Supplementary material to

Robust estimation of the conditional stable tail dependence function

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Abstract

This document contains the proofs of all the results given in the Appendix of the main part of the paper "Robust estimation of the conditional stable tail dependence function ", i.e., the technical lemmas and Theorem 4. Also Theorem 2 concerning the case of unknown margins is established.

1 Proof of Lemma 1

From Corollary 5.1 in Goegebeur et al. (2021), with a slight adjustment for the left continuity of $T_{n,1-t_j}(y|x_0)$, we have that $W_{n,1-t_j} \rightsquigarrow W_{1-t_j}$ in $\ell((0,T])$, for $j=1,\ldots,J$. To obtain the joint convergence of $(W_{n,1-t_1},\ldots,W_{n,1-t_J})$ we need to verify the finite dimensional convergence and joint tightness. We start by proving the finite dimensional convergence. Let

$$\mathbb{T}_n := \begin{bmatrix} T_{n,1-t_1}(y_1|x_0) \\ \vdots \\ T_{n,1-t_J}(y_J|x_0) \end{bmatrix} \quad \text{and} \quad \mathcal{T} := \begin{bmatrix} y_1 f_X(x_0) \\ \vdots \\ y_J f_X(x_0) \end{bmatrix}.$$

Since

$$\sqrt{kh_n^d}\,(\mathbb{T}_n - \mathcal{T}) = \sqrt{kh_n^d}\,(\mathbb{T}_n - \mathbb{E}(\mathbb{T}_n)) + \sqrt{kh_n^d}(\mathbb{E}(\mathbb{T}_n) - \mathcal{T}),$$

and $\sqrt{kh_n^d}(\mathbb{E}(\mathbb{T}_n) - \mathcal{T}) \to \mathbf{0}$, by Proposition 5.1 in Goegebeur et al. (2021) and our assumptions, it is sufficient to study the weak convergence of $\sqrt{kh_n^d}\,(\mathbb{T}_n - \mathbb{E}(\mathbb{T}_n))$. To this aim we use the Cramér-Wold device, and show that

$$\Lambda_n := \psi^T \sqrt{kh_n^d}\,(\mathbb{T}_n - \mathbb{E}(\mathbb{T}_n)) \rightsquigarrow N(0, \psi^T \Sigma \psi)$$

for all $\psi := (\psi_1, \dots, \psi_J)^T \in \mathbb{R}^J$, where the elements of Σ are as in the statement of Theorem 1. By a straightforward rearrangement of terms we have

$$\begin{aligned}\Lambda_n &= \sum_{i=1}^n \sqrt{\frac{h_n^d}{k}} \sum_{j=1}^J \psi_j \left[K_{h_n}(x_0 - X_i) \mathbb{1}_{\{\overline{F}_{Z_{1-t_j}}(Z_{1-t_j,i}|x_0) < \frac{k}{n} y_j\}} - \mathbb{E}\left(K_{h_n}(x_0 - X) \mathbb{1}_{\{\overline{F}_{Z_{1-t_j}}(Z_{1-t_j}|x_0) < \frac{k}{n} y_j\}} \right) \right] \\ &=: \sum_{i=1}^n V_{i,n}.\end{aligned}$$

Since $V_{1,n}, \dots, V_{n,n}$ are independent and identically distributed random variables, we have $\mathbb{V}ar(\Lambda_n) = n \mathbb{V}ar(V_{1,n})$, and hence

$$\mathbb{V}ar(\Lambda_n) = \sum_{j=1}^J \sum_{j'=1}^J \psi_j \psi_{j'} \frac{n h_n^d}{k} \mathbb{C}_{j,j'}, \tag{S1}$$

where

$$\mathbb{C}_{j,j'} := \mathbb{C}ov\left(K_{h_n}(x_0 - X) \mathbb{1}_{\{\overline{F}_{Z_{1-t_j}}(Z_{1-t_j}|x_0) < \frac{k}{n} y_j\}}, K_{h_n}(x_0 - X) \mathbb{1}_{\{\overline{F}_{Z_{1-t_{j'}}}(Z_{1-t_{j'}}|x_0) < \frac{k}{n} y_{j'}\}} \right).$$

We have

$$\begin{aligned}&\mathbb{E}\left(K_{h_n}^2(x_0 - X) \mathbb{1}_{\{\overline{F}_{Z_{1-t_j}}(Z_{1-t_j}|x_0) < \frac{k}{n} y_j, \overline{F}_{Z_{1-t_{j'}}}(Z_{1-t_{j'}}|x_0) < \frac{k}{n} y_{j'}\}} \right) \\ &\quad = \mathbb{E}\left(K_{h_n}^2(x_0 - X) \mathbb{P}\left(\overline{F}_{Z_{1-t_j}}(Z_{1-t_j}|x_0) < \frac{k}{n} y_j, \overline{F}_{Z_{1-t_{j'}}}(Z_{1-t_{j'}}|x_0) < \frac{k}{n} y_{j'} \middle| X \right) \right).\end{aligned}$$

Now let

$$\begin{aligned}m_n^{(1)} &:= \max\left(U_{Z_{1-t_j}}\left(\frac{n}{k y_j} \middle| x_0 \right), U_{Z_{1-t_{j'}}}\left(\frac{n}{k y_{j'}} \middle| x_0 \right) \right), \\ m_n^{(2)} &:= \max\left(\frac{t_j}{1-t_j} U_{Z_{1-t_j}}\left(\frac{n}{k y_j} \middle| x_0 \right), \frac{t_{j'}}{1-t_{j'}} U_{Z_{1-t_{j'}}}\left(\frac{n}{k y_{j'}} \middle| x_0 \right) \right).\end{aligned}$$

Then, for $x \in S_X$,

$$\mathbb{P}\left(\overline{F}_{Z_{1-t_j}}(Z_{1-t_j}|x_0) < \frac{k}{n} y_j, \overline{F}_{Z_{1-t_{j'}}}(Z_{1-t_{j'}}|x_0) < \frac{k}{n} y_{j'} \middle| X = x \right) = \mathbb{P}\left(Y^{(1)} > m_n^{(1)}, Y^{(2)} > m_n^{(2)} \middle| X = x \right).$$

We have

$$U_{Z_{1-t}}(y|x_0) = G_{1-t}(x_0) y [1 + a_{1-t}(y|x_0)],$$

where $a_{1-t}(.|x_0)$ is regularly varying with index $-\beta(x_0)$, and hence

$$\begin{aligned}m_n^{(1)} &= \frac{n}{k} \max\left(\frac{G_{1-t_j}(x_0)}{y_j} \left[1 + a_{1-t_j}\left(\frac{n}{k y_j} \middle| x_0 \right) \right], \frac{G_{1-t_{j'}}(x_0)}{y_{j'}} \left[1 + a_{1-t_{j'}}\left(\frac{n}{k y_{j'}} \middle| x_0 \right) \right] \right) \\ &=: \frac{n}{k} \widetilde{m}_n^{(1)}, \\ m_n^{(2)} &= \frac{n}{k} \max\left(\frac{t_j}{1-t_j} \frac{G_{1-t_j}(x_0)}{y_j} \left[1 + a_{1-t_j}\left(\frac{n}{k y_j} \middle| x_0 \right) \right], \frac{t_{j'}}{1-t_{j'}} \frac{G_{1-t_{j'}}(x_0)}{y_{j'}} \left[1 + a_{1-t_{j'}}\left(\frac{n}{k y_{j'}} \middle| x_0 \right) \right] \right) \\ &=: \frac{n}{k} \widetilde{m}_n^{(2)}.\end{aligned}$$

Using our model (1) for $(Y^{(1)}, Y^{(2)})$, gives then

$$
\begin{aligned}
&\mathbb{P}\left(Y^{(1)} > m_n^{(1)}, Y^{(2)} > m_n^{(2)} \middle| X = x\right) \\
&= \frac{k}{n} (\widetilde{m}_n^{(1)})^{-d_1(x)} (\widetilde{m}_n^{(2)})^{-d_2(x)} g\left(\frac{1}{\widetilde{m}_n^{(1)}}, \frac{1}{\widetilde{m}_n^{(2)}} \middle| x\right) \left[1 + \delta\left(\frac{k}{n}\frac{1}{\widetilde{m}_n^{(1)}}, \frac{k}{n}\frac{1}{\widetilde{m}_n^{(2)}} \middle| x\right)\right],
\end{aligned}
$$

and thus

$$
\begin{aligned}
&\mathbb{E}\left(K_{h_n}^2(x_0 - X) 1\!\!1_{\{\overline{F}_{Z_{1-t_j}}(Z_{1-t_j}|x_0) < \frac{k}{n} y_j, \overline{F}_{Z_{1-t_{j'}}}(Z_{1-t_{j'}}|x_0) < \frac{k}{n} y_{j'}\}}\right) \\
&= \frac{k}{n h_n^d} \int_{S_K} K^2(v) (\widetilde{m}_n^{(1)})^{-d_1(x_0 - h_n v)} (\widetilde{m}_n^{(2)})^{-d_2(x_0 - h_n v)} g\left(\frac{1}{\widetilde{m}_n^{(1)}}, \frac{1}{\widetilde{m}_n^{(2)}} \middle| x_0 - h_n v\right) \\
&\quad \times \left[1 + \delta\left(\frac{k}{n}\frac{1}{\widetilde{m}_n^{(1)}}, \frac{k}{n}\frac{1}{\widetilde{m}_n^{(2)}} \middle| x_0 - h_n v\right)\right] f_X(x_0 - h_n v) dv.
\end{aligned}
$$

Write

$$
\begin{aligned}
\delta\left(\frac{k}{n}\frac{1}{\widetilde{m}_n^{(1)}}, \frac{k}{n}\frac{1}{\widetilde{m}_n^{(2)}} \middle| x_0 - h_n v\right) &= \delta\left(\frac{k}{n}, \frac{k}{n} \middle| x_0\right) \frac{\delta\left(\frac{k}{n}, \frac{k}{n} \middle| x_0 - h_n v\right)}{\delta\left(\frac{k}{n}, \frac{k}{n} \middle| x_0\right)} \left[\xi\left(\frac{1}{\widetilde{m}_n^{(1)}}, \frac{1}{\widetilde{m}_n^{(2)}} \middle| x_0 - h_n v\right)\right. \\
&\quad \left. + \frac{\delta\left(\frac{k}{n}\frac{1}{\widetilde{m}_n^{(1)}}, \frac{k}{n}\frac{1}{\widetilde{m}_n^{(2)}} \middle| x_0 - h_n v\right)}{\delta\left(\frac{k}{n}, \frac{k}{n} \middle| x_0 - h_n v\right)} - \xi\left(\frac{1}{\widetilde{m}_n^{(1)}}, \frac{1}{\widetilde{m}_n^{(2)}} \middle| x_0 - h_n v\right)\right].
\end{aligned}
$$

