Supplementary Material
to the paper

Flexible asymmetric multivariate distributions based on two-piece univariate distributions

Jonas Baillien[1] – **Irène Gijbels**[2] – **Anneleen Verhasselt**[3]

Abstract This supplement contains the additional parts not provided in the main paper. These include:

- Explanation about the relation to Independent Component Analysis.
- Proofs of Lemmas 1, 2 and 3 of Section 3.2.
- Proofs of Propositions 1, 5 and 6 of Section 3.2.
- Some results on the extension to QBA-Student's t-distributed univariate components.
- Additional plots for the AIS-data example.
- In a separate R Markdown document the reader finds illustrative examples for the use of the R codes, provided on the GitHub platform at `https://github.com/Anonymous162222/LCQBA`.

S.1 Relation to Independent Component Analysis

Our method of generating multivariate distributions closely resembles reverse independent component analysis (ICA). We mix the sources to obtain a new distribution whereas ICA tries to unmix the data in order to obtain the source distributions. To this end, ICA imposes two main assumptions. The first is independence of the sources, the second is that at most one source is Gaussian (symmetric). The latter might seem odd, but is vital for identifiability of the mixing matrix $\mathbf{A}$. With these two assumptions in place, uniqueness of the model is still not

[1] Department of Mathematics and Leuven Statistics Research Center (LStat), KU Leuven, Celestijnenlaan 200B, Box 2400, B-3001 Leuven (Heverlee), Belgium
E-mail: jonas.baillien@kuleuven.be

[2] Corresponding Author, E-mail: irene.gijbels@kuleuven.be
Department of Mathematics and Leuven Statistics Research Center (LStat), KU Leuven, Celestijnenlaan 200B, Box 2400, B-3001 Leuven (Heverlee), Belgium

[3] Center for Statistics, Data Science Institute, Hasselt University,
Agoralaan-building D, B-3590 Diepenbeek, Belgium
E-mail: anneleen.verhasselt@uhasselt.be

guaranteed as scale and sign ambiguities as well as the permutation problem can still happen. The permutation problem arises because the order in which the sources appear is not fixed by the model and can thus be permuted, provided that the same permutation is applied to $\mathbf{A}$. ICA does not need the assumption of known sources to obtain $\mathbf{A}$, whereas we explicitly assume that the sources are indeed known. This simplifies the problem as more information is available, but in term also complicates it slightly more as for the purpose of statistical inference, the scale and sign ambiguity need to be resolved. In ICA, the shape of the set of sources is most important. This alleviates the need to know the scale and orientation of the sources.

For our purposes, ICA might be of interest as it can be incorporated in a two-step estimation where first $\mathbf{A}$ is estimated and in a second step the source parameters. For finding the mixing matrix, multiple methods have been proposed. The most renowned ones are JADE (Cardoso and Souloumiac (1993)) and FastICA (Hyvärinen (1999)) and adaptations thereof. For inference, strong asymptotic results, e.g. consistency and asymptotic normality, for the first step need to be available in order to show asymptotic results for the second step. Preferably, the conditions under which these asymptotic results hold are as weak as possible. Some of the most important asymptotic results in the ICA context are found in Bonhomme and Robin (2009), Ollila (2010), Ilmonen et al. (2012), Miettinen et al. (2015) and Gouriéroux et al. (2017). This makes that employing the proposed two-step procedure is possible.

Most ICA estimation techniques are semi-parametric however, whereas we opt for a fully parametric approach. As stated in Hyvärinen et al. (2001), once the source distributions are known, the likelihood function is entirely known. This creates an opportunity to switch from a semi-parametric estimation technique mostly used in the ICA methods, to a fully parametric technique in the form of maximum likelihood estimation. In our context, we estimate all model parameters simultaneously in one single step.

S.2 Proofs of theoretical results

S.2.1 Proof of Proposition 1

Only the proof for the marginal characteristic function is given. For the joint characteristic function a similar reasoning can be applied. Due to the independence of the Z_j's and the generating densities f_j $j = 1, \ldots, d$ containing neither a scale nor location parameter, we can write

$$\varphi_{X_k}(t) = \varphi_{\mu_{a,k} + \sum_{j=1}^{d} \mathbf{A}_{j,k} Z_j}(t) = e^{it\mu_{a,k}} \prod_{j=1}^{d} \varphi_{Z_j}(\mathbf{A}_{j,k} t).$$

The result then follows from the fact (Gijbels et al. (2019)) that

$$\varphi_{Z_j}(t) = 2\left[\alpha_j \varphi_j^+\left(\frac{-t}{1-\alpha_j}\right) + (1-\alpha_j)\varphi_j^+\left(\frac{t}{\alpha_j}\right)\right].$$

S.2.2 Proof of Lemma 1

We will use (15) and (16) from Section 3.2 to directly compute $\frac{\partial\left[(\mathbf{x}-\boldsymbol{\mu}_a)^T(\mathbf{A}^{-1})_{\cdot,j}\right]}{\partial \mathbf{A}_{k,l}}$ as

$$\begin{aligned}
\frac{\partial\left[(\mathbf{x}-\boldsymbol{\mu}_a)^T(\mathbf{A}^{-1})_{\cdot,j}\right]}{\partial \mathbf{A}_{k,l}} &= \frac{\partial}{\partial \mathbf{A}_{k,l}}\left[\sum_{i=1}^{d}(x_i-\mu_{a,i})(\mathbf{A}^{-1})_{i,j}\right] \\
&= \sum_{i=1}^{d}\frac{\partial}{\partial \mathbf{A}_{k,l}}\frac{(-1)^{i+j}\det(\mathbf{A}_{-j;-i})(x_i-\mu_{a,i})}{\det(\mathbf{A})} \\
&= \sum_{i=1}^{d}\frac{(-1)^{i+j}(x_i-\mu_{a,i})\det(\mathbf{A})\frac{\partial}{\partial \mathbf{A}_{k,l}}\det(\mathbf{A}_{-j;-i})}{(\det(\mathbf{A}))^2} \qquad \text{(S.1)} \\
&\quad -\sum_{i=1}^{d}\frac{(-1)^{i+j}\det(\mathbf{A}_{-j;-i})(x_i-\mu_{a,i})\frac{\partial}{\partial \mathbf{A}_{k,l}}\det(\mathbf{A})}{(\det(\mathbf{A}))^2}.
\end{aligned}$$

Employing (4), we have $x_i - \mu_{a,i} = \sum_{h=1}^{d}\mathbf{A}_{h,i}z_h$ and $z_h = (\mathbf{x}-\boldsymbol{\mu}_a)^T(\mathbf{A}^{-1})_{\cdot,h}$. Combining these two expressions gives $x_i-\mu_{a,i} = \sum_{h=1}^{d}\mathbf{A}_{h,i}(\mathbf{x}-\boldsymbol{\mu}_a)^T(\mathbf{A}^{-1})_{\cdot,h}$. Substituting this result in (S.1) and using (15), we get

$$\begin{aligned}
&\frac{\partial\left[(\mathbf{x}-\boldsymbol{\mu}_a)^T(\mathbf{A}^{-1})_{\cdot,j}\right]}{\partial \mathbf{A}_{k,l}} \\
&= \sum_{\substack{i=1\\ i\neq l}}^{d}\frac{(-1)^{i+j+k-1+\mathbb{1}\{j>k\}+l-1+\mathbb{1}\{i>l\}}\det(\mathbf{A}_{-j,-k;-i,-l})(x_i-\mu_{a,i})}{\det(\mathbf{A})}\mathbb{1}\{j\neq k\} \\
&\quad -\sum_{i=1}^{d}\frac{(-1)^{i+j+k+l}\det(\mathbf{A}_{-k;-l})\det(\mathbf{A}_{-j;-i})(x_i-\mu_{a,i})}{(\det(\mathbf{A}))^2} \\
&= \sum_{\substack{i=1\\ i\neq l}}^{d}\frac{(-1)^{i+j+k+\mathbb{1}\{j>k\}+l+\mathbb{1}\{i>l\}}\det(\mathbf{A}_{-j,-k;-i,-l})\sum_{h=1}^{d}\mathbf{A}_{h,i}(\mathbf{x}-\boldsymbol{\mu}_a)^T(\mathbf{A}^{-1})_{\cdot,h}}{\det(\mathbf{A})}\mathbb{1}\{j\neq k\} \\
&\quad -\frac{(-1)^{k+l}\det(\mathbf{A}_{-k;-l})}{\det(\mathbf{A})}\sum_{i=1}^{d}\frac{(-1)^{i+j}\det(\mathbf{A}_{-j;-i})(x_i-\mu_{a,i})}{\det(\mathbf{A})}.
\end{aligned}$$

By isolating the term for which $h=j$ and utilizing (14), the final result is obtained.

$$\frac{\partial\left[(\mathbf{x}-\boldsymbol{\mu}_a)^T(\mathbf{A}^{-1})_{\cdot,j}\right]}{\partial \mathbf{A}_{k,l}}$$

$$= \sum_{\substack{i=1\\i\neq l}}^{d} \sum_{\substack{h=1\\h\neq j}}^{d} \frac{(-1)^{i+j+k+\mathbb{1}\{j>k\}+l+\mathbb{1}\{i>l\}} \det(\mathbf{A}_{-j,-k;-i,-l}) \mathbf{A}_{h,i} (\mathbf{x}-\boldsymbol{\mu}_a)^T (\mathbf{A}^{-1})_{\cdot,h}}{\det(\mathbf{A})} \mathbb{1}\{j \neq k\}$$
$$+ \sum_{\substack{i=1\\i\neq l}}^{d} \frac{(-1)^{i+j+\mathbb{1}\{j>k\}+\mathbb{1}\{i>l\}} \det(\mathbf{A}_{-j,-k;-i,-l}) \mathbf{A}_{j,i} (-1)^{k+l} (\mathbf{x}-\boldsymbol{\mu}_a)^T (\mathbf{A}^{-1})_{\cdot,j}}{\det(\mathbf{A})} \mathbb{1}\{j \neq k\}$$
$$- \frac{(-1)^{k+l} \det(\mathbf{A}_{-k;-l})}{\det(\mathbf{A})} \sum_{i=1}^{d} (x_i - \mu_{a,i}) (\mathbf{A}^{-1})_{i,j}$$
$$= \sum_{\substack{i=1\\i\neq l}}^{d} \sum_{\substack{h=1\\h\neq j}}^{d} \frac{(-1)^{i+j+k+\mathbb{1}\{j>k\}+l+\mathbb{1}\{i>l\}} \det(\mathbf{A}_{-j,-k;-i,-l}) \mathbf{A}_{h,i} (\mathbf{x}-\boldsymbol{\mu}_a)^T (\mathbf{A}^{-1})_{\cdot,h}}{\det(\mathbf{A})} \mathbb{1}\{j \neq k\}$$
$$+ \frac{(-1)^{k+l} (\mathbf{x}-\boldsymbol{\mu}_a)^T (\mathbf{A}^{-1})_{\cdot,j} \det(\mathbf{A}_{-k;-l})}{\det(\mathbf{A})} \mathbb{1}\{j \neq k\}$$
$$- \frac{(-1)^{k+l} (\mathbf{x}-\boldsymbol{\mu}_a)^T (\mathbf{A}^{-1})_{\cdot,j} \det(\mathbf{A}_{-k;-l})}{\det(\mathbf{A})}.$$

Using the introduced notations (17) and (18), this can be simplified to

$$\frac{\partial \left[(\mathbf{x}-\boldsymbol{\mu}_a)^T (\mathbf{A}^{-1})_{\cdot,j} \right]}{\partial \mathbf{A}_{k,l}} = \begin{cases} D_{k,l} (\mathbf{x}-\boldsymbol{\mu}_a)^T (\mathbf{A}^{-1})_{\cdot,k} & \text{if } j = k \\ \sum_{\substack{h=1\\h\neq j}}^{d} B_{h,j}^{k,l} (\mathbf{x}-\boldsymbol{\mu}_a)^T (\mathbf{A}^{-1})_{\cdot,h} & \text{if } j \neq k. \end{cases}$$

S.2.3 Proof of Proposition 5

As the parameter vector contains three groups of parameters, $\boldsymbol{\alpha}$, $\boldsymbol{\mu}_a$ and $\mathbf{A}$, the proof is split up in three parts according to the different groups of parameters. The expectation of the score function is directly calculated for elements of each of these groups. To do this, an expression for the scores is needed. By taking the derivatives of (12) with respect to the desired parameters, we obtain

$$\frac{\partial \ell(\boldsymbol{\theta}; \mathbf{x})}{\partial \alpha_k} = \frac{1 - 2\alpha_k}{\alpha_k (1 - \alpha_k)}$$
$$+ \mathbb{1}\left\{ (\mathbf{x}-\boldsymbol{\mu_a})^T (\mathbf{A}^{-1})_{\cdot,k} \leq 0 \right\} \frac{f_k'(-(1-\alpha_k)(\mathbf{x}-\boldsymbol{\mu_a})^T (\mathbf{A}^{-1})_{\cdot,k})}{f_k(-(1-\alpha_k)(\mathbf{x}-\boldsymbol{\mu_a})^T (\mathbf{A}^{-1})_{\cdot,k})}$$
$$\cdot (\mathbf{x}-\boldsymbol{\mu_a})^T (\mathbf{A}^{-1})_{\cdot,k}$$
$$+ \mathbb{1}\left\{ (\mathbf{x}-\boldsymbol{\mu_a})^T (\mathbf{A}^{-1})_{\cdot,k} > 0 \right\} \frac{f_k'(\alpha_k (\mathbf{x}-\boldsymbol{\mu_a})^T (\mathbf{A}^{-1})_{\cdot,k})}{f_k(\alpha_k (\mathbf{x}-\boldsymbol{\mu_a})^T (\mathbf{A}^{-1})_{\cdot,k})}$$
$$\cdot (\mathbf{x}-\boldsymbol{\mu_a})^T (\mathbf{A}^{-1})_{\cdot,k}$$

$$\frac{\partial \ell(\boldsymbol{\theta}; \mathbf{x})}{\partial \mu_{a,k}}$$

$$= \sum_{j=1}^{d} \Bigg(\mathbb{1}\big\{(\mathbf{x}-\boldsymbol{\mu_a})^T(\mathbf{A}^{-1})_{\cdot,j} \le 0\big\} \, \frac{f_j'(-(1-\alpha_j)(\mathbf{x}-\boldsymbol{\mu_a})^T(\mathbf{A}^{-1})_{\cdot,j})}{f_j(-(1-\alpha_j)(\mathbf{x}-\boldsymbol{\mu_a})^T(\mathbf{A}^{-1})_{\cdot,j})}$$
$$\cdot(1-\alpha_j)(\mathbf{A}^{-1})_{k,j}$$
$$-\mathbb{1}\big\{(\mathbf{x}-\boldsymbol{\mu_a})^T(\mathbf{A}^{-1})_{\cdot,j} > 0\big\} \, \frac{f_j'(\alpha_j(\mathbf{x}-\boldsymbol{\mu_a})^T(\mathbf{A}^{-1})_{\cdot,j})\alpha_j(\mathbf{A}^{-1})_{k,j}}{f_j(\alpha_j(\mathbf{x}-\boldsymbol{\mu_a})^T(\mathbf{A}^{-1})_{\cdot,j})} \Bigg)$$

$$\frac{\partial \ell(\boldsymbol{\theta};\mathbf{x})}{\partial \mathbf{A}_{k,l}}$$
$$= -\frac{(-1)^{k+l}\det(\mathbf{A}_{-k;-l})}{\det(\mathbf{A})}$$
$$-\sum_{j=1}^{d} \Bigg(\mathbb{1}\Big\{(\mathbf{x}-\boldsymbol{\mu}_a)^T(\mathbf{A}^{-1})_{\cdot,j} > 0\Big\} \, \frac{f_j'(\alpha_j(\mathbf{x}-\boldsymbol{\mu}_a)^T(\mathbf{A}^{-1})_{\cdot,j})\alpha_j}{f_j(\alpha_j(\mathbf{x}-\boldsymbol{\mu}_a)^T(\mathbf{A}^{-1})_{\cdot,j})}$$
$$+\mathbb{1}\Big\{(\mathbf{x}-\boldsymbol{\mu}_a)^T(\mathbf{A}^{-1})_{\cdot,j} \le 0\Big\} \, \frac{f_j'(-(1-\alpha_j)(\mathbf{x}-\boldsymbol{\mu}_a)^T(\mathbf{A}^{-1})_{\cdot,j})(1-\alpha_j)}{f_j(-(1-\alpha_j)(\mathbf{x}-\boldsymbol{\mu}_a)^T(\mathbf{A}^{-1})_{\cdot,j})} \Bigg)$$
$$\cdot \frac{\partial \left[(\mathbf{x}-\boldsymbol{\mu}_a)^T(\mathbf{A}^{-1})_{\cdot,j}\right]}{\partial \mathbf{A}_{k,l}}.$$

