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Slash distributions, generalized convolutions, and extremes

M. Arendarczyk[1] · **T. J. Kozubowski**[2] · **A. K. Panorska**[2]

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Abstract

An α-slash distribution built upon a random variable X is a heavy tailed distribution corresponding to $Y = X/U^{1/\alpha}$, where U is standard uniform random variable, independent of X. We point out and explore a connection between α-slash distributions, which are gaining popularity in statistical practice, and generalized convolutions, which come up in the probability theory as generalizations of the standard concept of the convolution of probability measures and allow for the operation between the measures to be random itself. The stochastic interpretation of Kendall convolution discussed in this work brings this theoretical concept closer to statistical practice, and leads to new results for α-slash distributions connected with extremes. In particular, we show that the maximum of independent random variables with α-slash distributions is also a random variable with an α-slash distribution. Our theoretical results are illustrated by several examples involving standard and novel probability distributions and extremes.

Keywords Extreme value theory · Generalized convolution · Heavy tails · Kendall convolution · Pareto distribution · Slash distribution

✉ T. J. Kozubowski
tkozubow@unr.edu

M. Arendarczyk
marendar@math.uni.wroc.pl

A. K. Panorska
ania@unr.edu

1 Mathematical Institute, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

2 Department of Mathematics and Statistics, University of Nevada, 1664 N. Virginia Street, Mail Stop 84, Reno, NV 89557, USA

1 Introduction and notation

The concept of *slash* distributions, going back to the classical works on robustness (see Andrews et al., 1972; Rogers and Tukey, 1972), has experienced a rapid revival in recent years. The standard (also known as canonical or uniform) slash distribution (with $\alpha = 1$ in the formula below) arises as the distribution of the ratio

$$Y \stackrel{d}{=} \frac{X}{U^{1/\alpha}} \tag{1}$$

of a standard normal random variable X and an independent standard uniform random variable U (Andrews et al., 1972; Johnson et al., 1994, p. 63). Its more general version with $\alpha \in (0, 1)$ in (1) was studied by Rogers and Tukey (1972). Since then, this concept has been generalized in various ways, leading to a multitude of generalized probability distributions which are useful across many areas of application. In particular, multivariate slash distributions (see, e.g., Abanto-Valle et al., 2012; Arslan, 2008, 2009; Arslan and Genç, 2009; Cabral et al., 2012), obtained when the X in (1) is (either symmetric or skew) multivariate normal, proved useful in the linear modeling framework (Lachos, 2008; Lachos et al., 2010a, b; Lange and Sinsheimer, 1993; Liu, 1996; Wang and Genton, 2006). Here, alternative strategies based on slash distributions or similar *scale mixtures of Gaussian distributions* (or *normal/independent* distributions in the terminology of Lange and Sinsheimer, 1993) provide robustness against the underlying assumptions of normality in the classical framework. Further generalizations, where X in (1) is not necessarily normally distributed, abound in recent statistics literature, including an extension to discrete distributions (Jones, 2020) as well as one involving matrix-variate distributions (Bulut and Arslan, 2015).

The objective of this work is to point out and explore an unexpected connection between the concept of slash distribution in statistics and a branch of probability theory concerned with generalizations of the standard concept of convolution of probability measures. The latter, known as *generalized convolutions*, are commutative and associative binary operations on probability measures on $\mathbb{R}_+$ that have properties analogous to the standard operation of addition of independent random elements (see, e.g., Bingham, 1971; Kucharczak and Urbanik, 1974; Urbanik, 1964). The theory of generalized convolutions provides a unifying framework bringing together different limit schemes in probability theory, including the standard addition scheme as well as the extreme value theory, among others. In particular, the notion of the characteristic function is also extended in the theory of generalized convolutions to that of a generalized characteristic function, which has the same properties for its generalized convolution as the standard characteristic function has for the summation of random variables. An interesting property of the generalized convolutions is that they allow randomness in the operation between the probability measures (or random variables) itself (see Sect. 2, Proposition 4).

A comprehensive theory, with the focus on Lévy and additive processes in the context of generalized convolutions, has been recently developed in Borowiecka-Olszewska et al. 2015 (see also Jasiulis-Gołdyn et al. 2020a), where one can find numerous examples of this concept. It should be noted that, although there exist some results connected to applications (see, e.g., Jasiulis-Gołdyn et al., 2020b; Panorska, 1996, 1999), the great majority of the works on generalized convolutions are in the realm of theoretical probability, and the definitions of the convolutions are via probability measures, with little or no interpretation in terms of operation on random variables. This is the reason why the generalized convolutions are not very well known in applications. In this context we provide a new interpretation of two major generalized convolutions: the Kendall convolution (see, e.g., Arendarczyk et al., 2019; Borowiecka-Olszewska et al., 2015; Jasiulis-Gołdyn et al., 2020a, b; Jasiulis-Gołdyn and Misiewicz, 2011) and Kucharczak–Urbanik convolution (see, e.g., Borowiecka-Olszewska et al., 2015; Kucharczak and Urbanik, 1974), which involves the slash operation.

In short, the connection between the slash of X given by Y in (1) and Kendall convolution is that the cumulative distribution function (CDF) of Y is related to the generalized characteristic function corresponding to the Kendall convolution. More formally, this connection is provided by the fact that the CDF of the Y in (1) evaluated at $t \in \mathbb{R}_+$, with the distribution of X in (1) supported on $\mathbb{R}_+$, is given by the expectation $\mathbb{E}[h(X/t)]$, where $h(\cdot)$ can serve as the kernel in the integral representation of the generalized characteristic function (see Appendix A). This connection with the slash transformation (1) leads to a very natural interpretation of the Kendall convolution in terms of slash operation, which aids in the understanding of this abstract mathematical construction and may lead to applications. In turn, through the theory of generalized convolutions, one can establish certain new fundamental properties of the slash transformation related to extremes, presented in the subsequent sections. The main results connecting generalized convolutions with the slash transformation are given in Theorem 1 and Remark 4 for the Kendall convolution. In Sect. 4 we discuss possible generalization of these results.

In this context, note that since all slash distributions are heavy tailed (Jones, 2020), they provide an array of models for power law data. Such data have been commonly observed in finance, climate science, environmental science, hydrology, social studies, health care, demographics, and Internet traffic, among others (see., e.g., Embrechts et al., 1997 and Sornette, 2004, and the references therein). Due to the connection between slash laws and generalized convolutions, our results should bring generalized convolutions closer to the statistical practice and applications.

In this work, we restrict attention to distributions of nonnegative random variables, as the classical generalized convolutions have the same restriction. Following Jones (2020), a random variable Y given by the right-hand side in (1) will be said to have an α-slash distribution (derived from that of X). The main results of this paper are presented in Sect. 2, where we also provide more background information on the slash transformation (1). In particular, we prove that the maximum of independent random variables with α-slash distributions is also a random variable with an α-slash distribution, and provide the exact structure of the latter. In the process, we discuss the concept of "inverting" an α-slash distribution, leading to the recovery of the distribution of X in (1) from that of Y. Our results are illustrated by several examples of α-slash families from the literature presented in Sect. 3. Here, we also briefly mention an approximation method of Kendall generalized sums with a large number of

IID components. Since the background material on generalized convolutions is quite technical, we include it in an appendix (Appendix A), while the proofs of the main results are collected in Appendix B.

Notation Before concluding the introduction, we introduce the notation that shall be used throughout this paper. For a random variable X, the function $F_X(\cdot)$ denotes the CDF of X, while $\overline{F}_X(\cdot) = 1 - F_X(\cdot)$ denotes its survival function (SF). The expression $G_X^{(\alpha)}(\cdot)$ shall denote the CDF of the right-hand side in (1), that is the CDF of the α-slash transformation of X. Further, $a \vee b = \max\{a, b\}$, $a \wedge b = \min\{a, b\}$, and $(x)_+ := x \vee 0 = \max\{x, 0\}$. Similarly, $\bigvee_{i=1}^n a_i$ denotes the maximum of the $\{a_i\}$, $i = 1, \dots, n$, while $\bigwedge_{i=1}^n a_i$ denotes their minimum. Finally, the notation "$\stackrel{d}{=}$" indicates equality in distribution.

