# Supplemental material - Inference using exact distribution of test-statistics for random-effects meta-analysis

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## 1 Introduction

In this supplemental material, we provide the additional information via simulations and examples to investigate characteristics on the proposed almost-exact method. We also give the R program for probability density functions (PDFs) of between-study variance $\tau^2$ and $T_{DL}$, and an inference based on our almost-exact method. First, we show the approximation accuracy on Theorem 1 when the number of studies is small to large. Second, we show the accuracy of the confidence interval (CI) for the overall treatment effect $\theta$ under the other typical heterogeneity $I^2$. Finally, we show the R program for calculating $p$-value and CI of $\theta$.

## 2 Approximation accuracy of Theorem 1 when the number of studies is small

We show the approximation accuracy on Theorem 1 when the number of studies is very small to large. Theorem 1 needs an approximation, which depends on the number of studies, that the conditional random variable $T_{DL}|\hat{\tau}^2_{DL0}$ is approximately normally distributed. In this supplemental material we simulate the distribution of $T_{DL}$ when the number of studies $K$ is very small to large ($K = 2, 3, \cdots, 10, 15, 20, 30, 40, 50$). The case of $K = 2$ is too small situation in meta-analysis and $K = 50$ is sufficiently large. The step width of between-study heterogeneity $I^2$ is made smaller than that of the main text ($I^2 = 0, 0.05, \cdots, 0.90, 0.95$). The proposed almost-exact method is compared with standard normal distribution- and $t$-distribution-based methods.

In Table S.1, we provide the simulation result of the 97.5% points of $T_{DL}$. We also show the 97.5% points of $T_{DL}$ in Figure S.1 and S.2 because it is too long to read Table S.1. The simulation is close to Table 2 in the main text of the paper, but the setting is different. The almost-exact PDF of $T_{DL}$ becomes exact as $I^2$ is small to medium. However, the almost-exact PDF deviates the nominal 97.5% points when $I^2$ is large and the number of studies is very small ($K = 2, 3$). Thus, we can understand that the almost-exact PDF is highly accurate approximation even if the number of studies is the minimum. The standard normal distribution-based method trends slightly conservative in the case of small heterogeneity $I^2$, but it always deviates the nominal 97.5% points regardless of the number of studies. The $t$-distribution-based method always provides quite conservative trend. However, when the heterogeneity is less than medium ($I^2 \leq 0.5$), the $t$-distribution-based method trends too conservative. Further, in the case of small number of studies, the 97.5% points of the $t$-distribution is larger then simulation value.

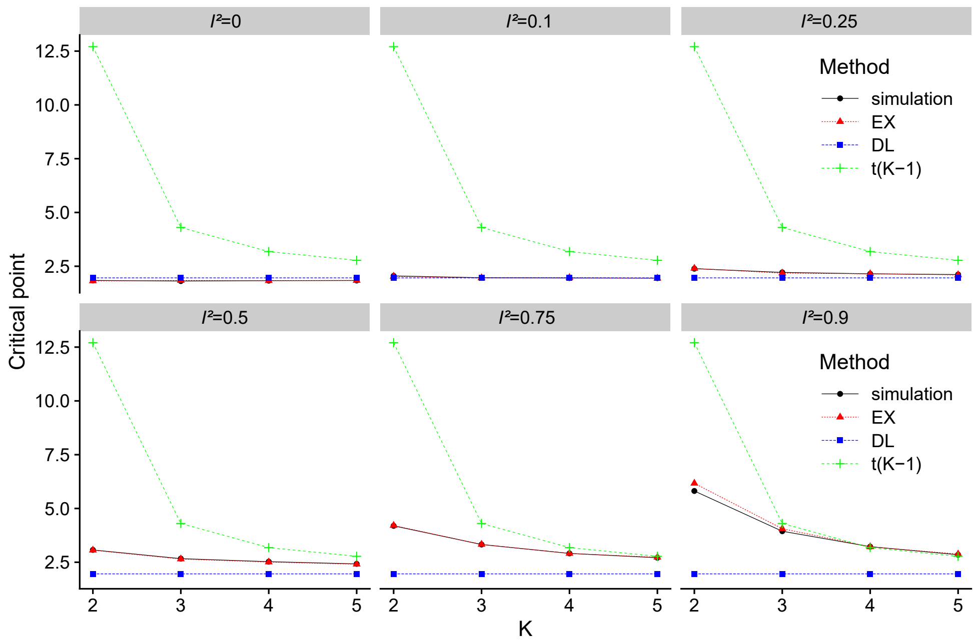


Figure S.1: The 97.5% point of the test statistic $T_{DL}$ for the overall treatment effect in the random effects meta-analysis when the number of studies is small ($K = 2, \cdots, 5$).

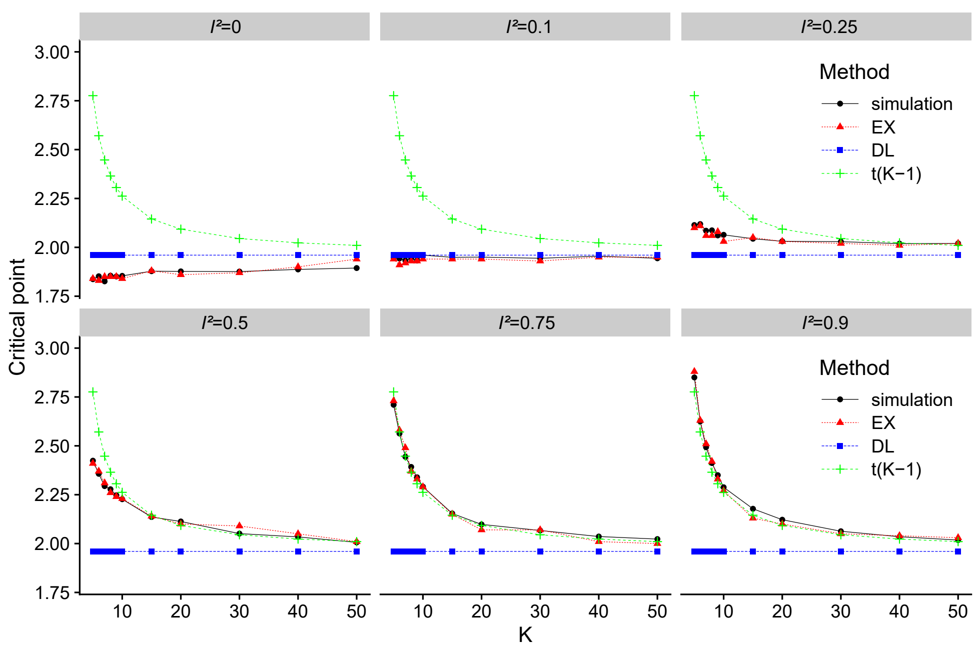


Figure S.2: The 97.5% point of the test statistic $T_{DL}$ for the overall treatment effect in the random effects meta-analysis when the number of studies is small to large ($K = 5, \cdots, 10, 15, 20, 30, 40, 50$).

Table S.1: The 97.5% point of the test statistic $T_{DL}$ for the overall treatment effect in the random effects meta-analysis. $100,000$ simulated values (SIM), the proposed method (EX), the standard normal distribution ($N(0,1)$) and $t$-distribution with $K-1$ degree of freedom ($t(K-1)$).

| $K$ | $I^2$ | SIM | EX | $N(0,1)$ | $t(K-1)$ |
|---|---|---|---|---|---|
| 2 | 0 | 1.84 | 1.82 | 1.96 | 12.71 |
| 2 | 0.05 | 1.93 | 1.93 | 1.96 | 12.71 |
| 2 | 0.1 | 2.05 | 2.03 | 1.96 | 12.71 |
| 2 | 0.15 | 2.16 | 2.16 | 1.96 | 12.71 |
| 2 | 0.2 | 2.28 | 2.26 | 1.96 | 12.71 |
| 2 | 0.25 | 2.39 | 2.40 | 1.96 | 12.71 |
| 2 | 0.3 | 2.51 | 2.52 | 1.96 | 12.71 |
| 2 | 0.35 | 2.65 | 2.65 | 1.96 | 12.71 |
| 2 | 0.4 | 2.77 | 2.79 | 1.96 | 12.71 |
| 2 | 0.45 | 2.92 | 2.92 | 1.96 | 12.71 |
| 2 | 0.5 | 3.07 | 3.06 | 1.96 | 12.71 |
| 2 | 0.55 | 3.29 | 3.26 | 1.96 | 12.71 |
| 2 | 0.6 | 3.44 | 3.43 | 1.96 | 12.71 |
| 2 | 0.65 | 3.65 | 3.69 | 1.96 | 12.71 |
| 2 | 0.7 | 3.87 | 3.92 | 1.96 | 12.71 |
| 2 | 0.75 | 4.19 | 4.21 | 1.96 | 12.71 |
| 2 | 0.8 | 4.61 | 4.57 | 1.96 | 12.71 |
| 2 | 0.85 | 5.04 | 5.17 | 1.96 | 12.71 |
| 2 | 0.9 | 5.81 | 6.17 | 1.96 | 12.71 |
| 2 | 0.95 | 7.13 | 9.07 | 1.96 | 12.71 |
| 3 | 0 | 1.81 | 1.84 | 1.96 | 4.30 |
| 3 | 0.05 | 1.90 | 1.89 | 1.96 | 4.30 |
| 3 | 0.1 | 1.97 | 1.96 | 1.96 | 4.30 |
| 3 | 0.15 | 2.06 | 2.06 | 1.96 | 4.30 |
| 3 | 0.2 | 2.13 | 2.14 | 1.96 | 4.30 |
| 3 | 0.25 | 2.21 | 2.18 | 1.96 | 4.30 |
| 3 | 0.3 | 2.28 | 2.29 | 1.96 | 4.30 |
| 3 | 0.35 | 2.36 | 2.39 | 1.96 | 4.30 |
| 3 | 0.4 | 2.48 | 2.51 | 1.96 | 4.30 |
| 3 | 0.45 | 2.57 | 2.57 | 1.96 | 4.30 |
| 3 | 0.5 | 2.67 | 2.65 | 1.96 | 4.30 |
| 3 | 0.55 | 2.79 | 2.78 | 1.96 | 4.30 |
| 3 | 0.6 | 2.89 | 2.87 | 1.96 | 4.30 |
| 3 | 0.65 | 2.97 | 3.02 | 1.96 | 4.30 |
| 3 | 0.7 | 3.15 | 3.17 | 1.96 | 4.30 |
| 3 | 0.75 | 3.33 | 3.33 | 1.96 | 4.30 |
| 3 | 0.8 | 3.51 | 3.49 | 1.96 | 4.30 |
| 3 | 0.85 | 3.68 | 3.70 | 1.96 | 4.30 |