By our model (1), the assumptions of Lemma 1, and the fact that $\delta_{0.5}(.|x)$ satisfies $(\mathcal{D}_{0.5})$ and $(\mathcal{H}_{0.5})$, one has that

$$
\delta\left(\frac{k}{n}\frac{1}{\widetilde{m}_n^{(1)}}, \frac{k}{n}\frac{1}{\widetilde{m}_n^{(2)}} \middle| x_0 - h_n v\right) = \delta\left(\frac{k}{n}, \frac{k}{n} \middle| x_0\right) \left[\xi\left(\frac{1}{\widetilde{m}_n^{(1)}}, \frac{1}{\widetilde{m}_n^{(2)}} \middle| x_0 - h_n v\right) + o(1)\right],
$$

where the $o(1)$ term is uniform in $v \in S_K$. Hence, by Lebesgue's dominated convergence theorem

$$
\begin{aligned}
&\mathbb{E}\left(K_{h_n}^2(x_0 - X) 1\!\!1_{\{\overline{F}_{Z_{1-t_j}}(Z_{1-t_j}|x_0) < \frac{k}{n} y_j, \overline{F}_{Z_{1-t_{j'}}}(Z_{1-t_{j'}}|x_0) < \frac{k}{n} y_{j'}\}}\right) \\
&= \frac{k}{n h_n^d} \left\{ \|K\|_2^2 f_X(x_0) \left(\max\left(\frac{G_{1-t_j}(x_0)}{y_j}, \frac{G_{1-t_{j'}}(x_0)}{y_{j'}}\right)\right)^{-d_1(x_0)} \right. \\
&\quad \times \left(\max\left(\frac{t_j}{1-t_j}\frac{G_{1-t_j}(x_0)}{y_j}, \frac{t_{j'}}{1-t_{j'}}\frac{G_{1-t_{j'}}(x_0)}{y_{j'}}\right)\right)^{-d_2(x_0)} \\
&\quad \left. \times g\left(\frac{1}{\max\left(\frac{G_{1-t_j}(x_0)}{y_j}, \frac{G_{1-t_{j'}}(x_0)}{y_{j'}}\right)}, \frac{1}{\max\left(\frac{t_j}{1-t_j}\frac{G_{1-t_j}(x_0)}{y_j}, \frac{t_{j'}}{1-t_{j'}}\frac{G_{1-t_{j'}}(x_0)}{y_{j'}}\right)} \middle| x_0\right) + o(1) \right\}.
\end{aligned}
$$

By taking into account that

$$\mathbb{E}\left(K_{h_n}(x_0-X)\mathbb{1}_{\{\overline{F}_{Z_{1-t_j}}(Z_{1-t_j}|x_0)<\frac{k}{n}y_j\}}\right) = \frac{k}{n}\,\mathbb{E}\left(T_{n,1-t_j}(y_j|x_0)\right) = \frac{k}{n}\,y_j f_X(x_0)(1+o(1)),$$

where the last step follows from Proposition 5.1 in Goegebeur et al. (2021), we have that

$$\begin{aligned}\frac{nh_n^d}{k}\mathbb{C}_{j,j'} \;\to\; & \|K\|_2^2 f_X(x_0)\left(\max\left(\frac{G_{1-t_j}(x_0)}{y_j},\frac{G_{1-t_{j'}}(x_0)}{y_{j'}}\right)\right)^{-d_1(x_0)} \\ & \times\left(\max\left(\frac{t_j}{1-t_j}\frac{G_{1-t_j}(x_0)}{y_j},\frac{t_{j'}}{1-t_{j'}}\frac{G_{1-t_{j'}}(x_0)}{y_{j'}}\right)\right)^{-d_2(x_0)} \\ & \times g\left(\frac{1}{\max\left(\frac{G_{1-t_j}(x_0)}{y_j},\frac{G_{1-t_{j'}}(x_0)}{y_{j'}}\right)},\frac{1}{\max\left(\frac{t_j}{1-t_j}\frac{G_{1-t_j}(x_0)}{y_j},\frac{t_{j'}}{1-t_{j'}}\frac{G_{1-t_{j'}}(x_0)}{y_{j'}}\right)}\,\middle|\,x_0\right),\end{aligned}$$

which shows the convergence of the variance in (S1).

To ensure the convergence in distribution of Λ_n to a normal random variable, we have to verify the Lyapounov condition for triangular arrays of random variables (Billingsley, 1995, p. 362). In the present context this simplifies to showing that $n\mathbb{E}|V_{1,n}|^3\to 0$. We have

$$\begin{aligned}\mathbb{E}|V_{1,n}|^3 \;\leqslant\; & \left(\frac{h_n^d}{k}\right)^{3/2}\left\{\mathbb{E}\left[\left(\sum_{j=1}^J|\psi_j|K_{h_n}(x_0-X)\mathbb{1}_{\{\overline{F}_{Z_{1-t_j}}(Z_{1-t_j}|x_0)<\frac{k}{n}y_j\}}\right)^3\right]\right. \\ & +3\mathbb{E}\left[\left(\sum_{j=1}^J|\psi_j|K_{h_n}(x_0-X)\mathbb{1}_{\{\overline{F}_{Z_{1-t_j}}(Z_{1-t_j}|x_0)<\frac{k}{n}y_j\}}\right)^2\right] \\ & \times\mathbb{E}\left(\sum_{j=1}^J|\psi_j|K_{h_n}(x_0-X)\mathbb{1}_{\{\overline{F}_{Z_{1-t_j}}(Z_{1-t_j}|x_0)<\frac{k}{n}y_j\}}\right) \\ & \left.+4\left[\mathbb{E}\left(\sum_{j=1}^J|\psi_j|K_{h_n}(x_0-X)\mathbb{1}_{\{\overline{F}_{Z_{1-t_j}}(Z_{1-t_j}|x_0)<\frac{k}{n}y_j\}}\right)\right]^3\right\}.\end{aligned}$$

With arguments similar to the ones used above when studying the covariance terms, one finds that $\mathbb{E}|V_{1,n}|^3 = O(1/(n\sqrt{kh_n^d}))$, and hence $n\mathbb{E}|V_{1,n}|^3\to 0$. This establishes the finite dimensional convergence. The joint tightness follows from the individual tightness (similarly to Lemma 1 in Bai and Taqqu, 2013). □

2 Proof of Lemma 2

Note that

$$\frac{\widehat{\overline{F}}_{Z_{1-t_j}}(u_n^{(j)}|x_0)}{\overline{F}_{Z_{1-t_j}}(u_n^{(j)}|x_0)} = \frac{1}{\frac{1}{n}\sum_{i=1}^n K_{h_n}(x_0-X_i)} \frac{\frac{1}{n}\sum_{i=1}^n K_{h_n}(x_0-X_i)\mathbb{1}_{\{Z_{1-t_j,i}>u_n^{(j)}\}}}{\overline{F}_{Z_{1-t_j}}(u_n^{(j)}|x_0)}$$

$$= \frac{1}{\frac{1}{n}\sum_{i=1}^n K_{h_n}(x_0-X_i)} \frac{\overline{F}_{Z_{1-t_j}}\left(U_{Z_{1-t_j}}(n/k|x_0)|x_0\right)}{\overline{F}_{Z_{1-t_j}}(u_n^{(j)}|x_0)} T_{n,1-t_j}\left(\frac{n}{k}\overline{F}_{Z_{1-t_j}}\left(u_n^{(j)}|x_0\right)\Big|x_0\right).$$

Then, by using Cramér-Wold device, we have for all $\psi := (\psi_1,\dots,\psi_J)^T \in \mathbb{R}^J$

$$\sum_{j=1}^J \psi_j \sqrt{nh_n^d \overline{F}_{Z_{1-t_j}}(u_n^{(j)}|x_0)} \left[\frac{\widehat{\overline{F}}_{Z_{1-t_J}}(u_n^{(j)}|x_0)}{\overline{F}_{Z_{1-t_J}}(u_n^{(j)}|x_0)} - 1\right]$$

$$= \sum_{j=1}^J \psi_j \sqrt{nh_n^d \overline{F}_{Z_{1-t_j}}(u_n^{(j)}|x_0)}$$

$$\times \left[\frac{1}{\frac{1}{n}\sum_{i=1}^n K_{h_n}(x_0-X_i)} \frac{\overline{F}_{Z_{1-t_j}}\left(U_{Z_{1-t_j}}(n/k|x_0)|x_0\right)}{\overline{F}_{Z_{1-t_j}}(u_n^{(j)}|x_0)} T_{n,1-t_j}\left(\frac{n}{k}\overline{F}_{Z_{1-t_j}}\left(u_n^{(j)}|x_0\right)\Big|x_0\right) - 1\right]$$

$$= \sum_{j=1}^J \psi_j \frac{\sqrt{kh_n^d}\left[T_{n,1-t_j}\left(\frac{n}{k}\overline{F}_{Z_{1-t_j}}\left(u_n^{(j)}|x_0\right)\Big|x_0\right) - \frac{n}{k}\overline{F}_{Z_{1-t_j}}(u_n^{(j)}|x_0)f_X(x_0)\right] - W_{1-t_j}\left(\frac{n}{k}\overline{F}_{Z_{1-t_j}}\left(u_n^{(j)}|x_0\right)\right)}{\frac{1}{n}\sum_{i=1}^n K_{h_n}(x_0-X_i)}$$

$$+ \sum_{j=1}^J \psi_j \sqrt{kh_n^d}\left(\sqrt{\frac{\overline{F}_{Z_{1-t_j}}\left(U_{Z_{1-t_j}}(n/k|x_0)|x_0\right)}{\overline{F}_{Z_{1-t_j}}(u_n^{(j)}|x_0)}} - 1\right)$$

$$\times \frac{T_{n,1-t_j}\left(\frac{n}{k}\overline{F}_{Z_{1-t_j}}\left(u_n^{(j)}|x_0\right)\Big|x_0\right) - \frac{n}{k}\overline{F}_{Z_{1-t_j}}(u_n^{(j)}|x_0)f_X(x_0)}{\frac{1}{n}\sum_{i=1}^n K_{h_n}(x_0-X_i)}$$

$$+ \sum_{j=1}^J \psi_j \frac{W_{1-t_j}\left(\frac{n}{k}\overline{F}_{Z_{1-t_j}}\left(u_n^{(j)}|x_0\right)\right) - W_{1-t_j}(1)}{\frac{1}{n}\sum_{i=1}^n K_{h_n}(x_0-X_i)} + \sum_{j=1}^J \psi_j \frac{W_{1-t_j}(1)}{\frac{1}{n}\sum_{i=1}^n K_{h_n}(x_0-X_i)}$$