The following notation is related to the generalized convolution theory (see Appendix A). We use $\mathcal{P}_+$ to denote the family of all probability measures on the Borel σ-algebra $\mathcal{B}(\mathbb{R}_+)$ with $\mathbb{R}_+ := [0, \infty)$. The quantity δ_x denotes the probability measure concentrated at the point $x \geq 0$. If $\lambda \in \mathcal{P}_+$ is the distribution of a random variable X and $a \in \mathbb{R}_+$, then $T_a\lambda$ denotes the distribution of aX, so that $(T_a\lambda)(A) := \lambda(A/a)$ where $A/x = \{ax^{-1} : a \in A\}$ for any $A \in \mathcal{B}(\mathbb{R}_+)$. Similarly, for a function $f : \mathbb{R}_+ \to \mathbb{R}$ and $a > 0$ we define the transformation T_a by $T_a f(t) := f(at)$. We use "$\Longrightarrow$" to denote weak convergence of probability measures. The Kendall convolution of measures $\nu_1, \nu_2 \in \mathcal{P}_+$ (see Definition 5 in Appendix A) is denoted by $\nu_1 *_\alpha \nu_2$, whereas $\nu_1 *_{\alpha,n} \nu_2$ denotes its extension known as the Kucharczak–Urbanik convolution (see Definition 4 in Appendix A). If X and Y are independent random variables with distributions $\lambda_1, \lambda_2 \in \mathcal{P}_+$, respectively, then $X \oplus Y$ shall denote a random variable with distribution $\lambda_1 *_\alpha \lambda_2$. Analogously, if $X_1, X_2, \dots, X_n$ are independent random variables with distributions $\lambda_1, \lambda_2, \dots, \lambda_n$, respectively, then $\bigoplus_{i=1}^n X_i$ denotes a random variable with distribution $\lambda_1 *_\alpha \lambda_2 *_\alpha \cdots *_\alpha \lambda_n$. We use $\Phi_\lambda^{(\alpha,n)}$ to denote the $*_{\alpha,n}$-generalized characteristic function (ChF) of the measure λ in the $(\mathcal{P}_+, *_{\alpha,n})$ algebra. Analogously, for a random variable X with distribution λ we use the notation $\Phi_X^{(\alpha,n)}$ for the generalized ChF of X. In the special case when $n = 1$ (corresponding to Kendall convolution) the index n in the ChF shall be omitted.

2 Main results

To set the stage, let us start with a formal definition of an α-slash transformation.

Definition 1 We say that a nonnegative random variable Y has an α-slash distribution if there exists another nonnegative random variable X and an independent of X random variable U with a uniform distribution on [0, 1], such that $Y \stackrel{d}{=} X/U^{\frac{1}{\alpha}}$ for some $\alpha > 0$.

While the concept of a slash distribution (or slash transformation) in Definition 1 extends to the distributions on the entire real line or on $\mathbb{R}^d$, we restrict our attention to the case of positive random variables, following the classical generalized convolution theory.

Remark 1 It should be noted that the effect of dividing by $U^{1/\alpha}$ in (1) is the same as that of multiplying by $1+R$, that is $Y \stackrel{d}{=} (1+R)X$ where $R = 1/U^{1/\alpha} - 1$ has the standard Pareto Type II (Lomax) distribution, given by the SF

$$\mathbb{P}(R > r) = \left(\frac{1}{1+r}\right)^{\alpha}, \quad r \in \mathbb{R}_+.$$

This gives an interpretation in terms of a growth rate, where $R = (Y - X)/X$ is a relative change from X to Y. In financial applications, X is the present value, R is the (stochastic) growth rate (interest rate), and Y is the accumulated value.

We now provide a key result that links the slash transformation (1) with generalized convolutions. Indeed, we show below that the CDF of an α-slash distribution obtained in (1) is directly related to the generalized characteristic function (24) corresponding to Kendall convolution $*_\alpha$ (see Appendix A, Definition 5).

Proposition 1 *For any random variable X with distribution λ supported on $\mathbb{R}_+$ and $t \geq 0, \alpha > 0$, we have*

$$G_X^{(\alpha)}(t) = \Phi_X^{(\alpha)}\left(\frac{1}{t}\right),$$

where $G_X^{(\alpha)}(\cdot)$ is the CDF of Y in (1) and $\Phi_X^{(\alpha)}(\cdot)$ is given by (24) in Appendix A.

Proof By Definition 1 of the α-slash distribution, we have

$$\begin{aligned} G_X^{(\alpha)}(t) &= \mathbb{P}\left(\frac{X}{U^{\frac{1}{\alpha}}} \leq t\right) = \mathbb{P}\left(U \geq \left(\frac{X}{t}\right)^{\alpha}\right) = \int_0^t \left(1 - \left(\frac{x}{t}\right)^{\alpha}\right) \mathrm{d}F_X(x) \\ &= \Phi_X^{(\alpha)}\left(\frac{1}{t}\right), \end{aligned}$$

where the last equality follows from (24) with t replaced by $1/t$. This completes the proof. □

An important question related to the slash transformation concerns "inverting" an α-slash distribution, that is finding the distribution of the underlying random variable X in (1) given that of the random variable Y. The result below shows that this is rather straightforward.

Lemma 1 *Let $F_X(\cdot)$ and $G_X^{(\alpha)}(\cdot)$ be the CDFs of the random variables X and Y in (1), respectively, where the distribution of X is supported on $\mathbb{R}_+$. Then,*

$$F_X(t) = G_X^{(\alpha)}(t) + \frac{t}{\alpha}\frac{\mathrm{d}}{\mathrm{d}t}G_X^{(\alpha)}(t) \tag{2}$$

for each $t > 0$ *at which* $G_X^{(\alpha)}(t)$ *is differentiable.*

Proof For any $t > 0$, integration by parts leads to

$$G_X^{(\alpha)}(t) = \int_0^t \left(1 - \left(\frac{x}{t}\right)^\alpha\right)\mathrm{d}F_X(x) = \alpha t^{-\alpha}\int_0^t s^{\alpha-1}F_X(s)\mathrm{d}s.$$

The proof is completed by differentiating both sides of the above equation with respect to t. □

Remark 2 It should be noted that in the case of the classical slash distribution based on normally distributed X in (1), the relation (2) appeared in Rogers and Tukey (1972). For an absolutely continuous distribution, it can also be found in Jones (2020). See also Jasiulis-Gołdyn et al. (2020b, Lemma 3), for an analogous result in the context of the generalized convolution theory.

Remark 3 It may be also of interest to know whether the distribution of a general random variable Y is an α-slash distribution for some $\alpha > 0$, that is whether the relation (1) holds with some X. This can be done by checking whether the quantity on the right-hand side of (2) with $G_X^{(\alpha)}(\cdot)$ replaced by the CDF of Y is a genuine CDF for some $\alpha > 0$. Generally this problem of "inverting" a distribution under the slash transformation may not have a solution, as a necessary condition for this to happen is that Y have infinite moments of order α and above (see Jones, 2020). For example, the exponential distribution does not have an inverse under the slash transformation, as its moments of positive order are all finite.

Following on from the previous remark we now establish our main result, which shows that the maximum of independent random variables Y_i with the α-slash distributions has an α-slash distribution as well, so that it always admits an inverse under the slash transformation. Moreover, that inverse is distributed as a generalized convolution of the distributions of X_i, where the $\{X_i\}$ are the inverses of the $\{Y_i\}$ under the slash transformation.

Theorem 1 *Let* $\alpha > 0, n \in \mathbb{N}$, *and let* $X_1, X_2, \dots, X_n$ *be independent, nonnegative random variables with distributions* $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathcal{P}_+$, *respectively. Further, let* $U_1, U_2, \dots, U_n$ *be independent standard uniform random variables, independent of the* $\{X_i\}$. *Then, there exists a random variable* X *with distribution* $\lambda = \lambda_1 *_\alpha \lambda_2 *_\alpha \cdots *_\alpha \lambda_n$ *and a standard uniform random variable* U, *independent of* X, *such that*

$$\bigvee_{i=1}^{n} \frac{X_i}{(U_i)^{\frac{1}{\alpha}}} \stackrel{d}{=} \frac{X}{U^{\frac{1}{\alpha}}}.$$

Remark 4 Let us emphasize that the distribution of X in Theorem 1 is the Kendall convolution $\lambda_1 *_\alpha \lambda_2 *_\alpha \cdots *_\alpha \lambda_n$ of the distributions $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the random variables $X_1, X_2, \ldots, X_n$. To account for the connection between the random variables $X_1, X_2, \ldots, X_n$ and X, we shall use the notation

$$X := \bigoplus_{i=1}^{n} X_i. \tag{3}$$

This result provides a new stochastic interpretation of the Kendall convolution (3), whose definition is rather abstract. The slash transformation plays a key role in this interpretation. Namely, the Kendall convolution of the distributions of random variables $\{X_i\}$ can be described as follows: (i) First, apply the slash transformation (1) to each random variable X_i with distribution λ_i to obtain its α-slash version, Y_i; (ii) Next, assuming independence of the $\{Y_i\}$, determine the probability distribution of their maximum, $Y = \max\{Y_1, \ldots, Y_n\}$; (iii) Finally, invert the distribution of Y under the slash transformation (1) using Lemma 1, leading to the distribution of X in (1). The resulting distribution of X is precisely that of (3).