The substitution $(\mathbf{x}-\boldsymbol{\mu}_a)(\mathbf{A}^{-1})_{\cdot,k} = z_k$ is used to calculate the expectation of the three expression above. This substitution follows directly from (4). The resulting expressions are then worked out further to show Proposition 5 holds. The first parameter group is $\boldsymbol{\alpha}$, the expectation of the score equations is

$$E\left[\frac{\partial \ell(\boldsymbol{\theta};\mathbf{X})}{\partial \alpha_k}\right]$$
$$= \int_{\mathbb{R}^d} \left(\frac{1-2\alpha_k}{\alpha_k(1-\alpha_k)} + \mathbb{1}\{z_k \le 0\} \, \frac{f_k'\left(-(1-\alpha_k)z_k\right) z_k}{f_k\left(-(1-\alpha_k)z_k\right)} + \mathbb{1}\{z_k > 0\} \, \frac{f_k'\left(\alpha_k z_k\right) z_k}{f_k\left(\alpha_k z_k\right)} \right)$$
$$\cdot \prod_{m=1}^{d} f_{Z_m}\left(z_m;\boldsymbol{\eta}_m\right) d\mathbf{z}$$

$$= \int_{\mathbb{R}^d} \left(\frac{1-2\alpha_k}{\alpha_k(1-\alpha_k)} + \mathbb{1}\{z_k \le 0\} \, \frac{f_k'\left(-(1-\alpha_k)z_k\right) z_k}{f_k\left(-(1-\alpha_k)z_k\right)} + \mathbb{1}\{z_k > 0\} \, \frac{f_k'\left(\alpha_k z_k\right) z_k}{f_k\left(\alpha_k z_k\right)} \right)$$
$$\cdot 2\alpha_k(1-\alpha_k)\Big(\mathbb{1}\{z_k \le 0\} \, f_k\left(-(1-\alpha_k)z_k\right) + \mathbb{1}\{z_k > 0\} \, f_k\left(\alpha_k z_k\right)\Big)$$
$$\cdot \prod_{\substack{m=1 \\ m\neq k}}^{d} f_{Z_m}\left(z_m;\boldsymbol{\eta}_m\right) d\mathbf{z}.$$

In the last step, the density of Z_k is written in its full formulation. Since f_{Z_m} is a density function and there are no other dependencies on z_m present in the integrand, these integrate to 1 for $m \neq k$. This results in

$$E\left[\frac{\partial \ell(\boldsymbol{\theta};\mathbf{X})}{\partial \alpha_k}\right]$$

$$= \int_{\mathbb{R}} \left(\frac{1-2\alpha_k}{\alpha_k(1-\alpha_k)} + \mathbb{1}\{z_k \leq 0\} \frac{f_k'\left(-(1-\alpha_k)z_k\right) z_k}{f_k\left(-(1-\alpha_k)z_k\right)} + \mathbb{1}\{z_k > 0\} \frac{f_k'\left(\alpha_k z_k\right) z_k}{f_k\left(\alpha_k z_k\right)} \right)$$

$$\cdot 2\alpha_k(1-\alpha_k)\Big(\mathbb{1}\{z_k \leq 0\} f_k\left(-(1-\alpha_k)z_k\right) + \mathbb{1}\{z_k > 0\} f_k\left(\alpha_k z_k\right)\Big) dz_k$$

$$= \frac{1-2\alpha_k}{\alpha_k(1-\alpha_k)} - \int_0^\infty \frac{2\alpha_k}{1-\alpha_k} u f_k'(u) du + \int_0^\infty \frac{2(1-\alpha_k)}{\alpha_k} v f_k'(v) dv$$

$$= \frac{1-2\alpha_k}{\alpha_k(1-\alpha_k)} + \frac{\alpha_k}{1-\alpha_k} - \frac{1-\alpha_k}{\alpha_k}$$

$$= 0,$$

where the substitutions $u = -(1-\alpha_k)z_k$ and $v = \alpha_k z_k$ have been applied to simplify the expression. These substitutions are used frequently throughout the remainder of the proof and that of Proposition 6. For the $\boldsymbol{\mu}_a$-group, we calculate

$$E\left[\frac{\partial \ell(\boldsymbol{\theta}; \mathbf{X})}{\partial \mu_{a,k}}\right]$$

$$= \int_{\mathbb{R}^d} \sum_{j=1}^{d} \left(\mathbb{1}\{z_j \leq 0\} \frac{f_j'\left(-(1-\alpha_j)z_j\right)(1-\alpha_j)(\mathbf{A}^{-1})_{k,j}}{f_j\left(-(1-\alpha_j)z_j\right)} \right.$$

$$\left. - \mathbb{1}\{z_j > 0\} \frac{f_j'\left(\alpha_j z_j\right) \alpha_j (\mathbf{A}^{-1})_{k,j}}{f_j\left(\alpha_j z_j\right)} \right) \prod_{m=1}^{d} f_{Z_m}\left(z_m; \boldsymbol{\eta}_m\right) d\mathbf{z},$$

each term of the summation is considered separately. By employing the same reasoning and substitutions as before, for the l-th term in the summation over j we have

$$\int_{\mathbb{R}^d} \left(\mathbb{1}\{z_l \leq 0\} \frac{f_l'\left(-(1-\alpha_l)z_l\right)(1-\alpha_l)(\mathbf{A}^{-1})_{k,l}}{f_l\left(-(1-\alpha_l)z_l\right)} - \mathbb{1}\{z_l > 0\} \frac{f_l'\left(\alpha_l z_l\right)\alpha_l(\mathbf{A}^{-1})_{k,l}}{f_l\left(\alpha_l z_l\right)} \right)$$

$$\cdot 2\alpha_l(1-\alpha_l)\Big(\mathbb{1}\{z_l \leq 0\} f_l\left(-(1-\alpha_l)z_l\right) + \mathbb{1}\{z_l > 0\} f_l\left(\alpha_l z_l\right)\Big)$$

$$\cdot \prod_{\substack{m=1 \\ m \neq l}}^{d} f_{Z_m}\left(z_m; \boldsymbol{\eta}_m\right) d\mathbf{z}$$

$$= 2\alpha_l(1-\alpha_l)(\mathbf{A}^{-1})_{k,l} \left(\int_{-\infty}^{0} f_l'\left(-(1-\alpha_l)z_l\right)(1-\alpha_l) dz_l - \int_0^\infty f_l'\left(\alpha_l z_l\right)\alpha_l dz_l \right)$$

$$= 2\alpha_l(1-\alpha_l)(\mathbf{A}^{-1})_{k,l} \left(\int_0^\infty f_l'(u) du - \int_0^\infty f_l'(v) dv \right)$$

$$= 0.$$

Since this result holds for each l, it follows that $E\left[\frac{\partial \ell(\boldsymbol{\theta};\mathbf{X})}{\partial \mu_{a,k}}\right] = 0$.

For $E\left[\frac{\partial \ell(\boldsymbol{\theta};\mathbf{X})}{\partial \mathbf{A}_{k,l}}\right]$ the expression for $\frac{\partial\left[(\mathbf{x}-\boldsymbol{\mu}_a)^T(\mathbf{A}^{-1})_{\cdot,j}\right]}{\partial \mathbf{A}_{k,l}}$ obtained in Lemma 1 is required. This together with the introduced notations (17) and (18) yields

$$
\begin{aligned}
& E\left[\frac{\partial \ell(\boldsymbol{\theta};\mathbf{X})}{\partial \mathbf{A}_{k,l}}\right] \\
&= \int_{\mathbb{R}^d} \Bigg\{ D_{k,l} z_k \Bigg(\mathbb{1}\{z_k > 0\} \frac{f_k'(\alpha_k z_k)\,\alpha_k}{f_k(\alpha_k z_k)} - \mathbb{1}\{z_k \le 0\} \frac{f_k'(-(1-\alpha_k)z_k)\,(1-\alpha_k)}{f_k(-(1-\alpha_k)z_k)} \Bigg) \\
&\quad + \Bigg(\sum_{\substack{j=1 \\ j \ne k}}^{d} \Bigg[\mathbb{1}\{z_j > 0\} \frac{f_j'(\alpha_j z_j)\,\alpha_j}{f_j(\alpha_j z_j)} - \mathbb{1}\{z_j \le 0\} \frac{f_j'(-(1-\alpha_j)z_j)\,(1-\alpha_j)}{f_j(-(1-\alpha_j)z_j)} \Bigg] \\
&\quad \cdot \sum_{\substack{h=1 \\ h \ne j}}^{d} B_{h,j}^{k,l} z_h \Bigg) \Bigg\} \prod_{m=1}^{d} f_{Z_m}(z_m;\boldsymbol{\eta}_m)\, d\mathbf{z} + D_{k,l}.
\end{aligned}
$$

Where the k-th term has been separated form the others due to Lemma 1. Next, each term in the summation over j is considered separately. For $j = s$, with $s \ne k$ this gives

$$
\begin{aligned}
& \int_{\mathbb{R}^d} \Bigg(\mathbb{1}\{z_s > 0\} \frac{f_s'(\alpha_s z_s)\,\alpha_s}{f_s(\alpha_s z_s)} - \mathbb{1}\{z_s \le 0\} \frac{f_s'(-(1-\alpha_s)z_s)\,(1-\alpha_s)}{f_s(-(1-\alpha_s)z_s)} \Bigg) \sum_{\substack{h=1 \\ h \ne s}}^{d} B_{h,s}^{k,l} z_h \\
&\quad \cdot 2\alpha_s(1-\alpha_s) \Big(\mathbb{1}\{z_s \le 0\} f_s(-(1-\alpha_s)z_s) + \mathbb{1}\{z_s > 0\} f_s(\alpha_s z_s) \Big) \\
&\quad \cdot \prod_{\substack{m=1 \\ m \ne s}}^{d} f_{Z_m}(z_m;\boldsymbol{\eta}_m)\, d\mathbf{z} \\
&= 2\alpha_s(1-\alpha_s) \Bigg(\int_0^{\infty} f_s'(\alpha_s z_s)\,\alpha_s dz_s - \int_{-\infty}^{0} f_s'(-(1-\alpha_s)z_s)\,(1-\alpha_s) dz_s \Bigg) \\
&\quad \cdot \sum_{\substack{h=1 \\ h \ne s}}^{d} B_{h,s}^{k,l} E_{Z_h}[Z_h] \\
&= 2\alpha_s(1-\alpha_s) \Bigg(\int_0^{\infty} f_s'(v)\, dv - \int_0^{\infty} f_s'(u)\, du \Bigg) \sum_{\substack{h=1 \\ h \ne s}}^{d} \underbrace{B_{h,s}^{k,l} E_{Z_h}[Z_h]}_{<\infty} \\
&= 0.
\end{aligned}
$$

In this, the previously introduced substitutions are used together with the result from Gijbels et al. (2019) that the expectation of Z_h is finite for $\alpha \in (0,1)$. By applying the same methodology as before to the term for which $j = k$

$$
\begin{aligned}
& D_{k,l} \int_{\mathbb{R}^d} \Bigg(\mathbb{1}\{z_k > 0\} \frac{f_k'(\alpha_k z_k)\,\alpha_k z_k}{f_k(\alpha_k z_k)} - \mathbb{1}\{z_k \le 0\} \frac{f_k'(-(1-\alpha_k)z_k)\,(1-\alpha_k) z_k}{f_k(-(1-\alpha_k)z_k)} \Bigg) \\
&\quad \cdot 2\alpha_k(1-\alpha_k) \Big(\mathbb{1}\{z_k \le 0\} f_k(-(1-\alpha_k)z_k) + \mathbb{1}\{z_k > 0\} f_k(\alpha_k z_k) \Big)
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \prod_{\substack{m=1\\m\neq k}}^{d} f_{Z_m}\left(z_m;\boldsymbol{\eta}_m\right) d\mathbf{z}\\
&= 2\alpha_k(1-\alpha_k)D_{k,l}\left(\int_0^{\infty} f_k'\left(\alpha_k z_k\right) z_k \alpha_k dz_k - \int_{-\infty}^{0} f_k'\left(-(1-\alpha_k)z_k\right) z_k (1-\alpha_k) dz_k\right)\\
&= D_{k,l}\left(2\alpha_k \int_0^{\infty} u f_k'(u)du + 2(1-\alpha_k)\int_0^{\infty} v f_k'(v)dv\right)\\
&= -D_{k,l}.
\end{aligned}
$$

The last step follows from Assumption (*N2*). Combining these results yields

$$
E\left[\frac{\partial \ell(\boldsymbol{\theta};\mathbf{X})}{\partial \mathbf{A}_{k,l}}\right] = \sum_{\substack{j=1\\j\neq k}}^{d} 0 - D_{k,l} + D_{k,l} = 0,
$$

which completes the proof.

S.2.4 Proof of Proposition 6

The proof of these entities is again obtained through direct computation. For this, the parameter space is divided into blocks involving interactions from different groups of parameters ($\boldsymbol{\alpha}$, $\boldsymbol{\mu}_a$ or $\mathbf{A}$). This creates six blocks for which an expression of the corresponding elements of the Fisher information matrix is derived. As in the proof of Proposition 5, the substitution $(\mathbf{x}-\boldsymbol{\mu}_a)(\mathbf{A}^{-1})_{\cdot,k} = z_k$ is applied. Also two other substitutions $u = -(1-\alpha_k)z_k$ and $v = \alpha_k z_k$ are used at the appropriate time, similar as in the proof of Proposition 5 above.

