Supplementary material for the manuscript "Bootstrap method for misspecified ergodic Lévy driven stochastic differential equation models"

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0.1 Proofs of auxiliary results for blocked sums

Proof of Lemma 19

Proof of (26): From the Lipschitz continuity of A, Jensen's inequality, and [Masuda(2013), Lemma 5.3], we have

$$\begin{aligned}
&E\left[\sum_{i=1}^{k_n}\left|\sum_{j\in B_{k_i}} R_{t_{j-1}}\int_{t_{j-1}}^{t_j}(A_s - A_{t_{j-1}})ds\right|^2\right] \\
&\lesssim \frac{nh_n}{k_n}\sum_{i=1}^{k_n}\sum_{j\in B_{k_i}} E\left[\left|R_{t_{j-1}}\right|^2\left[\int_{t_{j-1}}^{t_j} E^{j-1}\left[|X_s - X_{t_{j-1}}|^2\right]ds\right]\right] \\
&\lesssim \frac{n^2h_n^3}{k_n}.
\end{aligned}$$

Proof of (27): It is straightforward from Jensen's inequality.

Proof of (28): By applying Burkholder's inequality twice, it follows that

$$\begin{aligned}
&E\left[\sum_{i=1}^{k_n}\left|\sum_{j\in B_{k_i}} R_{t_{j-1}}\int_{t_{j-1}}^{t_j} C_{s-}dZ_s\right|^2\right] \\
&\lesssim \sum_{i=1}^{k_n}\sum_{j\in B_{k_i}} E\left[\left|R_{t_{j-1}}\right|^2 E^{j-1}\left[\left|\int_{t_{j-1}}^{t_j} C_{s-}dZ_s\right|^2\right]\right] \\
&\lesssim \sum_{i=1}^{k_n}\sum_{j\in B_{k_i}} E\left[\left|R_{t_{j-1}}\right|^2\int_{t_{j-1}}^{t_j} E^{j-1}\left[|C_s|^2\right]ds\right]
\end{aligned}$$

$$\lesssim T_n.$$

□

Proof of Lemma 20

Proof of (29): First we rewrite $\left(\int_{t_{j-1}}^{t_j} C_s dw_s\right)^2 - h_n C_{t_{j-1}}^2$ as

$$\begin{aligned}
&\left(\int_{t_{j-1}}^{t_j} C_s dw_s\right)^2 - h_n C_{t_{j-1}}^2 \\
&= \left(\int_{t_{j-1}}^{t_j} C_s dw_s\right)^2 - E^{j-1}\left[\left(\int_{t_{j-1}}^{t_j} C_s dw_s\right)^2\right] + E^{j-1}\left[\left(\int_{t_{j-1}}^{t_j} C_s dw_s\right)^2\right] - h_n C_{t_{j-1}}^2.
\end{aligned}$$

From Assumption 3 and Burkholder's inequality, there exists a positive constant K such that

$$\begin{aligned}
&E\left[\sum_{i=1}^{k_n}\left|\sum_{j\in B_{k_i}} R_{t_{j-1}}\left\{\left(\int_{t_{j-1}}^{t_j} C_s dw_s\right)^2 - E^{j-1}\left[\left(\int_{t_{j-1}}^{t_j} C_s dw_s\right)^2\right]\right\}\right|^2\right] \\
&\lesssim h_n \sum_{i=1}^{k_n}\sum_{j\in B_{k_i}} E\left[\left|R_{t_{j-1}}\right|^2 \int_{t_{j-1}}^{t_j} E^{j-1}\left[1 + |X_s - X_{t_{j-1}}|^K + |X_{t_{j-1}}|^K\right] ds\right] \\
&\lesssim n h_n^2.
\end{aligned}$$