The following result provides the CDF of $\bigoplus_{i=1}^{n} X_i$ in a convenient form.

Theorem 2 *Let $X_1, X_2, \ldots, X_n$ be independent nonnegative random variables. Then,*

$$F_{\bigoplus_{i=1}^{n} X_i}(t) = \left\{ 1 + \frac{t}{\alpha} \sum_{j=1}^{n} \frac{\frac{\mathrm{d}}{\mathrm{d}t} G_{X_j}^{(\alpha)}(t)}{G_{X_j}^{(\alpha)}(t)} \right\} \prod_{i=1}^{n} G_{X_i}^{(\alpha)}(t), \tag{4}$$

for each $t \in \mathbb{R}_+$ at which the $G_{X_i}^{(\alpha)}(t)$ are differentiable.

We also provide an alternative form of the CDF of $\bigoplus_{i=1}^{n} X_i$, which is more directly related to the generalized convolution theory, particularly the representation (20). While both of these results utilize Lemma 1 in their proofs, the latter differs in the way it handles the product of $G_{X_i}^{(\alpha)}(t)$ and its derivative, as can be seen in the proofs provided in Sects. 2 and 3, respectively.

Proposition 2 *Let $X_1, X_2, \ldots, X_n$ be independent random variables with distributions $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathcal{P}_+$, respectively. Then,*

$$F_{\bigoplus_{i=1}^{n} X_i}(t) = \int_0^\infty \int_0^\infty \cdots \int_0^\infty F_{\bigoplus_{i=1}^{n} x_i}(t)\, \lambda_1(\mathrm{d}x_1)\lambda_2(\mathrm{d}x_2) \cdots \lambda_n(\mathrm{d}x_n), \tag{5}$$

where

$$F_{\bigoplus_{i=1}^{n} x_i}(t) = \left[1 + \sum_{i=1}^{n-1} (-1)^i i \frac{\sum_{U \in \mathcal{U}_{i+1}^{(n)}} \prod_{j \in U} x_j}{t^{(i+1)\alpha}} \right] \mathbf{1}\left(t \geq \bigvee_{i=1}^{n} x_i \right), \quad x_i \in \mathbb{R}_+, \tag{6}$$

with $\mathcal{U}_i^{(n)}$ *denoting the family of all subsets of* $\{1, 2, \dots, n\}$ *with exactly i elements.*

For $n = 2$, the formula (5) reduces to the expression below, which is well-known in the theory of generalized convolutions (see, e.g., Jasiulis-Gołdyn et al., 2020b, Section 3.2):

$$F_{X\oplus Y}(t) = \int_0^\infty \int_0^\infty \left(1 - \frac{(xy)^\alpha}{t^{2\alpha}}\right)\mathbf{1}(t \geq x \vee y)\lambda_1(\mathrm{d}x)\lambda_2(\mathrm{d}y). \tag{7}$$

Note that, unless $x = y$, the distribution of $X \oplus Y$ in (7) is of mixed type, neither continuous nor discrete. The same is true for the general case of Proposition 2.

Remark 5 Let us observe that Theorem 1 is based on the idea that

$$G_X^{(\alpha)}(t)G_Y^{(\alpha)}(t) = G_{X\oplus Y}^{(\alpha)}(t), \tag{8}$$

where $X \oplus Y$ is a random variable with the distribution given by (7). Although this is a straightforward consequence of the properties of the $*_\alpha$-generalized characteristic function in the generalized convolution theory, we believe that a direct proof of (8) would be of interest to readers. Thus, we include it in Appendix B (see Sect. 4).

We conclude this section with a stochastic representation of $\bigoplus_{i=1}^n X_i$. We start with the case $n = 2$, where the result below is a direct consequence of the representation of the Kendall convolution [see (25) in Appendix A].

Proposition 3 *Let* X_1, X_2 *be independent, nonnegative random variables, and let* U_1, U_2 *be independent standard uniform random variables, independent of* X_1 *and* X_2*. Then*

$$X_1 \oplus X_2 \stackrel{d}{=} (X_1 \vee X_2)\left[\frac{1}{(U_1)^{\frac{1}{2\alpha}}}\mathbf{1}\left(U_2 \leq \left(\frac{X_1 \wedge X_2}{X_1 \vee X_2}\right)^\alpha\right)\right.$$
$$\left. +\mathbf{1}\left(U_2 > \left(\frac{X_1 \wedge X_2}{X_1 \vee X_2}\right)^\alpha\right)\right].$$

A generalization with $n \geq 2$ provided below can be established by induction. A special case of this result with *identically distributed* $\{X_i\}$, formulated in terms of a Markov process, can be found in Arendarczyk et al. (2019), Proposition 2.7 (see also Jasiulis-Gołdyn et al., 2020b).

Proposition 4 *Let* $X_i, i \in \mathbb{N}$*, be independent, nonnegative random variables and let* $U_{1,i}, U_{2,i}, i \in \mathbb{N}$*, be IID standard uniform random variables, independent of the* $\{X_i\}$*. Then*

$$\bigoplus_{i=1}^{n+1} X_i = \left(\bigoplus_{i=1}^{n} X_i \vee X_{n+1} \right) W,$$

where

$$W = \left\{ \frac{1}{(U_{1,n+1})^{\frac{1}{2\alpha}}} \mathbf{1} \left[U_{2,n+1} \leq \left(\frac{\bigoplus_{i=1}^{n} X_i \wedge X_{n+1}}{\bigoplus_{i=1}^{n} X_i \vee X_{n+1}} \right)^{\alpha} \right] \right.$$
$$\left. + \mathbf{1} \left[U_{2,n+1} > \left(\frac{\bigoplus_{i=1}^{n} X_i \wedge X_{n+1}}{\bigoplus_{i=1}^{n} X_i \vee X_{n+1}} \right)^{\alpha} \right] \right].$$

As shown by the above results, when Kendall convolution is seen through the lens of random variables $\{X_i\}$, the relevant operation involves an element of randomness (in this case connected with the uniform random variables) in addition to the $\{X_i\}$. Such a random effect in the operation on random variables goes back to the work of Kingman (1963), which sparked the subsequent research in this area and led to a mathematical development of the theory of generalized convolutions.

3 Examples

As noted by Jones (2020), the random variable Y obtained via the slash transformation in (1) can only have finite moments $\mathbb{E}(Y^r)$ of order $r < \alpha$. Thus, this transformation is a convenient way to generate a multitude of heavy tailed probability distributions that may be useful in applications. In this section we present several examples involving slash distributions and extremes, illustrating our main results.

3.1 Fréchet distribution and its slash-inverse

Let Y have a Fréchet distribution with shape parameter $\alpha > 0$ and scale parameter $\sigma > 0$, so that the CDF of Y is of the form

$$F_Y(t) = e^{-(t/\sigma)^{-\alpha}}, \quad t \in \mathbb{R}_+.$$

We shall denote this distribution by $\mathcal{F}(\alpha, \sigma)$. This is a heavy tailed distribution with tail index α, so its moments of order α and above do not exist. Thus, Y may arise in a slash transformation (1) with some nonnegative random variable X. Straightforward algebra involving Lemma 1 shows that this is indeed the case, and the variable X in (1) has the CDF of the form

$$F_X(t) = [1 + (t/\sigma)^{-\alpha}]e^{-(t/\sigma)^{-\alpha}}, \quad t \in \mathbb{R}_+. \tag{9}$$

This distribution, which is also heavy tailed with tail index 2α, shall be denoted by $\mathcal{ISF}(\alpha, \sigma)$ ($\mathcal{I}$nverse $\mathcal{S}$lash $\mathcal{F}$réchet). In the context of these two distributions, let us illustrate Theorem 1. Suppose that $X_1, \dots, X_n$ are independent with $X_i \sim \mathcal{ISF}(\alpha, \sigma_i)$, $i = 1, \dots, n$. Then, according to the above discussion, it follows that $Y_i = X_i / U_i^{1/\alpha} \sim \mathcal{F}(\alpha, \sigma_i)$, $i = 1, \dots, n$, where the $\{U_i\}$ are standard uniform random variables, independent of the $\{X_i\}$. By the well-known stability property of the Fréchet distribution with respect to the operation of taking maxima, we have the equality of distribution

$$\bigvee_{i=1}^{n} Y_i \stackrel{d}{=} Y \sim \mathcal{F}(\alpha, \sigma), \quad \sigma = \left(\sum_{i=1}^{n} \sigma_i^{\alpha} \right)^{1/\alpha}. \tag{10}$$

In turn, the variable Y in (10) is of the form (1) with $X \sim \mathcal{ISF}(\alpha, \sigma)$ with σ as in (10). Moreover, by Theorem 1, the distribution of this X is exactly the same as the Kendall convolution of the distributions of the $\{X_i\}$. Since the parameter σ here is a scale factor, we have the following relation:

$$\bigoplus_{i=1}^{n} X_i \stackrel{d}{=} \frac{\sigma}{\sigma_1} X_1 \sim \mathcal{ISF}(\alpha, \sigma), \quad \sigma = \left(\sum_{i=1}^{n} \sigma_i^{\alpha} \right)^{1/\alpha}.$$

This shows that the ISF distribution given by the CDF (9) is *stable* with respect to the Kendall convolution (see also Example 3.4 in Arendarczyk et al., 2019). The interpretation of this in terms of the slash transformation and its inverse is as follows. If we start with independent ISF variables with the same shape parameter α and convert them to their independent slash versions via (1), and subsequently calculate their maximum, then when this maximum is inverted with respect to the slash transformation, the resulting variable has an ISF distribution with shape parameter α as well. This shows the *stability* of the $\mathcal{ISF}(\alpha, \sigma)$ distribution given in (9) with respect to the max-slash operation.