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| $K$ | $I^2$ | SIM | EX | $N(0,1)$ | $t(K-1)$ |
|---|---|---|---|---|---|
| 3 | 0.9 | 3.94 | 4.06 | 1.96 | 4.30 |
| 3 | 0.95 | 4.19 | 4.57 | 1.96 | 4.30 |
| 4 | 0 | 1.83 | 1.83 | 1.96 | 3.18 |
| 4 | 0.05 | 1.88 | 1.91 | 1.96 | 3.18 |
| 4 | 0.1 | 1.95 | 1.97 | 1.96 | 3.18 |
| 4 | 0.15 | 2.03 | 2.03 | 1.96 | 3.18 |
| 4 | 0.2 | 2.08 | 2.10 | 1.96 | 3.18 |
| 4 | 0.25 | 2.15 | 2.15 | 1.96 | 3.18 |
| 4 | 0.3 | 2.21 | 2.20 | 1.96 | 3.18 |
| 4 | 0.35 | 2.30 | 2.29 | 1.96 | 3.18 |
| 4 | 0.4 | 2.36 | 2.33 | 1.96 | 3.18 |
| 4 | 0.45 | 2.42 | 2.43 | 1.96 | 3.18 |
| 4 | 0.5 | 2.53 | 2.51 | 1.96 | 3.18 |
| 4 | 0.55 | 2.60 | 2.57 | 1.96 | 3.18 |
| 4 | 0.6 | 2.69 | 2.69 | 1.96 | 3.18 |
| 4 | 0.65 | 2.75 | 2.76 | 1.96 | 3.18 |
| 4 | 0.7 | 2.83 | 2.86 | 1.96 | 3.18 |
| 4 | 0.75 | 2.91 | 2.91 | 1.96 | 3.18 |
| 4 | 0.8 | 3.00 | 3.01 | 1.96 | 3.18 |
| 4 | 0.85 | 3.14 | 3.13 | 1.96 | 3.18 |
| 4 | 0.9 | 3.23 | 3.22 | 1.96 | 3.18 |
| 4 | 0.95 | 3.28 | 3.34 | 1.96 | 3.18 |
| 5 | 0 | 1.84 | 1.84 | 1.96 | 2.78 |
| 5 | 0.05 | 1.89 | 1.88 | 1.96 | 2.78 |
| 5 | 0.1 | 1.95 | 1.94 | 1.96 | 2.78 |
| 5 | 0.15 | 1.98 | 1.99 | 1.96 | 2.78 |
| 5 | 0.2 | 2.07 | 2.04 | 1.96 | 2.78 |
| 5 | 0.25 | 2.11 | 2.10 | 1.96 | 2.78 |
| 5 | 0.3 | 2.19 | 2.16 | 1.96 | 2.78 |
| 5 | 0.35 | 2.21 | 2.25 | 1.96 | 2.78 |
| 5 | 0.4 | 2.30 | 2.25 | 1.96 | 2.78 |
| 5 | 0.45 | 2.37 | 2.35 | 1.96 | 2.78 |
| 5 | 0.5 | 2.42 | 2.41 | 1.96 | 2.78 |
| 5 | 0.55 | 2.50 | 2.51 | 1.96 | 2.78 |
| 5 | 0.6 | 2.53 | 2.51 | 1.96 | 2.78 |
| 5 | 0.65 | 2.60 | 2.58 | 1.96 | 2.78 |
| 5 | 0.7 | 2.62 | 2.66 | 1.96 | 2.78 |
| 5 | 0.75 | 2.71 | 2.73 | 1.96 | 2.78 |
| 5 | 0.8 | 2.75 | 2.79 | 1.96 | 2.78 |
| 5 | 0.85 | 2.79 | 2.80 | 1.96 | 2.78 |
| 5 | 0.9 | 2.85 | 2.88 | 1.96 | 2.78 |
| 5 | 0.95 | 2.87 | 2.91 | 1.96 | 2.78 |
| 6 | 0 | 1.85 | 1.83 | 1.96 | 2.57 |

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| $K$ | $I^2$ | SIM | EX | $N(0,1)$ | $t(K-1)$ |
|---|---|---|---|---|---|
| 6 | 0.05 | 1.88 | 1.88 | 1.96 | 2.57 |
| 6 | 0.1 | 1.94 | 1.91 | 1.96 | 2.57 |
| 6 | 0.15 | 2.00 | 2.00 | 1.96 | 2.57 |
| 6 | 0.2 | 2.05 | 2.05 | 1.96 | 2.57 |
| 6 | 0.25 | 2.12 | 2.11 | 1.96 | 2.57 |
| 6 | 0.3 | 2.16 | 2.13 | 1.96 | 2.57 |
| 6 | 0.35 | 2.19 | 2.22 | 1.96 | 2.57 |
| 6 | 0.4 | 2.25 | 2.27 | 1.96 | 2.57 |
| 6 | 0.45 | 2.31 | 2.30 | 1.96 | 2.57 |
| 6 | 0.5 | 2.36 | 2.37 | 1.96 | 2.57 |
| 6 | 0.55 | 2.39 | 2.44 | 1.96 | 2.57 |
| 6 | 0.6 | 2.45 | 2.44 | 1.96 | 2.57 |
| 6 | 0.65 | 2.50 | 2.49 | 1.96 | 2.57 |
| 6 | 0.7 | 2.52 | 2.57 | 1.96 | 2.57 |
| 6 | 0.75 | 2.56 | 2.58 | 1.96 | 2.57 |
| 6 | 0.8 | 2.57 | 2.57 | 1.96 | 2.57 |
| 6 | 0.85 | 2.62 | 2.62 | 1.96 | 2.57 |
| 6 | 0.9 | 2.63 | 2.63 | 1.96 | 2.57 |
| 6 | 0.95 | 2.62 | 2.63 | 1.96 | 2.57 |
| 7 | 0 | 1.83 | 1.85 | 1.96 | 2.45 |
| 7 | 0.05 | 1.90 | 1.88 | 1.96 | 2.45 |
| 7 | 0.1 | 1.93 | 1.92 | 1.96 | 2.45 |
| 7 | 0.15 | 1.99 | 1.97 | 1.96 | 2.45 |
| 7 | 0.2 | 2.04 | 2.04 | 1.96 | 2.45 |
| 7 | 0.25 | 2.09 | 2.06 | 1.96 | 2.45 |
| 7 | 0.3 | 2.12 | 2.12 | 1.96 | 2.45 |
| 7 | 0.35 | 2.18 | 2.20 | 1.96 | 2.45 |
| 7 | 0.4 | 2.23 | 2.23 | 1.96 | 2.45 |
| 7 | 0.45 | 2.27 | 2.26 | 1.96 | 2.45 |
| 7 | 0.5 | 2.29 | 2.31 | 1.96 | 2.45 |
| 7 | 0.55 | 2.34 | 2.40 | 1.96 | 2.45 |
| 7 | 0.6 | 2.38 | 2.34 | 1.96 | 2.45 |
| 7 | 0.65 | 2.42 | 2.42 | 1.96 | 2.45 |
| 7 | 0.7 | 2.44 | 2.42 | 1.96 | 2.45 |
| 7 | 0.75 | 2.44 | 2.49 | 1.96 | 2.45 |
| 7 | 0.8 | 2.47 | 2.43 | 1.96 | 2.45 |
| 7 | 0.85 | 2.48 | 2.51 | 1.96 | 2.45 |
| 7 | 0.9 | 2.49 | 2.51 | 1.96 | 2.45 |
| 7 | 0.95 | 2.50 | 2.51 | 1.96 | 2.45 |
| 8 | 0 | 1.86 | 1.85 | 1.96 | 2.37 |
| 8 | 0.05 | 1.89 | 1.86 | 1.96 | 2.37 |
| 8 | 0.1 | 1.94 | 1.93 | 1.96 | 2.37 |
| 8 | 0.15 | 2.00 | 1.98 | 1.96 | 2.37 |

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| $K$ | $I^2$ | SIM | EX | $N(0,1)$ | $t(K-1)$ |
|---|---|---|---|---|---|
| 8 | 0.2 | 2.03 | 2.01 | 1.96 | 2.37 |
| 8 | 0.25 | 2.09 | 2.06 | 1.96 | 2.37 |
| 8 | 0.3 | 2.13 | 2.12 | 1.96 | 2.37 |
| 8 | 0.35 | 2.18 | 2.15 | 1.96 | 2.37 |
| 8 | 0.4 | 2.20 | 2.23 | 1.96 | 2.37 |
| 8 | 0.45 | 2.24 | 2.24 | 1.96 | 2.37 |
| 8 | 0.5 | 2.28 | 2.26 | 1.96 | 2.37 |
| 8 | 0.55 | 2.31 | 2.31 | 1.96 | 2.37 |
| 8 | 0.6 | 2.35 | 2.33 | 1.96 | 2.37 |
| 8 | 0.65 | 2.37 | 2.32 | 1.96 | 2.37 |
| 8 | 0.7 | 2.36 | 2.37 | 1.96 | 2.37 |
| 8 | 0.75 | 2.39 | 2.37 | 1.96 | 2.37 |
| 8 | 0.8 | 2.37 | 2.42 | 1.96 | 2.37 |
| 8 | 0.85 | 2.41 | 2.42 | 1.96 | 2.37 |
| 8 | 0.9 | 2.41 | 2.42 | 1.96 | 2.37 |
| 8 | 0.95 | 2.40 | 2.40 | 1.96 | 2.37 |
| 9 | 0 | 1.85 | 1.85 | 1.96 | 2.31 |
| 9 | 0.05 | 1.91 | 1.90 | 1.96 | 2.31 |
| 9 | 0.1 | 1.93 | 1.93 | 1.96 | 2.31 |
| 9 | 0.15 | 1.99 | 2.00 | 1.96 | 2.31 |
| 9 | 0.2 | 2.02 | 2.03 | 1.96 | 2.31 |
| 9 | 0.25 | 2.06 | 2.08 | 1.96 | 2.31 |
| 9 | 0.3 | 2.12 | 2.10 | 1.96 | 2.31 |
| 9 | 0.35 | 2.15 | 2.13 | 1.96 | 2.31 |
| 9 | 0.4 | 2.19 | 2.19 | 1.96 | 2.31 |
| 9 | 0.45 | 2.22 | 2.23 | 1.96 | 2.31 |
| 9 | 0.5 | 2.25 | 2.24 | 1.96 | 2.31 |
| 9 | 0.55 | 2.26 | 2.27 | 1.96 | 2.31 |
| 9 | 0.6 | 2.30 | 2.30 | 1.96 | 2.31 |
| 9 | 0.65 | 2.31 | 2.34 | 1.96 | 2.31 |
| 9 | 0.7 | 2.32 | 2.33 | 1.96 | 2.31 |
| 9 | 0.75 | 2.34 | 2.33 | 1.96 | 2.31 |
| 9 | 0.8 | 2.35 | 2.34 | 1.96 | 2.31 |
| 9 | 0.85 | 2.34 | 2.34 | 1.96 | 2.31 |
| 9 | 0.9 | 2.35 | 2.33 | 1.96 | 2.31 |
| 9 | 0.95 | 2.38 | 2.38 | 1.96 | 2.31 |
| 10 | 0 | 1.85 | 1.84 | 1.96 | 2.26 |
| 10 | 0.05 | 1.89 | 1.89 | 1.96 | 2.26 |
| 10 | 0.1 | 1.96 | 1.94 | 1.96 | 2.26 |
| 10 | 0.15 | 1.97 | 2.00 | 1.96 | 2.26 |
| 10 | 0.2 | 2.01 | 2.02 | 1.96 | 2.26 |
| 10 | 0.25 | 2.06 | 2.03 | 1.96 | 2.26 |
| 10 | 0.3 | 2.10 | 2.08 | 1.96 | 2.26 |