$$-\sqrt{\frac{k}{n}} \frac{\sqrt{nh_n^d}\left[\frac{1}{n}\sum_{i=1}^n K_{h_n}(x_0-X_i) - f_X(x_0)\right]}{\frac{1}{n}\sum_{i=1}^n K_{h_n}(x_0-X_i)} \sum_{j=1}^J \psi_j \sqrt{\frac{\overline{F}_{Z_{1-t_j}}(u_n^{(j)}|x_0)}{\overline{F}_{Z_{1-t_j}}\left(U_{Z_{1-t_j}}(n/k|x_0)|x_0\right)}}$$

$$\rightsquigarrow \sum_{j=1}^J \psi_j \frac{W_{1-t_j}(1)}{f_X(x_0)},$$

under our assumptions combined with Lemma 1 and the Skorohod construction. □

3 Proof of Lemma 3

First, remark that

$$\mathbb{P}\left(\sqrt{kh_n^d}\left(\frac{\widehat{U}_{Z_{1-t_j}}(n/k|x_0)}{U_{Z_{1-t_j}}(n/k|x_0)}-1\right)\leqslant z_j, \forall j=1,\ldots,J\right)$$
$$=\mathbb{P}\left(\sqrt{nh_n^d\overline{F}_{Z_{1-t_j}}(a_n^{(j)}|x_0)}\left(\frac{\widehat{\overline{F}}_{Z_{1-t_j}}(a_n^{(j)}|x_0)}{\overline{F}_{Z_{1-t_j}}(a_n^{(j)}|x_0)}-1\right)\right.$$
$$\left.\leqslant \sqrt{nh_n^d\overline{F}_{Z_{1-t_j}}(a_n^{(j)}|x_0)}\left(\frac{\overline{F}_{Z_{1-t_j}}(U_{Z_{1-t_j}}(\frac{n}{k}|x_0)|x_0)}{\overline{F}_{Z_{1-t_j}}(a_n^{(j)}|x_0)}-1\right), \forall j=1,\ldots,J\right),$$

where $a_n^{(j)}:=U_{Z_{1-t_j}}(n/k|x_0)(1+z_j/\sqrt{kh_n^d})$. Then, using Lemma 2, it remains to show that, for all $j=1,\ldots,J$, we have

$$\sqrt{kh_n^d}\left(\frac{\overline{F}_{Z_{1-t_j}}(U_{Z_{1-t_j}}(n/k|x_0)|x_0)}{\overline{F}_{Z_{1-t_j}}(a_n^{(j)}|x_0)}-1\right) \longrightarrow c_j\in\mathbb{R}.$$

This convergence holds since, under our assumptions, we have

$$\begin{aligned}
&\sqrt{kh_n^d}\left(\frac{\overline{F}_{Z_{1-t_j}}(U_{Z_{1-t_j}}(n/k|x_0)|x_0)}{\overline{F}_{Z_{1-t_j}}(a_n^{(j)}|x_0)}-1\right)\\
&\quad = \quad z_j\,\frac{1+\delta_{1-t_j}(U_{Z_{1-t_j}}(n/k|x_0)|x_0)}{1+\delta_{1-t_j}(a_n^{(j)}|x_0)}\\
&\qquad -\sqrt{kh_n^d}\,\frac{\delta_{1-t_j}(U_{Z_{1-t_j}}(n/k|x_0)|x_0)}{1+\delta_{1-t_j}(a_n^{(j)}|x_0)}\left(\frac{\delta_{1-t_j}(a_n^{(j)}|x_0)}{\delta_{1-t_j}(U_{Z_{1-t_j}}(n/k|x_0)|x_0)}-1\right)\\
&\quad \longrightarrow \quad z_j.
\end{aligned}$$

Now, since

$$\sqrt{nh_n^d\overline{F}_{Z_{1-t_j}}(a_n^{(j)}|x_0)}\left(\frac{\overline{F}_{Z_{1-t_j}}(U_{Z_{1-t_j}}(n/k|x_0)|x_0)}{\overline{F}_{Z_{1-t_j}}(a_n^{(j)}|x_0)}-1\right) \longrightarrow z_j,$$

by Lemma 2 and continuity we deduce that

$$\mathbb{P}\left(\sqrt{kh_n^d}\left(\frac{\widehat{U}_{Z_{1-t_j}}(n/k|x_0)}{U_{Z_{1-t_j}}(n/k|x_0)}-1\right)\leqslant z_j, \forall j=1,\ldots,J\right)\longrightarrow \mathbb{P}\left(\frac{W_{1-t_j}(1)}{f_X(x_0)}\leqslant z_j, \forall j=1,\ldots,J\right).$$

This achieves the proof of Lemma 3. □

4 MDPD calculations and proof of Theorem 4

4.1 Derivatives

We need to compute the two first derivatives of the empirical divergence. Direct computations yield

$$\begin{aligned}
&\frac{d\widehat{\Delta}_{\alpha,1-t}(\delta_{1-t}|x_0)}{d\delta_{1-t}} \\
&= (1+\alpha)\Bigg\{\int_1^\infty h^\alpha(y;\delta_{1-t},\beta)\frac{d}{d\delta_{1-t}}h(y;\delta_{1-t},\beta)dy\frac{1}{k}\sum_{i=1}^n K_{h_n}(x_0-X_i)\mathbb{1}_{\{Z_{1-t,i}>\widehat{U}_{Z_{1-t}}(n/k|x_0)\}} \\
&\qquad -\frac{1}{k}\sum_{i=1}^n K_{h_n}(x_0-X_i)h^{\alpha-1}\left(\frac{Z_{1-t,i}}{\widehat{U}_{Z_{1-t}}(n/k|x_0)};\delta_{1-t},\beta\right) \\
&\qquad\qquad \times\frac{d}{d\delta_{1-t}}h\left(\frac{Z_{1-t,i}}{\widehat{U}_{Z_{1-t}(n/k|x_0)}};\delta_{1-t},\beta\right)\mathbb{1}_{\{Z_{1-t,i}>\widehat{U}_{Z_{1-t}(n/k|x_0)}\}}\Bigg\}
\end{aligned}$$

$$\begin{aligned}
&= (1+\alpha)\Bigg\{\frac{2\alpha\beta}{(1+2\alpha)(1+2\alpha+\beta)}\frac{1}{k}\sum_{i=1}^n K_{h_n}(x_0-X_i)\mathbb{1}_{\{Z_{1-t,i}>\widehat{U}_{Z_{1-t}}(n/k|x_0)\}} \\
&\qquad +\frac{1}{k}\sum_{i=1}^n K_{h_n}(x_0-X_i)\left(\frac{Z_{1-t,i}}{\widehat{U}_{Z_{1-t}}(n/k|x_0)}\right)^{-2\alpha}\mathbb{1}_{\{Z_{1-t,i}>\widehat{U}_{Z_{1-t}}(n/k|x_0)\}} \\
&\qquad -(1+\beta)\frac{1}{k}\sum_{i=1}^n K_{h_n}(x_0-X_i)\left(\frac{Z_{1-t,i}}{\widehat{U}_{Z_{1-t}}(n/k|x_0)}\right)^{-2\alpha-\beta}\mathbb{1}_{\{Z_{1-t,i}>\widehat{U}_{Z_{1-t}}(n/k|x_0)\}} \\
&\qquad +O_{\mathbb{P}}(\delta_{1-t})\Bigg\}
\end{aligned}$$

and

$$
\begin{aligned}
&\frac{d^2\widehat{\Delta}_{\alpha,1-t}(\delta_{1-t}|x_0)}{d\delta_{1-t}^2}\\
= \quad & (1+\alpha)\left\{\alpha\int_1^\infty h^{\alpha-1}(y;\delta_{1-t},\beta)\left(\frac{d}{d\delta_{1-t}}h(y;\delta_{1-t},\beta)\right)^2 dy\frac{1}{k}\sum_{i=1}^n K_{h_n}(x_0-X_i)1\!\!1_{\{Z_{1-t,i}>\widehat{U}_{Z_{1-t}}(n/k|x_0)\}}\right.\\
&\quad +\int_1^\infty h^{\alpha}(y;\delta_{1-t},\beta)\frac{d^2}{d\delta_{1-t}^2}h(y;\delta_{1-t},\beta)dy\frac{1}{k}\sum_{i=1}^n K_{h_n}(x_0-X_i)1\!\!1_{\{Z_{1-t,i}>\widehat{U}_{Z_{1-t}}(n/k|x_0)\}}\\
&\quad -\frac{\alpha-1}{k}\sum_{i=1}^n K_{h_n}(x_0-X_i)h^{\alpha-2}\left(\frac{Z_{1-t,i}}{\widehat{U}_{Z_{1-t}}(n/k|x_0)};\delta_{1-t},\beta\right)\\
&\qquad\qquad \times\left(\frac{d}{d\delta_{1-t}}h\left(\frac{Z_{1-t,i}}{\widehat{U}_{Z_{1-t}}(n/k|x_0)};\delta_{1-t},\beta\right)\right)^2 1\!\!1_{\{Z_{1-t,i}>\widehat{U}_{Z_{1-t}(n/k|x_0)}\}}\\
&\quad -\frac{1}{k}\sum_{i=1}^n K_{h_n}(x_0-X_i)h^{\alpha-1}\left(\frac{Z_{1-t,i}}{\widehat{U}_{Z_{1-t}}(n/k|x_0)};\delta_{1-t},\beta\right)\\
&\qquad\qquad \left.\times\frac{d^2}{d\delta_{1-t}^2}h\left(\frac{Z_{1-t,i}}{\widehat{U}_{Z_{1-t(n/k|x_0)}}};\delta_{1-t},\beta\right)1\!\!1_{\{Z_{1-t,i}>\widehat{U}_{Z_{1-t(n/k|x_0)}}\}}\right\}\\
= \quad & (1+\alpha)\left\{\frac{\alpha\beta^2(-7+\beta+4\alpha^2+2\alpha\beta)}{(1+2\alpha)(1+2\alpha+\beta)(1+2\alpha+2\beta)}\frac{1}{k}\sum_{i=1}^n K_{h_n}(x_0-X_i)1\!\!1_{\{Z_{1-t,i}>\widehat{U}_{Z_{1-t}}(n/k|x_0)\}}\right.\\
&-\frac{1+\alpha}{k}\sum_{i=1}^n K_{h_n}(x_0-X_i)\left(\frac{Z_{1-t,i}}{\widehat{U}_{Z_{1-t}}(n/k|x_0)}\right)^{-2\alpha}1\!\!1_{\{Z_{1-t,i}>\widehat{U}_{Z_{1-t}}(n/k|x_0)\}}\\
&+2(1+\alpha)(1+\beta)\frac{1}{k}\sum_{i=1}^n K_{h_n}(x_0-X_i)\left(\frac{Z_{1-t,i}}{\widehat{U}_{Z_{1-t}}(n/k|x_0)}\right)^{-2\alpha-\beta}1\!\!1_{\{Z_{1-t,i}>\widehat{U}_{Z_{1-t}}(n/k|x_0)\}}\\
&-[1+\alpha+2\beta+2\alpha\beta+\alpha\beta^2-\beta^2]\\
&\qquad\times\frac{1}{k}\sum_{i=1}^n K_{h_n}(x_0-X_i)\left(\frac{Z_{1-t,i}}{\widehat{U}_{Z_{1-t}}(n/k|x_0)}\right)^{-2\alpha-2\beta}1\!\!1_{\{Z_{1-t,i}>\widehat{U}_{Z_{1-t}}(n/k|x_0)\}}\\
&\left.+O_{\mathbb{P}}\left(\delta_{1-t}\right)\right\}.
\end{aligned}
$$

