Supplementary material to "Model averaging for semiparametric varying coefficient quantile regression models"

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In this supplementary material, we provide some additional numerical results that support our conclusions.

1 Additional Numerical Results for Case 1, Case 2, and Case 3

We present the simulation results for Cases 1, 2, and 3 with $n = 100, 300$. Under each setting, 500 replicates are simulated.

Case 1 (Different weight choice criteria).

The results of normalized QPE and t-test for the differences in QPE between CV-MA and alternatives when $n = 100$ and 300 are summarized in Table S1 and Table S2, where a positive t-statistic indicates that the estimator in the numerator produces a larger QPE than the estimator in the denominator. Besides, we plot the results of APE and optimality rate in Figure S1 and Figure S2 for $n = 100, 300$ with different error distributions, respectively.

Table S1: Simulation results for Case 1 with Setting I.

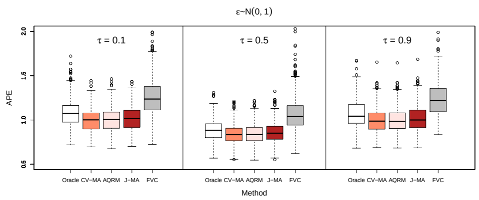
normalized QPE						
n	τ	Oracle	CV-MA	AQRM	J-MA	FVC
100	0.1	1.317	1.012	1.014	1.052	1.687
	0.2	1.146	1.013	1.014	1.041	1.381
	0.3	1.061	1.015	1.016	1.036	1.203
	0.4	1.097	1.017	1.018	1.042	1.300
	0.5	1.185	1.018	1.020	1.054	1.499
300	0.1	1.082	1.013	1.016	1.024	1.162
	0.2	1.039	1.010	1.014	1.019	1.079
	0.3	1.017	1.008	1.012	1.016	1.041
	0.4	1.024	1.009	1.014	1.021	1.061
	0.5	1.047	1.012	1.017	1.027	1.111

Paired t-test						
n	τ		$\frac{\text{FVC}}{\text{CV-MA}}$	$\frac{\text{Oracle}}{\text{CV-MA}}$	$\frac{\text{J-MA}}{\text{CV-MA}}$	$\frac{\text{AQRM}}{\text{CV-MA}}$
100	0.1	t	36.668	30.448	14.503	3.068
		p-value	0.000	0.000	0.000	0.002
	0.2	t	31.653	23.586	13.514	3.614
		p-value	0.000	0.000	0.000	0.000
	0.3	t	28.435	14.229	13.427	4.786
		p-value	0.000	0.000	0.000	0.000
	0.4	t	29.588	16.498	12.574	1.916
		p-value	0.000	0.000	0.000	0.056
	0.5	t	33.527	20.718	12.696	3.324
		p-value	0.000	0.000	0.000	0.001
300	0.1	t	31.435	22.877	11.031	2.662
		p-value	0.000	0.000	0.000	0.008
	0.2	t	27.168	16.280	13.722	5.171
		p-value	0.000	0.000	0.000	0.000
	0.3	t	21.646	8.722	16.055	7.257
		p-value	0.000	0.000	0.000	0.000
	0.4	t	26.687	11.174	19.974	7.107
		p-value	0.000	0.000	0.000	0.000
	0.5	t	27.708	16.039	17.252	5.569
		p-value	0.000	0.000	0.000	0.000

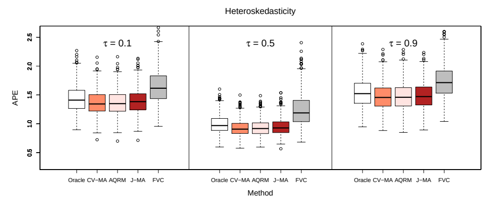
Table S2: Simulation results for Case 1 with Setting II.