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| $K$ | $I^2$ | SIM | EX | $N(0,1)$ | $t(K-1)$ |
|---|---|---|---|---|---|
| 10 | 0.35 | 2.13 | 2.16 | 1.96 | 2.26 |
| 10 | 0.4 | 2.17 | 2.15 | 1.96 | 2.26 |
| 10 | 0.45 | 2.21 | 2.20 | 1.96 | 2.26 |
| 10 | 0.5 | 2.23 | 2.23 | 1.96 | 2.26 |
| 10 | 0.55 | 2.24 | 2.25 | 1.96 | 2.26 |
| 10 | 0.6 | 2.27 | 2.25 | 1.96 | 2.26 |
| 10 | 0.65 | 2.29 | 2.23 | 1.96 | 2.26 |
| 10 | 0.7 | 2.28 | 2.28 | 1.96 | 2.26 |
| 10 | 0.75 | 2.29 | 2.29 | 1.96 | 2.26 |
| 10 | 0.8 | 2.32 | 2.28 | 1.96 | 2.26 |
| 10 | 0.85 | 2.29 | 2.33 | 1.96 | 2.26 |
| 10 | 0.9 | 2.29 | 2.27 | 1.96 | 2.26 |
| 10 | 0.95 | 2.32 | 2.29 | 1.96 | 2.26 |
| 15 | 0 | 1.88 | 1.88 | 1.96 | 2.15 |
| 15 | 0.05 | 1.90 | 1.90 | 1.96 | 2.15 |
| 15 | 0.1 | 1.95 | 1.94 | 1.96 | 2.15 |
| 15 | 0.15 | 1.98 | 1.98 | 1.96 | 2.15 |
| 15 | 0.2 | 2.02 | 1.98 | 1.96 | 2.15 |
| 15 | 0.25 | 2.04 | 2.05 | 1.96 | 2.15 |
| 15 | 0.3 | 2.09 | 2.07 | 1.96 | 2.15 |
| 15 | 0.35 | 2.10 | 2.10 | 1.96 | 2.15 |
| 15 | 0.4 | 2.10 | 2.11 | 1.96 | 2.15 |
| 15 | 0.45 | 2.16 | 2.12 | 1.96 | 2.15 |
| 15 | 0.5 | 2.14 | 2.14 | 1.96 | 2.15 |
| 15 | 0.55 | 2.16 | 2.20 | 1.96 | 2.15 |
| 15 | 0.6 | 2.16 | 2.16 | 1.96 | 2.15 |
| 15 | 0.65 | 2.16 | 2.17 | 1.96 | 2.15 |
| 15 | 0.7 | 2.17 | 2.18 | 1.96 | 2.15 |
| 15 | 0.75 | 2.15 | 2.15 | 1.96 | 2.15 |
| 15 | 0.8 | 2.15 | 2.13 | 1.96 | 2.15 |
| 15 | 0.85 | 2.18 | 2.16 | 1.96 | 2.15 |
| 15 | 0.9 | 2.18 | 2.13 | 1.96 | 2.15 |
| 15 | 0.95 | 2.16 | 2.16 | 1.96 | 2.15 |
| 20 | 0 | 1.88 | 1.86 | 1.96 | 2.09 |
| 20 | 0.05 | 1.90 | 1.91 | 1.96 | 2.09 |
| 20 | 0.1 | 1.95 | 1.94 | 1.96 | 2.09 |
| 20 | 0.15 | 1.99 | 1.99 | 1.96 | 2.09 |
| 20 | 0.2 | 2.03 | 2.02 | 1.96 | 2.09 |
| 20 | 0.25 | 2.03 | 2.03 | 1.96 | 2.09 |
| 20 | 0.3 | 2.07 | 2.05 | 1.96 | 2.09 |
| 20 | 0.35 | 2.08 | 2.06 | 1.96 | 2.09 |
| 20 | 0.4 | 2.10 | 2.07 | 1.96 | 2.09 |
| 20 | 0.45 | 2.10 | 2.08 | 1.96 | 2.09 |

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| $K$ | $I^2$ | SIM | EX | $N(0,1)$ | $t(K-1)$ |
|---|---|---|---|---|---|
| 20 | 0.5 | 2.11 | 2.10 | 1.96 | 2.09 |
| 20 | 0.55 | 2.13 | 2.13 | 1.96 | 2.09 |
| 20 | 0.6 | 2.11 | 2.08 | 1.96 | 2.09 |
| 20 | 0.65 | 2.10 | 2.12 | 1.96 | 2.09 |
| 20 | 0.7 | 2.11 | 2.10 | 1.96 | 2.09 |
| 20 | 0.75 | 2.10 | 2.07 | 1.96 | 2.09 |
| 20 | 0.8 | 2.11 | 2.11 | 1.96 | 2.09 |
| 20 | 0.85 | 2.11 | 2.11 | 1.96 | 2.09 |
| 20 | 0.9 | 2.12 | 2.10 | 1.96 | 2.09 |
| 20 | 0.95 | 2.10 | 2.08 | 1.96 | 2.09 |
| 30 | 0 | 1.88 | 1.87 | 1.96 | 2.05 |
| 30 | 0.05 | 1.92 | 1.91 | 1.96 | 2.05 |
| 30 | 0.1 | 1.94 | 1.93 | 1.96 | 2.05 |
| 30 | 0.15 | 1.98 | 1.99 | 1.96 | 2.05 |
| 30 | 0.2 | 2.01 | 2.00 | 1.96 | 2.05 |
| 30 | 0.25 | 2.03 | 2.02 | 1.96 | 2.05 |
| 30 | 0.3 | 2.04 | 2.01 | 1.96 | 2.05 |
| 30 | 0.35 | 2.05 | 2.03 | 1.96 | 2.05 |
| 30 | 0.4 | 2.05 | 2.01 | 1.96 | 2.05 |
| 30 | 0.45 | 2.06 | 2.08 | 1.96 | 2.05 |
| 30 | 0.5 | 2.05 | 2.09 | 1.96 | 2.05 |
| 30 | 0.55 | 2.02 | 2.06 | 1.96 | 2.05 |
| 30 | 0.6 | 2.08 | 2.04 | 1.96 | 2.05 |
| 30 | 0.65 | 2.05 | 2.03 | 1.96 | 2.05 |
| 30 | 0.7 | 2.06 | 2.04 | 1.96 | 2.05 |
| 30 | 0.75 | 2.07 | 2.07 | 1.96 | 2.05 |
| 30 | 0.8 | 2.05 | 2.07 | 1.96 | 2.05 |
| 30 | 0.85 | 2.05 | 2.04 | 1.96 | 2.05 |
| 30 | 0.9 | 2.06 | 2.05 | 1.96 | 2.05 |
| 30 | 0.95 | 2.06 | 2.08 | 1.96 | 2.05 |
| 40 | 0 | 1.89 | 1.90 | 1.96 | 2.02 |
| 40 | 0.05 | 1.93 | 1.90 | 1.96 | 2.02 |
| 40 | 0.1 | 1.96 | 1.95 | 1.96 | 2.02 |
| 40 | 0.15 | 1.99 | 1.99 | 1.96 | 2.02 |
| 40 | 0.2 | 2.00 | 2.00 | 1.96 | 2.02 |
| 40 | 0.25 | 2.02 | 2.01 | 1.96 | 2.02 |
| 40 | 0.3 | 2.03 | 2.01 | 1.96 | 2.02 |
| 40 | 0.35 | 2.03 | 2.01 | 1.96 | 2.02 |
| 40 | 0.4 | 2.04 | 2.01 | 1.96 | 2.02 |
| 40 | 0.45 | 2.02 | 2.04 | 1.96 | 2.02 |
| 40 | 0.5 | 2.03 | 2.05 | 1.96 | 2.02 |
| 40 | 0.55 | 2.03 | 2.04 | 1.96 | 2.02 |
| 40 | 0.6 | 2.04 | 2.03 | 1.96 | 2.02 |

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| $K$ | $I^2$ | SIM | EX | $N(0,1)$ | $t(K-1)$ |
|---|---|---|---|---|---|
| 40 | 0.65 | 2.02 | 2.04 | 1.96 | 2.02 |
| 40 | 0.7 | 2.03 | 2.03 | 1.96 | 2.02 |
| 40 | 0.75 | 2.04 | 2.01 | 1.96 | 2.02 |
| 40 | 0.8 | 2.04 | 2.02 | 1.96 | 2.02 |
| 40 | 0.85 | 2.04 | 2.00 | 1.96 | 2.02 |
| 40 | 0.9 | 2.04 | 2.04 | 1.96 | 2.02 |
| 40 | 0.95 | 2.03 | 2.00 | 1.96 | 2.02 |
| 50 | 0 | 1.89 | 1.94 | 1.96 | 2.01 |
| 50 | 0.05 | 1.92 | 1.91 | 1.96 | 2.01 |
| 50 | 0.1 | 1.94 | 1.95 | 1.96 | 2.01 |
| 50 | 0.15 | 1.96 | 1.97 | 1.96 | 2.01 |
| 50 | 0.2 | 2.00 | 2.01 | 1.96 | 2.01 |
| 50 | 0.25 | 2.02 | 2.02 | 1.96 | 2.01 |
| 50 | 0.3 | 2.00 | 2.02 | 1.96 | 2.01 |
| 50 | 0.35 | 2.03 | 2.05 | 1.96 | 2.01 |
| 50 | 0.4 | 2.02 | 2.01 | 1.96 | 2.01 |
| 50 | 0.45 | 2.02 | 2.05 | 1.96 | 2.01 |
| 50 | 0.5 | 2.01 | 2.01 | 1.96 | 2.01 |
| 50 | 0.55 | 2.02 | 2.04 | 1.96 | 2.01 |
| 50 | 0.6 | 2.03 | 2.01 | 1.96 | 2.01 |
| 50 | 0.65 | 2.03 | 2.00 | 1.96 | 2.01 |
| 50 | 0.7 | 2.02 | 1.99 | 1.96 | 2.01 |
| 50 | 0.75 | 2.02 | 2.00 | 1.96 | 2.01 |
| 50 | 0.8 | 2.03 | 2.00 | 1.96 | 2.01 |
| 50 | 0.85 | 2.01 | 2.01 | 1.96 | 2.01 |
| 50 | 0.9 | 2.02 | 2.03 | 1.96 | 2.01 |
| 50 | 0.95 | 2.03 | 2.02 | 1.96 | 2.01 |

*Last page*

## 3 The additional simulation of confidence interval of $\theta$

We show how the CI for the overall treatment effect $\theta$ is accurate via simulation in the main text. We here compare the CI for the additional other cases in this supplemental material. The simulation setting is given below. The number of studies is set to very small to large ($K = 2, 3, \cdots, 10, 15, 20, 30$). The heterogeneity $I^2$ is set to low to high, which is typical values but not included in the main text ($I^2 = 0, 0.250.5, 0.75, 0.95$). Other settings are the same as the simulation setting in Section 4.3 of the main text. Therefore, we compare the accuracy of the almost-exact method with those of DerSimonian–Laird method (DL), the restricted maximum likelihood method (REML), the method of Michael et al. (2019) and the method of Hartung (1999) (HKSJ). The almost-exact method needs a heterogeneity parameter for constructing the CI. We use the true $I^2$ (EX(I2)), the heterogeneity estimator $\hat{I}^2$ (EX(I2hat)) and conservative correction values with a vague prior distribution $U(0,1)$ (EX(I2c1)) and three type of more restricted prior distribution $U(0.5,1)$ (EX(I2c2)), $U(0.7,1)$ (EX(I2c3)), $U(0.9,1)$ (EX(I2c4)).