Since Itô's formula leads to

$$\begin{aligned}
E^{j-1}\left[\left(\int_{t_{j-1}}^{t_j} C_s dw_s\right)^2\right] &= \int_{t_{j-1}}^{t_j} E^{j-1}\left[C_s^2\right] ds \\
&= h_n C_{t_{j-1}}^2 + \int_{t_{j-1}}^{t_j}\left(\int_{t_{j-1}}^{s} E^{j-1}\left[\mathcal{A}C_u^2\right] du\right) ds,
\end{aligned}$$

in a similar manner to the proof of (26), we can obtain

$$E\left[\sum_{i=1}^{k_n}\left|\sum_{j\in B_{k_i}} R_{t_{j-1}}\left\{E^{j-1}\left[\left(\int_{t_{j-1}}^{t_j} C_s dw_s\right)^2\right] - h_n C_{t_{j-1}}^2\right\}\right|^2\right] \lesssim \frac{n^2 h_n^4}{k_n} = o\left(n h_n^2\right),$$

and thus (29).

Proof of (30): From Assumption 5, $\int_{\mathbb{R}} z^2 \nu_0(dz) = 1$. Hence, Itô's formula yields that

$$\left(\int_{t_{j-1}}^{t_j} C_{s-} dZ_s\right)^2 - h_n C_{t_{j-1}}^2$$

$$= \int_{t_{j-1}}^{t_j} \left(C_s^2 - C_{t_{j-1}}^2\right) ds + 2\int_{t_{j-1}}^{t_j} \left(\int_{t_{j-1}}^{s} C_{u-} dZ_u\right) C_{s-} dZ_s + \int_{t_{j-1}}^{t_j} \int_{\mathbb{R}} C_{s-}^2 z^2 \tilde{N}(ds, dz).$$

By taking a similar route to the proof of (29), we can easily observe that

$$E\left[\sum_{i=1}^{k_n} \left| \sum_{j \in B_{k_i}} R_{t_{j-1}} \int_{t_{j-1}}^{t_j} \left(C_s^2 - C_{t_{j-1}}^2\right) ds \right|^2\right] \lesssim \frac{n^2 h_n^{5/2}}{k_n}.$$

Since the second and third terms of the right-hand-side are martingale, the desired result directly follows from Burkholder's inequality and

$$\frac{n^2 h_n^{5/2}}{k_n T_n} = \frac{n h_n^{3/2}}{k_n} \lesssim \sqrt{n h_n^2} = o(1).$$

□

Proof of Lemma 21

Applying [Yoshida(2011), Lemma 4], we have

$$E\left[\left| \sum_{j \in B_{k_i}} \left(f_{t_{j-1}} - E[f_{t_{j-1}}]\right) \right|^2\right] \lesssim \sqrt{\frac{n}{k_n}}.$$

From Assumption 3, we also have

$$\begin{aligned} \left| \sum_{j \in B_{k_i}} E[f_{t_{j-1}}] \right|^2 &= \left| \sum_{j \in B_{k_i}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(y) \left[P_{t_{j-1}}(x, dy) - \pi_0(dy)\right] \eta(dx) \right|^2 \\ &\lesssim e^{-2a[(i-1)c_n + 1]h_n} \left| \sum_{j=0}^{c_n - 1} e^{-ajh_n} \right|^2 \\ &\lesssim 1. \end{aligned}$$

Thus we get the desired result. □

Proof of Lemma 22

For simplicity, we write

$$Y_{i,t} = \int_{(i-1)c_n h_n}^{t} \left[\sum_{j=(i-1)c_n+1}^{ic_n} f_{t_{j-1}}(w_s - w_{t_{j-1}}) \chi_j(s)\right] dw_s, \quad t \in ((i-1)c_n h_n, ic_n h_n].$$

From Itô's formula, it follows that

$$(Y_{i,ic_n h_n})^2 = 2\int_{(i-1)c_n h_n}^{ic_n h_n} Y_{i,s} dY_{i,s} + \sum_{j=(i-1)c_n+1}^{ic_n} \left[f_{t_{j-1}}^2 \int_{(j-1)h_n}^{jh_n} (w_s - w_{t_{j-1}})^2 ds\right].$$