For the $\boldsymbol{\alpha}$ - $\boldsymbol{\alpha}$ block we have

$$
\begin{aligned}
&E\left[\left(\frac{\partial \ell(\boldsymbol{\theta};\mathbf{X})}{\partial \alpha_k}\right)\left(\frac{\partial \ell(\boldsymbol{\theta};\mathbf{X})}{\partial \alpha_l}\right)\right]\\
&= \int_{\mathbb{R}^d}\left(\frac{1-2\alpha_k}{\alpha_k(1-\alpha_k)} + \mathbb{1}\{z_k\le 0\}\frac{f_k'\left(-(1-\alpha_k)z_k\right) z_k}{f_k\left(-(1-\alpha_k)z_k\right)} + \mathbb{1}\{z_k>0\}\frac{f_k'\left(\alpha_k z_k\right) z_k}{f_k\left(\alpha_k z_k\right)}\right)\\
&\quad\cdot\left(\frac{1-2\alpha_l}{\alpha_l(1-\alpha_l)} + \mathbb{1}\{z_l\le 0\}\frac{f_l'\left(-(1-\alpha_l)z_l\right) z_l}{f_l\left(-(1-\alpha_l)z_l\right)} + \mathbb{1}\{z_l>0\}\frac{f_l'\left(\alpha_l z_l\right) z_l}{f_l\left(\alpha_l z_l\right)}\right)\\
&\quad\cdot\prod_{m=1}^{d} f_{Z_m}\left(z_m;\boldsymbol{\eta}_m\right) d\mathbf{z}.
\end{aligned}
$$

First the case where $k = l$ is handled. Since the first two terms of the product only depend on the variable z_k, the integral over $\mathbb{R}^d$ can be split in d integrals over $\mathbb{R}$ (as was also done in the proof of Proposition 5). By the fact that f_{Z_m} is a density function, all integrals for which $m \neq k$ integrate to one and only the integral over z_k remains. We then get

$$
E\left[\left(\frac{\partial \ell(\boldsymbol{\theta};\mathbf{X})}{\partial \alpha_k}\right)\left(\frac{\partial \ell(\boldsymbol{\theta};\mathbf{X})}{\partial \alpha_k}\right)\right]
$$

$$= \left(\frac{1-2\alpha_k}{\alpha_k(1-\alpha_k)}\right)^2 + \int_{\mathbb{R}} \Bigg(\mathbb{1}\{z_k \le 0\}\, \frac{f_k'^2\left(-(1-\alpha_k)z_k\right) z_k^2}{f_k^2\left(-(1-\alpha_k)z_k\right)} + \mathbb{1}\{z_k > 0\}\, \frac{f_k'^2\left(\alpha_k z_k\right) z_k^2}{f_k^2\left(\alpha_k z_k\right)}$$
$$+\frac{2(1-2\alpha_k)}{\alpha_k(1-\alpha_k)}\left[\mathbb{1}\{z_k \le 0\}\, \frac{f_k'\left(-(1-\alpha_k)z_k\right) z_k}{f_k\left(-(1-\alpha_k)z_k\right)} + \mathbb{1}\{z_k > 0\}\, \frac{f_k'\left(\alpha_k z_k\right) z_k}{f_k\left(\alpha_k z_k\right)}\right]\Bigg)$$
$$\cdot f_{Z_k}\left(z_k; \boldsymbol{\eta}_k\right) dz_k$$

$$= \left(\frac{1-2\alpha_k}{\alpha_k(1-\alpha_k)}\right)^2 + 2\alpha_k(1-\alpha_k)\int_{-\infty}^{0} \frac{f_k'^2\left(-(1-\alpha_k)z_k\right) z_k^2}{f_k\left(-(1-\alpha_k)z_k\right)} dz_k$$
$$+2\alpha_k(1-\alpha_k)\int_{0}^{\infty} \frac{f_k'^2\left(\alpha_k z_k\right) z_k^2}{f_k\left(\alpha_k z_k\right)} dz_k + 4(1-2\alpha_k)\int_{-\infty}^{0} f_k'\left(-(1-\alpha_k)z_k\right) z_k dz_k$$
$$+4(1-2\alpha_k)\int_{0}^{\infty} f_k'\left(\alpha_k z_k\right) z_k dz_k$$

$$= \left(\frac{1-2\alpha_k}{\alpha_k(1-\alpha_k)}\right)^2 + \frac{2\alpha_k}{(1-\alpha_k)^2}\int_0^\infty \frac{f_k'^2(u)u^2}{f_k(u)}du + \frac{2(1-\alpha_k)}{\alpha_k^2}\int_0^\infty \frac{f_k'^2(v)v^2}{f_k(v)}dv$$
$$+\frac{4(1-2\alpha_k)}{\alpha_k(1-\alpha_k)}\left(-\frac{\alpha_k}{(1-\alpha_k)}\int_0^\infty f_k'(u)udu + \frac{(1-\alpha_k)}{\alpha_k}\int_0^\infty f_k'(v)vdv\right)$$

$$= \frac{(1-2\alpha_k)^2}{\alpha_k^2(1-\alpha_k)^2} + \frac{2\alpha_k^3\gamma_{k,3}}{\alpha_k^2(1-\alpha_k)^2} + \frac{2(1-\alpha_k)^3\gamma_{k,3}}{\alpha_k^2(1-\alpha_k)^2} - \frac{2(1-2\alpha_k)}{\alpha_k^2(1-\alpha_k)^2}$$

$$= \frac{2\left(\alpha_k^3 + (1-\alpha_k)^3\right)\gamma_{k,3} - (1-2\alpha_k)^2}{\alpha_k^2(1-\alpha_k)^2}.$$

The penultimate step is attained by using Assumptions (*N1*) and (*N2*). This results in the expression when $k = l$. The second case is when $k \neq l$. Since neither of both terms share a common z_j we get the product of the expected values, hence

$$E\left[\left(\frac{\partial \ell(\boldsymbol{\theta}; \mathbf{X})}{\partial \alpha_k}\right)\left(\frac{\partial \ell(\boldsymbol{\theta}; \mathbf{X})}{\partial \alpha_l}\right)\right]$$
$$= E\left[\left(\frac{\partial \ell(\boldsymbol{\theta}; \mathbf{X})}{\partial \alpha_k}\right)\right] E\left[\left(\frac{\partial \ell(\boldsymbol{\theta}; \mathbf{X})}{\partial \alpha_l}\right)\right]$$
$$= 0.$$

For the $\boldsymbol{\alpha}$ - $\boldsymbol{\mu}_a$ block we get

$$E\left[\left(\frac{\partial \ell(\boldsymbol{\theta}; \mathbf{X})}{\partial \alpha_k}\right)\left(\frac{\partial \ell(\boldsymbol{\theta}; \mathbf{X})}{\partial \mu_{a,l}}\right)\right]$$

$$= \int_{\mathbb{R}^d}\left(\frac{1-2\alpha_k}{\alpha_k(1-\alpha_k)} + \mathbb{1}\{z_k \le 0\}\, \frac{f_k'\left(-(1-\alpha_k)z_k\right) z_k}{f_k\left(-(1-\alpha_k)z_k\right)} + \mathbb{1}\{z_k > 0\}\, \frac{f_k'\left(\alpha_k z_k\right) z_k}{f_k\left(\alpha_k z_k\right)}\right)$$
$$\cdot \sum_{j=1}^{d}\Bigg(\mathbb{1}\{z_j \le 0\}\, \frac{f_j'\left(-(1-\alpha_j)z_j\right)(1-\alpha_j)(\mathbf{A}^{-1})_{l,j}}{f_j\left(-(1-\alpha_j)z_j\right)}$$

$$-\mathbb{1}\{z_j > 0\} \frac{f_j'(\alpha_j z_j)\, \alpha_j (\mathbf{A}^{-1})_{l,j}}{f_j(\alpha_j z_j)}\Bigg) \prod_{m=1}^{d} f_{Z_m}(z_m; \boldsymbol{\eta}_m)\, d\mathbf{z}.$$

First the term for which $j = k$ is isolated. Also $\frac{1-2\alpha_k}{\alpha_k(1-\alpha_k)} E\left[\frac{\partial \ell(\boldsymbol{\theta}; \mathbf{X})}{\partial \mu_{a,l}}\right]$ vanishes because of Proposition 5. Thus

$$
\begin{aligned}
&E\left[\left(\frac{\partial \ell(\boldsymbol{\theta}; \mathbf{X})}{\partial \alpha_k}\right)\left(\frac{\partial \ell(\boldsymbol{\theta}; \mathbf{X})}{\partial \mu_{a,l}}\right)\right] \\
&= \int_{\mathbb{R}^d} \left(\mathbb{1}\{z_k \le 0\} \frac{f_k'(-(1-\alpha_k) z_k)\, z_k}{f_k(-(1-\alpha_k) z_k)} + \mathbb{1}\{z_k > 0\} \frac{f_k'(\alpha_k z_k)\, z_k}{f_k(\alpha_k z_k)}\right) \\
&\quad \cdot \sum_{\substack{j=1 \\ j \ne k}}^{d} (\mathbf{A}^{-1})_{l,j} \left(\mathbb{1}\{z_j \le 0\} \frac{f_j'(-(1-\alpha_j) z_j)\,(1-\alpha_j)}{f_j(-(1-\alpha_j) z_j)} - \mathbb{1}\{z_j > 0\} \frac{f_j'(\alpha_j z_j)\, \alpha_j}{f_j(\alpha_j z_j)}\right) \\
&\quad \cdot \prod_{m=1}^{d} f_{Z_m}(z_m; \boldsymbol{\eta}_m)\, d\mathbf{z} \\
&\quad + \int_{\mathbb{R}^d} (\mathbf{A}^{-1})_{l,k} \left(\mathbb{1}\{z_k \le 0\} \frac{f_k'^2(-(1-\alpha_k) z_k)\,(1-\alpha_k) z_k}{f_k^2(-(1-\alpha_k) z_k)}\right. \\
&\quad \left. -\mathbb{1}\{z_k > 0\} \frac{f_k'^2(\alpha_k z_k)\, \alpha_k z_k}{f_k^2(\alpha_k z_k)}\right) \prod_{m=1}^{d} f_{Z_m}(z_m; \boldsymbol{\eta}_m)\, d\mathbf{z}.
\end{aligned}
$$

The first integral is zero due the calculations in the proof of Proposition 4. The second integral can be reduced to an integral only over z_k due to the others factors not depending on z_k and thus integrating to 1. This yields

$$
\begin{aligned}
&E\left[\left(\frac{\partial \ell(\boldsymbol{\theta}; \mathbf{X})}{\partial \alpha_k}\right)\left(\frac{\partial \ell(\boldsymbol{\theta}; \mathbf{X})}{\partial \mu_{a,l}}\right)\right] \\
&= 2(1-\alpha_k)\alpha_k (\mathbf{A}^{-1})_{l,k} \left(\int_{-\infty}^{0} \frac{f_k'^2(-(1-\alpha_k) z_k)\,(1-\alpha_k) z_k}{f_k(-(1-\alpha_k) z_k)} dz_k \right. \\
&\quad \left. - \int_{0}^{\infty} \frac{f_k'^2(\alpha_k z_k)\, \alpha_k z_k}{f_k(\alpha_k z_k)} dz_k \right) \\
&= -2\alpha_k (\mathbf{A}^{-1})_{l,k} \int_0^\infty \frac{f_k'^2(u)\, u}{f_k(u)} du - 2(1-\alpha_k)(\mathbf{A}^{-1})_{l,k} \int_0^\infty \frac{f_k'^2(v)\, v}{f_k(v)} dv \\
&= -2(\mathbf{A}^{-1})_{l,k} \gamma_{k,2}.
\end{aligned}
$$

Where the final equality is obtained by using Assumption (*N1*).

For the terms in $\boldsymbol{\alpha}$ - $\mathbf{A}$, the following terms need to be evaluated

$$E\left[\left(\frac{\partial \ell(\boldsymbol{\theta}; \mathbf{X})}{\partial \alpha_k}\right)\left(\frac{\partial \ell(\boldsymbol{\theta}; \mathbf{X})}{\partial \mathbf{A}_{l,m}}\right)\right]$$

$$= \int_{\mathbb{R}^d} \left(\frac{1-2\alpha_k}{\alpha_k(1-\alpha_k)} + \mathbb{1}\{z_k \le 0\} \frac{f_k'\left(-(1-\alpha_k)z_k\right) z_k}{f_k\left(-(1-\alpha_k)z_k\right)} + \mathbb{1}\{z_k > 0\} \frac{f_k'\left(\alpha_k z_k\right) z_k}{f_k\left(\alpha_k z_k\right)} \right)$$
$$\cdot \Bigg\{ \Bigg(\sum_{\substack{j=1 \\ j\neq l}}^{d} \left[\mathbb{1}\{z_j > 0\} \frac{f_j'\left(\alpha_j z_j\right)\alpha_j}{f_j\left(\alpha_j z_j\right)} - \mathbb{1}\{z_j \le 0\} \frac{f_j'\left(-(1-\alpha_j)z_j\right)(1-\alpha_j)}{f_j\left(-(1-\alpha_j)z_j\right)} \right]$$
$$\cdot \sum_{\substack{h=1 \\ h\neq j}}^{d} B_{h,j}^{l,m} z_h \Bigg) + \left(\mathbb{1}\{z_l > 0\} \frac{f_l'\left(\alpha_l z_l\right)\alpha_l}{f_l\left(\alpha_l z_l\right)} - \mathbb{1}\{z_l \le 0\} \frac{f_l'\left(-(1-\alpha_l)z_l\right)(1-\alpha_l)}{f_l\left(-(1-\alpha_l)z_l\right)} \right)$$
$$\cdot D_{l,m} z_l + D_{l,m} \Bigg\} \prod_{q=1}^{d} f_{Z_q}\left(z_q; \boldsymbol{\eta}_q\right) d\mathbf{z}.$$

A useful equality for this block is the following

$$\int_{\mathbb{R}} \left(\mathbb{1}\{z_j > 0\} \frac{f_j'\left(\alpha_j z_j\right)\alpha_j}{f_j\left(\alpha_j z_j\right)} - \mathbb{1}\{z_j \le 0\} \frac{f_j'\left(-(1-\alpha_j)z_j\right)(1-\alpha_j)}{f_j\left(-(1-\alpha_j)z_j\right)} \right) dz_j \tag{S.2}$$
$$= 2\alpha_j(1-\alpha_j) \int_{\mathbb{R}} \mathbb{1}\{z_j > 0\}\, \alpha_j f_j'\left(\alpha_j z_j\right) - \mathbb{1}\{z_j \le 0\}\,(1-\alpha_j) f_j'(-(1-\alpha_j)z_j)\, dz_j$$
$$= 2\alpha_j(1-\alpha_j) \left(\int_0^\infty f_j'(v)dv - \int_0^\infty f_j'(u)du \right)$$
$$= 0.$$

