Annals of the Institute of Statistical Mathematics (2023) 75:17–37
https://doi.org/10.1007/s10463-022-00832-8

Joint behavior of point processes of clusters and partial sums for stationary bivariate Gaussian triangular arrays

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Received: 15 October 2021 / Revised: 15 February 2022 / Accepted: 31 March 2022 /
Published online: 21 May 2022

Abstract
For Gaussian stationary triangular arrays, it is well known that the extreme values may occur in clusters. Here we consider the joint behaviors of the point processes of clusters and the partial sums of bivariate stationary Gaussian triangular arrays. For a bivariate stationary Gaussian triangular array, we derive the asymptotic joint behavior of the point processes of clusters and prove that the point processes and partial sums are asymptotically independent. As an immediate consequence of the results, one may obtain the asymptotic joint distributions of the extremes and partial sums. We illustrate the theoretical findings with a numeric example.

Keywords Bivariate stationary Gaussian triangular array · Point process of clusters · Partial sum · Joint behavior

1 Introduction

For a centered unit-variance stationary Gaussian triangular array $\{X_{n,s}, 1 \le s \le n\}$, the distributional behavior of the maximum $M_n = \max_{1\le s\le n} X_{n,s}$ has been studied extensively. Under the so called Berman condition about the correlation $\rho_{\ell,n} = \mathbb{E}\left(X_{n,s}X_{n,s+\ell}\right)$, M_n has the same asymptotic distribution as the maximum of n independent random variables (see e.g., Berman, 1964; Leadbetter et al., 1983). Mittal and Ylvisaker (1975) obtained the asymptotic distribution of M_n under some weaker conditions. However, it is not uncommon that large values may occur in clusters in practice, and as a result, the independence model may not be appropriate. For instance, displayed in Fig. 1 are the daily log returns of Amazon from October 6, 2011 to October 2, 2013. For clarity, we plotted the times on the horizontal axis only for those values which are greater than

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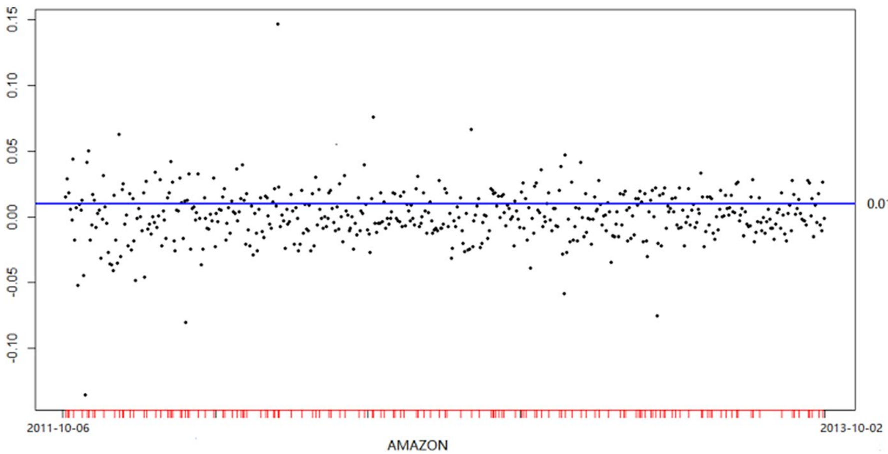

Fig. 1 Amazon daily log returns, from October 6, 2011 to October 2, 2013

0.01 and one observes the obvious tendency that large values occur in clusters. Hsing et al. (1996) first addressed the cluster issue by demonstrating that under appropriate conditions for the correlations, the locations of the extreme values are clustered. The asymptotic distribution of the extremes of a Gaussian triangular array was also developed in Hsing et al. (1996). Hashorva and Weng (2013) and Hashorva et al. (2015) generalized the results to bivariate and multivariate stationary Gaussian triangular arrays, respectively, and developed the limiting distributions with clustering information for the extremes. Furthermore, French and Davis (2013) and Ling (2019) extended the results to stationary random fields. However, the joint asymptotic behavior of the point processes of clusters and partial sums for the stationary bivariate Gaussian triangular array has not been well studied.