We provide the coverage probability of the 95% CI in Table S.2. Figures S.3 and S.4 plot the coverage probability when $K \leq 5$ and $K > 5$, respectively. We also provide the list of lengths of the 95% CI in Table S.3. Figures S.5 and S.6 plot the length when $K \leq 5$ and $K > 5$, respectively.

In all situations, EX(I2) keeps the coverage probability at the corresponding nominal significance level. EX(I2) is always exact as long as the between-study variance is known. The method of Michael et al. (2019) makes conservative CI in any cases, but it has wider CI in the large heterogeneity $I^2$ than our proposed CI. The coverage probability of EX(I2c4) is not stable when the number of study is medium. It is possible that probability $p(\hat{I}^2|I^2)$ is not calculated well by specifying an extreme prior distribution such as $U(0.9, 1)$. DL, REML and HKSJ converge to the nominal significance level when the number of studies is large. However, their convergence speed is slower than that of the almost-exact methods. On the other hand, the number of studies $K$ is larger, the length of 95% CI of Michael et al. (2019) is closer to EX(I2), but it is too long when $K = 2$. The CIs of the other methods are mutually close in the terms of lengths, but the difference appears in the coverage probabilities.

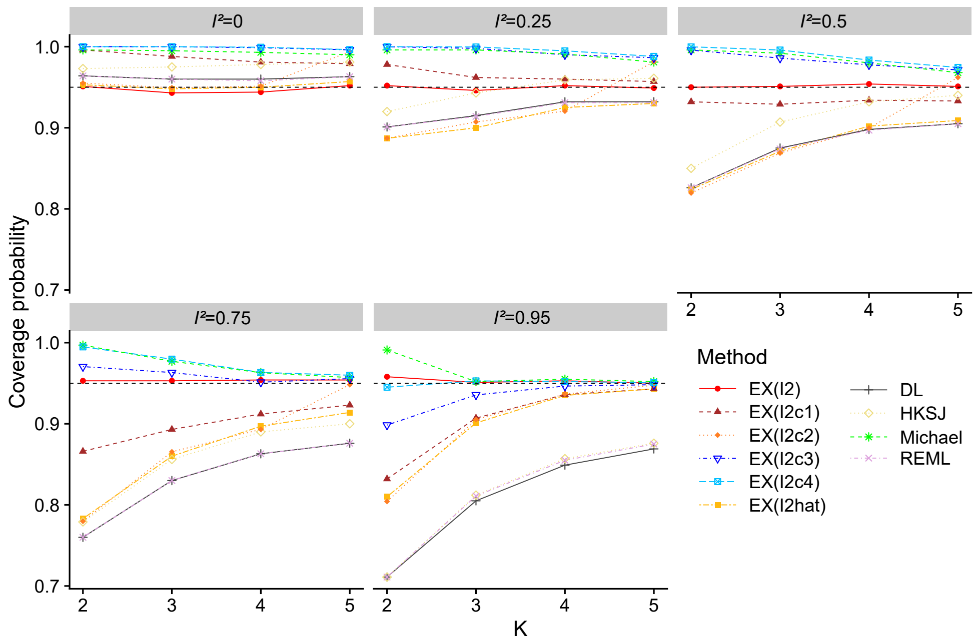


Figure S.3: The coverage probabilities of the 95% CI for overall treatment effect $\theta$ with 10,000 simulations in case of typical heterogeneity $I^2$. The number of studies is small ($K = 2, \cdots, 5$).

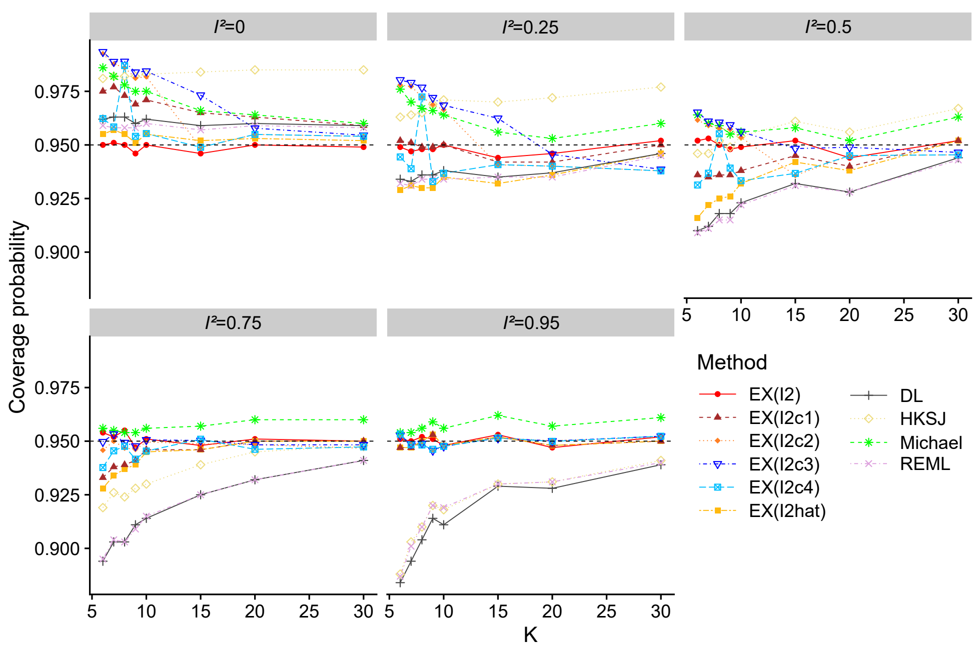


Figure S.4: The coverage probabilities of the 95% CI for overall treatment effect $\theta$ with 10,000 simulations in case of typical heterogeneity $I^2$. The number of studies is small to medium ($K = 5, \cdots, 10, 15, 20, 30$).

Table S.2: The coverage probability of the 95% CI for the overall treatment effect $\theta$ with $10,000$ simulations.

| $K$ | $I^2$ | EX(I2) | EX(I2c1) | EX(I2c2) | EX(I2c3) | EX(I2c4) | EX(I2hat) | Michael | DL | HKSJ | REML |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 0 | 0.951 | 0.996 | 0.955 | 1.000 | 1.000 | 0.953 | 0.996 | 0.964 | 0.973 | 0.964 |
| 2 | 0.25 | 0.952 | 0.978 | 0.887 | 1.000 | 1.000 | 0.887 | 0.996 | 0.901 | 0.920 | 0.901 |
| 2 | 0.5 | 0.950 | 0.932 | 0.820 | 0.996 | 1.000 | 0.824 | 0.996 | 0.826 | 0.850 | 0.826 |
| 2 | 0.75 | 0.953 | 0.866 | 0.780 | 0.971 | 0.995 | 0.783 | 0.997 | 0.760 | 0.779 | 0.760 |
| 2 | 0.95 | 0.958 | 0.832 | 0.804 | 0.898 | 0.945 | 0.810 | 0.991 | 0.711 | 0.711 | 0.711 |
| 3 | 0 | 0.943 | 0.988 | 0.949 | 1.000 | 1.000 | 0.948 | 0.995 | 0.960 | 0.975 | 0.960 |
| 3 | 0.25 | 0.946 | 0.962 | 0.907 | 0.998 | 1.000 | 0.900 | 0.996 | 0.915 | 0.943 | 0.914 |
| 3 | 0.5 | 0.951 | 0.929 | 0.869 | 0.986 | 0.996 | 0.871 | 0.992 | 0.875 | 0.907 | 0.874 |
| 3 | 0.75 | 0.953 | 0.893 | 0.865 | 0.963 | 0.980 | 0.860 | 0.977 | 0.830 | 0.856 | 0.830 |
| 3 | 0.95 | 0.951 | 0.907 | 0.905 | 0.936 | 0.953 | 0.901 | 0.952 | 0.805 | 0.812 | 0.811 |
| 4 | 0 | 0.944 | 0.981 | 0.951 | 0.999 | 0.999 | 0.950 | 0.993 | 0.960 | 0.978 | 0.958 |
| 4 | 0.25 | 0.952 | 0.960 | 0.921 | 0.990 | 0.995 | 0.925 | 0.991 | 0.932 | 0.959 | 0.931 |
| 4 | 0.5 | 0.954 | 0.934 | 0.900 | 0.978 | 0.984 | 0.902 | 0.980 | 0.898 | 0.932 | 0.897 |
| 4 | 0.75 | 0.954 | 0.912 | 0.893 | 0.951 | 0.963 | 0.897 | 0.963 | 0.863 | 0.890 | 0.863 |
| 4 | 0.95 | 0.953 | 0.936 | 0.937 | 0.947 | 0.952 | 0.935 | 0.955 | 0.849 | 0.857 | 0.855 |
| 5 | 0 | 0.952 | 0.979 | 0.994 | 0.996 | 0.996 | 0.957 | 0.990 | 0.963 | 0.981 | 0.963 |
| 5 | 0.25 | 0.949 | 0.957 | 0.982 | 0.986 | 0.988 | 0.930 | 0.981 | 0.932 | 0.961 | 0.931 |
| 5 | 0.5 | 0.951 | 0.933 | 0.962 | 0.971 | 0.974 | 0.909 | 0.968 | 0.905 | 0.940 | 0.905 |
| 5 | 0.75 | 0.954 | 0.923 | 0.948 | 0.956 | 0.960 | 0.914 | 0.957 | 0.876 | 0.900 | 0.876 |
| 5 | 0.95 | 0.950 | 0.943 | 0.948 | 0.948 | 0.951 | 0.943 | 0.952 | 0.869 | 0.876 | 0.875 |
| 6 | 0 | 0.950 | 0.975 | 0.993 | 0.993 | 0.962 | 0.955 | 0.986 | 0.962 | 0.981 | 0.959 |
| 6 | 0.25 | 0.949 | 0.952 | 0.978 | 0.980 | 0.944 | 0.929 | 0.976 | 0.934 | 0.963 | 0.932 |
| 6 | 0.5 | 0.952 | 0.936 | 0.962 | 0.965 | 0.931 | 0.916 | 0.964 | 0.910 | 0.946 | 0.909 |
| 6 | 0.75 | 0.954 | 0.933 | 0.946 | 0.950 | 0.938 | 0.928 | 0.956 | 0.894 | 0.919 | 0.895 |
| 6 | 0.95 | 0.951 | 0.947 | 0.952 | 0.953 | 0.953 | 0.947 | 0.954 | 0.884 | 0.888 | 0.887 |
| 7 | 0 | 0.951 | 0.977 | 0.988 | 0.989 | 0.958 | 0.957 | 0.982 | 0.963 | 0.982 | 0.958 |
| 7 | 0.25 | 0.947 | 0.951 | 0.977 | 0.979 | 0.939 | 0.931 | 0.970 | 0.933 | 0.964 | 0.931 |
| 7 | 0.5 | 0.953 | 0.935 | 0.959 | 0.961 | 0.937 | 0.922 | 0.960 | 0.912 | 0.946 | 0.911 |
| 7 | 0.75 | 0.952 | 0.938 | 0.950 | 0.953 | 0.946 | 0.934 | 0.955 | 0.903 | 0.926 | 0.904 |