First the case $k = l$ is handled. One finds that by (S.2), all products with $j \neq k$ integrate to 0. Using that $D_{k,m} E\left[\frac{\partial \ell(\boldsymbol{\theta};\mathbf{X})}{\partial \alpha_k}\right] = 0$,

$$E\left[\left(\frac{\partial \ell(\boldsymbol{\theta};\mathbf{X})}{\partial \alpha_k} \right) \left(\frac{\partial \ell(\boldsymbol{\theta};\mathbf{X})}{\partial \mathbf{A}_{k,m}} \right) \right]$$
$$= \int_{\mathbb{R}^d} \left(\frac{1-2\alpha_k}{\alpha_k(1-\alpha_k)} + \mathbb{1}\{z_k \le 0\} \frac{f_k'\left(-(1-\alpha_k)z_k\right) z_k}{f_k\left(-(1-\alpha_k)z_k\right)} + \mathbb{1}\{z_k > 0\} \frac{f_k'\left(\alpha_k z_k\right) z_k}{f_k\left(\alpha_k z_k\right)} \right)$$
$$\cdot \Bigg(\mathbb{1}\{z_k > 0\} \frac{f_k'\left(\alpha_k z_k\right)\alpha_k D_{k,m} z_k}{f_k\left(\alpha_k z_k\right)} - \mathbb{1}\{z_k \le 0\} \frac{f_k'\left(-(1-\alpha_k)z_k\right)(1-\alpha_k) D_{k,m} z_k}{f_k\left(-(1-\alpha_k)z_k\right)}$$
$$+ D_{k,m} \Bigg) \prod_{q=1}^{d} f_{Z_q}\left(z_q; \boldsymbol{\eta}_m\right) d\mathbf{z}$$
$$= -\frac{(1-2\alpha_k)D_{k,m}}{\alpha_k(1-\alpha_k)} + 2\alpha_k(1-\alpha_k)D_{k,m}$$
$$\cdot \int_{\mathbb{R}} \mathbb{1}\{z_k > 0\} \frac{f_k'^2\left(\alpha_k z_k\right)\alpha_k z_k^2}{f_k\left(\alpha_k z_k\right)} - \mathbb{1}\{z_k \le 0\} \frac{f_k'^2\left(-(1-\alpha_k)z_k\right)(1-\alpha_k) z_k^2}{f_k\left(-(1-\alpha_k)z_k\right)} dz_k$$
$$= -\frac{(1-2\alpha_k)D_{k,m}}{\alpha_k(1-\alpha_k)} + \frac{2(1-\alpha_k)D_{k,m}}{\alpha_k} \int_0^\infty \frac{f_k'^2(v)\, v^2}{f_k(v)} dv - \frac{2\alpha_k D_{k,m}}{1-\alpha_k} \int_0^\infty \frac{f_k'^2(u)\, u^2}{f_k(u)} du$$
$$= \frac{(1-2\alpha_k)(2\gamma_{k,3}-1)D_{k,m}}{\alpha_k(1-\alpha_k)}.$$

Next the case where $k \neq l$ is considered. By isolating the k-th term in the summation over j we get

$$E\left[\left(\frac{\partial \ell(\boldsymbol{\theta};\mathbf{X})}{\partial \alpha_k}\right)\left(\frac{\partial \ell(\boldsymbol{\theta};\mathbf{X})}{\partial \mathbf{A}_{l,m}}\right)\right]$$

$$= \int_{\mathbb{R}^d}\left(\frac{1-2\alpha_k}{\alpha_k(1-\alpha_k)} + \mathbb{1}\{z_k \leq 0\}\frac{f_k'\left(-(1-\alpha_k)z_k\right)z_k}{f_k\left(-(1-\alpha_k)z_k\right)} + \mathbb{1}\{z_k > 0\}\frac{f_k'\left(\alpha_k z_k\right)z_k}{f_k\left(\alpha_k z_k\right)}\right)$$
$$\left(\left[\mathbb{1}\{z_l > 0\}\frac{f_l'\left(\alpha_l z_l\right)\alpha_l D_{l,m} z_l}{f_l\left(\alpha_l z_l\right)} - \mathbb{1}\{z_l \leq 0\}\frac{f_l'\left(-(1-\alpha_l)z_l\right)(1-\alpha_l)D_{l,m}z_l}{f_k\left(-(1-\alpha_l)z_l\right)}\right.\right.$$
$$\left. + D_{l,m}\right] + \sum_{\substack{j=1\\ j\neq k,l}}^{d}\left[\mathbb{1}\{z_j > 0\}\frac{f_j'\left(\alpha_j z_j\right)\alpha_j}{f_j\left(\alpha_j z_j\right)} - \mathbb{1}\{z_j \leq 0\}\frac{f_j'\left(-(1-\alpha_j)z_j\right)(1-\alpha_j)}{f_j\left(-(1-\alpha_j)z_j\right)}\right]$$
$$\cdot \sum_{\substack{h=1\\ h\neq j}}^{d} B_{h,j}^{l,m} z_h + \left[\mathbb{1}\{z_k > 0\}\frac{f_k'\left(\alpha_k z_k\right)\alpha_k}{f_k\left(\alpha_k z_k\right)} - \mathbb{1}\{z_k \leq 0\}\frac{f_k'\left(-(1-\alpha_k)z_k\right)(1-\alpha_k)}{f_k\left(-(1-\alpha_k)z_k\right)}\right]$$
$$\left.\cdot \sum_{\substack{h=1\\ h\neq k}}^{d} B_{h,k}^{l,m} z_h\right)\prod_{q=1}^{d} f_{Z_q}\left(z_q;\boldsymbol{\eta}_q\right)d\mathbf{z}$$

$$= 2\alpha_k(1-\alpha_k)\sum_{\substack{h=1\\ h\neq k}}^{d} B_{h,k}^{l,m}\kappa_{h,1}\int_0^{\infty}\frac{f_k'^2\left(\alpha_k z_k\right)\alpha_k z_k}{f_k\left(\alpha_k z_k\right)}dz_k$$
$$-2\alpha_k(1-\alpha_k)\sum_{\substack{h=1\\ h\neq k}}^{d} B_{h,k}^{l,m}\kappa_{h,1}\int_{-\infty}^{0}\frac{f_k'^2\left(-(1-\alpha_k)z_k\right)(1-\alpha_k)z_k}{f_k\left(-(1-\alpha_k)z_k\right)}dz_k.$$

The last equality holds because terms involving $D_{l,m}$ integrate to zero as they do not depend on z_k and by Proposition 4. By (S.2), one can find that the terms involving the sum over j also integrate to 0. So by using (17), (18) and Assumption (*N1*), when $k \neq l$

$$E\left[\left(\frac{\partial \ell(\boldsymbol{\theta};\mathbf{X})}{\partial \alpha_k}\right)\left(\frac{\partial \ell(\boldsymbol{\theta};\mathbf{X})}{\partial \mathbf{A}_{l,m}}\right)\right]$$

$$= 2\alpha_k\sum_{\substack{h=1\\ h\neq k}}^{d} B_{h,k}^{l,m}\kappa_{h,1}\int_0^{\infty}\frac{f_k'^2\left(u\right)u}{f_k\left(u\right)}du + 2(1-\alpha_k)\sum_{\substack{h=1\\ h\neq k}}^{d} B_{h,k}^{l,m}\kappa_{h,1}\int_0^{\infty}\frac{f_k'^2\left(v\right)v}{f_k\left(v\right)}dv$$

$$= 2\gamma_{k,2}\sum_{\substack{h=1\\ h\neq k}}^{d} B_{h,k}^{l,m}\kappa_{h,1}.$$

For the $\boldsymbol{\mu}_a$ - $\boldsymbol{\mu}_a$ block, we get the following integral to solve

$$E\left[\left(\frac{\partial \ell(\boldsymbol{\theta};\mathbf{X})}{\partial \mu_{a,k}}\right)\left(\frac{\partial \ell(\boldsymbol{\theta};\mathbf{X})}{\partial \mu_{a,l}}\right)\right]$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^d} \left(\sum_{j=1}^{d} (\mathbf{A}^{-1})_{k,j} \left[\mathbb{1}\{z_j \leq 0\} \frac{f'_j\left(-(1-\alpha_j)z_j\right)(1-\alpha_j)}{f_j\left(-(1-\alpha_j)z_j\right)} - \mathbb{1}\{z_j > 0\} \frac{f'_j\left(\alpha_j z_j\right)\alpha_j}{f_j\left(\alpha_j z_j\right)} \right] \right) \\
&\quad \cdot \left(\sum_{q=1}^{d} (\mathbf{A}^{-1})_{l,q} \left[\mathbb{1}\{z_q \leq 0\} \frac{f'_q\left(-(1-\alpha_q)z_q\right)(1-\alpha_q)}{f_q\left(-(1-\alpha_q)z_q\right)} - \mathbb{1}\{z_q > 0\} \frac{f'_q\left(\alpha_q z_q\right)\alpha_q}{f_q\left(\alpha_q z_q\right)} \right] \right) \\
&\quad \cdot \prod_{m=1}^{d} f_{Z_m}\left(z_m; \boldsymbol{\eta}_m\right) d\mathbf{z}
\end{aligned}
$$

$$
\begin{aligned}
&E\left[\left(\frac{\partial \ell(\boldsymbol{\theta}; \mathbf{X})}{\partial \mu_{a,k}} \right) \left(\frac{\partial \ell(\boldsymbol{\theta}; \mathbf{X})}{\partial \mu_{a,l}} \right) \right] \\
&= \sum_{j=1}^{d} \int_{\mathbb{R}^d} \left(\mathbb{1}\{z_j \leq 0\} \frac{f'^2_j\left(-(1-\alpha_j)z_j\right)(1-\alpha_j)^2}{f^2_j\left(-(1-\alpha_j)z_j\right)} + \mathbb{1}\{z_j > 0\} \frac{f'^2_j\left(\alpha_j z_j\right)\alpha_j^2}{f^2_j\left(\alpha_j z_j\right)} \right) \\
&\quad \cdot (\mathbf{A}^{-1})_{k,j} (\mathbf{A}^{-1})_{l,j} \prod_{m=1}^{d} f_{Z_m}\left(z_m; \boldsymbol{\eta}_m\right) d\mathbf{z} \\
&\quad + \sum_{j=1}^{d} \sum_{\substack{q=1 \\ q\neq j}}^{d} \int_{\mathbb{R}^d} \left(\mathbb{1}\{z_j \leq 0\} \frac{f'_j\left(-(1-\alpha_j)z_j\right)(1-\alpha_j)}{f_j\left(-(1-\alpha_j)z_j\right)} - \mathbb{1}\{z_j > 0\} \frac{f'_j\left(\alpha_j z_j\right)\alpha_j}{f_j\left(\alpha_j z_j\right)} \right) \\
&\quad \cdot \left(\mathbb{1}\{z_q \leq 0\} \frac{f'_q\left(-(1-\alpha_q)z_q\right)(1-\alpha_q)}{f_q\left(-(1-\alpha_q)z_q\right)} - \mathbb{1}\{z_q > 0\} \frac{f'_q\left(\alpha_q z_q\right)\alpha_q}{f_q\left(\alpha_q z_q\right)} \right) \\
&\quad \cdot (\mathbf{A}^{-1})_{k,j} (\mathbf{A}^{-1})_{l,q} \prod_{m=1}^{d} f_{Z_m}\left(z_m; \boldsymbol{\eta}_m\right) d\mathbf{z}. \\
&= \sum_{j=1}^{d} 2\alpha_j(1-\alpha_j)(\mathbf{A}^{-1})_{k,j}(\mathbf{A}^{-1})_{l,j} \left(\int_{-\infty}^{0} \frac{f'^2_j\left(-(1-\alpha_j)z_j\right)(1-\alpha_j)^2}{f_j\left(-(1-\alpha_j)z_j\right)} dz_j \right. \\
&\quad \left. + \int_{0}^{\infty} \frac{f'^2_j\left(\alpha_j z_j\right)\alpha_j^2}{f_j\left(\alpha_j z_j\right)} dz_j \right) \\
&= \sum_{j=1}^{d} 2\alpha_j(1-\alpha_j)(\mathbf{A}^{-1})_{k,j}(\mathbf{A}^{-1})_{l,j} \left((1-\alpha_j) \int_{0}^{\infty} \frac{f'^2_j(u)}{f_j(u)} du + \alpha_j \int_{0}^{\infty} \frac{f'^2_j(v)}{f_j(v)} dv \right) \\
&= \sum_{j=1}^{d} 2\alpha_j(1-\alpha_j)(\mathbf{A}^{-1})_{k,j}(\mathbf{A}^{-1})_{l,j}\gamma_{j,1}.
\end{aligned}
$$

For the $\boldsymbol{\mu}_a$ - $\mathbf{A}$ block we have

$$
\begin{aligned}
&E\left[\left(\frac{\partial \ell(\boldsymbol{\theta}; \mathbf{X})}{\partial \mu_{a,k}} \right) \left(\frac{\partial \ell(\boldsymbol{\theta}; \mathbf{X})}{\partial \mathbf{A}_{l,m}} \right) \right] \\
&= \int_{\mathbb{R}^d} \left(\sum_{j=1}^{d} (\mathbf{A}^{-1})_{k,j} \left[\mathbb{1}\{z_j \leq 0\} \frac{f'_j\left(-(1-\alpha_j)z_j\right)(1-\alpha_j)}{f_j\left(-(1-\alpha_j)z_j\right)} - \mathbb{1}\{z_j > 0\} \frac{f'_j\left(\alpha_j z_j\right)\alpha_j}{f_j\left(\alpha_j z_j\right)} \right] \right) \\
&\quad \cdot \left\{ \left(\sum_{\substack{q=1 \\ q\neq l}}^{d} \left[\mathbb{1}\{z_q > 0\} \frac{f'_q\left(\alpha_q z_q\right)\alpha_q}{f_q\left(\alpha_q z_q\right)} - \mathbb{1}\{z_q \leq 0\} \frac{f'_q\left(-(1-\alpha_q)z_q\right)(1-\alpha_q)}{f_q\left(-(1-\alpha_q)z_q\right)} \right] \right. \right.
\end{aligned}
$$

$$\cdot \sum_{\substack{h=1\\h\neq q}}^{d} B_{h,q}^{l,m} z_h \Bigg) + \left(\mathbb{1}\{z_l > 0\} \frac{f_l'(\alpha_l z_l)\,\alpha_l}{f_l(\alpha_l z_l)} - \mathbb{1}\{z_l \leq 0\} \frac{f_l'(-(1-\alpha_l)z_l)\,(1-\alpha_l)}{f_l(-(1-\alpha_l)z_l)} \right)$$

$$\cdot D_{l,m} z_l + D_{l,m} \Bigg\} \prod_{r=1}^{d} f_{Z_r}(z_r; \boldsymbol{\eta}_r)\, d\mathbf{z}.$$

We tackle this by splitting the summation over j in its individual components. First the situation for which $j = l$ is looked at. In this, the integral is split into two terms, one involving z_l and a term not involving z_l.