For the univariate standard stationary Gaussian triangular array $\{X_{n,s}, 1 \le s \le n\}$, the exceedances point process is defined by

$$N_n(B) = \sum\nolimits_{s=1}^{n} I\left\{X_{n,s} > u_n(x), \frac{s}{n} \in B\right\},$$

for any real Borel set B on $(0, 1]$, where I is the indicator function and $u_n(x) = b_n + x/a_n$ with

$$a_n = \sqrt{2\log n}, \quad b_n = \sqrt{2\log n} - \frac{\log\log n + \log 4\pi}{2\sqrt{2\log n}}. \tag{1}$$

It is well known that $N_n(B)$ converges weakly to a Poisson process with intensity e^{-x} if the Berman condition holds. In particular, this implies that the number of exceedances of $u_n(x)$ by $X_{n,s}$ in the set $I_n = \{1, 2, \dots, n\}$ will have an asymptotic Poisson distribution. However, for a more general stationary sequence $\{\zeta_{n,s}, 1 \le s \le n\}$, the exceedances of $u_n(x)$ may tend to occur in clusters. One very simple means of defining the point of clusters is to take a sequence $r_n + \ell_n$ and consider events that occur within a distance of $r_n + \ell_n$ belong to the same cluster. More precisely, for any positive integers r_n and ℓ_n, let $q_n = [n/(r_n + \ell_n)]$, where $[x]$ denotes the integer part of the real number x. The point processes of clusters formed by $\{\zeta_{n,s}, 1 \le s \le n\}$ is defined by

$$\widehat{N}_n(B) = \sum\nolimits_{k=1}^{q_n} I\left\{ \cup_{s\in Q_k} \zeta_{n,s} > u_n(x), \frac{s}{n} \in B \right\},$$

where $Q_k = \{(k-1)(r_n + \ell_n) + 1, (k-1)(r_n + \ell_n) + 2, \cdots, k(r_n + \ell_n)\}$, and Q_k are intervals with length $r_n + \ell_n$. Leadbetter (1983) first studied the asymptotics of the point process of clusters for a stationary sequence and showed that the point process of clusters $\widehat{N}_n(B)$ also converges in distribution to a Poisson process. Furthermore, Wiśniewski (1996) considered the weak convergence of multivariate exceedances point processes formed by multivariate stationary Gaussian sequence and Peng et al. (2012) obtained the joint limiting distributions of exceedances point processes and partial sums of multivariate Gaussian sequences under some mild conditions.

In this paper we consider $\left\{ \left(X_{n,s}^{(1)}, X_{n,s}^{(2)} \right), 1 \le s \le n \right\}$, which is a centered bivariate stationary Gaussian triangular array with $\mathbb{E}[X_{n,s}^{(1)}] = \mathbb{E}[X_{n,s}^{(2)}] = 0$, $\mathrm{Var}[X_{n,s}^{(1)}] = \mathrm{Var}[X_{n,s}^{(2)}] = 1$. Let $\rho_{ij}(|k-\ell|, n)$ denote the correlation of $\left(X_{n,k}^{(i)}, X_{n,\ell}^{(j)} \right)$ for $i,j = 1,2$. Suppose that the correlation satisfies ($\delta_{ii}(0) = 0$)

$$\begin{cases} \lim_{n\to\infty}(1-\rho_{ij}(s,n))\log n = \delta_{ij}(s) \in (0,\infty], & \text{for } i,j=1,2, s=1,2\ldots \\ \lim_{n\to\infty}(1-\rho_{ij}(0,n))\log n = \delta_{ij}(0) \in (0,\infty], & \text{for } i,j=1,2, i \ne j. \end{cases} \tag{2}$$

Under appropriate conditions for correlations, Hashorva et al. (2015) proved that for all $x_1, x_2 \in R$,