*Table continued on next page*

*(Continued)*

| $K$ | $I^2$ | EX(I2) | EX(I2c1) | EX(I2c2) | EX(I2c3) | EX(I2c4) | EX(I2hat) | Michael | DL | HKSJ | REML |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 7 | 0.95 | 0.950 | 0.947 | 0.948 | 0.949 | 0.948 | 0.947 | 0.954 | 0.894 | 0.903 | 0.901 |
| 8 | 0 | 0.950 | 0.973 | 0.986 | 0.989 | 0.987 | 0.955 | 0.978 | 0.963 | 0.982 | 0.958 |
| 8 | 0.25 | 0.948 | 0.949 | 0.972 | 0.977 | 0.972 | 0.930 | 0.967 | 0.936 | 0.965 | 0.934 |
| 8 | 0.5 | 0.950 | 0.936 | 0.958 | 0.961 | 0.955 | 0.925 | 0.959 | 0.918 | 0.952 | 0.915 |
| 8 | 0.75 | 0.955 | 0.939 | 0.947 | 0.949 | 0.948 | 0.937 | 0.954 | 0.903 | 0.924 | 0.903 |
| 8 | 0.95 | 0.952 | 0.950 | 0.949 | 0.949 | 0.948 | 0.950 | 0.956 | 0.904 | 0.910 | 0.910 |
| 9 | 0 | 0.946 | 0.969 | 0.981 | 0.984 | 0.954 | 0.951 | 0.975 | 0.960 | 0.982 | 0.958 |
| 9 | 0.25 | 0.948 | 0.949 | 0.969 | 0.972 | 0.933 | 0.930 | 0.966 | 0.936 | 0.969 | 0.934 |
| 9 | 0.5 | 0.948 | 0.936 | 0.957 | 0.959 | 0.939 | 0.926 | 0.955 | 0.918 | 0.949 | 0.915 |
| 9 | 0.75 | 0.947 | 0.941 | 0.946 | 0.948 | 0.942 | 0.939 | 0.954 | 0.911 | 0.928 | 0.909 |
| 9 | 0.95 | 0.951 | 0.953 | 0.946 | 0.946 | 0.946 | 0.953 | 0.959 | 0.914 | 0.920 | 0.920 |
| 10 | 0 | 0.950 | 0.971 | 0.982 | 0.984 | 0.955 | 0.955 | 0.975 | 0.962 | 0.983 | 0.960 |
| 10 | 0.25 | 0.950 | 0.950 | 0.966 | 0.969 | 0.937 | 0.935 | 0.964 | 0.938 | 0.971 | 0.934 |
| 10 | 0.5 | 0.949 | 0.938 | 0.954 | 0.956 | 0.933 | 0.932 | 0.956 | 0.923 | 0.954 | 0.922 |
| 10 | 0.75 | 0.951 | 0.946 | 0.950 | 0.951 | 0.945 | 0.945 | 0.956 | 0.914 | 0.930 | 0.915 |
| 10 | 0.95 | 0.948 | 0.948 | 0.948 | 0.948 | 0.948 | 0.948 | 0.956 | 0.911 | 0.918 | 0.919 |
| 15 | 0 | 0.946 | 0.965 | 0.949 | 0.973 | 0.949 | 0.952 | 0.966 | 0.959 | 0.984 | 0.957 |
| 15 | 0.25 | 0.944 | 0.942 | 0.941 | 0.962 | 0.941 | 0.932 | 0.956 | 0.935 | 0.970 | 0.935 |
| 15 | 0.5 | 0.952 | 0.945 | 0.936 | 0.948 | 0.937 | 0.942 | 0.958 | 0.932 | 0.961 | 0.931 |
| 15 | 0.75 | 0.948 | 0.946 | 0.949 | 0.950 | 0.951 | 0.946 | 0.957 | 0.925 | 0.939 | 0.925 |
| 15 | 0.95 | 0.953 | 0.952 | 0.952 | 0.951 | 0.952 | 0.952 | 0.962 | 0.929 | 0.930 | 0.930 |
| 20 | 0 | 0.950 | 0.963 | 0.955 | 0.958 | 0.955 | 0.953 | 0.964 | 0.960 | 0.985 | 0.959 |
| 20 | 0.25 | 0.946 | 0.942 | 0.940 | 0.946 | 0.940 | 0.936 | 0.953 | 0.937 | 0.972 | 0.935 |
| 20 | 0.5 | 0.944 | 0.940 | 0.945 | 0.949 | 0.945 | 0.938 | 0.952 | 0.928 | 0.956 | 0.928 |
| 20 | 0.75 | 0.951 | 0.950 | 0.948 | 0.948 | 0.946 | 0.950 | 0.960 | 0.932 | 0.945 | 0.932 |
| 20 | 0.95 | 0.947 | 0.948 | 0.950 | 0.950 | 0.950 | 0.948 | 0.957 | 0.928 | 0.931 | 0.931 |
| 30 | 0 | 0.949 | 0.959 | 0.954 | 0.955 | 0.954 | 0.952 | 0.960 | 0.959 | 0.985 | 0.958 |
| 30 | 0.25 | 0.952 | 0.950 | 0.938 | 0.939 | 0.938 | 0.946 | 0.960 | 0.946 | 0.977 | 0.945 |
| 30 | 0.5 | 0.952 | 0.952 | 0.945 | 0.947 | 0.945 | 0.952 | 0.963 | 0.944 | 0.967 | 0.943 |
| 30 | 0.75 | 0.950 | 0.950 | 0.947 | 0.948 | 0.947 | 0.950 | 0.960 | 0.941 | 0.948 | 0.941 |
| 30 | 0.95 | 0.952 | 0.950 | 0.952 | 0.952 | 0.952 | 0.950 | 0.961 | 0.939 | 0.941 | 0.940 |

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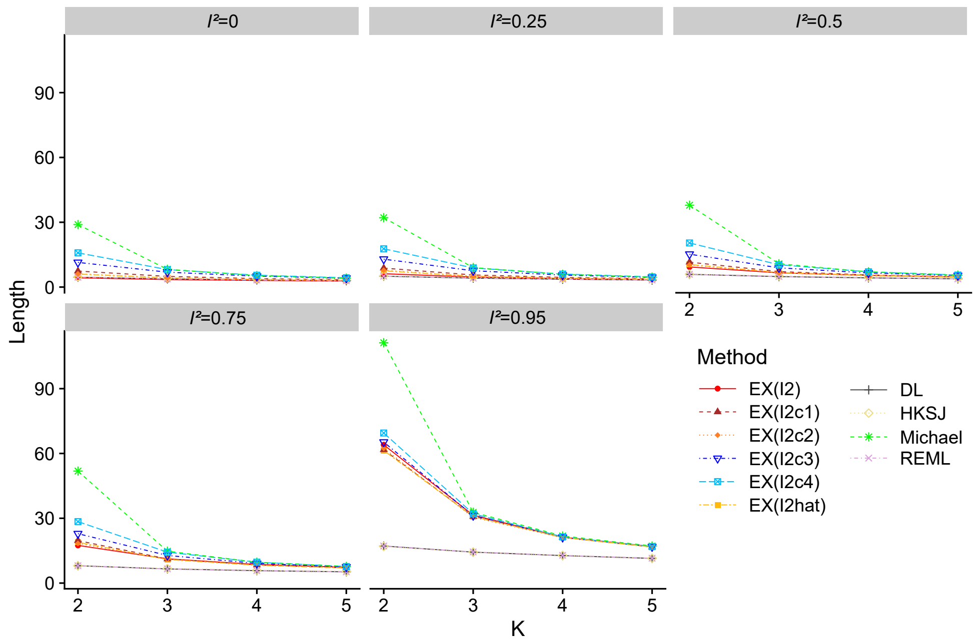


Figure S.5: The lengths of the 95% CI for overall treatment effect $\theta$ with 10,000 simulations in case of typical heterogeneity $I^2$. The number of studies is small ($K = 2, \cdots, 5$).

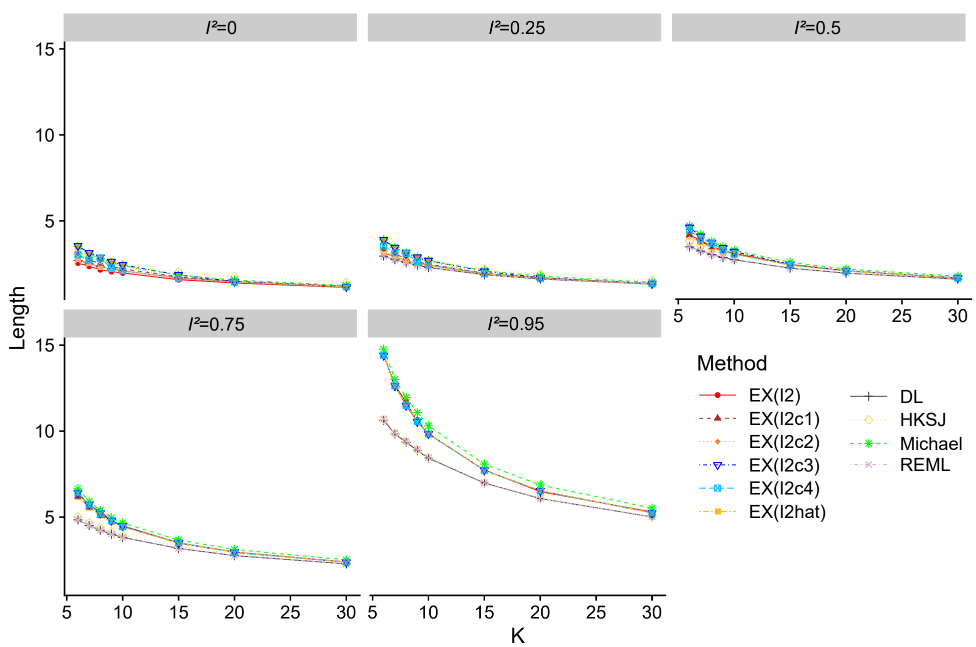


Figure S.6: The lengths of the 95% CI for overall treatment effect $\theta$ with 10,000 simulations in case of typical heterogeneity $I^2$. The number of studies is small to medium ($K = 5, \cdots, 10, 15, 20, 30$).