normalzed QPE						
n	τ	Oracle	CV-MA	AQRM	J-MA	FVC
100	0.1	1.262	1.016	1.017	1.042	1.595
	0.2	1.124	1.013	1.013	1.033	1.343
	0.3	1.056	1.014	1.015	1.026	1.214
	0.4	1.103	1.016	1.017	1.035	1.322
	0.5	1.177	1.020	1.021	1.045	1.489
300	0.1	1.072	1.009	1.013	1.023	1.141
	0.2	1.032	1.009	1.011	1.016	1.070
	0.3	1.014	1.010	1.012	1.017	1.039
	0.4	1.021	1.011	1.014	1.019	1.062
	0.5	1.048	1.011	1.017	1.027	1.118

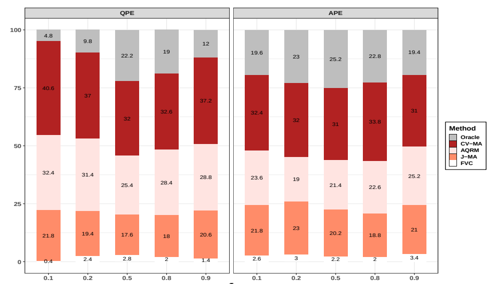
Paired t-test						
n	τ		$\frac{\text{FVC}}{\text{CV-MA}}$	$\frac{\text{Oracle}}{\text{CV-MA}}$	$\frac{\text{J-MA}}{\text{CV-MA}}$	$\frac{\text{AQRM}}{\text{CV-MA}}$
100	0.1	t	34.311	26.914	10.107	4.474
		p-value	0.000	0.000	0.000	0.000
	0.2	t	30.365	20.796	11.593	2.903
		p-value	0.000	0.000	0.000	0.004
	0.3	t	28.091	13.901	8.143	2.694
		p-value	0.000	0.000	0.000	0.007
	0.4	t	31.577	18.332	10.139	1.803
		p-value	0.000	0.000	0.000	0.072
	0.5	t	33.219	20.727	9.804	3.182
		p-value	0.000	0.000	0.000	0.002
300	0.1	t	32.024	24.245	15.378	4.267
		p-value	0.000	0.000	0.000	0.000
	0.2	t	27.067	14.511	10.878	3.300
		p-value	0.000	0.000	0.000	0.001
	0.3	t	21.072	4.162	15.334	5.721
		p-value	0.000	0.000	0.000	0.000
	0.4	t	25.009	7.326	13.404	4.804
		p-value	0.000	0.000	0.000	0.000
	0.5	t	29.488	16.459	17.459	7.783
		p-value	0.000	0.000	0.000	0.000



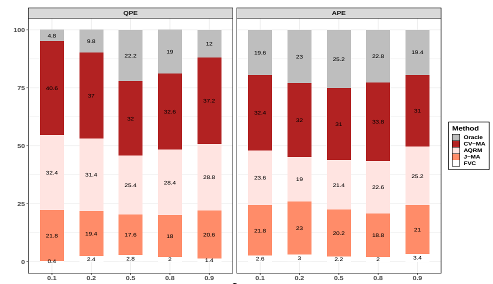
(a) Box-plot of APE for Setting I



(b) Box-plot of APE for Setting II

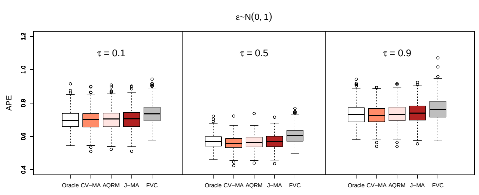


(c) Stacked bar plot of optimality rates for Setting I

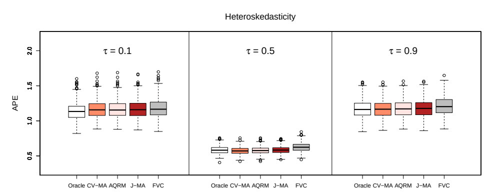


(d) Stacked bar plot of optimality rates for Setting II

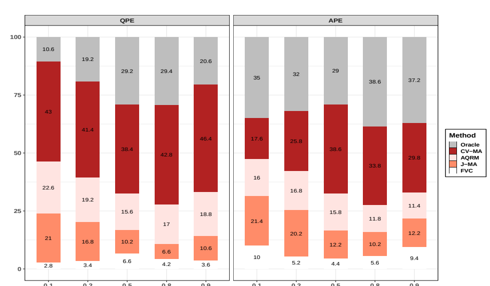
Figure S1: Simulation results for Case 1 with $n = 100$.