Remark 6 An important property of the *ISF* distribution, which can be useful in applications, is that it can serve as an approximation of the random variable $\bigoplus_{i=1}^{n} X_i$ for large enough n. Indeed, if $\{X_i\}$ is a sequence of IID random variables such that $Y_i = X_i / U_i^{1/\alpha}$ are in the maximum domain of attraction of the Fréchet distribution $\mathcal{F}(\beta, \sigma)$, then

$$\lim_{n \to \infty} \mathbb{P}\left(a_n \bigvee_{i=1}^{n} \frac{X_i}{(U_i)^{\frac{1}{\alpha}}} \le t \right) = e^{-t^{-\beta}} \sim \mathcal{F}(\beta, 1)$$

for a suitable sequence of norming constants $\{a_n\}$ (see, e.g., Embrechts et al., 1997). Note that, by Breiman's theorem (see Breiman, 1965), if there exists an $\varepsilon > 0$ such that $\mathbb{E}(X_1^{\varepsilon+\alpha}) < \infty$ then $\beta = \alpha$. We can now conclude from the results of this section that the ISF distribution with the CDF (9) can serve as an approximation of the distribution of $\bigoplus_{i=1}^{n} X_i$ for n large enough. Direct, formal proof of this observation can be obtained by application of a result by Bingham (see Arendarczyk et al., 2019, Remark 4.4 and Proposition 4.5, and Bingham, 1971) on domains of attraction for generalized convolutions.

3.2 Pareto distribution and its slash-inverse

The example below concerns the Pareto distribution with tail parameter $\alpha > 0$ and scale parameter $\sigma > 0$, given by the CDF

$$H(t) = \left(1 - \left(\frac{\sigma}{t}\right)^{\alpha}\right)_{+}, \quad t \in \mathbb{R}_{+}, \tag{11}$$

and denoted by $\mathcal{P}(\alpha, \sigma)$. This distribution arises as the distribution of Y under the slash transformation (1) when $X = \sigma$ with probability 1. In other words, the inverse of the Pareto distribution (11) under the slash transformation (1) is δ_{σ}, the probability measure concentrated at σ.

Assume that $Y_1, \dots, Y_n$ are independent random variables where $Y_i \sim \mathcal{P}(\alpha, \sigma_i)$, $i = 1, \dots, n$. Then each Y_i has an absolutely continuous distribution with the CDF and the PDF given by

$$F_{Y_i}(t) = 1 - \left(\frac{\sigma_i}{t}\right)^{\alpha}, \quad f_{Y_i}(t) = \frac{\alpha}{\sigma_i}\left(\frac{\sigma_i}{t}\right)^{\alpha+1}, \qquad t \geq \sigma_i,$$

respectively. Moreover, the CDF of the maximum $Y = \bigvee_{i=1}^{n} Y_i$ is given by

$$G_Y(t) = \prod_{i=1}^{n}\left(1 - \left(\frac{\sigma_i}{t}\right)^{\alpha}\right), \quad t \geq \bigvee_{i=1}^{n} \sigma_i.$$

Theorem 1 shows that this Y is a slash version of some nonnegative random variable X, where X and Y satisfy (1). By Theorem 2, the CDF of X is of the form (4) with

$$G_{X_i}^{(\alpha)}(t) = F_{Y_i}(t) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t} G_{X_i}^{(\alpha)}(t) = f_{Y_i}(t).$$

This leads to the formula

$$F_{\bigoplus_{i=1}^{n} X_i}(t) = \left\{1 + \sum_{i=1}^{n} \frac{\left(\frac{\sigma_i}{t}\right)^{\alpha}}{1 - \left(\frac{\sigma_i}{t}\right)^{\alpha}}\right\} \prod_{i=1}^{n}\left(1 - \left(\frac{\sigma_i}{t}\right)^{\alpha}\right), \quad t \geq \bigvee_{i=1}^{n} \sigma_i,$$

which, by Theorem 1, is the CDF of the Kendall convolution $*_{\alpha}$ of the $\{\delta_{\sigma_i}\}$. In the special case $n = 2$, the CDF of X simplifies to

$$F_X(t) = 1 - \left(\frac{\sigma_1}{t}\right)^{\alpha}\left(\frac{\sigma_2}{t}\right)^{\alpha}, \quad t \geq \sigma_1 \vee \sigma_2, \tag{12}$$

and coincides with formula (7) where $x = \sigma_1$ and $y = \sigma_2$. In summary, through an application of the slash transformation, we were able to identify the distribution of the Kendall convolution of the $\{\delta_{\sigma_i}\}$.

Remark 7 As noted earlier, the distribution in (12) is neither continuous nor discrete, unless $\sigma_1 = \sigma_2 = \sigma$, in which case the distribution of $X := X_1 \oplus X_2$ is absolutely continuous with $X \sim \mathcal{P}(2\alpha, \sigma)$. The case with general $n \geq 2$ is similar, where the distribution of $X := \bigoplus_{i=1}^n X_i$ is absolutely continuous when $\sigma_i = \sigma, i = 1, \dots, n$, although the latter is no longer Pareto if $n > 2$. A closer look at this case reveals that for general $n \geq 2$ and $\sigma_i = \sigma, i = 1, \dots, n$, the CDF of X simplifies to

$$F_X(t) = \left[1 - \left(\frac{\sigma}{t}\right)^\alpha\right]^n + n\left[1 - \left(\frac{\sigma}{t}\right)^\alpha\right]^{n-1}\left(\frac{\sigma}{t}\right)^\alpha = R_n(H(t)), \quad t \geq \sigma,$$

where $H(\cdot)$ is the Pareto CDF (11) and

$$R_n(u) = u^n + nu^{n-1}(1 - u), \quad u \in [0, 1], n \geq 2. \tag{13}$$

The function $R_n(\cdot)$ in (13) is the CDF of the beta distribution $\mathcal{B}(a, b)$ with parameters $a = n - 1$ and $b = 2$, and with the PDF $r_n(u) = n(n - 1)u^{n-2}(1 - u), u \in [0, 1], n \geq 2$. Thus, the random variable X admits the stochastic representation $X \stackrel{d}{=} H^{-1}(T)$, where $H^{-1}(\cdot)$ is the quantile function of the Pareto distribution (11) and $T \sim \mathcal{B}(n - 1, 2)$, which shows that the distribution of X is a particular case of a *beta-Pareto* distribution (see Akinsete et al., 2008). The PDF of X is of the form

$$f_X(t) = \frac{n(n-1)\alpha}{\sigma}\left(\frac{\sigma}{t}\right)^{2\alpha+1}\left[1 - \left(\frac{\sigma}{t}\right)^\alpha\right]^{n-2}, \quad t \geq \sigma,$$

and turns into the PDF of the Pareto distribution $\mathcal{P}(2\alpha, \sigma)$ when $n = 2$. The distribution of X can also be seen as that of the 2nd (upper) order statistic $X_{n-1:n}$ connected with a random sample of size n from the Pareto distribution $\mathcal{P}(\alpha, \sigma)$, which is why for $n = 2$ the variable X is the *minimum* of two Pareto $\mathcal{P}(\alpha, \sigma)$ variables, and consequently $X \sim \mathcal{P}(2\alpha, \sigma)$. This analysis leads to a useful interpretation of the Kendall convolution of n point masses at $\sigma > 0$, connecting this with a Pareto distribution as seen above. Under the ordinary convolution (summation), the result in this case would of course still be a point measure (concentrated at $n\sigma$).