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{P}\left(\max_{1\le s\le n} X_{n,s}^{(1)} \le u_n(x_1), \max_{1\le s\le n} X_{n,s}^{(2)} \le u_n(x_2) \right) \\ &\quad = \exp\left(-\vartheta_1(x_1,x_2)e^{-x_1} - \vartheta_2(x_1,x_2)e^{-x_2} \right), \end{aligned} \tag{3}$$

where

$$\begin{aligned} \vartheta_1(x_1,x_2) = \mathbb{P}\Big\{ & E/2 + \sqrt{\delta_{11}(s-1)}W_{s,1}^{(1)} \le \delta_{11}(s-1), E/2 \\ & + \sqrt{\delta_{21}(s-1)}W_{s,1}^{(2)} \le \delta_{21}(s-1) + \frac{x_2 - x_1}{2}, \\ & \delta_{11}(s-1) < \infty, \quad \delta_{21}(s-1) < \infty, \quad \text{for all } \ s \ge 2 \Big\}, \end{aligned} \tag{4}$$

and

$$\begin{aligned}\vartheta_2(x_1,x_2) = \mathbb{P}\Big\{ & E/2 + \sqrt{\delta_{12}(0)}W^{(1)}_{1,2} \le \delta_{12}(0) + \frac{x_1-x_2}{2}, E/2 \\ & + \sqrt{\delta_{12}(s-1)}W^{(1)}_{s,2} \le \delta_{12}(s-1) + \frac{x_1-x_2}{2}, \\ & E/2 + \sqrt{\delta_{22}(s-1)}W^{(2)}_{s,2} \le \delta_{22}(s-1), \\ & \delta_{12}(0)<\infty, \quad \delta_{12}(s-1)<\infty, \quad \delta_{22}(s-1)<\infty, \\ & \text{for all} \quad s\ge 2\Big\}.\end{aligned} \tag{5}$$

Here E is a standard exponential random variable and $\left\{W^{(t)}_{s,i}, t=1,2, \delta_{ti}(s-1)<\infty, s\ge 1\right\}$ are independent of E and jointly normal with zero means and correlation

$$\mathbb{E}\left(W^{(j)}_{s,i}W^{(t)}_{\ell,i}\right) = \frac{\delta_{ji}(s-1)+\delta_{ti}(\ell-1)-\delta_{jt}(|s-\ell|)}{2\sqrt{\delta_{ji}(s-1)\delta_{ti}(\ell-1)}}, \quad i=1,2,$$

where $j,t=1,2$ and $s,\ell\ge 1$ if $i\ne j$ and $i\ne t$, and $s,\ell\ge 2$ for $i=j$ or $i=t$. The results in Hsing et al. (1996) is a direct corollary by taking $x_2\to\infty$ in (3), that is

$$\lim_{n\to\infty}\mathbb{P}\left(\max_{1\le s\le n} X^{(1)}_{n,s} \le u_n(x_1)\right) = \exp\left(-\vartheta e^{-x_1}\right),$$

where

$$\begin{aligned}\vartheta &= \lim_{x_2\to+\infty}\vartheta_1(x_1,x_2) \\ &= \mathbb{P}\left\{E/2+\sqrt{\delta_{11}(s-1)}W^{(1)}_{s,1} \le \delta_{11}(s-1), \quad \delta_{11}(s-1)<\infty, \quad \text{for all} \quad s\ge 2\right\}.\end{aligned} \tag{6}$$

Note that $\frac{x_2-x_1}{2}\to\infty$ as $x_2\to\infty$ for any x_1, thus ϑ does not depend on x_1. Furthermore, we focus on the joint asymptotic behavior of the point processes of clusters formed by $\left\{\left(X^{(1)}_{n,s}, X^{(2)}_{n,s}\right), 1\le s\le n\right\}$ as well as their joint behavior with the partial sums under certain weak dependence conditions motivated by Leadbetter (1983) and Peng et al. (2012). As an immediate consequence of our results, one may obtain the asymptotic joint distributions of the extremes and partial sums for the univariate standard stationary Gaussian triangular arrays.

It is worth mentioning that the study of the joint behavior of the extremes and partial sums from a sequence of random variables has a long history, and recently there is renewed interest in it because of the increasing volume of environmental data with averages and extremes that are available to researchers. Random sums and maxima also arise naturally in different subjects such as finance, insurance, engineering, and energy modeling (see e.g., Kozubowski et al, 2011; Biondi et al, 2005 and Kozubowski and Panorska, 2005), and the use of partial sums is critical in the famous Hill estimator (see e.g., Hill, 1975 and Buitendag et al, 2020). Thus, studying of the joint behavior of extremes and partial sums is of both practical and theoretical significance. An early influential work is Chow and Teugels (1978), which focused on a sequence of independent and identically distributed random variables. Anderson and Turkman (1991, 1993, 1995) generalized the results to strong mixing sequences. For stationary Gaussian sequences, Ho and Hsing (1996), Ho and McCormick (1999), McCormick and Qi (2000) and Peng and Nadarajah (2003) showed that the maxima and partial sums are asymptotically independent (or dependent) if the Gaussian sequences are weakly (or strongly) dependent. James et al. (2007) considered the problem for multivariate stationary Gaussian sequences and Hu et al. (2009) extended the problem to the asymptotics of the point process of exceedances and the partial sum for a Gaussian triangular array.