Table S.3: The length of the 95% CI for the overall treatment effect $\theta$ with $10,000$ simulations.

| $K$ | $I^2$ | EX(I2) | EX(I2c1) | EX(I2c2) | EX(I2c3) | EX(I2c4) | EX(I2hat) | Michael | DL | HKSJ | REML |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 0 | 4.57 | 4.28 | 7.42 | 6.14 | 11.42 | 15.86 | 6.02 | 4.78 | 28.94 | 4.57 |
| 2 | 0.25 | 5.04 | 6.19 | 8.78 | 7.78 | 12.93 | 17.73 | 7.52 | 5.22 | 32.14 | 5.04 |
| 2 | 0.5 | 5.90 | 9.32 | 11.53 | 10.27 | 15.23 | 20.42 | 10.46 | 6.00 | 37.90 | 5.90 |
| 2 | 0.75 | 8.04 | 17.45 | 19.67 | 18.36 | 22.87 | 28.46 | 18.89 | 8.01 | 51.85 | 8.04 |
| 2 | 0.95 | 17.16 | 63.99 | 61.73 | 62.40 | 65.27 | 69.48 | 61.36 | 16.95 | 111.28 | 17.16 |
| 3 | 0 | 3.80 | 3.48 | 4.95 | 4.24 | 6.89 | 8.10 | 4.26 | 4.21 | 8.23 | 3.80 |
| 3 | 0.25 | 4.13 | 4.66 | 5.61 | 5.04 | 7.57 | 8.87 | 5.01 | 4.51 | 9.03 | 4.14 |
| 3 | 0.5 | 4.86 | 6.61 | 7.17 | 6.63 | 8.91 | 10.35 | 6.73 | 5.18 | 10.78 | 4.88 |
| 3 | 0.75 | 6.56 | 11.22 | 11.16 | 11.03 | 12.78 | 14.31 | 10.91 | 6.74 | 14.75 | 6.58 |
| 3 | 0.95 | 14.32 | 31.41 | 30.83 | 30.40 | 31.23 | 32.09 | 30.78 | 14.34 | 32.76 | 14.41 |
| 4 | 0 | 3.30 | 3.08 | 3.93 | 3.51 | 5.07 | 5.49 | 3.52 | 3.75 | 5.30 | 3.29 |
| 4 | 0.25 | 3.63 | 3.97 | 4.49 | 4.11 | 5.56 | 6.02 | 4.17 | 4.06 | 5.92 | 3.64 |
| 4 | 0.5 | 4.26 | 5.44 | 5.62 | 5.30 | 6.55 | 7.07 | 5.40 | 4.61 | 7.05 | 4.27 |
| 4 | 0.75 | 5.75 | 8.62 | 8.40 | 8.27 | 9.12 | 9.66 | 8.30 | 5.96 | 9.68 | 5.77 |
| 4 | 0.95 | 12.73 | 21.26 | 21.16 | 21.00 | 21.20 | 21.42 | 21.15 | 12.80 | 21.81 | 12.83 |
| 5 | 0 | 2.96 | 2.79 | 3.38 | 3.93 | 4.15 | 4.32 | 3.09 | 3.40 | 4.15 | 2.95 |
| 5 | 0.25 | 3.24 | 3.48 | 3.81 | 4.34 | 4.55 | 4.73 | 3.59 | 3.66 | 4.62 | 3.25 |
| 5 | 0.5 | 3.81 | 4.66 | 4.73 | 5.19 | 5.38 | 5.57 | 4.60 | 4.16 | 5.53 | 3.82 |
| 5 | 0.75 | 5.21 | 7.32 | 7.03 | 7.37 | 7.49 | 7.67 | 6.98 | 5.40 | 7.70 | 5.21 |
| 5 | 0.95 | 11.43 | 16.88 | 16.70 | 16.87 | 16.88 | 16.93 | 16.69 | 11.48 | 17.15 | 11.51 |
| 6 | 0 | 2.70 | 2.55 | 3.01 | 3.46 | 3.54 | 3.03 | 2.78 | 3.13 | 3.51 | 2.69 |
| 6 | 0.25 | 2.95 | 3.16 | 3.36 | 3.81 | 3.90 | 3.54 | 3.19 | 3.36 | 3.89 | 2.95 |
| 6 | 0.5 | 3.49 | 4.19 | 4.18 | 4.53 | 4.62 | 4.46 | 4.09 | 3.83 | 4.72 | 3.49 |
| 6 | 0.75 | 4.84 | 6.41 | 6.21 | 6.35 | 6.41 | 6.41 | 6.18 | 5.03 | 6.65 | 4.85 |
| 6 | 0.95 | 10.63 | 14.44 | 14.41 | 14.39 | 14.40 | 14.41 | 14.41 | 10.68 | 14.77 | 10.70 |
| 7 | 0 | 2.50 | 2.36 | 2.74 | 3.09 | 3.14 | 2.71 | 2.54 | 2.92 | 3.09 | 2.49 |
| 7 | 0.25 | 2.74 | 2.91 | 3.07 | 3.38 | 3.44 | 3.14 | 2.93 | 3.14 | 3.46 | 2.74 |
| 7 | 0.5 | 3.25 | 3.86 | 3.80 | 4.03 | 4.10 | 3.96 | 3.73 | 3.58 | 4.20 | 3.25 |
| 7 | 0.75 | 4.51 | 5.67 | 5.58 | 5.69 | 5.73 | 5.71 | 5.56 | 4.68 | 5.92 | 4.51 |

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*(Continued)*

| $K$ | $I^2$ | EX(I2) | EX(I2c1) | EX(I2c2) | EX(I2c3) | EX(I2c4) | EX(I2hat) | Michael | DL | HKSJ | REML |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 7 | 0.95 | 9.83 | 12.66 | 12.60 | 12.62 | 12.63 | 12.63 | 12.60 | 9.88 | 13.01 | 9.89 |
| 8 | 0 | 2.33 | 2.16 | 2.51 | 2.80 | 2.87 | 2.82 | 2.33 | 2.74 | 2.79 | 2.32 |
| 8 | 0.25 | 2.57 | 2.70 | 2.82 | 3.09 | 3.16 | 3.10 | 2.71 | 2.95 | 3.14 | 2.56 |
| 8 | 0.5 | 3.04 | 3.51 | 3.49 | 3.69 | 3.74 | 3.70 | 3.44 | 3.36 | 3.80 | 3.04 |
| 8 | 0.75 | 4.24 | 5.35 | 5.11 | 5.20 | 5.23 | 5.21 | 5.10 | 4.40 | 5.38 | 4.24 |
| 8 | 0.95 | 9.35 | 11.69 | 11.55 | 11.49 | 11.49 | 11.48 | 11.55 | 9.40 | 11.96 | 9.41 |
| 9 | 0 | 2.20 | 2.05 | 2.35 | 2.58 | 2.63 | 2.27 | 2.19 | 2.59 | 2.56 | 2.19 |
| 9 | 0.25 | 2.42 | 2.53 | 2.63 | 2.84 | 2.89 | 2.62 | 2.53 | 2.79 | 2.89 | 2.42 |
| 9 | 0.5 | 2.87 | 3.28 | 3.25 | 3.40 | 3.43 | 3.30 | 3.21 | 3.17 | 3.51 | 2.87 |
| 9 | 0.75 | 4.01 | 4.76 | 4.73 | 4.78 | 4.80 | 4.79 | 4.72 | 4.16 | 4.97 | 4.01 |
| 9 | 0.95 | 8.88 | 10.55 | 10.64 | 10.55 | 10.55 | 10.56 | 10.64 | 8.93 | 11.08 | 8.93 |
| 10 | 0 | 2.09 | 1.97 | 2.21 | 2.41 | 2.44 | 2.10 | 2.09 | 2.47 | 2.39 | 2.08 |
| 10 | 0.25 | 2.29 | 2.42 | 2.48 | 2.66 | 2.69 | 2.44 | 2.40 | 2.66 | 2.70 | 2.29 |
| 10 | 0.5 | 2.74 | 3.11 | 3.07 | 3.19 | 3.22 | 3.08 | 3.04 | 3.03 | 3.29 | 2.74 |
| 10 | 0.75 | 3.82 | 4.49 | 4.44 | 4.48 | 4.49 | 4.44 | 4.44 | 3.96 | 4.66 | 3.82 |
| 10 | 0.95 | 8.43 | 9.83 | 9.86 | 9.84 | 9.84 | 9.83 | 9.86 | 8.46 | 10.31 | 8.46 |
| 15 | 0 | 1.69 | 1.58 | 1.75 | 1.66 | 1.86 | 1.66 | 1.66 | 2.03 | 1.85 | 1.68 |
| 15 | 0.25 | 1.87 | 1.93 | 1.96 | 1.93 | 2.07 | 1.93 | 1.92 | 2.19 | 2.10 | 1.87 |
| 15 | 0.5 | 2.25 | 2.47 | 2.44 | 2.42 | 2.48 | 2.43 | 2.42 | 2.49 | 2.59 | 2.25 |
| 15 | 0.75 | 3.18 | 3.51 | 3.49 | 3.47 | 3.48 | 3.52 | 3.49 | 3.29 | 3.67 | 3.18 |
| 15 | 0.95 | 6.98 | 7.74 | 7.73 | 7.72 | 7.72 | 7.72 | 7.73 | 7.00 | 8.08 | 7.00 |
| 20 | 0 | 1.46 | 1.39 | 1.49 | 1.44 | 1.47 | 1.44 | 1.44 | 1.76 | 1.56 | 1.45 |
| 20 | 0.25 | 1.62 | 1.69 | 1.68 | 1.66 | 1.70 | 1.66 | 1.66 | 1.90 | 1.79 | 1.62 |
| 20 | 0.5 | 1.96 | 2.09 | 2.09 | 2.09 | 2.11 | 2.09 | 2.08 | 2.17 | 2.20 | 1.96 |
| 20 | 0.75 | 2.76 | 2.97 | 2.97 | 2.97 | 2.97 | 2.94 | 2.97 | 2.86 | 3.12 | 2.76 |
| 20 | 0.95 | 6.08 | 6.48 | 6.53 | 6.52 | 6.53 | 6.53 | 6.53 | 6.09 | 6.87 | 6.09 |
| 30 | 0 | 1.18 | 1.13 | 1.20 | 1.16 | 1.17 | 1.16 | 1.16 | 1.44 | 1.24 | 1.18 |
| 30 | 0.25 | 1.32 | 1.35 | 1.35 | 1.34 | 1.35 | 1.34 | 1.34 | 1.56 | 1.43 | 1.32 |
| 30 | 0.5 | 1.61 | 1.67 | 1.68 | 1.69 | 1.69 | 1.69 | 1.68 | 1.78 | 1.78 | 1.61 |
| 30 | 0.75 | 2.28 | 2.39 | 2.39 | 2.38 | 2.38 | 2.38 | 2.39 | 2.35 | 2.51 | 2.28 |
| 30 | 0.95 | 5.01 | 5.30 | 5.26 | 5.24 | 5.25 | 5.25 | 5.26 | 5.02 | 5.52 | 5.02 |

*Last page*

## 4 The simulation program

We provide the R program used for the simulations of the main text and the supplemental material. The PDF of $\hat{\tau}^2_{DL0}$ and the almost-exact PDF of $T_{DL}$ can be calculated by the R program in Listing 1. We also give the R program for computing CI and $p$-value.

