|$$
$$= O_P(n^{1/(2k)})\, o_P(n^{-1/4+\delta_0/2}) = o_P(n^{-1/4+\delta/2}),$$

since $\sup_{\omega \in (0,2\pi]} |y^{\tau}_{n,R}(\omega)| \le \sup_{\omega \in (0,2\pi]} |y^{\tau}_{n,U}(\omega)| + \sup_{\omega \in (0,2\pi]} |y^{\tau}_{n,R}(\omega) - y^{\tau}_{n,U}(\omega)| = O_P(n^{1/(2k)})$ holds for any $k \in \mathbb{N}$. Similarly, we obtain $\sup_{\omega \in (0,2\pi]} |y^{\tau_2}_{n,U}(\omega)||y^{\tau_1}_{n,R}(\omega) - y^{\tau_1}_{n,U}(\omega)| = o_P(n^{-1/4+\delta/2})$. This proves (39).

Since we have (37) and (39), using the triangular inequality straightforwardly produces (38). □

Lemma 8 *If assumption (C) is satisfied, then the CSDK is uniformly Hölder continuous, i.e.,*

$$|f_{q_{\tau_1},q_{\tau_2}}(\omega_1) - f_{q_{\tau_1},q_{\tau_2}}(\omega_2)| \le C|\omega_1 - \omega_2| \tag{40}$$

holds uniformly for all $(\tau_1, \tau_2) \in [0, 1]^2$.

Proof The assumption (C) implies that $|\mathbf{Cov}(\mathbb{I}_{A_1}(x_{t_1}), \mathbb{I}_{A_2}(x_{t_2}))| \le K\rho^{|t_1 - t_2|}$ holds for arbitrary intervals $A_1, A_2 \subset \mathbb{R}$ and arbitrary $t_1, t_2 \in \mathbb{Z}$, so does the $\gamma_k^U(\tau_1, \tau_2)$. By the triangular inequality, we obtain

$$|f_{q_{\tau_1},q_{\tau_2}}(\omega_1) - f_{q_{\tau_1},q_{\tau_2}}(\omega_2)| = \Big|\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\gamma_k^U(\tau_1, \tau_2)(\mathrm{e}^{-\mathrm{i}k\omega_1} - \mathrm{e}^{-\mathrm{i}k\omega_2})\Big|$$
$$\le \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}|\gamma_k^U(\tau_1, \tau_2)|\,|\mathrm{e}^{-\mathrm{i}k\omega_1}|\,|1 - \mathrm{e}^{\mathrm{i}k(\omega_1 - \omega_2)}| \le \frac{K}{2\pi}(\sum_{k=-\infty}^{\infty} k\rho^k)\,|\omega_1 - \omega_2| = C|\omega_1 - \omega_2|,$$

where the last inequality follows from the inequality $|\mathrm{e}^{\mathrm{i}a} - 1| \le |a|$. □

Proofs of Theorems 1 and 2

Proof of Theorem 1. Let

$$\widetilde{T}_M^{(\tau_1,\tau_2)} = \frac{\pi}{M}\sum_{m=1}^{M}\mathrm{Im} f_{q_{\tau_1},q_{\tau_2}}\big(\frac{m}{M}\pi\big) f_{q_{\tau_1},q_{\tau_2}}\big(\frac{m-1}{M}\pi\big),$$

and let

$$T_M^{(\tau_1,\tau_2)} = \frac{\pi}{M}\sum_{m=1}^{M}\Big(\mathrm{Im} f_{q_{\tau_1},q_{\tau_2}}\big(\frac{m}{M}\pi\big)\Big)^2.$$

We have the decomposition

$$\sqrt{M}\big(T_{n,M}^{(\tau_1,\tau_2)} - T^{(\tau_1,\tau_2)}\big) = \sqrt{M}\big(T_{n,M}^{(\tau_1,\tau_2)} - \widetilde{T}_{n,M}^{(\tau_1,\tau_2)}\big) + \sqrt{M}\big(\widetilde{T}_{n,M}^{(\tau_1,\tau_2)} - \widetilde{T}_M^{(\tau_1,\tau_2)}\big)$$
$$+ \sqrt{M}\big(\widetilde{T}_M^{(\tau_1,\tau_2)} - T_M^{(\tau_1,\tau_2)}\big) + \sqrt{M}\big(T_M^{(\tau_1,\tau_2)} - T^{(\tau_1,\tau_2)}\big),$$

where $\widetilde{T}_{n,M}^{(\tau_1,\tau_2)}$ is given by (12).