The paper is organized as follows. We present the main results in Sect. 2, and an illustrative numerical example in Sect. 3. Auxiliay results and the proofs are included in Sects. 4 and 5, respectively.

2 Main results

For a centered bivariate Gaussian triangular array $\left\{\left(X_{n,s}^{(1)}, X_{n,s}^{(2)}\right), 1 \le s \le n\right\}$, we are interested in the joint asymptotic behavior of the point process of clusters. In particular, under the same conditions as that in Hashorva et al. (2015), we first establish the joint limiting generating function of the numbers of cluster exceedances of $u_n(x_i)$, $i = 1, 2$, defined by

$$N_n^{(1)} = \sum_{k=1}^{q_n} I(\cup_{s\in Q_k} X_{n,s}^{(1)} > u_n(x_1)),$$
$$N_n^{(2)} = \sum_{k=1}^{q_n} I(\cup_{s\in Q_k} X_{n,s}^{(2)} > u_n(x_2)),$$

where $Q_k = \{(k-1)(r_n+\ell_n)+1, (k-1)(r_n+\ell_n)+2, \cdots, k(r_n+\ell_n)\}$ and $q_n = [n/(r_n+\ell_n)]$ for some positive integers r_n and ℓ_n. Here $[x]$ denotes the integer part of the real number x. A direct consequence of the result is that $N_n^{(1)}$ and $N_n^{(2)}$ are asymptotic dependent. The asymptotic distribution of the extreme values in Hsing et al. (1996) is also a direct corollary of the result. Based on the limiting distribution, we further study the joint behavior of the point processes of clusters and the partial sums of the stationary bivariate Gaussian triangular array. Throughout this paper, $u_n(x) = b_n + x/a_n$ with a_n, b_n are given in (1).

Theorem 1 *Let* $\left\{\left(X_{n,s}^{(1)}, X_{n,s}^{(2)}\right), 1 \le s \le n\right\}$ *be a centered bivariate stationary Gaussian triangular array with* $\mathbb{E}[X_{n,s}^{(1)}] = \mathbb{E}[X_{n,s}^{(2)}] = 0$, $\mathrm{Var}[X_{n,s}^{(1)}] = \mathrm{Var}[X_{n,s}^{(2)}] = 1$. *Suppose that the correlation functions* $\mathbb{E}\left(X_{n,k}^{(i)} X_{n,\ell}^{(j)}\right) = \rho_{ij}(|k-\ell|, n)$, $i,j = 1,2$, *satisfy* (2), *and there exist positive integers* r_n, ℓ_n *such that*

$$\frac{\ell_n}{r_n} \to 0, \quad \frac{r_n}{n} \to 0, \tag{7}$$

$$\lim_{n\to\infty} \frac{n^2}{r_n} \sum_{i,j=1}^{2} \sum_{s=\ell_n}^{n} |\rho_{ij}(s,n)| \exp\left\{ -\frac{2\log n - \log\log n}{1+|\rho_{ij}(s,n)|} \right\} = 0, \tag{8}$$

and

$$\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{i,j=1}^{n} \sum_{s=m}^{r_n} n^{-\frac{1-\rho_{ij}(s,n)}{1+\rho_{ij}(s,n)})} \frac{(\log n)^{-\rho_{ij}(s,n)/(1+\rho_{ij}(s,n))}}{\sqrt{1-\rho_{ij}^2(s,n)}} = 0. \tag{9}$$