Hence the left-hand-side of (32) can be rewritten as:

$$\frac{1}{nh_n^2}\sum_{i=1}^{k_n}\left\{\int_{(i-1)c_nh_n}^{ic_nh_n}\left[\sum_{j=(i-1)c_n+1}^{ic_n} f_{t_{j-1}}(w_s-w_{t_{j-1}})\chi_j(s)\right]dw_s\right\}^2$$
$$=\frac{2}{nh_n^2}\sum_{i=1}^{k_n}\int_{(i-1)c_nh_n}^{ic_nh_n} Y_{i,s}dY_{i,s}+\frac{1}{nh_n^2}\sum_{j=1}^{n}\left[f_{j-1}^2\int_{t_{j-1}}^{t_j}(w_s-w_{t_{j-1}})^2ds\right],$$

By utilizing Burkholder's inequality, we get

$$E\left[\left(\frac{1}{nh_n^2}\sum_{i=1}^{k_n}\int_{(i-1)c_nh_n}^{ic_nh_n} Y_{i,s}dY_{i,s}\right)^2\right]=O\left(\frac{1}{T_n}\right).$$

Moreover, Fubini's theorem and Jensen's inequality lead to

$$E^{j-1}\left[\int_{t_{j-1}}^{t_j}(w_s-w_{t_{j-1}})^2ds\right]=\frac{h_n^2}{2},$$
$$E^{j-1}\left[\left\{\int_{t_{j-1}}^{t_j}(w_s-w_{t_{j-1}})^2ds\right\}^2\right]\lesssim h_n^4.$$

Hence [Genon-Catalot and Jacod(1993), Lemma 9] and the ergodic theorem imply (32). Next we show (33). First we decompose $\left(\int_{(i-1)c_nh_n}^{ic_nh_n}\sum_{j=(i-1)c_n+1}^{ic_n} f_s\chi_j(s)dw_s\right)^2$ as:

$$\left(\int_{(i-1)c_nh_n}^{ic_nh_n}\sum_{j=(i-1)c_n+1}^{ic_n} f_s\chi_j(s)dw_s\right)^2$$
$$=\left(\int_{(i-1)c_nh_n}^{ic_nh_n}\sum_{j=(i-1)c_n+1}^{ic_n}(f_s-f_{t_{j-1}})\chi_j(s)dw_s+\sum_{j=(i-1)c_n+1}^{ic_n} f_{t_{j-1}}\Delta_j w\right)^2.$$

From the isometry property, Cauchy-Schwarz inequality, and Burkholder's inequality, we have

$$E\left[\left(\int_{(i-1)c_nh_n}^{ic_nh_n}\sum_{j=(i-1)c_n+1}^{ic_n}(f_s-f_{t_{j-1}})\chi_j(s)dw_s\right)^2\right]$$
$$=\sum_{j=(i-1)c_n+1}^{ic_n}\int_{t_{j-1}}^{t_j} E\left[(f_s-f_{t_{j-1}})^2\right]ds$$

$$\lesssim \sum_{j=(i-1)c_n+1}^{ic_n} \int_{t_{j-1}}^{t_j} \sqrt{E\left[|X_s - X_{t_{j-1}}|^4\right]} ds$$
$$\lesssim c_n h_n^2.$$

By the independence of the increments and the ergodic theorem, we also obtain

$$E\left[\left(\sum_{j=(i-1)c_n+1}^{ic_n} f_{t_{j-1}} \Delta_j w\right)^2\right] = h_n \sum_{j=(i-1)c_n+1}^{ic_n} E[f_{t_{j-1}}^2] \lesssim c_n h_n.$$