$$\int_{\mathbb{R}^d} (\mathbf{A}^{-1})_{k,l} \left(\mathbb{1}\{z_l \leq 0\} \frac{f_l'(-(1-\alpha_l)z_l)\,(1-\alpha_l)}{f_l(-(1-\alpha_l)z_l)} - \mathbb{1}\{z_l > 0\} \frac{f_l'(\alpha_l z_l)\,\alpha_l}{f_l(\alpha_l z_l)} \right)$$

$$\cdot \Bigg\{ \Bigg(\sum_{\substack{q=1\\q\neq l}}^{d} \left[\mathbb{1}\{z_q > 0\} \frac{f_q'(\alpha_q z_q)\,\alpha_q}{f_q(\alpha_q z_q)} - \mathbb{1}\{z_q \leq 0\} \frac{f_q'(-(1-\alpha_q)z_q)\,(1-\alpha_q)}{f_q(-(1-\alpha_q)z_q)} \right]$$

$$\cdot \sum_{\substack{h=1\\h\neq q}}^{d} B_{h,q}^{l,m} z_h \Bigg) + \left(\mathbb{1}\{z_l > 0\} \frac{f_l'(\alpha_l z_l)\,\alpha_l}{f_l(\alpha_l z_l)} - \mathbb{1}\{z_l \leq 0\} \frac{f_l'(-(1-\alpha_l)z_l)\,(1-\alpha_l)}{f_l(-(1-\alpha_l)z_l)} \right)$$

$$\cdot D_{l,m} z_l + D_{l,m} \Bigg\} \prod_{r=1}^{d} f_{Z_r}(z_r; \boldsymbol{\eta}_r)\, d\mathbf{z}$$

$$= \int_{\mathbb{R}^d} (\mathbf{A}^{-1})_{k,l} \left(\mathbb{1}\{z_l \leq 0\} \frac{f_l'(-(1-\alpha_l)z_l)\,(1-\alpha_l)}{f_l(-(1-\alpha_l)z_l)} - \mathbb{1}\{z_l > 0\} \frac{f_l'(\alpha_l z_l)\,\alpha_l}{f_l(\alpha_l z_l)} \right)$$

$$\cdot \Bigg(D_{l,m} z_l \left[\mathbb{1}\{z_l > 0\} \frac{f_l'(\alpha_l z_l)\,\alpha_l}{f_l(\alpha_l z_l)} - \mathbb{1}\{z_l \leq 0\} \frac{f_l'(-(1-\alpha_l)z_l)\,(1-\alpha_l)}{f_l(-(1-\alpha_l)z_l)} \right]$$

$$+ D_{l,m} \Bigg) \prod_{r=1}^{d} f_{Z_r}(z_r; \boldsymbol{\eta}_r)\, d\mathbf{z}$$

$$+ \sum_{\substack{q=1\\q\neq l}}^{d} \int_{\mathbb{R}^{d-1}} \left(\mathbb{1}\{z_q > 0\} \frac{f_q'(\alpha_q z_q)\,\alpha_q}{f_q(\alpha_q z_q)} - \mathbb{1}\{z_q \leq 0\} \frac{f_q'(-(1-\alpha_q)z_q)\,(1-\alpha_q)}{f_q(-(1-\alpha_q)z_q)} \right)$$

$$\cdot \Bigg(\sum_{\substack{h=1\\h\neq q,l}}^{d} B_{h,q}^{l,m} z_h \int_{\mathbb{R}} \left[\mathbb{1}\{z_l \leq 0\} \frac{f_l'(-(1-\alpha_l)z_l)\,(1-\alpha_l)}{f_l(-(1-\alpha_l)z_l)} - \mathbb{1}\{z_l > 0\} \frac{f_l'(\alpha_l z_l)\,\alpha_l}{f_l(\alpha_l z_l)} \right]$$

$$\cdot (\mathbf{A}^{-1})_{k,l} f_{Z_l}(z_l; \boldsymbol{\eta}_l) dz_l + B_{l,q}^{l,m} \int_{\mathbb{R}} f_{Z_l}(z_l; \boldsymbol{\eta}_l) \Bigg[\mathbb{1}\{z_l \leq 0\} \frac{f_l'(-(1-\alpha_l)z_l)\,(1-\alpha_l) z_l}{f_l(-(1-\alpha_l)z_l)}$$

$$- \mathbb{1}\{z_l > 0\} \frac{f_l'(\alpha_l z_l)\,\alpha_l z_l}{f_l(\alpha_l z_l)} \Bigg] (\mathbf{A}^{-1})_{k,l} dz_l \Bigg) \prod_{\substack{r=1\\r\neq l}}^{d} f_{Z_r}(z_r; \boldsymbol{\eta}_r)\, d\mathbf{z}_{-l}$$

$$= -2\alpha_l (1-\alpha_l) (\mathbf{A}^{-1})_{k,l} D_{l,m}$$

$$\cdot \left(\int_{-\infty}^{0} \frac{f_l'^2(-(1-\alpha_l)z_l)\,(1-\alpha_l)^2 z_l}{f_l(-(1-\alpha_l)z_l)} dz_l + \int_{0}^{\infty} \frac{f_l'^2(\alpha_l z_l)\,\alpha_l^2 z_l}{f_l(\alpha_l z_l)} dz_l \right)$$

$$
\begin{aligned}
&+\sum_{\substack{q=1\\q\neq l}}^{d} 2\alpha_q(1-\alpha_q)B^{l,m}_{l,q}\left(\int_{-\infty}^{0} f'_q\left(-(1-\alpha_q)z_q\right)(1-\alpha_q)dz_q+\int_0^{\infty} f'_q\left(\alpha_q z_q\right)\alpha_q dz_q\right)\\
&\quad\cdot 2\alpha_l(1-\alpha_l)(\mathbf{A}^{-1})_{k,l}\left(\int_{-\infty}^{0} f'_l\left(-(1-\alpha_l)z_l\right)(1-\alpha_l)z_l dz_l-\int_0^{\infty} f'_l\left(\alpha_l z_l\right)\alpha_l z_l dz_l\right)\\
&=2\alpha_l(1-\alpha_l)(\mathbf{A}^{-1})_{k,l}D_{l,m}\left(\int_0^{\infty}\frac{f'^2_l(u)\,u}{f_l(u)}du-\int_0^{\infty}\frac{f'^2_l(v)\,v}{f_l(v)}dv\right)\\
&-\sum_{\substack{q=1\\q\neq l}}^{d} 4\alpha_q(1-\alpha_q)(\mathbf{A}^{-1})_{k,l}B^{l,m}_{l,q}\left(-\int_0^{\infty} f'_q(u)\,du+\int_0^{\infty} f'_q(v)\,dv\right)\\
&\quad\cdot\left(\alpha_l\int_0^{\infty} f'_l(u)\,udu+(1-\alpha_l)\int_0^{\infty} f'_l(v)\,vdv\right)\\
&=0.
\end{aligned}
$$

The first step in this holds because by the calculations in the proof of Proposition 5, each term in the summation over h integrates to 0. In terms for which $j\neq l$, the integral is separated in components involving z_j and components not involving z_j. Due to calculations as in the proof of Proposition 5, terms involving $D_{l,m}$ integrate to zero and are hence omitted.

$$
\begin{aligned}
&\int_{\mathbb{R}^d}(\mathbf{A}^{-1})_{k,j}\left(\mathbb{1}\{z_j\leq 0\}\frac{f'_j\left(-(1-\alpha_j)z_j\right)(1-\alpha_j)}{f_j\left(-(1-\alpha_j)z_j\right)}-\mathbb{1}\{z_j>0\}\frac{f'_j\left(\alpha_j z_j\right)\alpha_j}{f_j\left(\alpha_j z_j\right)}\right)\\
&\quad\cdot\left(\sum_{\substack{q=1\\q\neq l,j}}^{d}\left[\mathbb{1}\{z_q>0\}\frac{f'_q\left(\alpha_q z_q\right)\alpha_q}{f_q\left(\alpha_q z_q\right)}-\mathbb{1}\{z_q\leq 0\}\frac{f'_q\left(-(1-\alpha_q)z_q\right)(1-\alpha_q)}{f_q\left(-(1-\alpha_q)z_q\right)}\right]\right.\\
&\quad\cdot\left[\sum_{\substack{h=1\\h\neq q,j}}^{d}B^{l,m}_{h,q}z_h+B^{l,m}_{j,q}z_j\right]+\left[\mathbb{1}\{z_j>0\}\frac{f'_j\left(\alpha_j z_j\right)\alpha_j}{f_j\left(\alpha_j z_j\right)}\right.\\
&\quad\left.\left.-\mathbb{1}\{z_j\leq 0\}\frac{f'_j\left(-(1-\alpha_j)z_j\right)(1-\alpha_j)}{f_j\left(-(1-\alpha_j)z_j\right)}\right]\sum_{\substack{h=1\\h\neq j}}^{d}B^{l,m}_{h,j}z_h\right)\prod_{r=1}^{d}f_{Z_r}\left(z_r;\boldsymbol{\eta}_r\right)d\mathbf{z}\\
&=\sum_{\substack{q=1\\q\neq l,j}}^{d}2\alpha_j(1-\alpha_j)(\mathbf{A}^{-1})_{k,j}\left(\left[\int_{-\infty}^{0}f'_j(-(1-\alpha_j)z_j)(1-\alpha_j)dz_j-\int_0^{\infty}f'_j(\alpha_j z_j)\alpha_j dz_j\right]\right.\\
&\quad\cdot 2\alpha_q(1-\alpha_q)\left[\int_0^{\infty}f'_q(\alpha_q z_q)\alpha_q dz_q-\int_{-\infty}^{0}f'_q(-(1-\alpha_q)z_q)(1-\alpha_q)dz_q\right]\\
&\quad\cdot\sum_{\substack{h=1\\h\neq q,j}}^{d}B^{l,m}_{h,q}\kappa_{h,1}+\left[\int_{-\infty}^{0}f'_j(-(1-\alpha_j)z_j)(1-\alpha_j)z_j dz_j-\int_0^{\infty}f'_j(\alpha_j z_j)\alpha_j z_j dz_j\right]B^{l,m}_{j,q}\\
&\quad\left.\cdot 2\alpha_q(1-\alpha_q)\left[-\int_{-\infty}^{0}f'_q(-(1-\alpha_q)z_q)(1-\alpha_q)dz_q+\int_0^{\infty}f'_q(\alpha_q z_q)\alpha_q dz_q\right]\right)\\
&-2\alpha_j(1-\alpha_j)(\mathbf{A}^{-1})_{k,j}\left(\int_0^{\infty}\frac{f'^2_j\left(\alpha_j z_j\right)\alpha_j^2}{f_j\left(\alpha_j z_j\right)}dz_j+\int_{-\infty}^{0}\frac{f'^2_j\left(-(1-\alpha_j)z_j\right)(1-\alpha_j)^2}{f_j\left(-(1-\alpha_j)z_j\right)}dz_j\right)
\end{aligned}
$$

$$\cdot \sum_{\substack{h=1 \\ h \neq j}}^{d} B_{h,j}^{l,m} \kappa_{h,1}$$

$$= \sum_{\substack{q=1 \\ q \neq l,j}}^{d} \Bigg(2\alpha_j(1-\alpha_j)(\mathbf{A}^{-1})_{k,j} \left[\int_0^{\infty} f_j'(u)du - \int_0^{\infty} f_j'(v)dv \right] \sum_{\substack{h=1 \\ h \neq q,j}}^{d} B_{h,q}^{l,m} \kappa_{h,1}$$

$$-2(\mathbf{A}^{-1})_{k,j} B_{j,q}^{l,m} \left[\alpha_j \int_0^{\infty} f_j'(u)udu + (1-\alpha_j) \int_0^{\infty} f_j'(v)vdv \right] \Bigg)$$

$$\cdot 2\alpha_q(1-\alpha_q) \left(\int_0^{\infty} f_q'(v)dv - \int_0^{\infty} f_q'(u)du \right)$$

$$-2\alpha_j(1-\alpha_j)^2(\mathbf{A}^{-1})_{k,j} \sum_{\substack{h=1 \\ h \neq j}}^{d} B_{h,j}^{l,m} \kappa_{h,1} \left((1-\alpha_j) \int_0^{\infty} \frac{f_j'^2(u)}{f_j(u)} du + \alpha_j \int_0^{\infty} \frac{f_j'^2(v)}{f_j(v)} dv \right)$$

$$= -2\alpha_j(1-\alpha_j)(\mathbf{A}^{-1})_{k,j} \gamma_{j,1} \sum_{\substack{h=1 \\ h \neq j}}^{d} B_{h,j}^{l,m} \kappa_{h,1}.$$

Thus by putting together all terms in the original summation over j we get as a final expression

$$E\left[\left(\frac{\partial \ell(\boldsymbol{\theta}; \mathbf{X})}{\partial \mu_{a,k}} \right) \left(\frac{\partial \ell(\boldsymbol{\theta}; \mathbf{X})}{\partial \mathbf{A}_{l,m}} \right) \right] = -2 \sum_{\substack{j=1 \\ j \neq l}}^{d} \alpha_j(1-\alpha_j)(\mathbf{A}^{-1})_{k,j} \gamma_{j,1} \sum_{\substack{h=1 \\ h \neq j}}^{d} B_{h,j}^{l,m} \kappa_{h,1}.$$

At last, we have arrived at the final block, which is the **A** - **A** block. By conveniently splitting the summations, this can be rewritten as

$$E\left[\left(\frac{\partial \ell(\boldsymbol{\theta}; \mathbf{X})}{\partial \mathbf{A}_{k,l}} \right) \left(\frac{\partial \ell(\boldsymbol{\theta}; \mathbf{X})}{\partial \mathbf{A}_{r,s}} \right) \right]$$

$$= \int_{\mathbb{R}^d} \Bigg\{ \left(\mathbb{1}\{z_k > 0\} \frac{f_k'(\alpha_k z_l)\, \alpha_k z_k}{f_k(\alpha_k z_k)} - \mathbb{1}\{z_k \leq 0\} \frac{f_k'(-(1-\alpha_k)z_k)\,(1-\alpha_k)z_k}{f_k(-(1-\alpha_k)z_k)} \right) D_{k,l}$$

$$+ \Bigg(\sum_{\substack{q=1 \\ q \neq k}}^{d} \left[\mathbb{1}\{z_q > 0\} \frac{f_q'(\alpha_q z_q)\, \alpha_q}{f_q(\alpha_q z_q)} - \mathbb{1}\{z_q \leq 0\} \frac{f_q'(-(1-\alpha_q)z_q)\,(1-\alpha_q)}{f_j(-(1-\alpha_q)z_q)} \right]$$

$$\cdot \sum_{\substack{m=1 \\ m \neq q}}^{d} B_{m,q}^{k,l} z_m \Bigg) + D_{k,l} \Bigg\}$$

$$\cdot \Bigg\{ \left(\mathbb{1}\{z_r > 0\} \frac{f_r'(\alpha_r z_r)\, \alpha_r z_r}{f_r(\alpha_r z_r)} - \mathbb{1}\{z_r \leq 0\} \frac{f_r'(-(1-\alpha_r)z_r)\,(1-\alpha_r)z_r}{f_r(-(1-\alpha_r)z_r)} \right) D_{r,s}$$

$$+ \Bigg(\sum_{\substack{j=1 \\ j \neq r}}^{d} \left[\mathbb{1}\{z_j > 0\} \frac{f_j'(\alpha_j z_j)\, \alpha_j}{f_j(\alpha_j z_j)} - \mathbb{1}\{z_j \leq 0\} \frac{f_j'(-(1-\alpha_j)z_j)\,(1-\alpha_j)}{f_j(-(1-\alpha_j)z_j)} \right]$$