To prove (9), it suffices to verify

$$\sqrt{M}\big(T_{n,M}^{(\tau_1,\tau_2)} - \widetilde{T}_{n,M}^{(\tau_1,\tau_2)}\big) \xrightarrow{P} 0, \tag{41}$$

$$\sqrt{M}\big(\widetilde{T}_{n,M}^{(\tau_1,\tau_2)} - \widetilde{T}_M^{(\tau_1,\tau_2)}\big) \rightsquigarrow N(0, V^{(\tau_1,\tau_2)}), \tag{42}$$

$$\sqrt{M}\big(\widetilde{T}_M^{(\tau_1,\tau_2)} - T_M^{(\tau_1,\tau_2)}\big) \longrightarrow 0 \tag{43}$$

and

$$\sqrt{M}\big(T_M^{(\tau_1,\tau_2)} - T^{(\tau_1,\tau_2)}\big) \to 0, \tag{44}$$

as n goes to infinity, where "$\xrightarrow{P}$" denotes convergence in probability.

Proof of (41). By the triangular inequality, we have

$$\begin{aligned}
&\big|\sqrt{M}\big(T_{n,M}^{(\tau_1,\tau_2)} - \widetilde{T}_{n,M}^{(\tau_1,\tau_2)}\big)\big| \\
&\quad \le \frac{\pi}{\sqrt{M}} \sum_{m=1}^{M} \big|\mathrm{Im}\, I_{n,R}^{\tau_1,\tau_2}\big(\frac{m-1}{M}\pi\big)\big|\big|\mathrm{Im}\, I_{n,R}^{\tau_1,\tau_2}\big(\frac{m}{M}\pi\big) - \mathrm{Im}\, I_{n,U}^{\tau_1,\tau_2}\big(\frac{m}{M}\pi\big)\big| \\
&\qquad + \frac{\pi}{\sqrt{M}} \sum_{m=1}^{M} \big|\mathrm{Im}\, I_{n,U}^{\tau_1,\tau_2}\big(\frac{m}{M}\pi\big)\big|\big|\mathrm{Im}\, I_{n,R}^{\tau_1,\tau_2}\big(\frac{m-1}{M}\pi\big) - \mathrm{Im}\, I_{n,U}^{\tau_1,\tau_2}\big(\frac{m-1}{M}\pi\big)\big| \\
&\quad =: B_{1,n,M} + B_{2,n,M}.
\end{aligned}$$

For any $\delta \in (0, 1/2)$, choosing a $\delta_0 \in (0, \delta)$ and taking $k = [2/(\delta - \delta_0)]$, it follows from Lemma 7 that

$$\begin{aligned}
B_{1,n,M} &\le \pi\sqrt{M} \sup_{\omega\in(0,2\pi]} |I_{n,R}^{\tau_1,\tau_2}(\omega)| \sup_{\omega\in(0,2\pi]} |I_{n,R}^{\tau_1,\tau_2}(\omega) - I_{n,U}^{\tau_1,\tau_2}(\omega)| \\
&= o_P(M^{1/2}/n^{1/4-\delta_0/2-1/k}) = o_P\Big(\sqrt{\frac{M}{n^{1/2-\delta}}}\Big)
\end{aligned}$$

holds. Similarly, we also have $B_{2,n,M} = o_P\Big(\sqrt{\frac{M}{n^{1/2-\delta}}}\Big)$. This proves (41).

Proof of (42). According to Lemma P4.5 of Brillinger (2001), to prove (42), it is required to check that the q-th-order cumulant of $\sqrt{M}\big(\widetilde{T}_{n,M}^{(\tau_1,\tau_2)} - \widetilde{T}_M^{(\tau_1,\tau_2)}\big)$ behaves in the manner required by the $N(0, V^{(\tau_1,\tau_2)})$, for each $q \ge 1$.

By (28), we know $\mathbf{E}\big[\sqrt{M}\big(\widetilde{T}_{n,M}^{(\tau_1,\tau_2)} - \widetilde{T}_M^{(\tau_1,\tau_2)}\big)\big] = O(\sqrt{M/n})$. This indicates that the first cumulant behaves in the manner required.