4.2 Asymptotic behavior of the derivatives

In this section we work under the assumptions of Theorem 1.

$$\sqrt{kh_n^d}\,\frac{d\widehat{\Delta}_{\alpha,1-t}(\delta_{1-t}|x_0)}{d\delta_{1-t}}\Big|_{\delta_{1-t}=\delta_{1-t}(\widehat{U}_{Z_{1-t}}(n/k|x_0)|x_0)}$$
$$= (1+\alpha)\left\{\frac{2\alpha\beta}{(1+2\alpha)(1+2\alpha+\beta)}\left(\sqrt{kh_n^d}\,[s_{n,1-t}(1|x_0)-1]\,f_X(x_0)+W_{1-t}(1)\right)\right.$$
$$+\frac{2\alpha\beta}{(1+2\alpha)(1+2\alpha+\beta)}\left(\sqrt{kh_n^d}\,[T_{n,1-t}(s_{n,1-t}(1|x_0)|x_0)-s_{n,1-t}(1|x_0)f_X(x_0)]-W_{1-t}\left(s_{n,1-t}(1|x_0)\right)\right)$$
$$+\frac{2\alpha\beta}{(1+2\alpha)(1+2\alpha+\beta)}\left[W_{1-t}\left(s_{n,1-t}(1|x_0)\right)-W_{1-t}(1)\right]+\sqrt{kh_n^d}\left[S_{n,1-t}(-2\alpha|x_0)-\frac{1}{1+2\alpha}f_X(x_0)\right]$$
$$-(1+\beta)\sqrt{kh_n^d}\left[S_{n,1-t}(-2\alpha-\beta|x_0)-\frac{1}{1+2\alpha+\beta}f_X(x_0)\right]$$
$$\left.+O_{\mathbb{P}}\left(\sqrt{kh_n^d}\,\delta_{1-t}\left(\widehat{U}_{Z_{1-t}}(n/k|x_0)\Big|x_0\right)\right)\right\}.$$

Now, remark that by Proposition B.1.10 in de Haan and Ferreira (2006), for n large, with arbitrary large probability, we have for $\varepsilon,\xi>0$

$$\begin{aligned}
&\sqrt{kh_n^d}\,\left|\delta_{1-t}\left(\widehat{U}_{Z_{1-t}}(n/k|x_0)\Big|x_0\right)\right|\\
&= \sqrt{kh_n^d}\,\left|\delta_{1-t}\left(U_{Z_{1-t}}(n/k|x_0)\Big|x_0\right)\right|\left|\frac{\delta_{1-t}\left(\widehat{U}_{Z_{1-t}}(n/k|x_0)\Big|x_0\right)}{\delta_{1-t}\left(U_{Z_{1-t}}(n/k|x_0)\Big|x_0\right)}\right|\\
&\leqslant \sqrt{kh_n^d}\,\left|\delta_{1-t}\left(U_{Z_{1-t}}(n/k|x_0)\Big|x_0\right)\right|\left[\varepsilon\left(\frac{\widehat{U}_{Z_{1-t}}(n/k|x_0)}{U_{Z_{1-t}}(n/k|x_0)}\right)^{-\beta(x_0)\pm\xi}+\left(\frac{\widehat{U}_{Z_{1-t}}(n/k|x_0)}{U_{Z_{1-t}}(n/k|x_0)}\right)^{-\beta(x_0)}\right]\\
&= o_{\mathbb{P}}(1).
\end{aligned}$$

Using Lemma 5.2 from Goegebeur et al. (2021), Lemma 1 and (16) in the proof of Theorem 3, both combined with the Skorohod construction, and Theorem 3, yields

$$\begin{aligned}
&\sqrt{kh_n^d}\,\frac{d\widehat{\Delta}_{\alpha,1-t}(\delta_{1-t}|x_0)}{d\delta_{1-t}}\Big|_{\delta_{1-t}=\delta_{1-t}(\widehat{U}_{Z_{1-t}}(n/k|x_0)|x_0)}\\
&\quad\rightsquigarrow -2\alpha(1+\alpha)\int_0^1\left[\frac{W_{1-t}(z)}{z}-W_{1-t}(1)\right]z^{2\alpha}\,dz\\
&\qquad\qquad+(1+\beta)(2\alpha+\beta)(1+\alpha)\int_0^1\left[\frac{W_{1-t}(z)}{z}-W_{1-t}(1)\right]z^{2\alpha+\beta}\,dz. \qquad \text{(S2)}
\end{aligned}$$

Also

$$
\begin{aligned}
&\frac{d^2\widehat{\Delta}_{\alpha,1-t}(\delta_{1-t}|x_0)}{d\delta_{1-t}^2}\Big|_{\delta_{1-t}=\delta_{1-t}(\widehat{U}_{Z_{1-t}}(n/k|x_0)|x_0)} \\
&= \frac{\beta^2(1+\alpha)(1+\beta+4\alpha^2+2\alpha\beta)}{(1+2\alpha)(1+2\alpha+\beta)(1+2\alpha+2\beta)} f_X(x_0) \\
&\quad + \frac{\alpha\beta^2(1+\alpha)(-7+\beta+4\alpha^2+2\alpha\beta)}{(1+2\alpha)(1+2\alpha+\beta)(1+2\alpha+2\beta)} \left[T_{n,1-t}(s_{n,1-t}(1|x_0)|x_0) - s_{n,1-t}(1|x_0) f_X(x_0)\right] \\
&\quad + \frac{\alpha\beta^2(1+\alpha)(-7+\beta+4\alpha^2+2\alpha\beta)}{(1+2\alpha)(1+2\alpha+\beta)(1+2\alpha+2\beta)} \left[s_{n,1-t}(1|x_0) - 1\right] f_X(x_0) \\
&\quad - (1+\alpha)^2 \left[S_{n,1-t}(-2\alpha|x_0) - \frac{f_X(x_0)}{1+2\alpha}\right] \\
&\quad + 2(1+\alpha)^2(1+\beta) \left[S_{n,1-t}(-2\alpha-\beta|x_0) - \frac{f_X(x_0)}{1+2\alpha+\beta}\right] \\
&\quad - (1+\alpha)\left[1+\alpha+2\beta+2\alpha\beta+\alpha\beta^2-\beta^2\right] \\
&\qquad\qquad \times \left[S_{n,1-t}(-2\alpha-2\beta|x_0) - \frac{f_X(x_0)}{1+2\alpha+2\beta}\right] \\
&\quad + O_{\mathbb{P}}\left(\delta_{1-t}\left(\widehat{U}_{Z_{1-t}}(n/k|x_0)\Big|x_0\right)\right) \\
&= \frac{\beta^2(1+\alpha)\left[1+\beta+4\alpha^2+2\alpha\beta\right]}{(1+2\alpha)(1+2\alpha+\beta)(1+2\alpha+2\beta)} f_X(x_0) + o_{\mathbb{P}}(1). \qquad \text{(S3)}
\end{aligned}
$$

4.3 Proof of Theorem 4

We limit the proof to deriving the asymptotic properties of a single MDPD estimator $\widehat{\delta}_{n,1-t}$. The joint asymptotic behavior of $(\widehat{\delta}_{n,1-t_1}, \ldots, \widehat{\delta}_{n,1-t_J})$ follows then from Theorem 3.

(i) Existence and consistency

The idea is to adjust the arguments used to prove existence and consistency of solutions of the likelihood estimating equations, see, for instance, Theorems 3.7 and 5.1 in Chapter 6 of Lehmann and Casella (1998), to the MDPD framework. To this aim, we start to prove that, for any $r > 0$ sufficiently small, we have

$$
\begin{aligned}
\mathbb{P}\left(\widehat{\Delta}_{\alpha,1-t}\left(\delta_{1-t}\left(\widehat{U}_{Z_{1-t}}(n/k|x_0)\Big|x_0\right)\Big|x_0\right) < \widehat{\Delta}_{\alpha,1-t}(\delta_{1-t}|x_0) \text{ for } \delta_{1-t} = \delta_{1-t}(\widehat{U}_{Z_{1-t}}(n/k|x_0)|x_0) \pm r\right) \\
\longrightarrow 1, \qquad \text{(S4)}
\end{aligned}
$$

as $n \to \infty$. By applying a Taylor series expansion, we have

$$
\begin{aligned}
&\widehat{\Delta}_{\alpha,1-t}(\delta_{1-t}|x_0) - \widehat{\Delta}_{\alpha,1-t}(\delta_{1-t}(\widehat{U}_{Z_{1-t}}(n/k|x_0)|x_0)|x_0) \\
&\quad= \left(\delta_{1-t} - \delta_{1-t}(\widehat{U}_{Z_{1-t}}(n/k|x_0)|x_0)\right) \left.\frac{d\widehat{\Delta}_{\alpha,1-t}(\delta_{1-t}|x_0)}{d\delta_{1-t}}\right|_{\delta_{1-t}=\delta_{1-t}(\widehat{U}_{Z_{1-t}}(n/k|x_0)|x_0)} \\
&\qquad+\frac{1}{2}\left(\delta_{1-t} - \delta_{1-t}(\widehat{U}_{Z_{1-t}}(n/k|x_0)|x_0)\right)^2 \left.\frac{d^2\widehat{\Delta}_{\alpha,1-t}(\delta_{1-t}|x_0)}{d\delta_{1-t}^2}\right|_{\delta_{1-t}=\delta_{1-t}(\widehat{U}_{Z_{1-t}}(n/k|x_0)|x_0)} \\
&\qquad+\frac{1}{6}\left(\delta_{1-t} - \delta_{1-t}(\widehat{U}_{Z_{1-t}}(n/k|x_0)|x_0)\right)^3 \left.\frac{d^3\widehat{\Delta}_{\alpha,1-t}(\delta_{1-t}|x_0)}{d\delta_{1-t}^3}\right|_{\delta_{1-t}=\widetilde{\delta}_{1-t}} \\
&\quad=: T_{1,n} + T_{2,n} + T_{3,n},
\end{aligned}
$$

where $\widetilde{\delta}_{1-t}$ is an intermediate value between δ_{1-t} and $\delta_{1-t}(\widehat{U}_{Z_{1-t}}(n/k|x_0)|x_0)$. According to (S2), we have $|T_{1,n}| \leqslant r^3$, whereas according to (S3), there exists $c > 0$ such that $T_{2,n} > cr^2$ with probability tending to 1. Additionally, tedious computations allow us to show that

$$
\sup_{\delta_{1-t}\in\left[\delta_{1-t}(\widehat{U}_{Z_{1-t}}(n/k|x_0)|x_0)-r;\delta_{1-t}(\widehat{U}_{Z_{1-t}}(n/k|x_0)|x_0)+r\right]} \left|\frac{d^3\widehat{\Delta}_{\alpha,1-t}(\delta_{1-t}|x_0)}{d\delta_{1-t}^3}\right| < M,
$$

with arbitrary large probability, from which we can deduce that $|T_{3,n}| \leqslant Mr^3/6$ with probability tending to 1.
Combining all these bounds, we deduce that with probability tending to 1, we have

$$
T_{1,n} + T_{2,n} + T_{3,n} > cr^2 - (1 + M/6)r^3,
$$

which yields (S4).