Remark 8 The results of this paper can also be used to obtain an explicit form of the Kendall convolution of Pareto distributions, using the slash transformation of a Pareto variable (11) rather than the inverse slash transformation of Pareto, discussed above. Such a slashed version of Pareto (with $\sigma = 1$) was considered by Felgueiras (2012), who termed the resulting distribution *extended slash Pareto* (ESP). Indeed, suppose that $X_1, \dots, X_n$ are independent random variables with Pareto $P(\alpha, \sigma_i)$ distributions, and let $Y_i = X_i/U_i$, where the $\{U_i\}$ are standard uniform random variables, independent of the $\{X_i\}$. Straightforward calculations show that the CDF of the maximum $Y = \bigvee_{i=1}^n Y_i$ is given by

$$F_Y(t) = \prod_{i=1}^{n} \left(1 - \frac{1 + \alpha \log(t/\sigma_i)}{(t/\sigma_i)^\alpha}\right), \quad t \geq \bigvee_{i=1}^{n} \sigma_i.$$

Theorem 1 shows that this Y is a slash version of some nonnegative random variable X, where X and Y satisfy (1). By Theorem 2, the CDF of X is of the form (4), which upon simplification produces

$$F_{\bigoplus_{i=1}^{n} X_i}(t) = \left\{1 + \sum_{i=1}^{n} \frac{\frac{\alpha \log(t/\sigma_i)}{(t/\sigma_i)^\alpha}}{1 - \frac{1+\alpha \log(t/\sigma_i)}{(t/\sigma_i)^\alpha}}\right\} \prod_{i=1}^{n} \left(1 - \frac{1 + \alpha \log(t/\sigma_i)}{(t/\sigma_i)^\alpha}\right), \quad t \geq \bigvee_{i=1}^{n} \sigma_i.$$

By Theorem 1, this is the CDF of the Kendal convolution $*_\alpha$ of Pareto distributions $\mathcal{P}(\alpha, \sigma_i), i = 1, \dots, n$. Unlike the case of ordinary convolution of Pareto distributions, where the relevant CDF is not known explicitly, for Kendall convolution the formula for the CDF is explicit.

3.3 Exponential distribution and its slash version

Let $X_1, \dots, X_n$ be independent random variables where each X_i is exponentially distributed with scale parameter $\sigma_i > 0$, so that

$$F_{X_i}(t) = 1 - e^{-t/\sigma_i}, \quad t \in \mathbb{R}_+. \tag{14}$$

Standard calculations involving the CDFs of $Y_i = X_i/U_i$, where the $\{U_i\}$ are standard uniform random variables, independent of the $\{X_i\}$ (cf. Jones, 2020), show that the CDF of $Y := \bigvee_{i=1}^{n} Y_i$ is of the form

$$F_Y(t) = \prod_{i=1}^{n} \left[1 - \frac{\sigma_i}{t}\left(1 - e^{-t/\sigma_i}\right)\right], \quad t \in \mathbb{R}_+.$$

Theorem 1 guarantees that this Y is a slash version of some nonnegative random variable X, where X and Y satisfy (1) with $\alpha = 1$. By Theorem 2, the CDF of this X is of the form (4), which upon simplification produces

$$F_{\bigoplus_{i=1}^{n} X_i}(t) = \left\{1 + \frac{1}{\alpha} \sum_{i=1}^{n} \frac{\frac{\sigma_i}{t}\left[1 - e^{-t/\sigma_i}\left(1 + \frac{t}{\sigma_i}\right)\right]}{1 - \frac{\sigma_i}{t}\left(1 - e^{-t/\sigma_i}\right)}\right\}$$
$$\times \prod_{i=1}^{n} \left[1 - \frac{\sigma_i}{t}\left(1 - e^{-t/\sigma_i}\right)\right], \quad t \in \mathbb{R}_+.$$

By Theorem 1, this is the CDF of the Kendall convolution $*_1$ of n heterogeneous exponential distributions with the CDFs given by (14). In contrast, under the ordinary convolution connected with the usual operation of addition of independent random variables, the sum of heterogeneous exponential variables follows the *hypo-exponential* distribution, also known as the generalized Erlang distribution (see, e.g., Johnson et al., 1994, p. 552). If the $\{X_i\}$ have the same exponential distribution with a common scale parameter σ, then the CDF of their Kendall convolution $\bigoplus_{i=1}^n X_i$ simplifies to

$$F_{\bigoplus_{i=1}^n X_i}(t) = \left[1 - \frac{\sigma}{t}(1 - e^{-t/\sigma})\right]^{n-1} \times \left[1 - ne^{-t/\sigma} + (n-1)\frac{\sigma}{t}(1 - e^{-t/\sigma})\right], \quad t \in \mathbb{R}_+,$$

while the usual convolution is a gamma distribution with shape parameter n (and scale σ).

4 Extensions

Theorem 1 is a special case of a general representation, where for some positive random variable T we have the equality in distribution

$$\bigvee_{i=1}^m \frac{X_i}{T_i} \stackrel{d}{=} \frac{X}{T}. \tag{15}$$

Here, the $\{X_i\}$ are arbitrary independent random variables with distributions $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathcal{P}_+$, respectively, independent of the $\{T_i\}$, which are IID copies of T, and X is a random variable with distribution $\lambda \in \mathcal{P}_+$, independent of T. By the results of Sect. 2 the random variable $T \stackrel{d}{=} U^{1/\alpha}$, with U being standard uniform and $\alpha > 0$, is a special case where (15) is true. As we show in the sequel, there exist other random variables T for which this representation is also true, as well as those for which this representation does not hold. As shown in the result below, one particular class of variables that satisfy this representation incorporates the distributions on $\mathbb{R}_+$ with SFs $h(t)$ that are *quasi-stable* (that is, h is a bounded, continuous function such that $\forall a, b \in \mathbb{R}_+$ the function $(T_a h)(T_b h)$ belongs to the closed convex hull of the set $\{T_x h : x \in \mathbb{R}_+\}$) and such that $h(t) = 1 - t^q L(t)$ where $q > 0$ and $L(\cdot)$ is slowly varying at the origin (or, equivalently, $1/T$ is a regularly varying random variable).

Theorem 3 *Let $X_1, X_2, \ldots, X_m$ be independent and nonnegative random variables. Further, let $T_1, T_2, \ldots, T_m$ be IID random variables, independent of the $\{X_i\}$, with a common SF $h(u) = \mathbb{P}(T_i > u)$ for $i = 1, 2, \ldots, m$. Assume that the function $h(\cdot)$ satisfies the following assumptions*:

(A1) *$h(\cdot)$ is quasi-stable*,
(A2) *There exists $q > 0$ such that $h(u) = 1 - u^q L(u)$ where $L(\cdot)$ is slowly varying at* 0.

Then, there exists a random variable X and a random variable T with survival function $h(\cdot)$ and independent of X, such that (15) holds.

Remark 9 Let us note that, by the results of Kucharczak and Urbanik (1974), a SF $h(\cdot)$ that satisfies the conditions $(A1)$ and $(A2)$ of Theorem 3 can serve as a kernel of the generalized ChF (21) corresponding to some generalized convolution algebra $(\mathcal{P}_+, *)$. In turn, if the kernel $h(\cdot)$ of the generalized ChF (21) is a genuine SF of some random variable T, then, according to Theorem 3, the property (15) holds with that T. In addition to the case where $h(\cdot)$ is the SF of $U^{1/\alpha}$ with U standard uniform and arbitrary $\alpha > 0$, there are many other known examples of such an $h(\cdot)$. These include the SF of $G^{1/\alpha}$ with arbitrary $\alpha > 0$, where G has standard gamma distribution with shape parameter less than or equal to 1, corresponding to the Kucharczak generalized convolution (see, e.g., Jasiulis-Gołdyn et al., 2021) or the SF of $B^{1/\alpha}$ with arbitrary $\alpha > 0$, where B has the beta distribution $\mathcal{B}(1, \beta)$ with $\beta = n \in \mathbb{N}$, corresponding to the Kucharczak–Urbanik generalized convolution (see Kucharczak and Urbanik, 1974), discussed below. We also note that if the parameter β of $B \sim \mathcal{B}(1, \beta)$ is not an integer, then the SF of B is not quasi-stable (see Corollary 2 in Kucharczak and Urbanik, 1974), in which case Theorem 3 is not applicable. Some other examples can be found in Jasiulis-Gołdyn et al. (2021).