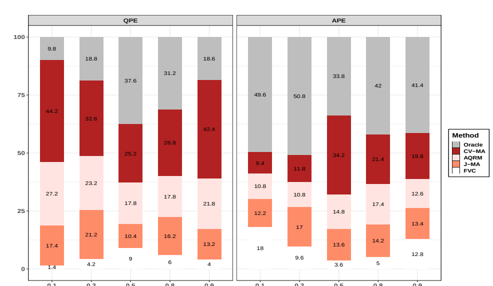
(a) Box-plot of APE for Setting I



(b) Box-plot of APE for Setting II



(c) Stacked bar plot of optimality rates for Setting I



(d) Stacked bar plot of optimality rates for Setting II

Figure S2: Simulation results for Case 1 with $n = 300$.

Case 2 (Model misspecification). In this case, we assess the performances of methods (a)-(e) when model is misspecified. The results of normalized QPE and t-test with $n = 100, 300$ are presented in Table S3 and Table S4. Figure S3 and Figure S4 show the results of APE and optimality rate.

Table S3: Simulation results for Case 2 with Setting I.

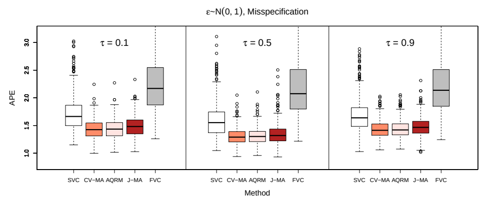
normalized QPE						
n	τ	SVC	CV-MA	AQRM	J-MA	FVC
100	0.1	1.534	1.022	1.024	1.068	2.326
	0.2	1.339	1.016	1.017	1.045	1.900
	0.3	1.213	1.013	1.013	1.034	1.614
	0.4	1.308	1.023	1.024	1.050	1.820
	0.5	1.460	1.039	1.040	1.079	2.158
300	0.1	1.214	1.011	1.011	1.028	1.549
	0.2	1.115	1.007	1.008	1.021	1.329
	0.3	1.061	1.006	1.006	1.014	1.216
	0.4	1.107	1.009	1.009	1.019	1.327
	0.5	1.204	1.011	1.012	1.029	1.542

Paired t-test						
n	τ		$\frac{\text{FVC}}{\text{CV-MA}}$	$\frac{\text{SVC}}{\text{CV-MA}}$	$\frac{\text{J-MA}}{\text{CV-MA}}$	$\frac{\text{AQRM}}{\text{CV-MA}}$
100	0.1	t	29.751	26.764	8.484	3.826
		p-value	0.000	0.000	0.000	0.000
	0.2	t	29.742	23.521	8.263	2.560
		p-value	0.000	0.000	0.000	0.011
	0.3	t	33.674	23.439	7.688	1.844
		p-value	0.000	0.000	0.000	0.066
	0.4	t	29.105	19.262	7.125	0.996
		p-value	0.000	0.000	0.000	0.320
	0.5	t	28.985	19.070	6.185	2.220
		p-value	0.000	0.000	0.000	0.027
300	0.1	t	28.817	21.872	8.886	0.776
		p-value	0.000	0.000	0.000	0.438
	0.2	t	27.227	19.477	10.309	1.794
		p-value	0.000	0.000	0.000	0.073
	0.3	t	29.459	21.508	9.513	2.205
		p-value	0.000	0.000	0.000	0.028
	0.4	t	30.526	21.113	6.355	1.776
		p-value	0.000	0.000	0.000	0.076
	0.5	t	32.418	21.907	8.373	3.343
		p-value	0.000	0.000	0.000	0.001