From (29), we obtain that for each $k \in \mathbb{N}$,

$$\begin{aligned}
&\mathbf{E}\Big[\Big(\sum_{m=1}^{M} \mathrm{Im}\, I_{n,M}^{\tau_1,\tau_2}\big(\frac{m}{M}\pi\big)\, \mathrm{Im}\, I_{n,M}^{\tau_1,\tau_2}\big(\frac{m-1}{M}\pi\big)\Big)^2\Big] \\
&\qquad = \mathbf{E}\Big[\Big(\sum_{m=1}^{M} \mathrm{Im}\, I^{\tau_1,\tau_2}\big(\frac{m}{M}\pi\big)\, \mathrm{Im}\, I^{\tau_1,\tau_2}\big(\frac{m-1}{M}\pi\big)\Big)^2\Big] + O\big(\frac{M^2}{\sqrt{n}}\big)
\end{aligned}$$

holds. This implies that

$$\mathbf{Var}\big(\sqrt{M}\widetilde{T}_{n,M}^{(\tau_1,\tau_2)}\big) = V_M^{(\tau_1,\tau_2)} + O\big(\frac{M}{n^{1/2}}\big), \tag{45}$$

with

$$V_M^{(\tau_1,\tau_2)} = \frac{\pi^2}{M}\sum_{m=1}^{M}\mathbf{Var}\big(\mathrm{Im}\, I^{\tau_1,\tau_2}(\frac{m}{M}\pi)\mathrm{Im}\, I^{\tau_1,\tau_2}(\frac{m-1}{M}\pi)\big)$$
$$+ 2\frac{\pi^2}{M}\sum_{m=2}^{M}\mathbf{Cov}\big(\mathrm{Im}\, I^{\tau_1,\tau_2}(\frac{m}{M}\pi)\mathrm{Im}\, I^{\tau_1,\tau_2}(\frac{m-1}{M}\pi), \mathrm{Im}\, I^{\tau_1,\tau_2}(\frac{m-1}{M}\pi)\mathrm{Im}\, I^{\tau_1,\tau_2}(\frac{m-2}{M}\pi)\big).$$

Employing Lemma 3 in combination with straightforward derivations yields

$$\begin{aligned} V_M^{(\tau_1,\tau_2)} =& \frac{\pi^2}{M}\sum_{m=1}^{M}\Big\{A_{\tau_1,\tau_2}\big(\frac{m}{M}\pi\big)A_{\tau_1,\tau_2}\big(\frac{m-1}{M}\pi\big) \\ &- \big(\mathrm{Im} f_{q_{\tau_1},q_{\tau_2}}\big(\frac{m}{M}\pi\big)\big)^2\big(\mathrm{Im} f_{q_{\tau_1},q_{\tau_2}}\big(\frac{m-1}{M}\pi\big)\big)^2\Big\} \\ &+ 2\frac{\pi^2}{M}\sum_{m=2}^{M}\Big\{\mathrm{Im} f_{q_{\tau_1},q_{\tau_2}}\big(\frac{m}{M}\pi\big)A_{\tau_1,\tau_2}\big(\frac{m-1}{M}\pi\big)\mathrm{Im} f_{q_{\tau_1},q_{\tau_2}}\big(\frac{m-2}{M}\pi\big) \\ &- \mathrm{Im} f_{q_{\tau_1},q_{\tau_2}}\big(\frac{m}{M}\pi\big)\big(\mathrm{Im} f_{q_{\tau_1},q_{\tau_2}}\big(\frac{m-1}{M}\pi\big)\big)^2\mathrm{Im} f_{q_{\tau_1},q_{\tau_2}}\big(\frac{m-2}{M}\pi\big)\Big\}, \end{aligned} \tag{46}$$

where $A_{\tau_1,\tau_2}(\cdot)$ is defined in (22). As M goes to infinity, we have an alternative expression of (46) as

$$\begin{aligned} V_M^{(\tau_1,\tau_2)} =& \frac{\pi^2}{M}\sum_{m=1}^{M}\Big\{\Big(A_{\tau_1,\tau_2}\big(\frac{m}{M}\pi\big)\Big)^2 - \Big(\mathrm{Im} f_{q_{\tau_1},q_{\tau_2}}\big(\frac{m}{M}\pi\big)\Big)^4\Big\} \\ &+ 2\frac{\pi^2}{M}\sum_{m=2}^{M}\Big\{\Big(\mathrm{Im} f_{q_{\tau_1},q_{\tau_2}}\big(\frac{m}{M}\pi\big)\Big)^2 A_{\tau_1,\tau_2}\big(\frac{m}{M}\pi\big) - \Big(\mathrm{Im} f_{q_{\tau_1},q_{\tau_2}}\big(\frac{m}{M}\pi\big)\Big)^4\Big\} + o(1) \\ =& V^{(\tau_1,\tau_2)} + o(1). \end{aligned} \tag{47}$$

Combining (45) with (47) shows that the second-order cumulant of $\sqrt{M}\big(\widetilde{T}_{n,M}^{(\tau_1,\tau_2)} - \widetilde{T}_M^{(\tau_1,\tau_2)}\big)$ behaves in the manner required.