Remark 10 By taking the reciprocals of the random variables in (15), we can state this relation equivalently in terms of the minimum,

$$\bigwedge_{i=1}^{m} \frac{T_i}{X_i} \overset{d}{=} \frac{T}{X}. \tag{16}$$

In particular, (16) holds with exponentially distributed $\{T_i\}$, since the SF of the standard exponential distribution, $h(t) = e^{-t}$, satisfies both conditions $(A1)$ and $(A2)$ of Theorem 3. Indeed, this $h(\cdot)$ is the kernel of the generalized ChF of standard convolution (which is the Laplace transform), and the variables in the denominators in (16) satisfy the distributional equality $X \overset{d}{=} X_1 + \cdots + X_m$. Thus, we recover the well-known property stating that the distribution of the minimum of independent random variables whose distributions are scale mixtures of exponential distributions is also a mixture of exponential distributions (see, e.g., Part 2 of Theorem 1 in Hesselager et al., 1998). More generally, the alternative formulation of Theorem 3 via (16) makes a direct connection with the concept of *scale mixtures* (of exponential or other distributions, depending on the nature of the random variables $\{T_i\}$). Scale mixtures are important stochastic models in a variety of fields, including actuarial and financial applications, where exponential mixtures with their key property of infinite divisibility are particularly popular (see, e.g., Choy and Chan, 2003; Hesselager et al., 1998; Jewell, 1982; Klugmann et al., 2012). In financial applications, the extremes of $\{T_i/X_i\}$ can be related to the extreme risk of financial portfolios.

Remark 11 The class of all distributions of T that satisfy (15) is unknown, and the characterization of this class is an open problem. In particular, it is not known if the converse of Theorem 3 holds. An example of T having a continuous distribution for which (15) does not generally hold is one where T has the standard Fréchet distribution with shape parameter $\alpha = 1$, $T \sim \mathcal{F}(1, 1)$, so that $1/T$ is standard exponential. Indeed, if we set $m = 2$ and let X_1 and X_2 be IID random variables taking on the value of 1 with probability 1, then the CDF of the left-hand side in (15) is of the form $F_L(t) = (1 - e^{-t})^2 = 1 - 2e^{-t} + e^{-2t}$, $t \geq 0$. Thus, if the relation (15) was true for some nonnegative random variable X then we would necessarily have $\mathbb{P}(X = 0) = 0$, and the CDF of the right-hand side in (15) would be of the form $F_R(t) = 1 - \psi(t)$, $t \geq 0$, where $\psi(\cdot)$ is the Laplace transform (LT) of the random variable $1/X$, as can be seen by straightforward calculation involving standard conditioning arguments. Consequently, (15) would imply that $F_L(t) = F_R(t)$, $t \geq 0$, leading to $\psi(t) = 2e^{-t} - e^{-2t}$, $t \geq 0$. While this function is strictly decreasing in t with $\psi(0) = 1$, it is not completely monotone (since its second derivative is negative for t close to zero), and thus this function is not a legitimate LT (see, e.g., Feller, 1981, Section XIII.4). We conclude that the relation (15) does not generally hold with $T \sim \mathcal{F}(1, 1)$. Another interesting open problem relates to the case of discrete T in (15): are there any non-trivial discrete random variables T for which the representation holds? In connection with the latter, we note that this representation does not hold if the distribution of T is supported on a finite set of points in $\mathbb{R}_+$. Indeed, suppose that the random variable T takes on two values, $1/a$ and $1/b$, with probabilities p and $1 - p$, respectively, for some $0 < a < b$ and $p \in (0, 1)$ (if T takes on more than two values, the arguments are similar). Further, set $m = 2$ and let X_1 and X_2 be IID random variables taking on the value of $1/c$ with probability 1 for some $c > b$. It is easy to see that the random variable on the left-hand side in (15) takes on the values of a/c and b/c with probabilities p^2 and $1 - p^2$, respectively. Thus, in order for (15) to be true for some X, this X must take on the value of $1/c$ with probability 1. However, for this X the random variable X/T takes on the values of a/c and b/c with probabilities p and $1 - p$, respectively. Hence, the relation (15) fails to hold with such X_1, X_2, and T.

As discussed in Remark 9, the survival function $h(t) = (1 - t^\alpha)^n$, $t \in [0, 1]$, of the random variable $B^{1/\alpha}$ with arbitrary $\alpha > 0$, where B has the beta distribution $\mathcal{B}(1, n)$ with $n \in \mathbb{N}$, is the kernel in the generalized ChF of the Kucharczak–Urbanik distribution, given by (23). This immediately leads to the following result, which generalizes Proposition 1.

Corollary 1 *Let X be a random variable with distribution $\lambda \in \mathcal{P}_+$ and let $Y \stackrel{d}{=} X/B^{1/\alpha}$, where $\alpha > 0$ and B is an independent of X beta $\mathcal{B}(1, n)$ random variable with $n \in \mathbb{N}$. Then the CDF of Y can be written as*

$$\mathbb{P}(Y \leq t) = \Phi_\lambda^{(\alpha,n)}\left(\frac{1}{t}\right), \quad t \in \mathbb{R}_+,$$

where $\Phi_\lambda^{(\alpha,n)}(\cdot)$ is the ChF (23) corresponding to Kucharczak–Urbanik convolution.

We now provide an extension of Theorem 1, which reduces to the latter for $n = 1$.

Theorem 4 *Let $\alpha > 0$, $m, n \in \mathbb{N}$, and let $X_1, X_2 \ldots, X_m$ be independent random variables with distributions $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathcal{P}_+$, respectively. Further, let $B_1, B_2, \ldots, B_m$ be IID beta $\mathcal{B}(1, n)$ random variables, independent of the $\{X_i\}$. Then, there exists a random variable X with distribution $\lambda = \lambda_1 *_{\alpha,n} \lambda_2 *_{\alpha,n} \cdots *_{\alpha,n} \lambda_m$ and a beta $\mathcal{B}(1, n)$ random variable B, independent of X, such that*

$$\bigvee_{i=1}^{m} \frac{X_i}{(B_i)^{\frac{1}{\alpha}}} \stackrel{d}{=} \frac{X}{B^{\frac{1}{\alpha}}}. \tag{17}$$

We conclude this section with an equivalent version of Theorem 4 written in terms of uniform random variables. Note that $B \sim \mathcal{B}(1, n)$ admits the stochastic representation

$$B \stackrel{d}{=} \bigwedge_{j=1}^{n} U_j, \tag{18}$$

where the $\{U_i\}$ are IID standard uniform random variables. Consequently,

$$\frac{X}{B^{\frac{1}{\alpha}}} \stackrel{d}{=} \bigvee_{i=1}^{n} \frac{X}{U_i^{1/\alpha}}. \tag{19}$$

In view of the above we have the following alternative version of Theorem 4.

Corollary 2 *Let $X_1, \ldots, X_m$ be independent random variables with distributions $\lambda_1, \ldots, \lambda_m \in \mathcal{P}_+$, respectively and let $U_{i,j}$, $i = 1, \ldots, m, j = 1, \ldots, n$, be IID standard uniform random variables, independent of the $\{X_i\}$. Then, there exists a random variable X with distribution $\lambda_1 *_{\alpha,n} \cdots *_{\alpha,n} \lambda_m$ and IID standard uniform random variables $U_1, \ldots, U_n$, independent of X, such that*

$$\bigvee_{i=1}^{m} \bigvee_{j=1}^{n} \frac{X_i}{(U_{i,j})^{\frac{1}{\alpha}}} \stackrel{d}{=} \bigvee_{k=1}^{n} \frac{X}{(U_k)^{\frac{1}{\alpha}}}.$$

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Appendix A: Basic facts on generalized convolutions

In this section we recall some basic facts on generalized convolutions and two crucial examples of Kendall and Kucharczak–Urbanik convolutions that serve as the main tools for analyzing extremes of random variables with α-slash distributions. We refer to Urbanik (1964) for the background material on the theory of generalized convolutions and the corresponding characteristic functions as well as to more recent works such as Borowiecka-Olszewska et al. (2015) or Jasiulis-Gołdyn et al.

(2020b, 2021) and the references therein for new interesting results and extensive examples.

We start with the definition of generalized convolution (see, e.g., Urbanik, 1964; Borowiecka-Olszewska et al., 2015, Section 2.1).

Definition 2 A commutative and associative operation $*: \mathcal{P}_+^2 \to \mathcal{P}_+$ is called *generalized convolution* if it satisfies the following conditions:

(i) $\delta_0 * \lambda = \lambda$ for all $\lambda \in \mathcal{P}_+$;
(ii) $(p\mu + (1-p)\nu) * \lambda = p(\mu * \lambda) + (1-p)(\nu * \lambda)$ for all $p \in [0,1], \mu, \nu, \lambda \in \mathcal{P}_+$;
(iii) $T_x\mu * T_x\nu = T_x(\mu * \nu)$, for all $x > 0, \mu, \nu \in \mathcal{P}_+$;
(iv) If $\mu_n \Longrightarrow \mu$ for $\mu_n, \mu \in \mathcal{P}_+$, then $\mu_n * \lambda \Longrightarrow \mu * \lambda$ for all $\lambda \in \mathcal{P}_+$;
(v) There exists a sequence $\{c_n\}_{n=1}^{\infty}, c_n > 0$ and $\mu \in \mathcal{P}_+, \mu \neq \delta_0$ such that $T_{c_n}\delta_1^{*n} \Longrightarrow \mu$, where δ_1^{*n} denotes the convolution of n identical measures δ_1.