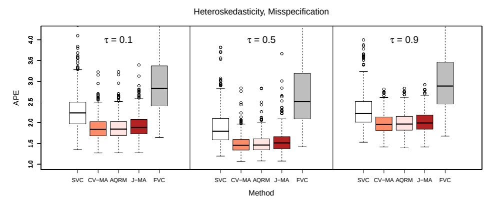
Table S4: Simulation results for Case 2 with Setting II.

normalized QPE						
n	τ	SVC	CV-MA	AQRM	J-MA	FVC
100	0.1	1.589	1.026	1.028	1.062	2.381
	0.2	1.374	1.015	1.017	1.051	1.952
	0.3	1.227	1.014	1.015	1.040	1.666
	0.4	1.334	1.018	1.020	1.058	1.909
	0.5	1.506	1.027	1.030	1.069	2.268
300	0.1	1.239	1.012	1.013	1.033	1.578
	0.2	1.129	1.008	1.009	1.023	1.344
	0.3	1.069	1.006	1.007	1.016	1.220
	0.4	1.117	1.008	1.009	1.023	1.350
	0.5	1.217	1.010	1.011	1.030	1.576

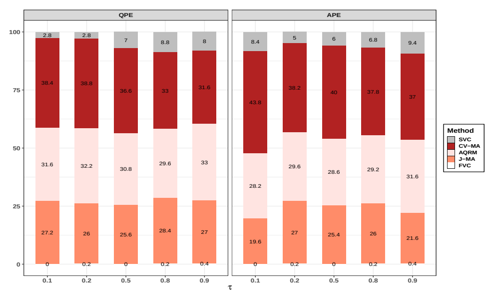
Paired t-test						
n	τ		$\frac{\text{FVC}}{\text{CV-MA}}$	$\frac{\text{SVC}}{\text{CV-MA}}$	$\frac{\text{J-MA}}{\text{CV-MA}}$	$\frac{\text{AQRM}}{\text{CV-MA}}$
100	0.1	t	29.344	26.641	6.73	5.889
		p-value	0.000	0.000	0.000	0.000
	0.2	t	28.406	23.151	9.082	5.047
		p-value	0.000	0.000	0.000	0.000
	0.3	t	29.874	25.036	8.023	5.229
		p-value	0.000	0.000	0.000	0.000
	0.4	t	28.06	22.519	9.44	4.438
		p-value	0.000	0.000	0.000	0.000
	0.5	t	29.069	22.155	8.143	5.182
		p-value	0.000	0.000	0.000	0.000
300	0.1	t	31.946	24.153	7.498	2.791
		p-value	0.000	0.000	0.000	0.005
	0.2	t	31.513	21.865	8.793	4.301
		p-value	0.000	0.000	0.000	0.000
	0.3	t	32.889	20.756	9.443	3.101
		p-value	0.000	0.000	0.000	0.002
	0.4	t	28.824	20.873	9.537	2.548
		p-value	0.000	0.000	0.000	0.011
	0.5	t	29.374	22.957	8.508	1.550
		p-value	0.000	0.000	0.000	0.122