Finally, we prove the equality

$$\mathrm{cum}_q(\sqrt{M}\widetilde{T}_{n,M}^{(\tau_1,\tau_2)}) = O\big(\frac{M^{q/2}}{n^{2q-4}}\big) \tag{48}$$

holds for general $q \geq 3$.

Since the equality

$$\mathrm{cum}_q(\sqrt{M}\widetilde{T}^{(\tau_1,\tau_2)}_{n,M}) = \frac{\pi^q}{M^{q/2}} \sum_{m_1=1}^{M} \cdots \sum_{m_q=1}^{M} \Big\{ \mathrm{cum}\Big(\mathrm{Im}\, I^{\tau_1,\tau_2}_{n,U}\big(\frac{m_1}{M}\pi\big)\, \mathrm{Im}\, I^{\tau_1,\tau_2}_{n,U}\big(\frac{m_1-1}{M}\pi\big), \\ \cdots, \mathrm{Im}\, I^{\tau_1,\tau_2}_{n,U}\big(\frac{m_q}{M}\pi\big)\, \mathrm{Im}\, I^{\tau_1,\tau_2}_{n,U}\big(\frac{m_q-1}{M}\pi\big)\Big)\Big\}$$

holds, to prove (48), it suffices to check

$$\mathrm{cum}\big(\mathrm{Im}\, I^{\tau_1,\tau_2}_{n,U}(\omega_1)\,\mathrm{Im}\, I^{\tau_1,\tau_2}_{n,U}(\omega_2), \dots, \mathrm{Im}\, I^{\tau_1,\tau_2}_{n,U}(\omega_{2q-1})\,\mathrm{Im}\, I^{\tau_1,\tau_2}_{n,U}(\omega_{2q})\big) = O\big(n^{4-2q}\big) \tag{49}$$

holds uniformly for $\omega_1, \dots, \omega_{2q} \in (0, \pi]$. According to (30), to check (49), we need only to verify

$$\mathrm{cum}\big(y^{\tau_1}_{n,U}(\omega_1)y^{\tau_2}_{n,U}(\omega_1)\, y^{\tau_1}_{n,U}(\omega_2)y^{\tau_2}_{n,U}(\omega_2), \dots, y^{\tau_1}_{n,U}(\omega_{2q-1})y^{\tau_2}_{n,U}(\omega_{2q-1})\, y^{\tau_1}_{n,U}(\omega_{2q})y^{\tau_2}_{n,U}(\omega_{2q})\big) \\ = O\big(n^{4-2q}\big) \tag{50}$$

holds uniformly for $\omega_1, \dots, \omega_{2q} \in (0, \pi]$. Since we have established (26), using Theorem 2.3.2 of Brillinger (2001) yields (50). This proves (48) and indicates that the q-th-order ($q \geq 3$) cumulant of $\sqrt{M}\big(\widetilde{T}^{(\tau_1,\tau_2)}_{n,M} - \widetilde{T}^{(\tau_1,\tau_2)}_{M}\big)$ behaves in the manner required.

Now, all the cumulants of $\sqrt{M}\big(\widetilde{T}^{(\tau_1,\tau_2)}_{n,M} - \widetilde{T}^{(\tau_1,\tau_2)}_{M}\big)$ behave in the manner required by the $N(0, V^{(\tau_1,\tau_2)})$. This proves (42).

Proof of (43). Employing Lemmas 1 and 8 in combination with the triangular inequality, we have that

$$\sqrt{M}\big|\widetilde{T}^{(\tau_1,\tau_2)}_{M} - T^{(\tau_1,\tau_2)}_{M}\big| \\ \leq \frac{\pi}{\sqrt{M}} \sum_{m=1}^{M} \big|\mathrm{Im}f_{q_{\tau_1},q_{\tau_2}}\big(\frac{m}{M}\pi\big)\big|\, \Big|\mathrm{Im}f_{q_{\tau_1},q_{\tau_2}}\big(\frac{m}{M}\pi\big) - \mathrm{Im}f_{q_{\tau_1},q_{\tau_2}}\big(\frac{m-1}{M}\pi\big)\Big| \tag{51} \\ \leq C\frac{1}{\sqrt{M}} \longrightarrow 0,$$

as M goes to infinity.