To complete the proof, we adjust the line of argumentation of Theorem 3.7 in Chapter 6 of Lehmann and Casella (1998). To this aim, for any $r > 0$ sufficiently small, we define

$$
S_n(r) := \left\{\widehat{\Delta}_{\alpha,1-t}\left(\delta_{1-t}\left(\widehat{U}_{Z_{1-t}}(n/k|x_0)\middle|x_0\right)\middle|x_0\right) < \widehat{\Delta}_{\alpha,1-t}(\delta_{1-t}|x_0) \text{ for } \delta_{1-t} = \delta_{1-t}(\widehat{U}_{Z_{1-t}}(n/k|x_0)|x_0) \pm r\right\}.
$$

For $v \in S_n(r)$, since $\widehat{\Delta}_{\alpha,1-t}\left(\delta_{1-t}|x_0\right)$ is differentiable with respect to δ_{1-t}, there exists

$$
\widetilde{\delta}_{n,1-t} \in \left(\delta_{1-t}(\widehat{U}_{Z_{1-t}}(n/k|x_0)|x_0) - r; \delta_{1-t}(\widehat{U}_{Z_{1-t}}(n/k|x_0)|x_0) + r\right)
$$

where $\widehat{\Delta}_{\alpha,1-t}\left(\delta_{1-t}|x_0\right)$ achieves a local minimum.
According to (S4), we have $\mathbb{P}(S_n(r)) \to 1$ for any small enough r, and hence there exists a sequence $r_n \downarrow 0$ such that $\mathbb{P}(S_n(r_n)) \to 1$ as $n \to \infty$. Now, let $\widehat{\delta}_{n,1-t} := \widetilde{\delta}_{n,1-t}$ if $v \in S_n(r_n)$ and arbitrary otherwise. Since $v \in S_n(r_n)$ implies

$$
\left.\frac{d\widehat{\Delta}_{\alpha,1-t}(\delta_{1-t}|x_0)}{d\delta_{1-t}}\right|_{\delta_{1-t}=\widehat{\delta}_{n,1-t}} = 0,
$$

we have

$$\mathbb{P}\left(\left.\frac{d\widehat{\Delta}_{\alpha,1-t}(\delta_{1-t}|x_0)}{d\delta_{1-t}}\right|_{\delta_{1-t}=\widehat{\delta}_{n,1-t}} = 0\right) \geqslant \mathbb{P}(S_n(r_n)) \longrightarrow 1,$$

as $n \to \infty$, which establishes the existence part.

Concerning now the consistency part, note that for any $r > 0$ and n large enough such that $r_n \leqslant r$, we have

$$\begin{aligned}\mathbb{P}\left(\left|\widehat{\delta}_{n,1-t} - \delta_{1-t}(\widehat{U}_{Z_{1-t}}(n/k|x_0)|x_0)\right| < r\right) &\geqslant \mathbb{P}\left(\left|\widehat{\delta}_{n,1-t} - \delta_{1-t}(\widehat{U}_{Z_{1-t}}(n/k|x_0)|x_0)\right| < r_n\right) \\ &\geqslant \mathbb{P}(S_n(r_n)) \longrightarrow 1,\end{aligned}$$

as $n \to \infty$, whence the consistency of the estimator sequence.

(ii) Asymptotic normality

By definition we have $\left.\frac{d\widehat{\Delta}_{\alpha,1-t}(\delta_{1-t}|x_0)}{d\delta_{1-t}}\right|_{\delta_{1-t}=\widehat{\delta}_{n,1-t}} = 0$. Thus a Taylor series expansion around $\delta_{1-t}(\widehat{U}_{Z_{1-t}}(n/k|x_0)|x_0)$ combined with the boundedness of the third derivative of $\widehat{\Delta}_{\alpha,1-t}(\delta_{1-t}|x_0)$ with respect to δ_{1-t} leads to

$$\begin{aligned}0 = &\left.\frac{d\widehat{\Delta}_{\alpha,1-t}(\delta_{1-t}|x_0)}{d\delta_{1-t}}\right|_{\delta_{1-t}=\delta_{1-t}(\widehat{U}_{Z_{1-t}}(n/k|x_0)|x_0)} \\ &+\left.\frac{d^2\widehat{\Delta}_{\alpha,1-t}(\delta_{1-t}|x_0)}{d\delta_{1-t}^2}\right|_{\delta_{1-t}=\delta_{1-t}(\widehat{U}_{Z_{1-t}}(n/k|x_0)|x_0)} \left(\widehat{\delta}_{n,1-t} - \delta_{1-t}(\widehat{U}_{Z_{1-t}}(n/k|x_0)|x_0)\right)(1+o_{\mathbb{P}}(1))\end{aligned}$$

from which we deduce that

$$\begin{aligned}&\sqrt{kh_n^d}\left(\widehat{\delta}_{n,1-t} - \delta_{1-t}(\widehat{U}_{Z_{1-t}}(n/k|x_0)|x_0)\right) \\ &\quad = -\left(\left.\frac{d^2\widehat{\Delta}_{\alpha,1-t}(\delta_{1-t}|x_0)}{d\delta_{1-t}^2}\right|_{\delta_{1-t}=\delta_{1-t}(\widehat{U}_{Z_{1-t}}(n/k|x_0)|x_0)}\right)^{-1} \\ &\qquad\qquad \times\sqrt{kh_n^d}\left.\frac{d\widehat{\Delta}_{\alpha,1-t}(\delta_{1-t}|x_0)}{d\delta_{1-t}}\right|_{\delta_{1-t}=\delta_{1-t}(\widehat{U}_{Z_{1-t}}(n/k|x_0)|x_0)} (1+o_{\mathbb{P}}(1)) \\ &\quad \rightsquigarrow c\left\{2\alpha\int_0^1\left[\frac{W_{1-t}(z)}{z} - W_{1-t}(1)\right]z^{2\alpha}\,dz\right. \\ &\qquad\qquad \left. -(1+\beta)(2\alpha+\beta)\int_0^1\left[\frac{W_{1-t}(z)}{z} - W_{1-t}(1)\right]z^{2\alpha+\beta}\,dz\right\}. \;\square\end{aligned}$$

5 Auxiliary results in case of unknown margins

We introduce the same key statistic as $S_{n,1-t}(s|x_0)$ but, this time, defined with Z_{1-t} replaced by $\breve{Z}_{1-t}$, that is

$$\breve{S}_{n,1-t}(s|x_0) := \frac{1}{k}\sum_{i=1}^{n} K_{h_n}(x_0 - X_i)\left(\frac{\breve{Z}_{1-t,i}}{\widehat{U}_{\breve{Z}_{1-t}}(n/k|x_0)}\right)^s 1\!\!1_{\{\breve{Z}_{1-t,i}>\widehat{U}_{\breve{Z}_{1-t}}(n/k|x_0)\}}.$$

Similarly as in (10) of the present paper, assuming that $F_{Z_{1-t}}(y|x_0)$ is strictly increasing in y, we can show that

$$\breve{S}_{n,1-t}(s|x_0) = \breve{T}_{n,1-t}\left(\breve{s}_{n,1-t}(1|x_0)|x_0\right) + \int_0^1 \breve{T}_{n,1-t}\left(\breve{s}_{n,1-t}(z|x_0)|x_0\right)\, s\, z^{-1-s}\, dz,$$

where

$$\begin{aligned}
\breve{T}_{n,1-t}(y|x_0) &:= \frac{1}{k}\sum_{i=1}^{n} K_{h_n}(x_0 - X_i) 1\!\!1_{\{\overline{F}_{Z_{1-t}}(\breve{Z}_{1-t,i}|x_0)<\frac{k}{n}y\}}, y \in (0, T],\\
\breve{s}_{n,1-t}(z|x_0) &:= \frac{n}{k}\overline{F}_{Z_{1-t}}\left(z^{-1}\widehat{U}_{\breve{Z}_{1-t}}(n/k|x_0)\middle|x_0\right).
\end{aligned}$$

The aim of the next theorem is to show the joint weak convergence of $\breve{S}_{n,1-t_j}(s_j|x_0)$, $j = 1, \ldots, M$.

Theorem S1 *Under the same conditions as Theorem 2, we have for $s_1, \ldots, s_M < 0$,*

$$\sqrt{kh_n^d}\begin{pmatrix} \breve{S}_{n,1-t_1}(s_1|x_0) - \frac{1}{1-s_1}f_X(x_0) \\ \vdots \\ \breve{S}_{n,1-t_M}(s_M|x_0) - \frac{1}{1-s_M}f_X(x_0)\end{pmatrix} \rightsquigarrow \begin{pmatrix} s_1\int_0^1 \left[\frac{W_{1-t_1}(z)}{z} - W_{1-t_1}(1)\right] z^{-s_1}dz \\ \vdots \\ s_M\int_0^1 \left[\frac{W_{1-t_M}(z)}{z} - W_{1-t_M}(1)\right] z^{-s_M}dz\end{pmatrix},$$

where the processes W_{1-t_j}, $j = 1, \ldots, M$, are as in Theorem 1.

Note that this limiting process is the same as the one obtained in Theorem 3, with additional conditions needed to measure the discrepancy between the conditional distribution function $F_j(y|x)$ and its empirical kernel version $\widehat{F}_{n,j}(y|x)$, $j = 1, 2$, uniformly in (x, y).