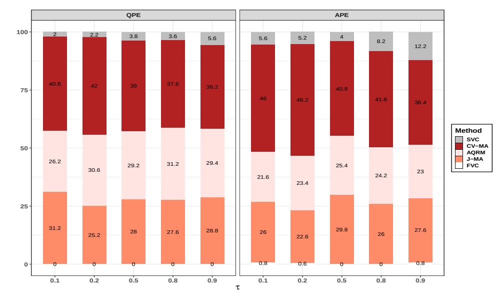
(a) Box-plot of APE for Setting I



(b) Box-plot of APE for Setting II

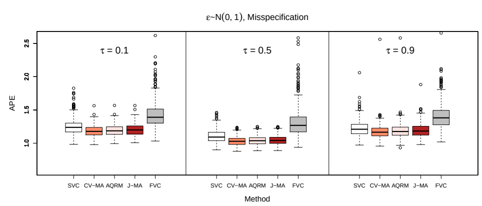


(c) Stacked bar plot of optimality rates for Setting I

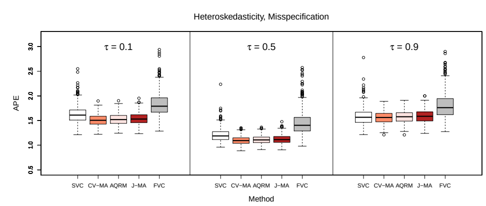


(d) Stacked bar plot of optimality rates for Setting II

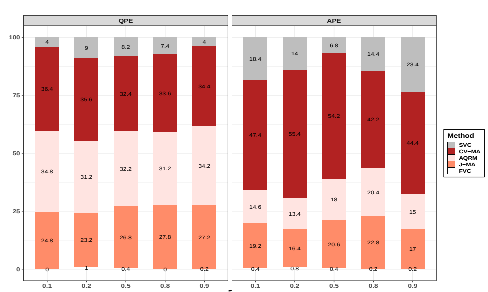
Figure S3: Simulation results for Case 2 with $n = 100$.



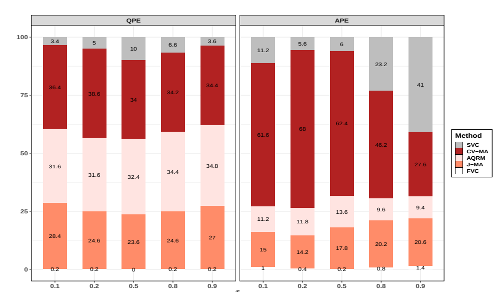
(a) Box-plot of APE for Setting I



(b) Box-plot of APE for Setting II



(c) Stacked bar plot of optimality rates for Setting I



(d) Stacked bar plot of optimality rates for Setting II

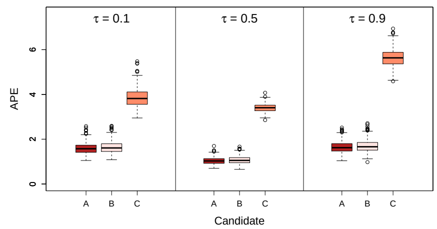
Figure S4: Simulation results for Case 2 with $n = 300$.

Case 3 (Different candidate models). In this case, we examine the performances of different candidate models. The results of normalized QPE and t-test for heteroscedastic case with $n = 100, 300$ are presented in Table S5. The results of APE and optimality rate with $n = 100, 300$ are shown in Figure S5.

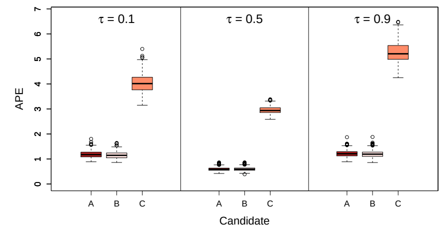
Table S5: Simulation results for Case 3 with Setting II.

normalzed QPE				
n	τ	Candidate A	Candidate B	Candidate C
100	0.1	1.003	1.231	1.877
	0.2	1.004	1.105	2.289
	0.3	1.009	1.028	2.283
	0.4	1.009	1.043	2.578
	0.5	1.010	1.093	2.433
300	0.1	1.003	1.048	2.700
	0.2	1.005	1.019	2.962
	0.3	1.006	1.007	2.696
	0.4	1.007	1.009	2.985
	0.5	1.007	1.023	2.807