Proof of (44). According to the first mean value theorem for definite integrals (e.g., Comenetz 2002, pp. 159), we obtain

$$\sqrt{M}\big(T^{(\tau_1,\tau_2)}_{M} - T^{(\tau_1,\tau_2)}\big) \\ = \sqrt{M} \sum_{m=1}^{M} \Big\{ \big(\mathrm{Im}f_{q_{\tau_1},q_{\tau_2}}\big(\frac{m}{M}\pi\big)\big)^2 \frac{\pi}{M} - \int_{\frac{m-1}{M}\pi}^{\frac{m}{M}\pi} \big(\mathrm{Im}f_{q_{\tau_1},q_{\tau_2}}(\omega)\big)^2 \,\mathrm{d}\omega \Big\} \\ = \frac{\pi}{\sqrt{M}} \sum_{m=1}^{M} \Big(\big(\mathrm{Im}f_{q_{\tau_1},q_{\tau_2}}\big(\frac{m}{M}\pi\big)\big)^2 - \big(\mathrm{Im}f_{q_{\tau_1},q_{\tau_2}}(\xi_m)\big)^2 \Big),$$

where $\xi_m \in (\frac{m-1}{M}\pi, \frac{m}{M}\pi)$, $m = 1, \dots, M$. Then, by the triangular inequality and the Hölder continuity (40), we have

$$\sqrt{M}\left|T_M^{(\tau_1,\tau_2)} - T^{(\tau_1,\tau_2)}\right| \leq C\frac{1}{\sqrt{M}}\sum_{m=1}^{M}\left\{\left(\left|\mathrm{Im}f_{q_{\tau_1},q_{\tau_2}}\left(\frac{m}{M}\pi\right)\right| + \left|\mathrm{Im}f_{q_{\tau_1},q_{\tau_2}}(\xi_m)\right|\right)\right.$$
$$\left.\times\left|\mathrm{Im}f_{q_{\tau_1},q_{\tau_2}}\left(\frac{m}{M}\pi\right) - \mathrm{Im}f_{q_{\tau_1},q_{\tau_2}}(\xi_m)\right|\right\}$$
$$\leq C\frac{\pi}{\sqrt{M}}\sum_{m=1}^{M}\left|\frac{m}{M}\pi - \xi_m\right| \leq C\frac{1}{\sqrt{M}} \longrightarrow 0, \tag{52}$$

as M goes to infinity. This completes the proof.

Proof of Theorem 2. Under the assumptions of Theorem 2, it follows from Theorem 3.5 of Kley et al. (2016) that $\hat{G}_{n,R}(\tau_1,\tau_2;\omega)$ is a consistent estimator of $f_{q_{\tau_1},q_{\tau_2}}(\omega)$ for each pair of levels (τ_1,τ_2). Then, Combining Theorem 1, the continuous mapping theorem and the definition of definite integral, we obtain the results of Theorem 2.

References

Beare, B. K., Seo, J. (2014). Time irreversible copula-based Markov models. *Economic Theory, 30*, 923–960.

Birr, S., Volgushev, S., Kley, T., Dette, H., Hallin, M. (2017). Quantile spectral analysis for locally stationary time series. *Journal of the Royal Statistical Society*: *Series B (Statistical Methodology), 79*, 1619–1643.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics, 31*, 307–327.

Bradley, R. C. (2005). Basic properties of strong mixing conditions. A survey and some open questions. *Probability Surveys, 2,* 107–144.

Brillinger, D. R. (2001). *Time series*: *Data analysis and theory*. Philadephia: SIAM.

Carrasco, M., Chen, X. (2002). Mixing and moment properties of various GARCH and stochastic volatility models. *Economic Theory, 18,* 17–39.

Chen, Y. (2003). Testing serial independence against time irreversibility. *Studies in Nonlinear Dynamics and Econometrics, 7,* 1–28.

Chen, Y., Kuan, C. (2002). Time irreversibility and EGARCH effects in US stock index returns. *Journal of Applied Econometrics, 17*, 565–578.

Chen, Y., Chou, R. Y., Kuan, C. (2000). Testing time reversibility without moment restrictions. *Journal of Econometrics, 95*, 199–218.

Comenetz, M. (2002). *Calculus*: *The elements*. Singapore: World Scientific Publishing Company.