Hence Cauchy-Schwarz inequality yields that

$$\frac{1}{T_n}\sum_{i=1}^{k_n}\left(\int_{(i-1)c_nh_n}^{ic_nh_n} \sum_{j=(i-1)c_n+1}^{ic_n} f_s \chi_j(s) dw_s\right)^2$$
$$= \frac{1}{T_n}\sum_{i=1}^{k_n}\left(\sum_{j=(i-1)c_n+1}^{ic_n} f_{t_{j-1}} \Delta_j w\right)^2 + O_p\left(\sqrt{h_n}\right)$$
$$= \frac{1}{T_n}\sum_{i=1}^{k_n}\left(\int_{(i-1)c_nh_n}^{ic_nh_n} \sum_{j=(i-1)c_n+1}^{ic_n} f_{t_{j-1}} \chi_j(s) dw_s\right)^2 + O_p\left(\sqrt{h_n}\right).$$

Then, by mimicking the proof of (32), we can easily get (33). □

Proof of Lemma 23

First we remark that since for all $0 \le s \le t$,

$$\int_s^t \int_{\mathbb{R}} E\left[(f(X_{u-}, z))^2\right] \nu_0(dz) du < \infty,$$

the stochastic integral $\int_t^s \int f(X_{s-}, z)\tilde{N}(ds, dz)$ is well defined (cf. [Applebaum(2009)]). We take a similar way to the proof of the previous lemma. Let

$$Y_{i,t} = \int_{(i-1)c_nh_n}^{t} \int f(X_{s-}, z)\tilde{N}(ds, dz), \quad t \in ((i-1)c_nh_n, ic_nh_n].$$

By applying Itô's formula, we have

$$(Y_{i,ic_nh_n})^2 = 2\int_{(i-1)c_nh_n}^{ic_nh_n} Y_{i,s-} dY_{i,s} + \int_{(i-1)c_nh_n}^{ic_nh_n} \int_{\mathbb{R}} (f(X_{s-}, z))^2 \tilde{N}(ds, dz)$$
$$+ \int_{(i-1)c_nh_n}^{ic_nh_n} \int_{\mathbb{R}} (f(X_s, z))^2 \nu_0(dz) ds.$$

Hence it follows that

$$\frac{1}{T_n}\sum_{i=1}^{k_n}\left(\int_{(i-1)c_nh_n}^{ic_nh_n} \int f(X_{s-}, z)\tilde{N}(ds, dz)\right)^2$$

$$= \frac{1}{T_n}\left(2\sum_{i=1}^{k_n}\int_{(i-1)c_nh_n}^{ic_nh_n} Y_{i,s}dY_{i,s} + \int_0^{T_n}\int_{\mathbb{R}} (f(X_{s-},z))^2 \tilde{N}(ds,dz) + \int_0^{T_n}\int_{\mathbb{R}} (f(X_s,z))^2 \nu_0(dz)ds\right).$$

From the isometry property, we can easily observe that

$$E\left[\left\{\frac{1}{T_n}\left(\sum_{i=1}^{k_n}\int_{(i-1)c_nh_n}^{ic_nh_n} Y_{i,s}dY_{i,s} + \int_0^{T_n}\int_{\mathbb{R}} (f(X_{s-},z))^2 \tilde{N}(ds,dz)\right)\right\}^2\right] = O\left(\frac{1}{T_n}\right),$$

so that

$$\frac{1}{T_n}\sum_{i=1}^{k_n}\left(\int_{(i-1)c_nh_n}^{ic_nh_n}\int f(X_{s-},z)\tilde{N}(ds,dz)\right)^2 = \frac{1}{T_n}\int_0^{T_n}\int_{\mathbb{R}} (f(X_s,z))^2 \nu_0(dz)ds + o_p(1).$$

Since we can pick a positive constant ϵ such that

$$\int_{\mathbb{R}} (f(x,z))^2 \nu_0(dz) \lesssim \left(1+|x|^C\right)\int_{\mathbb{R}} \left(|z|^{\beta+\epsilon} \vee |z|^{2\delta}\right)\nu_0(dz) \lesssim 1+|x|^C,$$

for all $x \in \mathbb{R}$, the desired result follows from the ergodic theorem. □

References

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