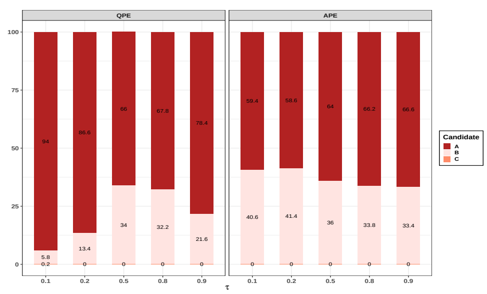
Paired t-test				
n	τ		$\frac{\text{Candidate B}}{\text{Candidate A}}$	$\frac{\text{Candidate C}}{\text{Candidate A}}$
100	0.1	t	26.635	78.161
		p-value	0.000	0.000
	0.2	t	21.666	134.214
		p-value	0.000	0.000
	0.3	t	9.513	213.919
		p-value	0.000	0.000
	0.4	t	10.746	181.1 00
		p-value	0.000	0.000
	0.5	t	14.306	116.889
		p-value	0.000	0.000
300	0.1	t	17.631	137.936
		p-value	0.000	0.000
	0.2	t	11.245	181.331
		p-value	0.000	0.000
	0.3	t	2.062	284.896
		p-value	0.000	0.040
	0.4	t	1.160	325.482
		p-value	0.000	0.246
	0.5	t	7.958	211.230
		p-value	0.000	0.000



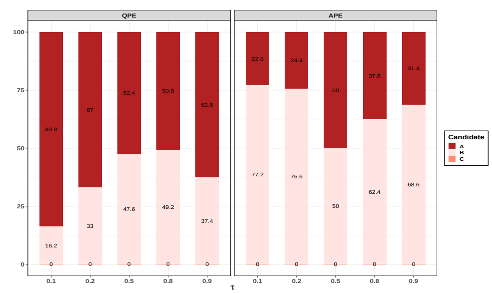
(a) Box-plot of APE with $n = 100$



(b) Box-plot of APE with $n = 300$



(c) Stacked bar plot of optimality rates with $n = 100$



(d) Stacked bar plot of optimality rates with $n = 300$

Figure S5: Simulation results for Case 3 with Setting II.

2 Asymptotic Properties of the Final Estimator

In this section, we conduct simulation studies to investigate the asymptotic properties of the CV-MA estimator.

2.1 Asymptotic Distribution of the Final Estimator

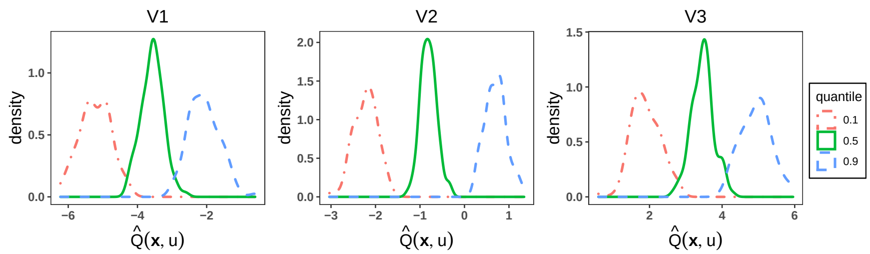
In this simulation, the data are generated from the semiparametric varying coefficient model with Case 1 mentioned in the main paper. We set $n = 200$. To explore the asymptotic distribution of CV-MA estimator. We fix three values for U and $\mathbf{X} = (X_1, X_2, \cdots, X_{10})^\top$.