To prove Theorem S1, we need, as a preliminary result, the weak convergence of statistics $\breve{T}_{n,1-t}(y|x_0)$. This is the aim of the next proposition. Let

$$\breve{W}_{n,1-t_j} := \left\{\sqrt{kh_n^d}\left(\breve{T}_{n,1-t_j}(y|x_0) - yf_X(x_0)\right); y \in (0, T]\right\},$$

for $j = 1, \ldots, J$.

Proposition S1 *Assume that there exists $b > 0$ such that $f_X(x) \geqslant b, \forall x \in S_X \subset \mathbb{R}^d$, f_X is bounded, $(\mathcal{D}_{1-t_j})$, $(\mathcal{H}_{1-t_j})$ for $j = 1, \ldots, J$, $(\mathcal{D}_{0.5})$, $(\mathcal{H}_{0.5})$, $(\mathcal{K}_2)$, $(\mathcal{F}_m)$ hold, and that $y \mapsto F_{Z_{1-t_j}}(y|x_0)$, $j = 1, \ldots, J$, are strictly increasing at $x_0 \in Int(S_X)$ non-empty. Consider sequences $k \to \infty$, $h_n \to 0$ and $c_n \to 0$ as $n \to \infty$, such that $k/n \to 0$, $kh_n^d \to \infty$, $h_n^{\eta_{\varepsilon_{1-t_1}} \wedge \cdots \wedge \eta_{\varepsilon_{1-t_J}} \wedge \eta_{\varepsilon_{0.5}}} \log \frac{n}{k} \to 0$, $\sqrt{kh_n^d} h_n^{\eta_{f_X} \wedge \eta_{G_{1-t_1}} \wedge \cdots \wedge \eta_{G_{1-t_J}}} \to 0$, $\sqrt{kh_n^d} |\delta_{1-t_j}(U_{Z_{1-t_j}}(\frac{n}{k}|x_0)|x_0)| h_n^{\eta_{C_{1-t_j}}} \to 0$ and $\sqrt{kh_n^d} |\delta_{1-t_j}(U_{Z_{1-t_j}}(\frac{n}{k}|x_0)|x_0)| h_n^{\eta_{\varepsilon_{1-t_j}}} \log \frac{n}{k} \to 0$ for $j = 1, \ldots, J$. Under conditions (8) and (9), we have*

$$\left(\breve{W}_{n,1-t_1}, \ldots, \breve{W}_{n,1-t_J}\right) \rightsquigarrow \left(W_{1-t_1}, \ldots, W_{1-t_J}\right),$$

in $\ell^J((0,T])$, for any $T > 0$.

Then, the next ingredients required in order to proof the weak convergence of $\breve{S}_{n,1-t}(s|x_0)$ are lemmas similar to Lemma 3 from the present paper and Lemma 5.2 in Goegebeur et al. (2021), but this time for the variable $\breve{Z}_{1-t}$ instead of Z_{1-t}. This is the aim of Lemmas S2 and S3 below for which we need first to show the weak convergence of $\widehat{\overline{F}}_{\breve{Z}_{1-t}}(.|x_0)$ correctly normalized.

Lemma S1 *Under the assumptions of Proposition S1, for any sequence $u_n^{(j)}$ satisfying*

$$\sqrt{kh_n^d} \left(\frac{\overline{F}_{Z_{1-t_j}}(U_{Z_{1-t_j}}(n/k|x_0)|x_0)}{\overline{F}_{Z_{1-t_j}}(u_n^{(j)}|x_0)} - 1 \right) \to c_j \in \mathbb{R},$$

as $n \to \infty$, $j = 1, \ldots, J$, we have

$$\begin{pmatrix} \sqrt{nh_n^d \overline{F}_{Z_{1-t_1}}(u_n^{(1)}|x_0)} \left(\frac{\widehat{\overline{F}}_{\breve{Z}_{1-t_1}}(u_n^{(1)}|x_0)}{\overline{F}_{Z_{1-t_1}}(u_n^{(1)}|x_0)} - 1 \right) \\ \vdots \\ \sqrt{nh_n^d \overline{F}_{Z_{1-t_J}}(u_n^{(J)}|x_0)} \left(\frac{\widehat{\overline{F}}_{\breve{Z}_{1-t_J}}(u_n^{(J)}|x_0)}{\overline{F}_{Z_{1-t_J}}(u_n^{(J)}|x_0)} - 1 \right) \end{pmatrix} \rightsquigarrow \frac{1}{f_X(x_0)} \begin{pmatrix} W_{1-t_1}(1) \\ \vdots \\ W_{1-t_J}(1) \end{pmatrix}.$$

Lemma S2 *Assume that there exists $b > 0$ such that $f_X(x) \geqslant b, \forall x \in S_X \subset \mathbb{R}^d$, f_X is bounded, $(\mathcal{D}_{1-t_j})$, $(\mathcal{H}_{1-t_j})$, for $j = 1, \ldots, J$, $(\mathcal{D}_{0.5})$, $(\mathcal{H}_{0.5})$, $(\mathcal{K}_2)$, $(\mathcal{F}_m)$ hold, and that $y \mapsto F_{Z_{1-t_j}}(y|x_0)$, $j = 1, \ldots, J$, are strictly increasing at $x_0 \in Int(S_X)$ non-empty. Consider sequences $k \to \infty$, $h_n \to 0$ and $c_n \to 0$ as $n \to \infty$, such that $k/n \to 0$, $kh_n^d \to \infty$, $h_n^{\eta_{\varepsilon_{1-t_1}} \wedge \cdots \wedge \eta_{\varepsilon_{1-t_J}} \wedge \eta_{\varepsilon_{0.5}}} \log \frac{n}{k} \to 0$, $\sqrt{kh_n^d} h_n^{\eta_{f_X} \wedge \eta_{G_{1-t_1}} \wedge \cdots \wedge \eta_{G_{1-t_J}}} \to 0$, $\sqrt{kh_n^d} |\delta_{1-t_j}(U_{Z_{1-t_j}}(\frac{n}{k}|x_0)|x_0)| \to 0$, $j = 1, \ldots, J$. Under conditions (8) and (9), we have*

$$\sqrt{kh_n^d} \begin{pmatrix} \frac{\widehat{U}_{\breve{Z}_{1-t_1}}(n/k|x_0)}{U_{Z_{1-t_1}}(n/k|x_0)} - 1 \\ \vdots \\ \frac{\widehat{U}_{\breve{Z}_{1-t_J}}(n/k|x_0)}{U_{Z_{1-t_J}}(n/k|x_0)} - 1 \end{pmatrix} \rightsquigarrow \frac{1}{f_X(x_0)} \begin{pmatrix} W_{1-t_1}(1) \\ \vdots \\ W_{1-t_J}(1) \end{pmatrix}.$$

From Lemma S2, we can show now the uniform convergence in probability of $\breve{s}_{n,1-t}(z|x_0)$ towards z for any $z \in (0, T]$.

Lemma S3 *Under the assumptions of Lemma S2, for any $T > 0$, we have*

$$\sup_{z\in(0,T]} |\breve{s}_{n,1-t}(z|x_0) - z| = o_{\mathbb{P}}(1).$$

5.1 Proof of the auxiliary results in case of unknown margins

Proof of Proposition S1. Firstly, we consider the weak convergence of a single process $\{\sqrt{kh_n^d}(\breve{T}_{n,1-t}(y|x_0) - yf_X(x_0)); y \in (0, T]\}$, where for simplicity of notation we have ignored the index j from t. We use the decomposition

$$\begin{aligned}
&\sqrt{kh_n^d}\left(\breve{T}_{n,1-t}(y|x_0) - yf_X(x_0)\right) \\
&\quad = \sqrt{kh_n^d}\left(T_{n,1-t}(y|x_0) - yf_X(x_0)\right) \\
&\qquad + \sqrt{kh_n^d}\left(\breve{T}_{n,1-t}(y|x_0) - T_{n,1-t}(y|x_0) - \mathbb{E}\left[\breve{T}_{n,1-t}(y|x_0) - T_{n,1-t}(y|x_0)\right]\right) \\
&\qquad + \sqrt{kh_n^d}\mathbb{E}\left[\breve{T}_{n,1-t}(y|x_0) - T_{n,1-t}(y|x_0)\right] \\
&\quad =: \sum_{i=1}^{3} Q_{i,1-t}(y|x_0).
\end{aligned} \tag{S5}$$

According to Lemma 1, we have

$$Q_{1,1-t}(y|x_0) \rightsquigarrow W_{1-t}(y), \tag{S6}$$

in $\ell((0, T])$.

The next step consists in showing that

$$\sup_{y\in(0,T]} |Q_{2,1-t}(y|x_0)| = o_{\mathbb{P}}(1). \tag{S7}$$

To this aim, we will make use of empirical process theory with changing function classes, see for instance van der Vaart and Wellner (1996). We start by introducing some notation. Let P be the distribution measure of $(Y^{(1)}, Y^{(2)}, X)$, and denote the expected value under P as $Pf := \int f dP$ for any real-valued measurable function $f : \mathbb{R}^2 \times \mathbb{R}^d \to \mathbb{R}$. For a function class $\mathcal{F}$, define now the covering number $N(\mathcal{F}, L_2(Q), \tau)$ as the minimal number of $L_2(Q)$-balls of radius τ needed to cover the class of functions $\mathcal{F}$ and the uniform entropy integral as

$$J(\delta, \mathcal{F}, L_2) := \int_0^\delta \sqrt{\log \sup_{Q\in\mathcal{Q}} N(\mathcal{F}, L_2(Q), \tau\|F\|_{Q,2})}\, d\tau,$$

where $\mathcal{Q}$ is the set of all probability measures Q for which $0 < \|F\|_{Q,2}^2 := \int F^2 dQ < \infty$ and F is an envelope function for the class $\mathcal{F}$.