The pair $(\mathcal{P}_+, *)$ is called the generalized convolution algebra.

Remark 12 It is well known (see, e.g., Borowiecka-Olszewska et al., 2015, Section 2.1) that a generalized convolution $*$ is uniquely determined by the convolution of point-mass measures $\delta_x * \delta_y, x, y \geq 0$. That is, for every $\lambda_1, \lambda_2 \in \mathcal{P}_+$ and $A \in \mathcal{B}(\mathbb{R}_+)$ we have

$$(\lambda_1 * \lambda_2)(A) = \int_0^\infty \int_0^\infty \left(\delta_x * \delta_y\right)(A)\ \lambda_1(\mathrm{d}x)\lambda_2(\mathrm{d}y). \tag{20}$$

The main technical tool in the study of generalized convolutions, which plays an analogous role to that of the Laplace transform for ordinary convolutions, is a *generalized* characteristic function (see, e.g, Urbanik, 1964, Section 4; Borowiecka-Olszewska et al., 2015, Definition 2.2), defined below.

Definition 3 We say that the generalized convolution algebra $(\mathcal{P}_+, *)$ admits a characteristic function if there exists one-to-one correspondence between probability measures $\lambda \in \mathcal{P}_+$ and functions $\Phi_\lambda(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ such that

(i) $\Phi_{p\mu+(1-p)\nu} = p\Phi_\mu + (1-p)\Phi_\nu$ for all $\mu, \nu \in \mathcal{P}_+, p \in [0,1]$;
(ii) $\Phi_{\mu*\nu} = \Phi_\mu\Phi_\nu$ for all $\mu, \nu \in \mathcal{P}_+$;
(iii) $\Phi_{T_x\mu}(t) = \Phi_\mu(xt)$ for all $x, t \geq 0, \mu \in \mathcal{P}_+$;
(iv) The uniform convergence of Φ_{λ_n} on every bounded interval is equivalent to the weak convergence of λ_n.

The function $\Phi_\lambda(\cdot)$ is termed the $*$-generalized characteristic function of measure λ. Analogously, if a random variable X has distribution λ, then $\Phi_X(\cdot)$ denotes the $*$-generalized characteristic function of that λ.

If $(\mathcal{P}_+, *)$ admits a generalized characteristic function, then $\Phi_\lambda(t)$ is an integral transform of the form (see Urbanik, 1964, Theorem 6)

$$\Phi_\lambda(t) = \int_0^\infty h(tx)\lambda(\mathrm{d}x) \tag{21}$$

for some kernel $h(\cdot)$. As shown in Kucharczak and Urbanik (1974), a function $h : \mathbb{R}_+ \to \mathbb{R}$ is a kernel of the characteristic function corresponding to a generalized convolution algebra if and only if it is *quasi-stable* (that is, h is a bounded, continuous function such that $\forall a, b \in \mathbb{R}_+$ the function $(T_a h)(T_b h)$ belongs to the closed convex hull of the set $\{T_x h : x \in \mathbb{R}_+\}$) and, in addition, $h(t) = 1 - t^q L(t)$ for some $q > 0$ and a slowly varying (at the origin) function $L(\cdot)$. Further, there must exist a probability measure $\mu \in \mathcal{P}_+$ such that $\limsup_{t\to\infty} \int_0^\infty h(tx)\mu(\mathrm{d}x) < 1$.

A particular example of a function $h(\cdot)$ that satisfies these conditions, which plays a central role in our work, is the function (see, for example function f_6 in Kucharczak and Urbanik, 1974)

$$h(t) = (1 - t^\alpha)_+^n, \quad t \in [0, 1], \alpha > 0, n \in \mathbb{N}. \tag{22}$$

We refer to Kucharczak and Urbanik (1974) for other examples and general discussion on the connections between quasi-stable functions and generalized convolutions.

Generalized convolution corresponding to the characteristic function with kernel (22) and its special case with parameter $n = 1$ plays a crucial role in proving the results of Sect. 2.

Definition 4 Let $\alpha > 0, n \in \mathbb{N}$. A generalized convolution $*_{\alpha,n}$ with the characteristic function

$$\Phi_\lambda^{(\alpha,n)}(t) = \int_0^\infty (1 - x^\alpha t^\alpha)_+^n \lambda(\mathrm{d}x), \tag{23}$$

that is with the kernel given by (22), is called Kucharczak–Urbanik convolution (see, e.g., Kucharczak and Urbanik, 1974, p. 268; Borowiecka-Olszewska et al., 2015, Example 2.8).

Definition 5 Let $\alpha > 0, n \in \mathbb{N}$. A generalized convolution $*_\alpha$ with the characteristic function

$$\Phi_\lambda^{(\alpha)}(t) = \int_0^\infty (1 - x^\alpha t^\alpha)_+ \lambda(\mathrm{d}x) \tag{24}$$

is called Kendall convolution (see, e.g., Urbanik, 1988; Borowiecka-Olszewska et al., 2015, Example 2.4).

As discussed in Remark 12, the generalized convolution of measures $\lambda_1, \lambda_2 \in \mathcal{P}_+$ can be uniquely defined by the convolution of point mass measures, which in the case of Kendall convolution, has the following form (see, e.g., Example 2.4 in Borowiecka-Olszewska et al., 2015)

$$\delta_x *_\alpha \delta_y = T_{x\vee y}\left\{ \left(\frac{x \wedge y}{x \vee y}\right)^\alpha \pi_{2\alpha} + \left(1 - \left(\frac{x \wedge y}{x \vee y}\right)^\alpha\right)\delta_1 \right\}. \tag{25}$$

In the above expression, the quantity $T_{x\vee y}$ is the shift operator, $\pi_{2\alpha}$ denotes the standard Pareto distribution (scale 1) with parameter 2α, and 0/0 is assumed to be 0.

Appendix B: Proofs

This section contains proofs of our main results.

Appendix B.1: Proof of Theorem 1

Let $t \geq 0$. Observe that, due to the independence of $X_i, U_i, i = 1, 2, \dots, n$, we have

$$\mathbb{P}\left(\bigvee_{i=1}^{n} \frac{X_i}{(U_i)^{\frac{1}{\alpha}}} \leq t\right) = \prod_{i=1}^{n} \mathbb{P}\left(\frac{X_i}{(U_i)^{\frac{1}{\alpha}}} \leq t\right) = \prod_{i=1}^{n} G_{X_i}^{(\alpha)}(t).$$

By Proposition 1, we have

$$\prod_{i=1}^{n} G_{X_i}^{(\alpha)}(t) = \prod_{i=1}^{n} \Phi_{X_i}^{(\alpha)}\left(\frac{1}{t}\right).$$

Due to Definition 5, combined with Property (ii) in Definition 3 (see Appendix A), applied to the Kendall convolution $*_\alpha$, there exists a random variable X with distribution $\lambda_1 *_\alpha \lambda_2 *_\alpha \cdots *_\alpha \lambda_n$ such that

$$\prod_{i=1}^{n} \Phi_{X_i}^{(\alpha)}\left(\frac{1}{t}\right) = \Phi_{X}^{(\alpha)}\left(\frac{1}{t}\right).$$

Finally, due to Proposition 1, there exists a uniform random variable U, independent of X, such that

$$\Phi_{X}^{(\alpha)}\left(\frac{1}{t}\right) = \mathbb{P}\left(\frac{X}{U^{\frac{1}{\alpha}}} \leq t\right).$$

This completes the proof. □

Appendix B.2: Proof of Theorem 2

For $i = 1, \dots, n$, let $Y_1, \dots, Y_n$ be independent α-slash versions of the $\{X_i\}$, with the CDFs

$$F_{Y_i}(t) = G_{X_i}^{(\alpha)}(t), \tag{26}$$

respectively. By the independence of the $\{Y_i\}$, their maximum $Y := \bigvee_{i=1}^{n} Y_i$ has the CDF of the form

$$F_Y(t) = \prod_{i=1}^{n} F_{Y_i}(t), \quad t \in \mathbb{R}_+, \tag{27}$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t} F_Y(t) = F_Y(t) \sum_{i=1}^{n} \frac{\frac{\mathrm{d}}{\mathrm{d}t} F_{Y_i}(t)}{F_{Y_i}(t)}, \quad t \in \mathbb{R}_+. \tag{28}$$