$$\cdot \sum_{\substack{h=1\\h\neq j}}^{d} B_{h,j}^{r,s} z_h\Bigg) + D_{r,s}\Bigg\} \prod_{p=1}^{d} f_{Z_p}\left(z_p;\boldsymbol{\eta}_p\right) d\mathbf{z}$$

$$= \int_{\mathbb{R}^d} \sum_{\substack{q=1\\q\neq k}}^{d} \sum_{\substack{j=1\\j\neq r}}^{d} \left(\mathbb{1}\{z_q > 0\} \frac{f_q'\left(\alpha_q z_q\right)\alpha_q}{f_q\left(\alpha_q z_q\right)} - \mathbb{1}\{z_q \leq 0\} \frac{f_q'\left(-(1-\alpha_q)z_q\right)(1-\alpha_q)}{f_j\left(-(1-\alpha_q)z_q\right)} \right)$$
$$\cdot \left(\mathbb{1}\{z_j > 0\} \frac{f_j'\left(\alpha_j z_j\right)\alpha_j}{f_j\left(\alpha_j z_j\right)} - \mathbb{1}\{z_j \leq 0\} \frac{f_j'\left(-(1-\alpha_j)z_j\right)(1-\alpha_j)}{f_j\left(-(1-\alpha_j)z_j\right)} \right) \tag{S.3}$$
$$\cdot \left(\sum_{\substack{m=1\\m\neq q}}^{d} B_{m,q}^{k,l} z_m \right) \left(\sum_{\substack{h=1\\h\neq j}}^{d} B_{h,j}^{r,s} z_h \right) \prod_{p=1}^{d} f_{Z_p}\left(z_p;\boldsymbol{\eta}_p\right) d\mathbf{z}$$

$$+ \int_{\mathbb{R}^d} \sum_{\substack{q=1\\q\neq k}}^{d} \left(\mathbb{1}\{z_q > 0\} \frac{f_q'\left(\alpha_q z_q\right)\alpha_q}{f_q\left(\alpha_q z_q\right)} - \mathbb{1}\{z_q \leq 0\} \frac{f_q'\left(-(1-\alpha_q)z_q\right)(1-\alpha_q)}{f_j\left(-(1-\alpha_q)z_q\right)} \right)$$
$$\cdot \left(\mathbb{1}\{z_r > 0\} \frac{f_r'\left(\alpha_r z_r\right)\alpha_r z_r}{f_r\left(\alpha_r z_r\right)} - \mathbb{1}\{z_r \leq 0\} \frac{f_r'\left(-(1-\alpha_r)z_r\right)(1-\alpha_r)z_r}{f_r\left(-(1-\alpha_r)z_r\right)} \right) \tag{S.4}$$
$$\cdot D_{r,s} \left(\sum_{\substack{m=1\\m\neq q}}^{d} B_{m,q}^{k,l} z_m \right) \prod_{p=1}^{d} f_{Z_p}\left(z_p;\boldsymbol{\eta}_p\right) d\mathbf{z}$$

$$+ \int_{\mathbb{R}^d} \sum_{\substack{j=1\\j\neq r}}^{d} \left(\mathbb{1}\{z_j > 0\} \frac{f_j'\left(\alpha_j z_j\right)\alpha_j}{f_j\left(\alpha_j z_j\right)} - \mathbb{1}\{z_j \leq 0\} \frac{f_j'\left(-(1-\alpha_j)z_j\right)(1-\alpha_j)}{f_j\left(-(1-\alpha_j)z_j\right)} \right)$$
$$\cdot \left(\mathbb{1}\{z_k > 0\} \frac{f_k'\left(\alpha_k z_k\right)\alpha_k z_k}{f_k\left(\alpha_k z_k\right)} - \mathbb{1}\{z_k \leq 0\} \frac{f_k'\left(-(1-\alpha_k)z_k\right)(1-\alpha_k)z_k}{f_k\left(-(1-\alpha_k)z_k\right)} \right) \tag{S.5}$$
$$\cdot D_{k,l} \left(\sum_{\substack{h=1\\h\neq j}}^{d} B_{h,j}^{r,s} z_h \right) \prod_{p=1}^{d} f_{Z_p}\left(z_p;\boldsymbol{\eta}_p\right) d\mathbf{z}$$

$$+ \int_{\mathbb{R}^d} \left(\mathbb{1}\{z_k > 0\} \frac{f_k'\left(\alpha_k z_l\right)\alpha_k z_k}{f_k\left(\alpha_k z_k\right)} - \mathbb{1}\{z_k \leq 0\} \frac{f_k'\left(-(1-\alpha_k)z_k\right)(1-\alpha_k)z_k}{f_k\left(-(1-\alpha_k)z_k\right)} \right)$$
$$\cdot \left(\mathbb{1}\{z_r > 0\} \frac{f_r'\left(\alpha_r z_r\right)\alpha_r z_r}{f_r\left(\alpha_r z_r\right)} - \mathbb{1}\{z_r \leq 0\} \frac{f_r'\left(-(1-\alpha_r)z_r\right)(1-\alpha_r)z_r}{f_r\left(-(1-\alpha_r)z_r\right)} \right) \tag{S.6}$$
$$\cdot D_{k,l} D_{r,s} \prod_{p=1}^{d} f_{Z_p}\left(z_p;\boldsymbol{\eta}_p\right) d\mathbf{z}$$

$$-D_{k,l} D_{r,s}.$$

The term $-D_{k,l}D_{r,s}$ requires some explanation. By Proposition 4. $D_{k,l}E\left[\frac{\partial \ell(\boldsymbol{\theta};\mathbf{X})}{\partial \mathbf{A}_{r,s}}\right] = 0$ and $D_{r,s}E\left[\frac{\partial \ell(\boldsymbol{\theta};\mathbf{X})}{\partial \mathbf{A}_{k,l}}\right] = 0$. But if we perform both these products, we need the $D_{k,l}D_{r,s}$ twice (as one of both is completely removed from the equation by performing this product). However, $D_{k,l}D_{r,s}$ is only available once, thus to compensate we need to subtract it once, therefor the minus sign. Next, each of the four integrals (S.3)−(S.6) is calculated individually. Starting with the integral (S.3) and by isolating the term for $j = q$ in the summation over j.

$$
\begin{aligned}
&\text{(S.3)}\\
&= \int_{\mathbb{R}^d} \sum_{\substack{q=1\\q\neq k,r}}^{d} \left(\mathbb{1}\{z_q \leq 0\} \frac{f_q'^2\left(-(1-\alpha_q)z_q\right)(1-\alpha_q)^2}{f_q^2\left(-(1-\alpha_q)z_q\right)} + \mathbb{1}\{z_q > 0\} \frac{f_q'^2\left(\alpha_q z_q\right)\alpha_q^2}{f_q^2\left(\alpha_q z_q\right)} \right)\\
&\quad \cdot \left(\sum_{\substack{m=1\\m\neq j}}^{d} \sum_{\substack{h=1\\h\neq q}}^{d} B_{m,q}^{k,l} z_m B_{h,q}^{r,s} z_h \right)\\
&\quad + \sum_{\substack{q=1\\q\neq k}}^{d} \sum_{\substack{j=1\\j\neq q,r}}^{d} \left(\mathbb{1}\{z_q > 0\} \frac{f_q'\left(\alpha_q z_q\right)\alpha_q}{f_q\left(\alpha_q z_q\right)} - \mathbb{1}\{z_q \leq 0\} \frac{f_q'\left(-(1-\alpha_q)z_q\right)(1-\alpha_q)}{f_j\left(-(1-\alpha_q)z_q\right)} \right)\\
&\quad \cdot \left(\mathbb{1}\{z_j > 0\} \frac{f_j'\left(\alpha_j z_j\right)\alpha_j}{f_j\left(\alpha_j z_j\right)} - \mathbb{1}\{z_j \leq 0\} \frac{f_j'\left(-(1-\alpha_j)z_j\right)(1-\alpha_j)}{f_j\left(-(1-\alpha_j)z_j\right)} \right)\\
&\quad \cdot \left(B_{j,q}^{k,l} z_j B_{q,j}^{r,s} z_q + \sum_{\substack{m=1\\m\neq q,j}}^{d} B_{m,q}^{k,l} z_m B_{q,j}^{r,s} z_q + \sum_{\substack{h=1\\h\neq q,j}}^{d} B_{j,q}^{k,l} z_j B_{h,j}^{r,s} z_h \right.\\
&\quad \left. + \sum_{\substack{m=1\\m\neq q,j}}^{d} \sum_{\substack{h=1\\h\neq q,j}}^{d} B_{m,q}^{k,l} z_m B_{h,j}^{r,s} z_h \right) \prod_{p=1}^{d} f_{Z_p}\left(z_p; \boldsymbol{\eta}_p\right) d\mathbf{z}\\
&= \sum_{\substack{q=1\\j\neq k,r}}^{d} 2\alpha_q(1-\alpha_q) \left(\int_{-\infty}^{0} \frac{f_q'^2\left(-(1-\alpha_q)z_q\right)(1-\alpha_q)^2}{f_q\left(-(1-\alpha_q)z_q\right)} dz_q + \int_{0}^{\infty} \frac{f_q'^2\left(\alpha_q z_q\right)\alpha_q^2}{f_q\left(\alpha_q z_q\right)} dz_q \right)\\
&\quad \cdot \left(\sum_{\substack{m=1\\m\neq q}}^{d} \sum_{\substack{h=1\\h\neq q,m}}^{d} B_{m,q}^{k,l} \kappa_{m,1} B_{h,q}^{r,s} \kappa_{h,1} + \sum_{\substack{g=1\\g\neq q}}^{d} B_{g,q}^{k,l} B_{g,q}^{r,s} \kappa_{g,2} \right)\\
&\quad + \sum_{\substack{q=1\\q\neq k}}^{d} \sum_{\substack{j=1\\j\neq q,r}}^{d} 4\alpha_q(1-\alpha_q)\alpha_j(1-\alpha_j) B_{q,j}^{r,s} B_{j,q}^{k,l}\\
&\quad \cdot \left(\int_{0}^{\infty} f_q'(\alpha_q z_q)\alpha_q z_q dz_q - \int_{-\infty}^{0} f_q'(-(1-\alpha_q)z_q)(1-\alpha_q) z_q dz_q \right)\\
&\quad \cdot \left(\int_{0}^{\infty} f_j'(\alpha_j z_j)\alpha_j z_j dz_j - \int_{-\infty}^{0} f_j'(-(1-\alpha_j)z_j)(1-\alpha_j) z_j dz_j \right)\\
&\quad + \sum_{\substack{q=1\\q\neq k}}^{d} \sum_{\substack{j=1\\j\neq q,r}}^{d} 4\alpha_q(1-\alpha_q)\alpha_j(1-\alpha_j) B_{q,j}^{r,s} \left(\sum_{\substack{m=1\\m\neq q,j}}^{d} B_{m,q}^{k,l} \kappa_{m,1} \right)\\
&\quad \cdot \left(\int_{0}^{\infty} f_q'(\alpha_q z_q)\alpha_q z_q dz_q - \int_{-\infty}^{0} f_q'(-(1-\alpha_q)z_q)(1-\alpha_q) z_q dz_q \right)\\
&\quad \cdot \left(\int_{0}^{\infty} f_j'(\alpha_j z_j)\alpha_j dz_j - \int_{-\infty}^{0} f_j'(-(1-\alpha_j)z_j)(1-\alpha_j) dz_j \right)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{\substack{q=1\\q\neq k}}^{d}\sum_{\substack{j=1\\j\neq q,r}}^{d} 4\alpha_j(1-\alpha_j)\alpha_q(1-\alpha_q)B^{k,l}_{j,q}\Bigg(\sum_{\substack{h=1\\h\neq q,j}}^{d} B^{r,s}_{h,j}\kappa_{h,1}\Bigg)\\
&\quad\cdot\left(\int_0^\infty f'_j(\alpha_j z_j)\alpha_j z_j dz_j-\int_{-\infty}^0 f'_j(-(1-\alpha_j)z_j)(1-\alpha_j)z_j dz_j\right)\\
&\quad\cdot\left(\int_0^\infty f'_q(\alpha_q z_q)\alpha_q dz_q-\int_{-\infty}^0 f'_q(-(1-\alpha_q)z_q)(1-\alpha_q)dz_q\right)\\
&+\sum_{\substack{q=1\\q\neq k}}^{d}\sum_{\substack{j=1\\j\neq q,r}}^{d} 4\alpha_q(1-\alpha_q)\alpha_j(1-\alpha_j)\\
&\quad\cdot\left(\int_0^\infty f'_q(\alpha_q z_q)\alpha_q dz_q-\int_{-\infty}^0 f'_q(-(1-\alpha_q)z_q)(1-\alpha_q)dz_q\right)\\
&\quad\cdot\left(\int_0^\infty f'_j(\alpha_j z_j)\alpha_j dz_j-\int_{-\infty}^0 f'_j(-(1-\alpha_j)z_j)(1-\alpha_j)dz_j\right)\\
&\quad\cdot\Bigg(\sum_{\substack{g=1\\g\neq q,j}}^{d} B^{k,l}_{g,q}B^{r,s}_{g,j}\kappa_{g,2}+\sum_{\substack{m=1\\m\neq q,j}}^{d}\sum_{\substack{h=1\\h\neq m,q,j}}^{d} B^{k,l}_{m,q}\kappa_{m,1}B^{r,s}_{h,j}\kappa_{h,1}\Bigg).
\end{aligned}
$$

In this, only the first and second summations give a non-zero result. The third to fifth summations integrate to zero. This is because for fixed q only the last term depends on z_j and this term integrates to zero as previously shown in (S.2). By Assumptions (*N1*) and (*N2*) this eventually results in

(S.3)

$$
\begin{aligned}
&=\sum_{\substack{q=1\\q\neq k,r}}^{d} 2\alpha_q(1-\alpha_q)\Bigg(\Bigg[(1-\alpha_q)\int_0^\infty \frac{f'^2_q(u)}{f_q(u)}du+\alpha_q\int_0^\infty \frac{f'^2_q(v)}{f_q(v)}dv\Bigg]\\
&\qquad\cdot\Bigg[\sum_{\substack{m=1\\m\neq q}}^{d}\sum_{\substack{h=1\\h\neq q,m}}^{d} B^{k,l}_{m,q}\kappa_{m,1}B^{r,s}_{h,q}\kappa_{h,1}+\sum_{\substack{g=1\\g\neq q}}^{d} B^{k,l}_{g,q}B^{r,s}_{g,q}\kappa_{g,2}\Bigg]\Bigg)\\
&\quad+\sum_{\substack{q=1\\q\neq k}}^{d}\sum_{\substack{j=1\\j\neq q,r}}^{d} 4B^{r,s}_{q,j}B^{k,l}_{j,q}\Bigg(\Bigg[\alpha_q\int_0^\infty f'_q(u)udu+(1-\alpha_q)\int_0^\infty f'_q(v)vdv\Bigg]\\
&\qquad\cdot\Bigg[\alpha_j\int_0^\infty f'_j(u)udu+(1-\alpha_j)\int_0^\infty f'_j(v)vdv\Bigg]\Bigg)\\
&=2\sum_{\substack{q=1\\q\neq k,r}}^{d}\alpha_q(1-\alpha_q)\gamma_{q,1}\Bigg(\sum_{\substack{m=1\\m\neq q}}^{d}\sum_{\substack{h=1\\h\neq q,m}}^{d} B^{k,l}_{m,q}\kappa_{m,1}B^{r,s}_{h,q}\kappa_{h,1}+\sum_{\substack{g=1\\g\neq q}}^{d} B^{k,l}_{g,q}B^{r,s}_{g,q}\kappa_{g,2}\Bigg)
\end{aligned}
$$

$$+\left(\sum_{\substack{q=1\\q\neq k}}^{d}\sum_{\substack{j=1\\j\neq q,r}}^{d} B_{j,q}^{k,l}B_{q,j}^{r,s}\right).$$