Let

$$\mathcal{I}_n := \{g_{y,\boldsymbol{\delta},n} : y \in (0,T], \boldsymbol{\delta} \in H\}$$

where $H := \left\{\boldsymbol{\delta} = (\delta_1, \delta_2);\ \boldsymbol{\delta} : \mathbb{R} \times \mathbb{R} \times S_X \to \mathbb{R}^2\right\}$, and

$$\begin{aligned} g_{y,\boldsymbol{\delta},n}(v_1,v_2,u) &:= \sqrt{\frac{nh_n^d}{k}} K_{h_n}(x_0-u) q_{y,\boldsymbol{\delta},n}(v_1,v_2,u), \\ q_{y,\boldsymbol{\delta},n}(v_1,v_2,u) &:= \mathbb{1}_{\{\overline{F}_{Z_{1-t}}(Z_{\boldsymbol{\delta}}(v_1,v_2,u)|x_0) < \frac{k}{n} y\}}, \end{aligned}$$

with

$$Z_{\boldsymbol{\delta}}(v_1,v_2,u) \ := \ \min\left(\frac{1}{|1-\delta_1(v_1,v_2,u)|}, \frac{1-t}{t}\frac{1}{|1-\delta_2(v_1,v_2,u)|}\right).$$

For convenience, denote $\boldsymbol{\delta}_n := \left(\widehat{F}_{n,1}, \widehat{F}_{n,2}\right)$ and $\boldsymbol{\delta}_0 := (F_1, F_2)$. According to Lemma 3.1 in Escobar-Bach et al. (2018a), if $r_n := \max(\sqrt{|\log c_n|^q / n c_n^d}, c_n^\eta)$, we have $r_n^{-1}|\boldsymbol{\delta}_n - \boldsymbol{\delta}_0|$ converges in probability towards the null function $H_0 := \{0\}$ in H, endowed with the norm $\|\boldsymbol{\delta}\|_H := \|\delta_1\|_\infty + \|\delta_2\|_\infty$ for any $\boldsymbol{\delta} \in H$. We want to apply Theorem 2.3 in van der Vaart and Wellner (2007). To this aim, we consider the class

$$\mathcal{E}_n(y,b) := \{g_{y,\boldsymbol{\delta}_0 + r_n\boldsymbol{\delta},n} - g_{y,\boldsymbol{\delta}_0,n} : \boldsymbol{\delta} \in H, \|\boldsymbol{\delta}\|_H \leqslant b\},$$

with envelope function given by

$$\begin{aligned} G_n(y,b)(v_1,v_2,u) &:= \sqrt{\frac{nh_n^d}{k}}\, K_{h_n}(x_0-u) \mathbb{1}_{\left\{\frac{1}{U_{Z_{1-t}}(n/(ky)|x_0)} \in [1-F_1(v_1|u) - r_n b; 1-F_1(v_1|u)+r_n b]\right\}} \\ &\quad + \sqrt{\frac{nh_n^d}{k}}\, K_{h_n}(x_0-u) \mathbb{1}_{\left\{\frac{1}{U_{Z_{1-t}}(n/(ky)|x_0)} \in \left[\frac{t}{1-t}(1-F_2(v_2|u) - r_n b); \frac{t}{1-t}(1-F_2(v_2|u)+r_n b)\right]\right\}} \end{aligned}$$

since

$$\begin{aligned} &|g_{y,\boldsymbol{\delta}_0+r_n\boldsymbol{\delta},n}(v_1,v_2,u) - g_{y,\boldsymbol{\delta}_0,n}(v_1,v_2,u)| \\ &= \sqrt{\frac{nh_n^d}{k}}\, K_{h_n}(x_0-u) |q_{y,\boldsymbol{\delta}_0+r_n\boldsymbol{\delta},n}(v_1,v_2,u) - q_{y,\boldsymbol{\delta}_0,n}(v_1,v_2,u)| \\ &= \sqrt{\frac{nh_n^d}{k}}\, K_{h_n}(x_0-u) \left| \mathbb{1}_{\{Z_{\boldsymbol{\delta}_0+r_n\boldsymbol{\delta}}(v_1,v_2,u) > U_{Z_{1-t}}(n/(ky)|x_0)\}} - \mathbb{1}_{\{Z_{\boldsymbol{\delta}_0}(v_1,v_2,u) > U_{Z_{1-t}}(n/(ky)|x_0)\}} \right| \\ &\leqslant \sqrt{\frac{nh_n^d}{k}}\, K_{h_n}(x_0-u) \mathbb{1}_{\left\{U_{Z_{1-t}}(n/(ky)|x_0) \in [\min(Z_{\boldsymbol{\delta}_0+r_n\boldsymbol{\delta}}(v_1,v_2,u), Z_{\boldsymbol{\delta}_0}(v_1,v_2,u)), \max(Z_{\boldsymbol{\delta}_0+r_n\boldsymbol{\delta}}(v_1,v_2,u), Z_{\boldsymbol{\delta}_0}(v_1,v_2,u))]\right\}} \end{aligned}$$

$$
\begin{aligned}
&\leqslant \sqrt{\frac{nh_n^d}{k}}\, K_{h_n}(x_0-u) \\
&\quad \times \Bigg\{ \mathbb{1}_{\{\frac{1}{U_{Z_{1-t}}(n/(ky)|x_0)} \in [\min(|1-F_1(v_1|u)-r_n\delta_1|, 1-F_1(v_1|u)), \max(|1-F_1(v_1|u)-r_n\delta_1|, 1-F_1(v_1|u))]} \\
&\qquad + \mathbb{1}_{\left\{\frac{1}{U_{Z_{1-t}}(n/(ky)|x_0)} \in \left[\frac{t}{1-t}\min(|1-F_2(v_2|u)-r_n\delta_2|, 1-F_2(v_2|u)), \frac{t}{1-t}\max(|1-F_2(v_2|u)-r_n\delta_2|, 1-F_2(v_2|u))\right]\right\}} \Bigg\} \\
&\leqslant G_n(y,b)(v_1,v_2,u).
\end{aligned}
$$

We need to show
Assertion 1: $\sup_{y\in(0,T]} \sqrt{n} P G_n(y,b_n) \longrightarrow 0$ for every $b_n \to 0$
and
Assertion 2: Let $\overline{G}_n$ be an envelope for $\{G_n(y,b) : y \in (0,T]\}$, we have

A) $P\overline{G}_n^2 = O(1)$;

B) $P\overline{G}_n^2\{\overline{G}_n \geqslant \varepsilon\sqrt{n}\} \to 0$, $\forall \varepsilon > 0$;

C) $\sup_{y\in(0,T]} PG_n^2(y,b) \to 0$;

D) $J(d_n, \{G_n(y,b) : y \in (0,T]\}, L_2) \to 0$, $\forall d_n \searrow 0$.

Proof of Assertion 1. Remark that

$$
\begin{aligned}
&\sqrt{n} P G_n(y,b_n) \\
&= n\sqrt{\frac{h_n^d}{k}} \Bigg\{ \mathbb{E}\left[K_{h_n}(x_0-X)\, \mathbb{E}\left\{ \mathbb{1}_{\left\{\frac{1}{U_{Z_{1-t}}(n/(ky)|x_0)} \in [1-F_1(Y^{(1)}|X)-r_nb_n, 1-F_1(Y^{(1)}|X)+r_nb_n]\right\}} \Big| X \right\} \right] \\
&\qquad + \mathbb{E}\left[K_{h_n}(x_0-X)\, \mathbb{E}\left\{ \mathbb{1}_{\left\{\frac{1}{U_{Z_{1-t}}(n/(ky)|x_0)} \in \left[\frac{t}{1-t}\left(1-F_2(Y^{(2)}|X)-r_nb_n\right), \frac{t}{1-t}\left(1-F_2(Y^{(2)}|X)+r_nb_n\right)\right]\right\}} \Big| X \right\} \right] \Bigg\} \\
&\leqslant n\sqrt{\frac{h_n^d}{k}}\, 4 r_n b_n \mathbb{E}\left[K_{h_n}(x_0-X)\right].
\end{aligned}
$$

Thus Assertion 1 is satisfied as soon as $r_n n\sqrt{\frac{h_n^d}{k}} \to 0$.

Proof of Assertion 2 A). Define

$$
\begin{aligned}
\overline{G}_n^{(1)}(b)(v_1,u) &:= \sqrt{\frac{nh_n^d}{k}} K_{h_n}(x_0-u)\mathbb{1}_{\left\{1-F_1(v_1|u) \leqslant \frac{1}{U_{Z_{1-t}}(n/(kT)|x_0)} + r_n b\right\}}, \\
\overline{G}_n^{(2)}(b)(v_2,u) &:= \sqrt{\frac{nh_n^d}{k}} K_{h_n}(x_0-u)\mathbb{1}_{\left\{1-F_2(v_2|u) \leqslant \frac{1-t}{t}\frac{1}{U_{Z_{1-t}}(n/(kT)|x_0)} + r_n b\right\}},
\end{aligned}
$$

and set $\overline{G}_n(b)(v_1, v_2, u) := \overline{G}_n^{(1)}(b)(v_1, u) + \overline{G}_n^{(2)}(b)(v_2, u)$. This yields

$$
\begin{aligned}
P\overline{G}_n^2 &\leqslant 2\,\frac{nh_n^d}{k}\mathbb{E}\left\{K_{h_n}^2(x_0 - X)\mathbb{P}\left(1 - F_1(Y^{(1)}|X) \leqslant \frac{1}{U_{Z_{1-t}}(n/(kT)|x_0)} + r_n b\Big|X\right)\right\} \\
&\quad +2\,\frac{nh_n^d}{k}\mathbb{E}\left\{K_{h_n}^2(x_0 - X)\mathbb{P}\left(1 - F_2(Y^{(2)}|X) \leqslant \frac{1-t}{t}\,\frac{1}{U_{Z_{1-t}}(n/(kT)|x_0)} + r_n b\Big|X\right)\right\} \\
&\leqslant C\left[\frac{n}{k}\frac{1}{U_{Z_{1-t}}(n/(kT)|x_0)} + \frac{n}{k} r_n b\right] \\
&= O(1),
\end{aligned}
$$

since $kh_n^d \to \infty$ and $r_n n\sqrt{\frac{h_n^d}{k}} \to 0$. This achieves the proof of Assertion 2 A).

Proof of Assertion 2 B). Clearly, we have

$$
\overline{G}_n^{2+\alpha} \leqslant C\left[\left(\overline{G}_n^{(1)}\right)^{2+\alpha} + \left(\overline{G}_n^{(2)}\right)^{2+\alpha}\right].
$$

Thus $\forall \varepsilon > 0$, we have

$$
\begin{aligned}
&P\overline{G}_n^2\{\overline{G}_n \geqslant \varepsilon\sqrt{n}\} \\
&\leqslant \frac{1}{n^{\alpha/2}\varepsilon^{\alpha}}\mathbb{E}\left(\overline{G}_n^{2+\alpha}\right) \\
&\leqslant \frac{C}{\varepsilon^{\alpha}}\frac{n}{k}\frac{1}{(kh_n^d)^{\alpha/2}} \\
&\quad \times\left\{\mathbb{E}\left[\frac{1}{h_n^d}K^{2+\alpha}\left(\frac{x_0 - X}{h_n}\right)\mathbb{P}\left(1 - F_1(Y^{(1)}|X) \leqslant \frac{1}{U_{Z_{1-t}}(n/(kT)|x_0)} + r_n b\Big|X\right)\right]\right. \\
&\quad \left. + \mathbb{E}\left[\frac{1}{h_n^d}K^{2+\alpha}\left(\frac{x_0 - X}{h_n}\right)\mathbb{P}\left(1 - F_2(Y^{(2)}|X) \leqslant \frac{1-t}{t}\frac{1}{U_{Z_{1-t}}(n/(kT)|x_0)} + r_n b\Big|X\right)\right]\right\} \\
&\leqslant \frac{C}{\varepsilon^{\alpha}}\frac{n}{k}\frac{1}{(kh_n^d)^{\alpha/2}}\left[\frac{1}{U_{Z_{1-t}}(n/(kT)|x_0)} + r_n b\right]
\end{aligned}
$$

which tends again to 0 under our assumptions.