Theorem 1 guarantees that Y is a slash version of some nonnegative random variable X, where X and Y satisfy (1). Finally, an application of Lemma 1 combined with (28) shows that the CDF of $X = \bigoplus_{i=1}^{n} X_i$ is of the form

$$F_X(t) = F_Y(t) + \frac{t}{\alpha} \frac{\mathrm{d}}{\mathrm{d}t} F_Y(t) = F_Y(t) \left\{ 1 + \frac{t}{\alpha} \sum_{i=1}^{n} \frac{\frac{\mathrm{d}}{\mathrm{d}t} F_{Y_i}(t)}{F_{Y_i}(t)} \right\}, \quad t \in \mathbb{R}_+.$$

The proof is completed by substituting (27) and (26) in the above expression. □

Appendix B.3: Proof of Proposition 2

In view of (20), in order to prove Proposition 2 it is enough to prove (6). Upon applying Theorem 1 to deterministic $\{X_i\}$, i.e., $X_i = x_i$, $i = 1, 2, \ldots, n$, we obtain

$$G^{(\alpha)}_{\bigoplus_{i=1}^{n} x_i}(t) = \prod_{i=1}^{n} \left[1 - \left(\frac{x_i}{t} \right)^{\alpha} \right] \mathbf{1}(t \geq x_i).$$

Hence, by Lemma 1, we have

$$\begin{aligned}
F_{\bigoplus_{i=1}^{n} x_i}(t) &= \prod_{i=1}^{n} \left[1 - \left(\frac{x_i}{t} \right)^{\alpha} \right] \mathbf{1}(t \geq x_i) + \frac{t}{\alpha} \frac{\mathrm{d}}{\mathrm{d}t} \prod_{i=1}^{n} \left[1 - \left(\frac{x_i}{t} \right)^{\alpha} \right] \mathbf{1}(t \geq x_i) \\
&= \left[1 + \sum_{i=1}^{n} (-1)^i \frac{\sum_{U \in \mathcal{U}_i^{(n)}} \prod_{j \in U} x_j}{t^{i\alpha}} + \sum_{i=1}^{n} (-1)^{i+1} i \frac{\sum_{U \in \mathcal{U}_i^{(n)}} \prod_{j \in U} x_j}{t^{i\alpha}} \right] \\
&\quad \times \mathbf{1}\left(t \geq \bigvee_{i=1}^{n} x_i \right) \\
&= \left[1 + \sum_{i=2}^{n} (-1)^{i+1} (i-1) \frac{\sum_{U \in \mathcal{U}_i^{(n)}} \prod_{j \in U} x_j}{t^{i\alpha}} \right] \mathbf{1}\left(t \geq \bigvee_{i=1}^{n} x_i \right).
\end{aligned}$$

The proof is completed by replacing $i - 1$ with i in the above summation. □

Appendix B.4: Direct proof of property (8)

First, we show that for any $x, y, t > 0$, we have $G^{(\alpha)}_x(t) G^{(\alpha)}_y(t) = G^{(\alpha)}_{x \oplus y}(t)$, that is

$$\left(1-\left(\frac{x}{t}\right)^{\alpha}\right)_{+}\left(1-\left(\frac{y}{t}\right)^{\alpha}\right)_{+}=\int_0^t\left(1-\left(\frac{s}{t}\right)^{\alpha}\right)\mathrm{d}F_{x\oplus y}(s). \tag{29}$$

Upon integrating by parts, we find the right-hand side of the above equation to be

$$\int_0^t\left(1-\left(\frac{s}{t}\right)^{\alpha}\right)\mathrm{d}F_{x\oplus y}(s)=\frac{\alpha}{t^{\alpha}}\int_0^t s^{\alpha-1}F_{x\oplus y}(s)\mathrm{d}s \tag{30}$$

$$\begin{aligned}&=\frac{\alpha}{t^{\alpha}}\int_0^t s^{\alpha-1}\left(1-\frac{(xy)^{\alpha}}{s^{2\alpha}}\right)\mathbf{1}(s\geq x\vee y)\mathrm{d}s\\ &=I_1-I_2,\end{aligned} \tag{31}$$

where we have

$$I_1=\frac{\alpha}{t^{\alpha}}\int_{x\vee y}^t s^{\alpha-1}\mathrm{d}s=1-\frac{(x\vee y)^{\alpha}}{t^{\alpha}} \tag{32}$$

and

$$I_2=\frac{\alpha(xy)^{\alpha}}{t^{\alpha}}\int_{x\vee y}^t s^{-\alpha-1}\mathrm{d}s=-\frac{(xy)^{\alpha}}{t^{2\alpha}}+\frac{(x\wedge y)^{\alpha}}{t^{\alpha}}. \tag{33}$$

By combining (32) and (33) with (31), we obtain

$$\begin{aligned}\int_0^t\left(1-\left(\frac{s}{t}\right)^{\alpha}\right)\mathrm{d}F_{x\oplus y}(s)&=1-\frac{(x\vee y)^{\alpha}}{t^{\alpha}}-\frac{(x\wedge y)^{\alpha}}{t^{\alpha}}+\frac{(xy)^{\alpha}}{t^{2\alpha}}\\ &=\left(1-\left(\frac{x}{t}\right)^{\alpha}\right)_{+}\left(1-\left(\frac{y}{t}\right)^{\alpha}\right)_{+},\end{aligned}$$

producing (29). For general X with distribution $\lambda_1\in\mathcal{P}_+$ and Y with distribution $\lambda_2\in\mathcal{P}_+$, integration by parts leads to

$$G^{(\alpha)}_{X\oplus Y}(t)=\int_0^t\left(1-\left(\frac{s}{t}\right)^{\alpha}\right)\mathrm{d}F_{X\oplus Y}(s)=\frac{\alpha}{t^{\alpha}}\int_0^t s^{\alpha-1}F_{X\oplus Y}(s)\mathrm{d}s.$$

Due to (20), the last expression is equivalent to

$$\frac{\alpha}{t^{\alpha}}\int_0^t s^{\alpha-1}\int_0^{\infty}\int_0^{\infty}F_{x\oplus y}(s)\mathrm{d}F_X(x)\mathrm{d}F_Y(y)\mathrm{d}s,$$

which, by Fubini's theorem, is equivalent to

$$\int_0^{\infty}\int_0^{\infty}\left\{\frac{\alpha}{t^{\alpha}}\int_0^t s^{\alpha-1}F_{x\oplus y}(s)\mathrm{d}s\right\}\mathrm{d}F_X(x)\mathrm{d}F_Y(y)=\int_0^{\infty}\int_0^{\infty}G^{(\alpha)}_{x\oplus y}(t)\mathrm{d}F_X(x)\mathrm{d}F_Y(y) \tag{34}$$

$$
\begin{aligned}
&= \int_0^\infty \int_0^\infty G_x^{(\alpha)}(t) G_y^{(\alpha)}(t) \mathrm{d}F_X(x) \mathrm{d}F_Y(y) \\
&= \int_0^\infty G_x^{(\alpha)}(t) \mathrm{d}F_X(x) \int_0^\infty G_y^{(\alpha)}(t) \mathrm{d}F_Y(y),
\end{aligned}
\tag{35}
$$

where (34) is by (30) and (35) is by (29). This completes the proof. □

Appendix B.5: Proof of Theorem 3

We provide only a sketch of the proof of Theorem 3, as it is similar to the proof of Theorem 1. Let $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathcal{P}_+$ be distributions of random variables $X_1, X_2, \dots, X_n$, respectively. First, observe that for any probability measure μ we have $\limsup_{x\to\infty} \int_0^\infty h(xt)\mu(dx) < 1$. Thus, due to Theorem 2 in Kucharczak and Urbanik (1974), the function $h(u)$ satisfies all the conditions for being a kernel of a generalized ChF for some generalized convolution $*$. Hence, $\Phi_{X_i}\left(\frac{1}{t}\right) := \mathbb{P}\left(\frac{X_i}{T_i} \le t\right) = \int_0^\infty h\left(\frac{x}{t}\right)\lambda_i(dx)$ is a proper generalized characteristic function at $\frac{1}{t}$ for the generalized convolution $*$. Then, there exists a random variable X with distribution $\lambda_1 * \lambda_2 * \dots * \lambda_n$ such that $\prod_{i=1}^n \Phi_{X_i}\left(\frac{1}{t}\right) = \Phi_X\left(\frac{1}{t}\right)$. Thus, there exists a random variable T, independent of X and with survival function $h(u)$, such that

$$
\mathbb{P}\left(\bigvee_{i=1}^n \frac{X_i}{T_i} \le t\right) = \prod_{i=1}^n \Phi_{X_i}\left(\frac{1}{t}\right) = \Phi_X\left(\frac{1}{t}\right) = \mathbb{P}\left(\frac{X}{T} \le t\right).
$$

This completes the proof. □

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