Since (S.4) and (S.5) are identical up to indexation, they give the same value. Here, the third integral is worked out. Suppose first the situation where $k=r$, in that case, (S.5) becomes

$$\begin{aligned}
&(\text{S.5})\\
&= D_{k,l}\sum_{\substack{j=1\\j\neq k}}^{d}\int_{\mathbb{R}^d}\left(\mathbb{1}\{z_j>0\}\frac{f_j'(\alpha_j z_j)\,\alpha_j}{f_j(\alpha_j z_j)}-\mathbb{1}\{z_j\leq 0\}\frac{f_j'(-(1-\alpha_j)z_j)\,(1-\alpha_j)}{f_j(-(1-\alpha_j)z_j)}\right)\\
&\cdot\left(\mathbb{1}\{z_k>0\}\frac{f_k'(\alpha_k z_k)\,\alpha_k z_k}{f_k(\alpha_k z_k)}-\mathbb{1}\{z_k\leq 0\}\frac{f_k'(-(1-\alpha_k)z_k)\,(1-\alpha_k)z_k}{f_k(-(1-\alpha_k)z_k)}\right)\\
&\cdot\left(\sum_{\substack{h=1\\h\neq j}}^{d} B_{h,j}^{r,s}z_h\right)\prod_{p=1}^{d} f_{Z_p}(z_p;\boldsymbol{\eta}_p)\,d\mathbf{z}\\
&= 0.
\end{aligned}$$

The latter follows due to z_j only appearing in the first term, a similar calculation as in the proof of Proposition 5 (for $\boldsymbol{\mu}_a$) yields the result. When $k\neq r$ in (S.5), isolating the term with $j=k$ results in

$$\begin{aligned}
&(\text{S.5})\\
&= D_{k,l}\sum_{\substack{j=1\\j\neq k,r}}^{d}\int_{\mathbb{R}^d}\left(\mathbb{1}\{z_j>0\}\frac{f_j'(\alpha_j z_j)\,\alpha_j}{f_j(\alpha_j z_j)}-\mathbb{1}\{z_j\leq 0\}\frac{f_j'(-(1-\alpha_j)z_j)\,(1-\alpha_j)}{f_j(-(1-\alpha_j)z_j)}\right)\\
&\cdot\left(\mathbb{1}\{z_k>0\}\frac{f_k'(\alpha_k z_k)\,\alpha_k z_k}{f_k(\alpha_k z_k)}-\mathbb{1}\{z_k\leq 0\}\frac{f_k'(-(1-\alpha_k)z_k)\,(1-\alpha_k)z_k}{f_k(-(1-\alpha_k)z_k)}\right)\\
&\cdot\left(\sum_{\substack{h=1\\h\neq j}}^{d} B_{h,j}^{r,s}z_h\right)\prod_{p=1}^{d} f_{Z_p}(z_p;\boldsymbol{\eta}_p)\,d\mathbf{z}\\
&+D_{k,l}\int_{\mathbb{R}^d}\left(\mathbb{1}\{z_k>0\}\frac{f_k'(\alpha_k z_k)\,\alpha_k}{f_k(\alpha_k z_k)}-\mathbb{1}\{z_k\leq 0\}\frac{f_k'(-(1-\alpha_k)z_k)\,(1-\alpha_k)}{f_k(-(1-\alpha_k)z_k)}\right)\\
&\cdot\left(\mathbb{1}\{z_k>0\}\frac{f_k'(\alpha_k z_k)\,\alpha_k z_k}{f_k(\alpha_k z_k)}-\mathbb{1}\{z_k\leq 0\}\frac{f_k'(-(1-\alpha_k)z_k)\,(1-\alpha_k)z_k}{f_k(-(1-\alpha_k)z_k)}\right)\\
&\cdot\left(\sum_{\substack{h=1\\h\neq k}}^{d} B_{h,j}^{r,s}z_h\right)\prod_{p=1}^{d} f_{Z_p}(z_p;\boldsymbol{\eta}_p)\,d\mathbf{z}.
\end{aligned}$$

The first integral is zero for the same reason as mentioned for $k=r$. So for $k\neq r$ it holds that

(S.5)

$$= 2\alpha_k(1-\alpha_k)\Bigg(\int_{-\infty}^{0} \frac{f_k'^2\left(-(1-\alpha_k)z_k\right)(1-\alpha_k)^2 z_k}{f_k\left(-(1-\alpha_k)z_k\right)} dz_k$$

$$+ \int_0^\infty \frac{f_k'^2\left(\alpha_k z_k\right)\alpha_k^2 z_k}{f_k\left(\alpha_k z_k\right)} dz_k\Bigg) D_{k,l}\Bigg(\sum_{\substack{h=1\\h\neq k}}^{d} B_{h,j}^{r,s}\kappa_{h,1}\Bigg)$$

$$= 2\alpha_k(1-\alpha_k)\Bigg(\int_0^\infty \frac{f_k'^2(v)\,v}{f_k(v)}dv - \int_0^\infty \frac{f_k'^2(u)\,u}{f_k(u)}du\Bigg) D_{k,l}\Bigg(\sum_{\substack{h=1\\h\neq k}}^{d} B_{h,j}^{r,s}\kappa_{h,1}\Bigg)$$

$$= 0.$$

Ultimately, for the fourth and final integral (S.6), suppose first that $k = r$.

$$(\text{S.6})$$

$$= \Bigg(\int_{-\infty}^{0} \frac{f_k'^2\left(-(1-\alpha_k)z_k\right)(1-\alpha_k)^2 z_k^2}{f_k\left(-(1-\alpha_k)z_k\right)} dz_k + \int_0^\infty \frac{f_k'^2\left(\alpha_k z_k\right)\alpha_k^2 z_k^2}{f_k\left(\alpha_k z_k\right)} dz_k\Bigg)$$

$$\cdot 2\alpha_k(1-\alpha_k)D_{k,l}D_{r,s}$$

$$= D_{k,l}D_{r,s}\Bigg(2\alpha_k\int_0^\infty \frac{f_k'^2(u)\,u^2}{f_k(u)}du + 2(1-\alpha_k)\int_0^\infty \frac{f_k'^2(v)\,v^2}{f_k(v)}dv\Bigg)$$

$$= 2D_{k,l}D_{r,s}\gamma_{k,3}.$$

Now suppose that $k \neq r$. Due to Proposition 5 (the calculation for $\mathbf{A}$ under the situation $j = k$), the integral equals 1, hence we get as a result that (S.6) $= D_{k,l}D_{r,s}$ when $k \neq r$.

Combining the four integrals with the remaining term $-D_{k,l}D_{r,s}$ we have neglected so far, we arrive at the proposed expression.

S.2.5 Proof of Lemma 2

First note that $\boldsymbol{\Psi}(\mathbf{x};\boldsymbol{\theta})$ is continuous a.e. in $\mathbf{x}$. The points in which it is not continuous are the union hyperplanes $\mathbf{D}$ defined by

$$\mathbf{D} = \bigcup_{j=1}^{d}\left\{(\mathbf{x}-\boldsymbol{\mu}_a)^T(\mathbf{A}^{-1})_{\cdot,j} = 0\right\}.$$

Note that $\mathbf{D}$ has measure zero and that all hyperplanes of discontinuity pass through $\mathbf{x} = \boldsymbol{\mu}_a$.

Since $\boldsymbol{\Psi}(\mathbf{x};\boldsymbol{\theta})$ is continuous a.e., it is measurable. This is because for any $\mathbf{c} \in \mathbb{R}^d$,

$$\boldsymbol{\Psi}^{-1}(\mathbf{x};\boldsymbol{\theta})(]\mathbf{c},\infty[) = \{\boldsymbol{\Psi}^{-1}(\mathbf{x};\boldsymbol{\theta})(]\mathbf{c},\infty[)\cap(\mathbb{R}^d\backslash\mathbf{D})\}\cup\{\boldsymbol{\Psi}^{-1}(\mathbf{x};\boldsymbol{\theta})(]c,\infty[)\cap\mathbf{D}\}.$$

The first subset is measurable since $\boldsymbol{\Psi}(\mathbf{x};\boldsymbol{\theta})$ is continuous on the subset and according to Billingsley (1995), a continuous function is measurable. The second subset is subset of $\mathbf{D}$, which has measure zero, thereby, it is measurable. As the union of measurable sets is measurable, the lemma holds.

S.2.6 Proof of Lemma 3

From Proposition 5 it holds that $\boldsymbol{\lambda}(\boldsymbol{\theta}_0) = \mathbf{0}$, so it suffices to prove that $\forall \epsilon > 0, \exists \delta > 0 : \|\boldsymbol{\theta} - \boldsymbol{\theta}_0\| < \delta \implies \|\boldsymbol{\lambda}(\boldsymbol{\theta})\| < \epsilon$. What follows is a rough sketch. The concept is given for the derivative of $\ell(\boldsymbol{\theta};\mathbf{x})$ with respect to $\mu_{a,k}$. Analogous one can construct a proof for the derivative with respect to $\mathbf{A}_{k,l}$. In the following, $\boldsymbol{\theta}_0$ stands for the true parameter whereas $\boldsymbol{\theta}$ is in the aforementioned neighborhood of $\boldsymbol{\theta}_0$.

$$\begin{aligned}
&E\left[\frac{\partial \ell(\boldsymbol{\theta};\mathbf{X})}{\partial \mu_{a,k}}\right] \\
&= \int_{\mathbb{R}^d} \sum_{j=1}^{d} \Bigg(-\mathbb{1}\Big\{(\mathbf{x}-\boldsymbol{\mu}_a)^T(\mathbf{A}^{-1})_{\cdot,j} > 0\Big\} \frac{f_j'\left(\alpha_j(\mathbf{x}-\boldsymbol{\mu}_a)^T(\mathbf{A}^{-1})_{\cdot,j}\right)\alpha_j(\mathbf{A}^{-1})_{k,j}}{f_j\left(\alpha_j(\mathbf{x}-\boldsymbol{\mu}_a)^T(\mathbf{A}^{-1})_{\cdot,j}\right)} \\
&\quad + \mathbb{1}\Big\{(\mathbf{x}-\boldsymbol{\mu}_a)^T(\mathbf{A}^{-1})_{\cdot,j} \le 0\Big\} \frac{f_j'\left(-(1-\alpha_j)(\mathbf{x}-\boldsymbol{\mu}_a)^T(\mathbf{A}^{-1})_{\cdot,j}\right)(1-\alpha_j)(\mathbf{A}^{-1})_{k,j}}{f_j\left(-(1-\alpha_j)(\mathbf{x}-\boldsymbol{\mu}_a)^T(\mathbf{A}^{-1})_{\cdot,j}\right)} \Bigg) \\
&\quad \cdot \frac{1}{|\det(\mathbf{A}_0)|} \prod_{m=1}^{d} f_{Z_m}\left((\mathbf{x}-\boldsymbol{\mu}_{a,0})^T(\mathbf{A}_0^{-1})_{\cdot,m};\boldsymbol{\eta}_{m,0}\right) d\mathbf{x} \\
&< \int_{\mathbb{R}^d} \sum_{j=1}^{d} (\mathbf{A}^{-1})_{k,j} \Bigg(-\mathbb{1}\Big\{(\mathbf{x}-\boldsymbol{\mu}_a)^T(\mathbf{A}^{-1})_{\cdot,j} > 0\Big\} \\
&\quad \cdot \frac{(1-\tilde{\epsilon}_{j,1}) f_j'\left(\alpha_{j,0}(\mathbf{x}-\boldsymbol{\mu}_{a,0})^T(\mathbf{A}_0^{-1})_{\cdot,j}^{-1}\right)\alpha_j}{(1+\epsilon_{j,1}) f_j\left(\alpha_{j,0}(\mathbf{x}-\boldsymbol{\mu}_{a,0})^T(\mathbf{A}_0^{-1})_{\cdot,j}^{-1}\right)} \\
&\quad + \mathbb{1}\Big\{(\mathbf{x}-\boldsymbol{\mu}_a)^T(\mathbf{A}^{-1})_{\cdot,j} \le 0\Big\} \\
&\quad \cdot \frac{(1+\tilde{\epsilon}_{j,2}) f_j'\left(-(1-\alpha_{j,0})(\mathbf{x}-\boldsymbol{\mu}_{a,0})^T(\mathbf{A}_0^{-1})_{\cdot,j}^{-1}\right)(1-\alpha_j)}{(1-\epsilon_{j,1}) f_j\left(-(1-\alpha_{j,0})(\mathbf{x}-\boldsymbol{\mu}_{a,0})^T(\mathbf{A}_0^{-1})_{\cdot,j}^{-1}\right)} \Bigg) \\
&\quad \cdot \frac{1}{|\det(\mathbf{A}_0)|} \prod_{m=1}^{d} f_{Z_m}\left((\mathbf{x}-\boldsymbol{\mu}_{a,0})^T(\mathbf{A}_0^{-1})_{\cdot,m};\boldsymbol{\eta}_{m,0}\right) d\mathbf{x},
\end{aligned}$$

with $\tilde{\epsilon}_{j,i}, \epsilon_{j,i} > 0$, $j = 1, \ldots, d; i = 1, 2$. The latter inequality follows from continuity of both f_j and f_j' on either the positive or the negative half real line (as these are the univariate symmetric densities). This allows to bound $f_j^{(')}$ in the following way with $f_{j,0}^{(')}$ the density (or its derivative) evaluated using the true parameter

$$(1-\epsilon_{j,1})f_{j,0} < f_j < (1+\epsilon_{j,2})f_{j,0}$$
$$(1-\tilde{\epsilon}_{j,1})f'_{j,0} < f'_j < (1+\tilde{\epsilon}_{j,2})f'_{j,0}.$$

As we can now apply the substitutions used in Proposition 5, the integrals can be calculated but there is still a problem present, namely the discrepancy between the indicator functions used in the true densities and the one in front of the fractions in the summation over j. This is the part where the integration comes to the rescue. As $\boldsymbol{\theta}$ is arbitrarily close to $\boldsymbol{\theta}_0$ the area over which to integrate the discrepancy between indicators is also arbitrarily small and the functions to integrate are bounded, hence they can be bounded above by any positive constant, say $\tilde{\epsilon}_{j,3}$ as long as we take $\boldsymbol{\theta}$ sufficiently close to $\boldsymbol{\theta}_0$. This leaves us with the following bound on $E\left[\frac{\partial \ell(\boldsymbol{\theta};\mathbf{X})}{\partial \mu_{a,k}}\right]$

$$\begin{aligned}
&E\left[\frac{\partial \ell(\boldsymbol{\theta};\mathbf{X})}{\partial \mu_{a,k}}\right]\\
&< 2\sum_{j=1}^{d}\left(\frac{1+\tilde{\epsilon}_{j,2}}{1-\epsilon_{j,1}}\alpha_{j,0}(1-\alpha_j)-\frac{1-\tilde{\epsilon}_{j,2}}{1+\epsilon_{j,1}}\alpha_j(1-\alpha_{j,0})\right)\mathbf{A}_{k,j}\int_0^\infty f'_j(u)du+\tilde{\epsilon}_{j,3}\\
&= 2\sum_{j=1}^{d}\left(\frac{1+\tilde{\epsilon}_{j,2}}{1-\epsilon_{j,1}}(1+\epsilon_{j,3})-\frac{1-\tilde{\epsilon}_{j,2}}{1+\epsilon_{j,1}}(1+\epsilon_{j,4})\right)\alpha_j(1-\alpha_j)\mathbf{A}_{k,j}\int_0^\infty f'_j(u)du+\tilde{\epsilon}_{j,3}\\
&< \epsilon,
\end{aligned}$$

as all terms can be bounded and all $\tilde{\epsilon}_{j,\cdot}$'s and $\epsilon_{j,\cdot}$'s can be taken arbitrarily small, this completes the proof.