V1: $u = 0.25$, $\mathbf{x} = (1, x_{0.25}, \cdots, x_{0.25})^\top$, where $x_{0.25}$ is the 0.25-th quantile of $N(0, 1)$;

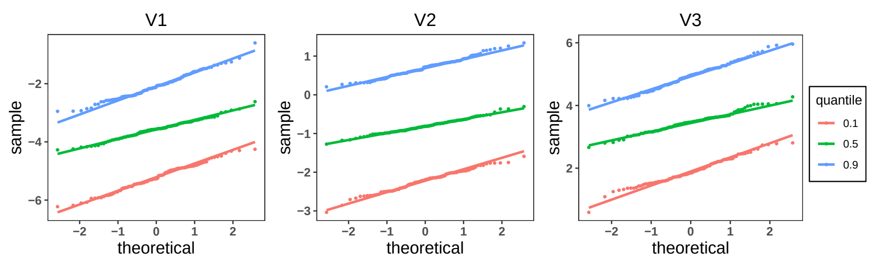
V2: $u = 0.5$, $\mathbf{x} = (1, 0, \cdots, 0)^\top$;

V3: $u = 0.75$, $\mathbf{x} = (1, x_{0.75}, \cdots, x_{0.75})^\top$, where $x_{0.75}$ is the 0.75-th quantile of $N(0, 1)$.

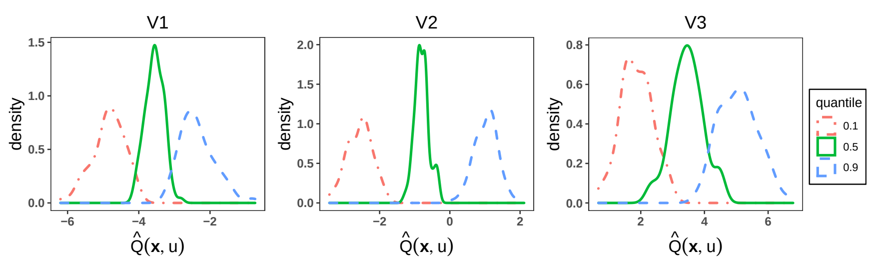
Based on 500 replications, we obtain the density plots for the final estimator $\widehat{Q}_\tau(\mathbf{x}, u)$, see Figure S6 (a) and (c) for different error distributions and different quantiles. To further investigate the distribution of the final estimator, we plot Q-Q (quantile-quantile) plots of a sample of $\widehat{Q}_\tau(\mathbf{x}, u)$ versus a normal distribution in (b) and (d) of Figure S6.



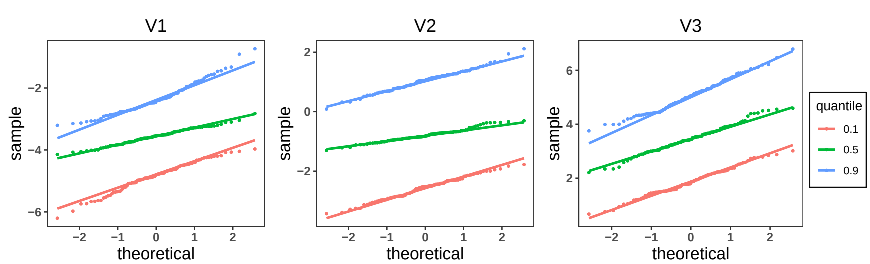
(a) density plots for Setting I



(b) Q-Q plots for Setting I



(c) density plots for Setting II



(d) Q-Q plots for Setting II

Figure S6: The density plots and Q-Q plots of the CV-MA estimator

From Figure S6, we observe that the shape of the density function of $\widehat{Q}_\tau(\mathbf{x}, u)$ looks like a bell, which indicates that the distribution of the final estimator is approaching to a normal distribution. Also, the Q-Q plots suggest that the sample data of the final estimator fit the normal distribution well. A more detailed discussion about the asymptotic distribution of $\widehat{Q}_\tau(\mathbf{x}, u)$ is of interest and deserves further research.