Listing 1: R program for numerical calculation of Theorem 1 including CI and $p$-value.

```
# Probability density function of between-study variance
pdf_tau2dl = function(K,sig2i,tau2,nsim.chi = 100000, n=1000){
  s1 = sum(1/sig2i)
  s2 = sum(1/sig2i^2)
  V = matrix(0,ncol=K,nrow=K)
  for(i in 1:K){
    for(j in 1:K){
      V[i,j] = -1/s1 + (s2/s1^2 - (1/sig2i[i]+1/sig2i[j])/s1)*
          tau2
    }
    V[i,i] = (sig2i[i] - 1/s1) + (1 + s2/s1^2 - 2/sig2i[i]/s1)*
        tau2
  }
  W = diag(x=1/sig2i,nrow=K,ncol=K)
  U = V %*% W
  lambda = eigen(U)$value
  chi_sim = matrix(0, ncol=K, nrow=nsim.chi)
  for(i in 1:K){
    chi_sim[,i] = rchisq(n=nsim.chi, df=1)
  }
  Q_chis = chi_sim %*% lambda
  Fq = ecdf(Q_chis)
  qmin = min(Q_chis)
  qmax = max(Q_chis)
  h = (qmax-qmin)/n
  fq = numeric(n+1)
  x = 0:n * h
  for(i in 1:(length(x))){
    fq[i] = (Fq(x[i]+h)-Fq(x[i]))/h
  }
  ftau2 = (s1-s2/s1) * fq
  x_ftau2 = (x - (K-1)) / (s1-s2/s1)
  return_val = list(x=x_ftau2, y=ftau2, q=x, fq=fq, Fq=Fq)
  return(return_val)
}


g_monte_all = function(zi, x, K, sig2i, tau2){
  s1 = sum(1/sig2i)
  s2 = sum(1/sig2i^2)
  len.zi = length(zi[,1])
  len.x = length(x)
  x = t(matrix(x, ncol=len.zi, nrow=len.x))
  theta_w = matrix(apply(t(zi)/sig2i[-K], 2, sum), nrow=len.zi,
      ncol=len.x)
  theta_w2 = matrix(apply(t(zi^2)/sig2i[-K], 2, sum), nrow=len.
     zi, ncol=len.x)
  c = theta_w^2/(s1*sig2i[K])^2 - (1-1/(s1*sig2i[K]))/sig2i[K
     ]*(theta_w2 - theta_w^2/s1 - (K-1))
  d = (1-1/(s1*sig2i[K]))/sig2i[K] * (s1-s2/s1)
  fztau2x = numeric(length(x))
  cdx = c + d*x
  J = matrix(0, nrow=len.zi, ncol=len.x)
```

```
  J[cdx>=0] = abs((s1-s2/s1) / (2*sqrt(cdx[cdx>=0])))
  g_sqrts = theta_w^2/(s1*sig2i[K])^2 - (1-1/(s1*sig2i[K]))/
     sig2i[K]*(theta_w2-theta_w^2/s1-((s1-s2/s1)*x+(K-1)))
  gL = matrix(0, ncol=len.x, nrow=len.zi)
  gL[g_sqrts>=0] = (theta_w[g_sqrts>=0]/(s1*sig2i[K]) - sqrt(
     g_sqrts[g_sqrts>=0])) / ((1-1/(s1*sig2i[K]))/sig2i[K])
  gU = matrix(0, ncol=len.x, nrow=len.zi)
  gU[g_sqrts>=0] = (theta_w[g_sqrts>=0]/(s1*sig2i[K]) + sqrt(
     g_sqrts[g_sqrts>=0])) / ((1-1/(s1*sig2i[K]))/sig2i[K])
  fztau2x = J * (dnorm(gL, mean=0, sd=sqrt(sig2i[K]+tau2)) +
     dnorm(gU, mean=0, sd=sqrt(sig2i[K]+tau2))) / 2
  fztau2x1 = J * (gL*dnorm(gL, mean=0, sd=sqrt(sig2i[K]+tau2))
     + gU*dnorm(gU, mean=0, sd=sqrt(sig2i[K]+tau2))) / 2
  fztau2x2 = J * (gL^2*dnorm(gL, mean=0, sd=sqrt(sig2i[K]+tau2)
     ) + gU^2*dnorm(gU, mean=0, sd=sqrt(sig2i[K]+tau2))) / 2
  res = list(fztau2=as.big.matrix(t(fztau2x)), g1=as.big.matrix
     (t(fztau2x1)), g2=as.big.matrix(t(fztau2x2)))
  return(res)
}
v2tdls_fun = function(cov, x, sig2i, K){
  v.tdl = numeric(length(x))
  xt = apply(cbind(x,0), 1, max)
  vxts = 1 / (t(matrix(sig2i, ncol=length(x), nrow=K)) + matrix
     (xt, ncol=K, nrow=length(x)))
  vxt = apply(vxts, 1, sum)
  for(i.x in 1:length(x)){
    v.tdl[i.x] = sum(cov[[i.x]] * (vxts[i.x,] %*% t(vxts[i.x,])
       )) / vxt[i.x]
  }
  return(v.tdl)
}

# Probability density function of test statistic
pdf_ftdl = function(t, K, sig2i, tau2, theta, nsim=10000, h
   =0.01){
  s1 = sum(1/sig2i); s2 = sum(1/sig2i^2)
  sim = sapply(1:(K-1), function(k){rnorm(nsim, mean=theta, sd=
     sqrt(sig2i[k]+tau2))})
  ftau2dl = pdf_tau2dl(K=K, sig2i=sig2i, tau2=tau2)
  x = ftau2dl$x
  gs = g_monte_all(zi=sim, x=x, K=K, sig2i=sig2i, tau2=tau2)
  gs_x = apply(gs$fztau2[,], 1, sum)
  gs2_x = apply(gs$g2[,], 1, sum)
  gs_meanx = gs$fztau2[,] %*% sim / gs_x
  gs1_meanx = apply(gs$g1[,], 1, sum) / gs_x
  gsx = list(gs_x=gs_x, gs2_x=gs2_x, gs_meanx=gs_meanx,
     gs1_meanx=gs1_meanx)
  cov.monte = lapply(1:length(x), function(i.x){
    cov.monte = matrix(0, ncol=K, nrow=K)
    for(i.k in 1:K){
      if(i.k!=K){
        v.thetai = sim[,i.k] - gsx$gs_meanx[i.x,i.k]
      }else{
        v.thetai = gs$g1[i.x,] - gsx$gs1_meanx[i.x]
      }
      for(j.k in 1:K){
        if(j.k==K && i.k==K){
          cov.monte[i.k,j.k] = sum(gs$g2[i.x,])/sum(gs$fztau2[i
             .x,])
        }else if(i.k==K){
          v.thetaj = sim[,j.k] - gsx$gs_meanx[i.x,j.k]
          cov.monte[i.k,j.k] = sum(v.thetaj*gs$g1[i.x,]) / sum(
```

```
              gs$fztau2[i.x,])
        }else if(j.k==K){
          cov.monte[i.k,j.k] = sum(v.thetai*gs$g1[i.x,])/sum(
              gs$fztau2[i.x,])
        }else{
          v.thetaj = sim[,j.k] - gsx$gs_meanx[i.x,j.k]
          cov.monte[i.k,j.k] = sum(v.thetai*v.thetaj*gs$fztau2[
              i.x,])/sum(gs$fztau2[i.x,])
        }}}
    return(cov.monte)
  })
  v2 = v2tdls_fun(cov=cov.monte, x=x, sig2i=sig2i, K=K)
  v2[is.nan(v2)] = v2[length(v2[is.nan(v2)])+1]
  ftdly = numeric(length(t))
  for(i.x in 1:length(x)){
    m = theta * sqrt(sum(1/(sig2i+max(0,x[i.x]))))
    ftdly = ftdly + dnorm(x=t, mean=m, sd=sqrt(v2[i.x])) *
      ftau2dl$y[i.x] * (x[2]-x[1])
  }
  return(list(x=t, y=ftdly))
}


# p-value and confidence interval
test_exactDL = function(yi, vi, alpha=0.05, method="exactDL",
    side=2, a=0, b=1){
  K = length(vi); s1 = sum(1/vi); s2 = sum(1/vi^2); sig2t = (K
     -1)*s1 / (s1^2-s2)
  thetabar = sum(yi/vi) / sum(1/vi)
  Q = sum((yi - thetabar)^2/vi)
  tau2dl = max(0, (Q-(K-1))/(s1-s2/s1))
  thetadl = sum(yi/(vi+tau2dl)) / sum(1/(vi+tau2dl))
  TDL = sum(yi/(vi+tau2dl)) / sqrt(sum(1/(vi+tau2dl)))
  I2hat = tau2dl / (sig2t + tau2dl)
  vdl = 1 / sqrt(sum(1/(vi+tau2dl)))

  if(method=="exactDL"){
    t = -2000:2000/100
    ftdl = pdf_ftdl(t=t, K=K, sig2i=vi, tau2=tau2dl, theta=0)
    Ftdl = sapply(1:length(ftdl$y), function(i){sum(ftdl$y[1:i
        ])}) * (ftdl$x[2]-ftdl$x[1])
    tq = t[which.min(abs(Ftdl - (1-alpha/2)))]
  }else if(method=="I2c"){
    nsim = 10000
    I2sim = 0:99/100
    pI2sim = sapply(I2sim, function(I2x){
      h = I2sim[2]-I2sim[1]
      tau2sim = sig2t * I2x / (1-I2x)
      thetahat = sapply(1:K, function(k){rnorm(n=nsim, mean=0,
          sd=sqrt(vi[k]+tau2sim))})
      thetabari = thetahat %*% (1/vi) / s1
      Qi = sapply(1:nsim, function(i){sum((thetahat[i,]-
          thetabari[i])^2/vi)})
      tau2dl0i = (Qi - (K-1))/(s1-s2/s1)
      tau2dli = sapply(tau2dl0i, function(x){max(0,x)})
      I2dli = floor(1/h * tau2dli / (sig2t+tau2dli)) * h
      PI2 = ecdf(I2dli)
      pI2 = sapply(I2sim, function(x){
        return( (PI2(x+h) - PI2(x))/h )
      })
      pI2[1] = (1 - sum(pI2[-1]) * h) / h
      return(pI2)
```

```
    })
    pre_pI2 = dunif(I2sim, min=a, max=b)
    I2x = sapply(I2sim, function(y){max(I2hat, y)})
    I2c = sum(I2x * pre_pI2 * pI2sim) / sum(pre_pI2 * pI2sim)
    if(is.nan(I2c)==T){ I2c=I2hat }
    tau2c = sig2t * I2c / (1-I2c)
    t = -2000:2000/100
    ftdl = pdf_ftdl(t=t, K=K, sig2i=vi, tau2=tau2c, theta=0)
    Ftdl = sapply(1:length(ftdl$y), function(i){sum(ftdl$y[1:i
       ])}) * (ftdl$x[2]-ftdl$x[1])
    tq = t[which.min(abs(Ftdl - (1-alpha/2)))]

  }else{
    print("Error: The method is not existing")
  }
  ciL = thetadl - tq * vdl
  ciU = thetadl + tq * vdl
  if(side==1 || side==2){
    pval = side * (1 - Ftdl[which.min(abs(t-abs(TDL)))])
  }else{
    print(paste("Variable 'side' is appropriate for 1 or 2"))
  }
  return(list(theta=thetadl, sd=vdl, tau2=tau2dl, zval=TDL, ciL
     =ciL, ciU=ciU, pval=pval))
}
```