S.3 Example when extra parameters are involved

A classical example of a distribution with other parameters than a location or scale parameter is the Student's t-distribution. The extra parameter is the degrees of freedom (ν) which governs the kurtosis of the distribution. To check whether asymptotic normality under the presence of Student's t-distributed random variables still holds, the conditions of Theorem 2 need to be fulfilled. If this is not the case, the theory developed here becomes invalid. As consistency is a key property and some non-trivial assumptions are imposed, this should be checked first.

The assumptions of Proposition 4 still hold, but we need to check whether or not the parameters are identifiable. For the parameters $\boldsymbol{\alpha}$, $\boldsymbol{\mu}_a$ and $\mathbf{A}$ this is the same as done in Proposition 2, of course if the assumptions are valid. Rests there only the parameter for the degrees of freedom for $f_{Z_j}(Z_j;\alpha_j,\nu_j)$. However, this does not pose a problem as for a regular Student's t-distribution, the degrees of freedom is uniquely determinable. Hence, by Proposition 4 the estimator is consistent.

Asymptotic normality
Using the same notation as in Theorem 2, we check whether $\boldsymbol{\Psi}(\mathbf{x};\boldsymbol{\xi})$ is continuous in $\mathbf{x}$, $\boldsymbol{\lambda}(\boldsymbol{\xi})$ is continuous in ν_j, $E\left[\frac{\partial \ell(\boldsymbol{\xi};\mathbf{X})}{\partial \nu_j}\right]_{\boldsymbol{\xi}=\boldsymbol{\xi}_0} = 0$ and further whether $E\left[\left(\frac{\partial \ell(\boldsymbol{\xi};\mathbf{X})}{\partial \nu_j}\right)\left(\frac{\partial \ell(\boldsymbol{\xi};\mathbf{X})}{\partial \xi_j}\right)\right]_{\boldsymbol{\xi}=\boldsymbol{\xi}_0} < \infty$, $\xi_j \in \boldsymbol{\Xi}$. For this we need the partial derivative of the log-likelihood with respect to ν_j

$$
\begin{aligned}
&\frac{\partial \ell(\boldsymbol{\xi};\mathbf{x})}{\partial \nu_j} \\
&= -\frac{1}{2\nu_j} - \mathbb{1}\left\{(\mathbf{x}-\boldsymbol{\mu})^T(\mathbf{A}^{-1})_{\cdot,j} \leq 0\right\} \frac{1}{2} \ln\left(1 + \frac{(1-\alpha_j)(\mathbf{x}-\boldsymbol{\mu})^T(\mathbf{A}^{-1})_{\cdot,j}}{\nu_j}\right) \\
&\quad + \mathbb{1}\left\{(\mathbf{x}-\boldsymbol{\mu})^T(\mathbf{A}^{-1})_{\cdot,j} \leq 0\right\} \frac{\nu_j+1}{2\nu_j^2} \frac{(1-\alpha_j)^2((\mathbf{x}-\boldsymbol{\mu})^T(\mathbf{A}^{-1})_{\cdot,j})^2}{1+\frac{(1-\alpha_j)^2((\mathbf{x}-\boldsymbol{\mu})^T(\mathbf{A}^{-1})_{\cdot,j})^2}{\nu_j}} \\
&\quad - \mathbb{1}\left\{(\mathbf{x}-\boldsymbol{\mu})^T(\mathbf{A}^{-1})_{\cdot,j} > 0\right\} \frac{1}{2} \ln\left(1 + \frac{\alpha_j(\mathbf{x}-\boldsymbol{\mu})^T(\mathbf{A}^{-1})_{\cdot,j}}{\nu_j}\right) \\
&\quad + \mathbb{1}\left\{(\mathbf{x}-\boldsymbol{\mu})^T(\mathbf{A}^{-1})_{\cdot,j} > 0\right\} \frac{\nu_j+1}{2\nu_j^2} \frac{\alpha_j^2((\mathbf{x}-\boldsymbol{\mu})^T(\mathbf{A}^{-1})_{\cdot,j})^2}{1+\frac{\alpha_j^2((\mathbf{x}-\boldsymbol{\mu})^T(\mathbf{A}^{-1})_{\cdot,j})^2}{\nu_j}} - \frac{1}{2}D(\nu_j),
\end{aligned}
$$

with $D(\nu_j) = \psi\left(\frac{\nu_j}{2}\right) - \psi\left(\frac{\nu_j+1}{2}\right)$ and $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ the digamma function (i.e. the first derivative of the log-gamma function. In the same way, the trigamma function is defined as the second derivative of the log-gamma function etc.). It is immediately clear that $\frac{\partial \ell(\boldsymbol{\xi};\mathbf{x})}{\partial \nu_j}$ is continuous in $\mathbf{x}$ for fixed $\boldsymbol{\xi}$. Since $\nu_j > 0$, we have that $D(\nu_j)$ is continuous since the gamma function and natural logarithm are smooth functions for positive arguments. Hence, $\frac{\partial \ell(\boldsymbol{\xi};\mathbf{x})}{\partial \nu_j}$, and in extension also $\boldsymbol{\lambda}(\boldsymbol{\xi})$, are continuous functions in ν_j. Lemma 2 and Lemma 3 thus apply. In Gijbels et al. (2019), it is shown that $E\left[\frac{\partial \ell(\boldsymbol{\xi};\mathbf{X})}{\partial \nu_j}\right]_{\boldsymbol{\xi}=\boldsymbol{\xi}_0} = 0$. Applying the results concerning QBA-Student's t-distribution from Gijbels et al. (2019) and a similar calculation as in the proof of Proposition 6, the following Lemma is obtained.

Lemma S.1 *The elements of the Fisher Information matrix that are associated with the degrees of freedom ν_j of the univariate Student's t-distribution $f_{Z_j}(Z_j;\boldsymbol{\eta}_j)$ are given by*

$$
E\left[\left(\frac{\partial \ell(\boldsymbol{\xi};\mathbf{X})}{\partial \nu_k}\right)\left(\frac{\partial \ell(\boldsymbol{\xi};\mathbf{X})}{\partial \nu_j}\right)\right] = \begin{cases} \frac{1}{2}\frac{\partial}{\partial \nu_j}D(\nu_j) - \frac{\nu_j+5}{2\nu_j(\nu_j+1)(\nu_j+3)} & \text{if } j = k \\ 0 & \text{if } j \neq k. \end{cases}
$$

$$E\left[\left(\frac{\partial \ell(\boldsymbol{\xi};\mathbf{X})}{\partial \alpha_k}\right)\left(\frac{\partial \ell(\boldsymbol{\xi};\mathbf{X})}{\partial \nu_j}\right)\right] = \begin{cases} \frac{2(1-2\alpha_k)}{\alpha_k(1-\alpha_k)(\nu_j+1)(\nu_j+3)} & if\ j=k \\ 0 & if\ j \neq k. \end{cases}$$

$$E\left[\left(\frac{\partial \ell(\boldsymbol{\xi};\mathbf{X})}{\partial \mu_k}\right)\left(\frac{\partial \ell(\boldsymbol{\xi};\mathbf{X})}{\partial \nu_j}\right)\right] = 0$$

$$E\left[\left(\frac{\partial \ell(\boldsymbol{\xi};\mathbf{X})}{\partial \mathbf{A}_{k,l}}\right)\left(\frac{\partial \ell(\boldsymbol{\xi};\mathbf{X})}{\partial \nu_j}\right)\right] = \begin{cases} \frac{2(-1)^{j+l+1}det(\mathbf{A}_{-j;-l})}{(\nu_j+1)(\nu_j+3)det(\mathbf{A})} & if\ j=k \\ 0 & if\ j \neq k. \end{cases}$$

The Fisher-information matrix is then obtained by combining these four block with the other six from Proposition 6. From Lemma S.1, it also follows that

$$E\left[\left(\frac{\partial \ell(\boldsymbol{\xi};\mathbf{X})}{\partial \nu_j}\right)\left(\frac{\partial \ell(\boldsymbol{\xi};\mathbf{X})}{\partial \xi_j}\right)\right]_{\boldsymbol{\xi}=\boldsymbol{\xi}_0} < \infty, \qquad i,j=i,\ldots,d.$$

Hence, by Theorem 2, asymptotic normality for the parameter estimates obtained by maximum likelihood holds.

S.4 Additional plots for the AIS-data analysis

Posterior distributions for the Bayesian parameter estimations in the QBA logistic-normal model, using respectively uniform and normal priors, are shown in Figures S.1 and S.2.

References

Billingsley, P. (1995). *Probability and Measure*. Wiley Series in Probability and Statistics. Wiley.

Bonhomme, S. and Robin, J.-M. (2009). Consistent noisy independent component analysis. *Journal of Econometrics*, 149(1):12 – 25.

Cardoso, J. F. and Souloumiac, A. (1993). Blind beamforming for non-gaussian signals. *IEEE Proceedings F - Radar and Signal Processing*, 140(6):362–370.

Gijbels, I., Karim, R., and Verhasselt, A. (2019). On quantile-based asymmetric family of distributions: Properties and inference. *International Statistical Review*, 87(3):471–504.

Gouriéroux, C., Monfort, A., and Renne, J. (2017). Statistical inference for independent component analysis: Application to structural VAR models. *Journal of Econometrics*, 196(1):111–126.

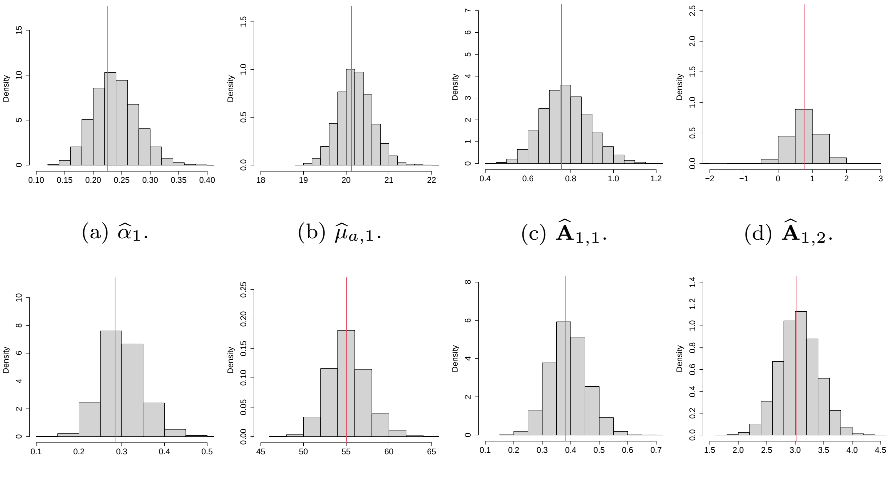
Hyvärinen, A. (1999). Fast and robust fixed-point algorithms for independent component analysis. *IEEE Transactions on Neural Networks*, 10(3):626–634.

Hyvärinen, A., Karhunen, J., and Oja, E. (2001). *ICA by Maximum Likelihood Estimation*, chapter 9, pages 203–219. John Wiley & Sons, Ltd.

Ilmonen, P., Nordhausen, K., Oja, H., and Ollila, E. (2012). On asymptotics of ICA estimators and their performance indices. *arXiv preprint arXiv:1212.3953*.

Miettinen, J., Taskinen, S., Nordhausen, K., and Oja, H. (2015). Fourth moments and independent component analysis. *Statistical Science*, 30(3):372–390.

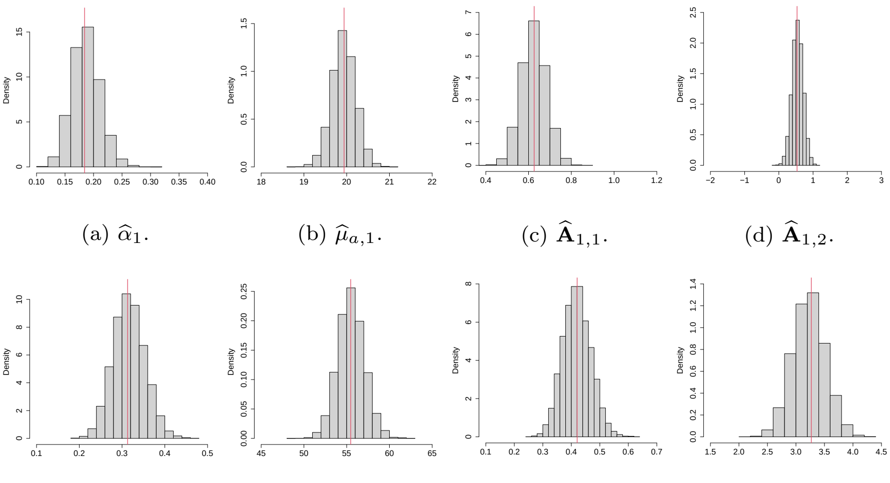
Ollila, E. (2010). The deflation-based fastica estimator: Statistical analysis revisited. *IEEE Transactions on Signal Processing*, 58(3):1527–1541.



(a) $\widehat{\alpha}_1$. (b) $\widehat{\mu}_{a,1}$. (c) $\widehat{\mathbf{A}}_{1,1}$. (d) $\widehat{\mathbf{A}}_{1,2}$.

(e) $\widehat{\alpha}_2$. (f) $\widehat{\mu}_{a,2}$. (g) $\widehat{\mathbf{A}}_{2,1}$. (h) $\widehat{\mathbf{A}}_{2,2}$.

Fig. S.1: *AIS-data: Histograms of posterior distribution using uniform priors, the red line represents the mode of the posterior.*



(a) $\widehat{\alpha}_1$. (b) $\widehat{\mu}_{a,1}$. (c) $\widehat{\mathbf{A}}_{1,1}$. (d) $\widehat{\mathbf{A}}_{1,2}$.

(e) $\widehat{\alpha}_2$. (f) $\widehat{\mu}_{a,2}$. (g) $\widehat{\mathbf{A}}_{2,1}$. (h) $\widehat{\mathbf{A}}_{2,2}$.

Fig. S.2: *AIS-data: Histograms of posterior distribution using normal priors, the red line represents the mode of the posterior.*