Darolles, S., Florens, J. P., Gouriéroux, C. (2004). Kernel-based nonlinear canonical analysis and time reversibility. *Journal of Econometrics, 119*, 323–353.

Dette, H., Hildebrandt, T. (2012). A note on testing hypotheses for stationary processes in the frequency domain. *Journal of Multivariate Analysis, 104*, 101–114.

Dette, H., Paparoditis, E. (2009). Bootstrapping frequency domain tests in multivariate time series with an application to comparing spectral densities. *Journal of the Royal Statistical Society*: *Series B* (*Statistical Methodology*), *71*, 831–857.

Dette, H., Kinsvater, T., Vetter, M. (2011). Testing non-parametric hypotheses for stationary processes by estimating minimal distances. *Journal of Time Series Analysis, 32*, 447–461.

Dette, H., Preuss, P., Vetter, M. (2011). A measure of stationarity in locally stationary processes with applications to testing. *Journal of the American Statistical Association, 106*, 1113–1124.

Dette, H., Hallin, M., Kley, T., Volgushev, S. (2015). Of copulas, quantiles, ranks and spectra: An L_1 -approach to spectral analysis. *Bernoulli, 21*, 781–831.

Eichler, M. (2008). Testing nonparametric and semiparametric hypotheses in vector stationary process. *Journal of Multivariate Analysis, 99*, 968–1009.

Fryzlewicz, P., Rao, S. S. (2011). Mixing properties of ARCH and time-varying ARCH processes. *Bernoulli, 17*, 320–346.
Gasser, T., Műller, H. G., Mammitzsch, V. (1985). Kernels for nonparametric curve estimation. *Journal of the Royal Statistical Society*: *Series B* (*Statistical Methodology*), *47*, 238–252.
Hinich, M. J., Rothman, R. (1998). Frequency-domain test of time reversibility. *Macroeconomic Dynamics, 2*, 72–88.
Hong, Y. (2000). Generalized spectral tests for serial dependence. *Journal of the Royal Statistical Society*: *Series B* (*Statistical Methodology*), *62*, 557–574.
Jentsch, C., Pauly, M. (2015). Testing equality of spectral densities using randomization techniques. *Bernoulli, 21*, 697–739.
Kley, T., Volgushev, S., Dette, H., Hallin, M. (2015). Supplement to "Quantile spectral processes: Asymptotic analysis and inference''. *Bernoulli*. https://doi.org/10.3150/15-BEJ711SUPP.
Kley, T., Volgushev, S., Dette, H., Hallin, M. (2016). Quantile spectral process: Asymptotic analysis and inference. *Bernoulli, 22*, 1770–1807.
Koenker, R., Xiao, Z. (2006). Quantile autoregression. *Journal of the American Statistical Association, 101*, 980–990.
Mokkadem, A. (1988). Mixing properties of ARMA processes. *Stochastic Processes and their Applications, 29*, 309–315.
Nagao, H. (1973). On some test criteria for covariance matrix. *The Annals of Statistics, 1*, 700–709.
Paparoditis, E., Politis, D. N. (2002). The local bootstrap for Markov processes. *Journal of Statistical Planning and Inference, 108*, 301–328.
Psaradakis, Z. (2008). Assessing time reversibility under minimal assumptions. *Journal of Time Series Analysis, 29*, 881–905.
Racine, J. S., Maasoumi, E. (2007). A versatile and robust metric entropy test of time-reversibility, and other hypotheses. *Journal of Econometrics, 138*, 547–567.
Ramsey, J. B., Rothman, P. (1996). Time irreversibility and bussiness cycle asymmetry. *Journal of Money, Credit and Banking, 28*, 3–20.
Sharifdoost, M., Mahmoodi, S., Pasha, E. (2009). A statistical test for time reversibility of stationary finite state Markov chains. *Applied Mathematical Sciences, 52*, 2563–2574.
Silverman, B. (1986). *Density estimation for statistics and data analysis*. London: Chapman & Hall.
Tong, H. (1990). *Nonlinear time series-a dynamic system approach*. Oxford: Oxford University Press.
Wild, P., Foster, J., Hinich, M. J. (2014). Testing for non-linear and time irreversible probabilistic structure in high frequency financial time series data. *Journal of the Royal Statistical Society*: *Series A* (*Statistics in Society*), *177*, 643–659.
Zhang, S. (2019). Bayesian copula spectral analysis for stationary time series. *Computational Statistics and Data Analysis, 133,* 166–179.