Proof of Assertion 2 C). Following the lines of proof of Assertion 1, we have

$$
\sup_{y\in(0,T]} PG_n^2(y, b) \leqslant C r_n\frac{n}{k} \to 0.
$$

Proof of Assertion 2 D). We follow the lines of proof of Theorem 2.2 in Escobar-Bach et al. (2018b). The class of functions on $[0, 1]$

$$
\left\{u \to \mathbb{1}_{\{y_1 \leqslant 1-u \leqslant y_2\}},\ \ y_1 < y_2\right\},
$$

is a VC-class. This allows us to prove that there exist positive constants C and V such that

$$
\sup_{Q\in\mathcal{Q}} N(\{G_n(y, b) : y \in (0, T]\}, L_2(Q), \tau\|\overline{G}_n\|_{Q,2}) \leqslant C\left(\frac{1}{\tau}\right)^V,
$$

from which Assertion 2 D) follows. This achieves the proof of (S7).

It remains to show that

$$\sup_{y\in(0,T]} |Q_{3,1-t}(y|x_0)| = o(1). \tag{S8}$$

Note that

$$\begin{aligned}
&\sqrt{kh_n^d}\left|\mathbb{E}\left[\breve{T}_{n,1-t}(y|x_0) - T_{n,1-t}(y|x_0)\right]\right| \\
&= n\sqrt{\frac{h_n^d}{k}}\left|\mathbb{E}\left[K_{h_n}(x_0 - X)\left(\mathbb{1}_{\{1-F_{Z_{1-t}}(\breve{Z}_{1-t}|x_0)<\frac{k}{n}y\}} - \mathbb{1}_{\{1-F_{Z_{1-t}}(Z_{1-t}|x_0)<\frac{k}{n}y\}}\right)\right]\right| \\
&\leqslant \sqrt{n}\sqrt{\frac{nh_n^d}{k}}\mathbb{E}\left[K_{h_n}(x_0 - X)\left|\mathbb{1}_{\{1-F_{Z_{1-t}}(\breve{Z}_{1-t}|x_0)<\frac{k}{n}y\}} - \mathbb{1}_{\{1-F_{Z_{1-t}}(Z_{1-t}|x_0)<\frac{k}{n}y\}}\right|\right] \\
&= \sqrt{n}\sqrt{\frac{nh_n^d}{k}}\mathbb{E}\left[K_{h_n}(x_0 - X)\left|\mathbb{1}_{\{1-F_{Z_{1-t}}(Z_{\delta_n}|x_0)<\frac{k}{n}y\}} - \mathbb{1}_{\{1-F_{Z_{1-t}}(Z_{\delta_0}|x_0)<\frac{k}{n}y\}}\right|\right] \\
&\leqslant \sqrt{n}PG_n(y,b),
\end{aligned}$$

for n large enough. This implies that

$$\sup_{y\in(0,T]}\sqrt{kh_n^d}\left|\mathbb{E}\left[\breve{T}_{n,1-t}(y|x_0) - T_{n,1-t}(y|x_0)\right]\right| \leqslant \sup_{y\in(0,T]}\sqrt{n}PG_n(y,b) \longrightarrow 0$$

by Assertion 1 since it is clear that $b_n \to 0$ can be replaced by any fixed value b as soon as $r_n n\sqrt{\frac{h_n^d}{k}} \to 0$ holds.

Combining (S6), (S7) and (S8) we have shown that

$$\sqrt{kh_n^d}\left(\breve{T}_{n,1-t}(y|x_0) - yf_X(x_0)\right) \rightsquigarrow W_{1-t}(y),$$

in $\ell((0,T])$.

Secondly, to obtain the joint weak convergence of $(\breve{W}_{n,1-t_1},\ldots,\breve{W}_{n,1-t_J})$ we need to establish the finite dimensional weak convergence together with tightness. The finite dimensional weak convergence can be shown by the Cramér-Wold device, as in the proof of Lemma 1 along with decomposition (S5). The joint tightness follows from the individual tightness (similarly to Lemma 1 in Bai and Taqqu, 2013). □

Proof of Lemma S1. This proof is omitted since it is similar to the proof of Lemma 2, by using Proposition S1 instead of Lemma 1 combining with the Skorohod construction (keeping the same notation). □

Proof of Lemma S2. Again, this proof is omitted since it is similar to the proof of Lemma 3, by using Lemma S1 instead of Lemma 2. □

Proof of Lemma S3. By definition, we have

$$
\begin{aligned}
|\breve{s}_{n,1-t}(z|x_0) - z| &= \left| \frac{\overline{F}_{Z_{1-t}}\left(z^{-1}\widehat{U}_{\breve{Z}_{1-t}}(n/k|x_0)\Big|x_0\right)}{\overline{F}_{Z_{1-t}}\left(U_{Z_{1-t}}(n/k|x_0)|x_0\right)} - z \right| \\
&= \left| z\left(\frac{\widehat{U}_{\breve{Z}_{1-t}}(n/k|x_0)}{U_{Z_{1-t}}(n/k|x_0)}\right)^{-1} \frac{1+\delta_{1-t}\left(z^{-1}\widehat{U}_{\breve{Z}_{1-t}}(n/k|x_0)\Big|x_0\right)}{1+\delta_{1-t}\left(U_{Z_{1-t}}(n/k|x_0)|x_0\right)} - z \right| \\
&\leqslant z \left| \left(\frac{\widehat{U}_{\breve{Z}_{1-t}}(n/k|x_0)}{U_{Z_{1-t}}(n/k|x_0)}\right)^{-1} - 1 \right| \left| \frac{1+\delta_{1-t}\left(z^{-1}\widehat{U}_{\breve{Z}_{1-t}}(n/k|x_0)\Big|x_0\right)}{1+\delta_{1-t}\left(U_{Z_{1-t}}(n/k|x_0)|x_0\right)} \right| \\
&\quad + z\left| \frac{1+\delta_{1-t}\left(z^{-1}\widehat{U}_{\breve{Z}_{1-t}}(n/k|x_0)\Big|x_0\right)}{1+\delta_{1-t}\left(U_{Z_{1-t}}(n/k|x_0)|x_0\right)} - 1 \right| \\
&\leqslant z \left| \left(\frac{\widehat{U}_{\breve{Z}_{1-t}}(n/k|x_0)}{U_{Z_{1-t}}(n/k|x_0)}\right)^{-1} - 1 \right| \left| \frac{1+\delta_{1-t}\left(z^{-1}\widehat{U}_{\breve{Z}_{1-t}}(n/k|x_0)|x_0\right)}{1+\delta_{1-t}\left(U_{Z_{1-t}}(n/k|x_0)|x_0\right)} \right| \\
&\quad + z\left| \frac{\delta_{1-t}\left(U_{Z_{1-t}}(n/k|x_0)|x_0\right)}{1+\delta_{1-t}\left(U_{Z_{1-t}}(n/k|x_0)|x_0\right)} \right| \left\{ \left| \left(z^{-1}\frac{\widehat{U}_{\breve{Z}_{1-t}}(n/k|x_0)}{U_{Z_{1-t}}(n/k|x_0)}\right)^{-\beta(x_0)} - 1 \right| \right. \\
&\quad \left. + \left| \frac{\delta_{1-t}\left(z^{-1}\widehat{U}_{\breve{Z}_{1-t}}(n/k|x_0)|x_0\right)}{\delta_{1-t}\left(U_{Z_{1-t}}(n/k|x_0)|x_0\right)} - \left(z^{-1}\frac{\widehat{U}_{\breve{Z}_{1-t}}(n/k|x_0)}{U_{Z_{1-t}}(n/k|x_0)}\right)^{-\beta(x_0)} \right| \right\}.
\end{aligned}
$$

By Proposition B.1.10 in de Haan and Ferreira (2006), for n large, with arbitrary large probability, we have for $\varepsilon, \xi > 0$ and $z \leqslant T$

$$\begin{aligned}
|\breve{s}_{n,1-t}(z|x_0) - z| &\leqslant C\, z \left| \left(\frac{\widehat{U}_{\breve{Z}_{1-t}}(n/k|x_0)}{U_{Z_{1-t}}(n/k|x_0)} \right)^{-1} - 1 \right| \\
&\quad + C\, \varepsilon\, z \left| \delta_{1-t}\left(U_{Z_{1-t}}(n/k|x_0)|x_0 \right) \right| \left(z^{-1} \frac{\widehat{U}_{\breve{Z}_{1-t}}(n/k|x_0)}{U_{Z_{1-t}}(n/k|x_0)} \right)^{-\beta(x_0)\pm\xi} \\
&\quad + C\, z \left| \delta_{1-t}\left(U_{Z_{1-t}}(n/k|x_0)|x_0 \right) \right| \left\{ z^{\beta(x_0)} \left(\frac{\widehat{U}_{\breve{Z}_{1-t}}(n/k|x_0)}{U_{Z_{1-t}}(n/k|x_0)} \right)^{-\beta(x_0)} + 1 \right\} \\
&\leqslant C\, T \left| \left(\frac{\widehat{U}_{\breve{Z}_{1-t}}(n/k|x_0)}{U_{Z_{1-t}}(n/k|x_0)} \right)^{-1} - 1 \right| \\
&\quad + C\, \varepsilon\, T^{1+\beta(x_0)\pm\xi} \left| \delta_{1-t}\left(U_{Z_{1-t}}(n/k|x_0)|x_0 \right) \right| \left(\frac{\widehat{U}_{\breve{Z}_{1-t}}(n/k|x_0)}{U_{Z_{1-t}}(n/k|x_0)} \right)^{-\beta(x_0)\pm\xi} \\
&\quad + C\, T \left| \delta_{1-t}\left(U_{Z_{1-t}}(n/k|x_0)|x_0 \right) \right| \left\{ T^{\beta(x_0)} \left(\frac{\widehat{U}_{\breve{Z}_{1-t}}(n/k|x_0)}{U_{Z_{1-t}}(n/k|x_0)} \right)^{-\beta(x_0)} + 1 \right\}.
\end{aligned}$$

Using Lemma S2, Lemma S3 follows. □

5.2 Proof of Theorem S1.

This proof is omitted since it follows exactly the same lines of proof as those for Theorem 3 but with the auxiliary results in case of unknown margins. □

5.3 Proof of Theorem 2.

This proof is omitted since it follows exactly the same lines of proof as those for Theorem 1 with the auxiliary results in case of unknown margins. □

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