2.2 Comparisons of MSE, Bias and Variance

In this subsection, we investigate the performances of different methods in terms of MSE, Bias and Variance. In this simulation, data are generated from the semiparametric varying coefficient model with $n = 200$ under Case 1 mentioned in the main paper. We consider the mean of mean squared error (MSE) with the testing sample, defined as

$$\text{MSE} = \frac{1}{N}\sum_{l=1}^{N}\frac{1}{n_{\text{test}}}\sum_{i=1}^{n_{\text{test}}}\left(\widehat{Q}_\tau^l(\mathbf{X}_i, U_i) - Q_\tau(\mathbf{X}_i, U_i)\right)^2,$$

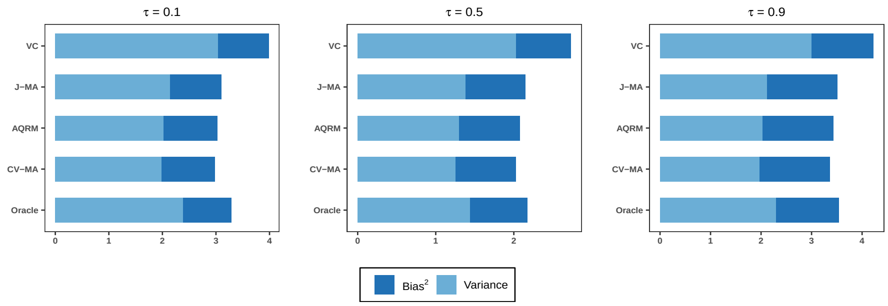
it can be decomposed into Variance and Bias^2, where

$$\text{Bias}^2 = \frac{1}{n_{\text{test}}}\sum_{i=1}^{n_{\text{test}}}\left(\bar{\widehat{Q}}_\tau(\mathbf{X}_i, U_i) - Q_\tau(\mathbf{X}_i, U_i)\right)^2,$$

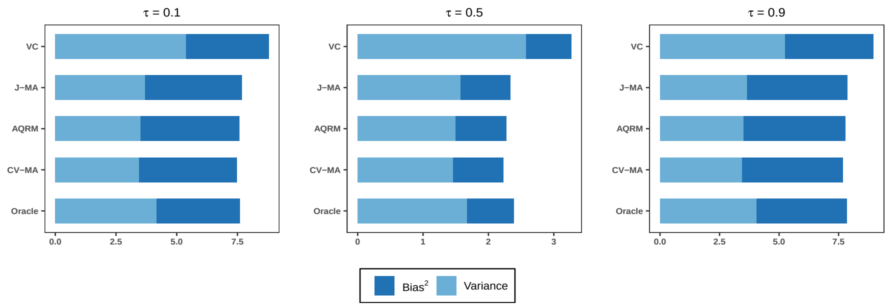
$$\text{Variance} = \frac{1}{N}\sum_{l=1}^{N}\frac{1}{n_{\text{test}}}\sum_{i=1}^{n_{\text{test}}}\left(\widehat{Q}_\tau^l(\mathbf{X}_i, U_i) - \bar{\widehat{Q}}_\tau(\mathbf{X}_i, U_i)\right)^2,$$

where $\bar{\widehat{Q}}_\tau(\mathbf{X}_i, U_i) = \frac{1}{N}\sum_{l=1}^{N}\widehat{Q}_\tau^l(\mathbf{X}_i, U_i)$ and $\widehat{Q}_\tau^l(\mathbf{X}_i, U_i)$ is the estimate of $Q_\tau(\mathbf{X}_i, U_i)$ in the l-th replication.

We provide a comparison of bias component and variance component of MSE with $\tau = 0.1, 0.5, 0.9$ for different error distributions in Figure S7. As shown in Figure S7, the CV-MA has an edge over other methods including the Oracle in terms of Variance in almost all cases, resulting in competitive performance in MSE. Not surprisingly, the VC method always produces the largest Variance. For the result of Bias^2, it is complicated that the Oracle method and VC method may perform better than our proposed CV-MA sometimes. However, for our proposed method is designed to minimize the QPE, we are not so hang-up about the optimality in terms of Bias^2 and Variance.



(a) MSE decomposition for Setting I



(b) MSE decomposition for Setting II

Figure S7: The results of MSE comparisons.