# 5 Two solution of the equation (7)

In Appendix A.1, we have showed the solution of the equation (7). The equation (7) can be transformed into the following equation.

$$
\begin{aligned}
\sum_{k=1}^{K} \frac{(\hat{\theta}_k - \bar{\theta}_0)^2}{\sigma_k^2} &= \sum_{k=1}^{K} \frac{\hat{\theta}_k^2}{\sigma_k^2} - \gamma_0^{(1)} \bar{\theta}_0^2 \\
&= \left( \frac{\hat{\theta}_{k'}^2}{\sigma_{k'}^2} + \sum_{k \neq k'} \frac{\hat{\theta}_k^2}{\sigma_k^2} \right) - \frac{1}{\gamma_0^{(1)}} \left( \frac{\hat{\theta}_{k'}}{\sigma_{k'}^2} + \sum_{k \neq k'} \frac{\hat{\theta}_{k'}}{\sigma_{k'}^2} \right)^2 \\
&= \left( \frac{\hat{\theta}_{k'}^2}{\sigma_{k'}^2} + \gamma_{0,k'}^{(1)} v_{\text{-}k'}^2 \right) - \frac{1}{\gamma_0^{(1)}} \left( \frac{\hat{\theta}_{k'}^2}{\sigma_{k'}^4} + 2\gamma_{0,\text{-}k'}^{(1)} \bar{\theta}_{0,\text{-}k'} \frac{\hat{\theta}_{k'}}{\sigma_{k'}^2} + (\gamma_{0,\text{-}k'}^{(1)})^2 \bar{\theta}_{0,\text{-}k'}^2 \right) \\
&= \frac{1}{\sigma_{k'}^2 \gamma_0^{(1)}} \left( \gamma_0^{(1)} - \frac{1}{\sigma_{k'}^2} \right) \hat{\theta}_{k'}^2 - 2 \frac{\gamma_{0,\text{-}k'}^{(1)} \hat{\theta}_{k'}}{\gamma_0^{(1)} \sigma_{k'}^2} \bar{\theta}_{0,\text{-}k'} - \frac{(\gamma_{0,\text{-}k'}^{(1)})^2}{\gamma_0^{(1)}} \bar{\theta}_{0,\text{-}k'}^2 + \gamma_{0,\text{-}k'}^{(1)} v_{\text{-}k'}^2 \\
&= \frac{\gamma_{0,\text{-}k'}^{(1)}}{\sigma_{k'}^2 \gamma_0^{(1)}} \hat{\theta}_{k'}^2 - 2 \frac{\gamma_{0,\text{-}k'}^{(1)}}{\sigma_{k'}^2 \gamma_0^{(1)}} \hat{\theta}_{k'} \bar{\theta}_{0,\text{-}k'} - \frac{(\gamma_{0,\text{-}k'}^{(1)})^2}{\gamma_0^{(1)}} \bar{\theta}_{0,\text{-}k'}^2 + \gamma_{0,\text{-}k'}^{(1)} v_{\text{-}k'}^2 \\
&= \frac{\gamma_{0,\text{-}k'}^{(1)}}{\sigma_{k'}^2 \gamma_0^{(1)}} \left( \hat{\theta}_{k'} - \bar{\theta}_{0,\text{-}k'} \right)^2 + \frac{\gamma_{0,\text{-}k'}^{(1)}}{\sigma_{k'}^2 \gamma_0^{(1)}} \bar{\theta}_{0,\text{-}k'}^2 - \frac{(\gamma_{0,\text{-}k'}^{(1)})^2}{\gamma_0^{(1)}} \bar{\theta}_{0,\text{-}k'}^2 + \gamma_{0,\text{-}k'}^{(1)} v_{\text{-}k'}^2 \\
&= \left( \gamma_0^{(1)} - \frac{\gamma_0^{(2)}}{\gamma_0^{(1)}} \right) x + (K-1)
\end{aligned}
$$

Thus, we can get following equation.

$$\hat{\theta}_{k'} = \bar{\theta}_{0,\text{-}k'} \pm \sqrt{\bar{\theta}^2_{0,\text{-}k'} + \sigma^2_{k'}\gamma^{(1)}_{0,\text{-}k'}\left\{\bar{\theta}^2_{0,\text{-}k'} - \frac{\gamma^{(1)}_0}{\gamma^{(1)}_{0,\text{-}k'}}v^2_{\text{-}k'} + \left(\frac{(\gamma^{(1)}_0)^2 - \gamma^{(2)}_0}{(\gamma^{(1)}_{0,\text{-}k'})^2}\right)x + (K-1)\frac{\gamma^{(1)}_0}{(\gamma^{(1)}_{0,\text{-}k'})^2}\right\}}$$

# 6 Derivation details for $f_{\hat{\theta}_K|\hat{\tau}^2_u}$

We show the details for deriving $f_{\hat{\theta}_K|\hat{\tau}^2_u}$ in this section. For simplicity, we assume $k' = K$. The PDF of $\hat{\theta}_K|\hat{\tau}^2_u$ is

$$\begin{aligned}
f_{\hat{\theta}_K|\hat{\tau}^2_u}(z_K|x) &= f_{\hat{\theta}_K,\hat{\tau}^2_u}(z_K, x)/f_{\hat{\tau}^2_u}(x) \\
&= \int \cdots \int_\Omega f_{\hat{\theta}_1,\cdots,\hat{\theta}_{K-2},\hat{\tau}^2_u,\hat{\theta}_K}(z_1, \ldots, z_{K-2}, x, z_K)dz_1 \cdots dz_{K-2}/f_{\hat{\tau}^2_u}(x)
\end{aligned}$$

where the integral region $\Omega$ is $\Omega = \{(\hat{\theta}_1, \ldots, \hat{\theta}_K) \in \mathbb{R}^n|\hat{\tau}^2_u = x\}$. From the solution of Eq. (7), $\hat{\tau}^2_u \leq x$ and $(\hat{\theta}_{K-1} - \bar{z}_{\text{-}(K-1)})^2 \leq \xi_{\text{-}(K-1)}(\mathbf{z})$ are equivalent, where $\bar{z}_{\text{-}(K-1)} = \bar{\theta}_{0,\text{-}(K-1)}(z_1, \ldots, z_{K-2}, z_K)$ and $\xi_{\text{-}(K-1)}(\mathbf{z}) = \xi_{\text{-}(K-1)}(z_1, \ldots, z_{K-2}, x, z_K)$. Therefore, the distribution function of $(\hat{\theta}_1, \ldots, \hat{\theta}_{K-2}, \hat{\tau}^2_u, \hat{\theta}_K)$ is

$$\begin{aligned}
&F_{\hat{\theta}_1,\ldots,\hat{\theta}_{K-2},\hat{\tau}^2_u,\hat{\theta}_K}(z_1, \ldots, z_{K-2}, x, z_K) \\
&= \Pr(\hat{\theta}_1 \leq z_1, \cdots, \hat{\theta}_{K-2} \leq z_{K-2}, \hat{\tau}^2_u \leq x, \hat{\theta}_K \leq z_K) \\
&= \Pr\left(\hat{\theta}_1 \leq z_1, \cdots, \hat{\theta}_{K-2} \leq z_{K-2}, \bar{z}_{\text{-}(K-1)} - \xi_{\text{-}(K-1)}(\mathbf{z}) \leq \hat{\theta}_{K-1} \leq \bar{z}_{\text{-}(K-1)} + \xi_{\text{-}(K-1)}(\mathbf{z}), \hat{\theta}_K \leq z_K\right) \\
&= F_{\hat{\theta}_1,\ldots,\hat{\theta}_K}(z_1, \ldots, z_{K-2}, \bar{z}_{\text{-}(K-1)} + \xi_{\text{-}(K-1)}(\mathbf{z}), z_K) - F_{\hat{\theta}_1,\ldots,\hat{\theta}_K}(z_1, \ldots, z_{K-2}, \bar{z}_{\text{-}K} - \xi_{\text{-}(K-1)}(\mathbf{z}), z_K).
\end{aligned}$$

Let $\phi_k$ be the PDF of $N(\theta, \sigma^2_k + \tau^2)$. The PDF of $(\hat{\theta}_1, \ldots, \hat{\theta}_{K-2}, \hat{\tau}^2_u, \hat{\theta}_K)$ can be expressed as

$$\begin{aligned}
&f_{\hat{\theta}_1,\ldots,\hat{\theta}_{K-2},\hat{\tau}^2_u,\hat{\theta}_K}(z_1, \ldots, z_{K-2}, x, z_K) \\
&= f_{\hat{\theta}_1,\ldots,\hat{\theta}_K}(z_1, \ldots, z_{K-2}, \bar{z}_{\text{-}(K-1)} + \xi_{\text{-}(K-1)}(\mathbf{z}), z_K)\left|\frac{\partial}{\partial x}(\bar{z}_{\text{-}(K-1)} + \xi_{\text{-}(K-1)}(\mathbf{z}), z_K)\right| \\
&+ f_{\hat{\theta}_1,\ldots,\hat{\theta}_K}(z_1, \ldots, z_{K-2}, \bar{z}_{\text{-}(K-1)} - \xi_{\text{-}(K-1)}(\mathbf{z}))\left|\frac{\partial}{\partial x}(\bar{z}_{\text{-}(K-1)} - \xi_{\text{-}(K-1)}(\mathbf{z}), z_K)\right| \\
&= \left(\prod_{k=1}^{K-2}\phi_k(z_k)\right)\left\{\phi_{K-1}(\bar{z}_{\text{-}(K-1)} + \xi_{\text{-}(K-1)}(\mathbf{z})) + \phi_{K-1}(\bar{z}_{\text{-}(K-1)} - \xi_{\text{-}(K-1)}(\mathbf{z}))\right\}\phi_K(z_K)\frac{\sigma^2_{K-1}((\gamma^{(1)}_0)^2 - \gamma^{(2)}_0)x}{\xi_{\text{-}(K-1)}(\mathbf{z})}.
\end{aligned}$$

# References


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