Then the limiting generating function of $N_n^{(1)}$ *and* $N_n^{(2)}$ *is*

$$\lim_{n\to\infty} \mathbb{E}\left(\omega^{N_n^{(1)}} z^{N_n^{(2)}}\right) = \exp\{-(1-\omega)\vartheta e^{-x_1} - (1-z)\vartheta e^{-x_2} + (1-\omega)(1-z) \left[\vartheta e^{-x_1} - \vartheta_1(x_1,x_2)e^{-x_1} + \vartheta e^{-x_2} - \vartheta_2(x_1,x_2)e^{-x_2}\right]\}, \tag{10}$$

for $0 < \omega < 1$ *and* $0 < z < 1$, *where* $\vartheta_1(x_1,x_2)$, $\vartheta_2(x_1,x_2)$ *and* ϑ *are given in* (4), (5) *and* (6), *respectively.*

Corollary 1 *Let* $\{X_{n,s}^{(1)}, 1 \le s \le n\}$ *be a centered stationary Gaussian triangular array with variance one and define the cluster point process*

$$N_n^{(1)}(B) = \sum_{k=1}^{q_n} I\left\{ \cup_{s\in Q_k}, X_{n,s}^{(1)} > u_n(x_1), \frac{s}{n} \in B \right\},$$

for any real Borel set B on (0, 1]. *Under the conditions of Theorem* 1, *the cluster point process* $N_n^{(1)}(B)$ *converges to N in distribution, where N is a Poisson process with intensity* ϑe^{-x_1}.

Remark 1 Hsing et al. (1996) proved that if (2) holds, then the conditions (7), (8) and (9) are satisfied if

$$\lim_{n\to\infty} \sum_{1\le i,j\le 2} \max_{\ell_n \le s \le n} \left|\rho_{ij}(s,n)\right| \log n = 0 \ \ \textit{for some} \ \ \ell_n = o(n) \tag{11}$$

and

$$\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{i,j=1}^{2} \sum_{s=m}^{\ell_n} n^{-\frac{1-\rho_{ij}(s,n)}{1+\rho_{ij}(s,n)}} \frac{(\log n)^{-\rho_{ij}(s,n)/(1+\rho_{ij}(s,n))}}{\sqrt{1-\rho_{ij}^2(s,n)}} = 0. \tag{12}$$

Next, we study the relation between the point process of clusters and the partial sum. The point process of clusters formed by the bivariate stationary Gaussian triangular array $\left\{\left(X_{n,s}^{(1)}, X_{n,s}^{(2)}\right), 1 \le s \le n\right\}$ is defined by

$$\mathbf{N}_n(\mathbf{B}, \mathbf{x}) = \sum_{i=1}^{2} \sum_{k=1}^{q_n} I\left\{ U_{s\in Q_k} X_{n,s}^{(i)} > u_n(x_i), \frac{s}{n} \in B_i \right\}$$

for $\mathbf{x} \in R^2$, $\mathbf{B} = \cup_{i=1}^{2}(B_i \times \{i\})$ where B_i are Borel sets on (0, 1].

Theorem 2 *Let* $\left\{(X_{n,s}^{(1)}, X_{n,s}^{(2)}), 1 \le s \le n\right\}$ *be a centered bivariate stationary Gaussian triangular array with variance one and* $\mathbf{S_n} = (S_{n1}, S_{n2})$ *with* $S_{ni} = \sum_{s=1}^{n} X_{n,s}^{(i)}$, $i = 1, 2$. *Assume that the correlation* $\rho_{ij}(|s-\ell|, n) = \mathbb{E}\left(X_{n,s}^{(i)} X_{n,\ell}^{(j)}\right)$ *satisfies*

$$\lim_{n\to\infty} \frac{\log n}{n^2} \sum_{s=1}^{n} \sum_{\ell=1}^{n} |\rho_{ij}(|s-\ell|, n)| = 0, \tag{13}$$

for $i, j = 1, 2$. *Then* $\mathbf{N}_n(\mathbf{B}, \mathbf{x})$ *and* $\mathbf{S_n}$ *are asymptotically independent.*

Based on the asymptotic distribution in the Corollary 1, we can easily obtain the asymptotic joint behavior of the extremes and partial sums. For example, the following corollary is an immediate consequence of the Corollary 1 and Theorem 2.

Corollary 2 *Let* $\{X_{n,s}^{(1)}, 1 \le s \le n\}$ *be a centered stationary Gaussian triangular array with variance one and* $M_{n1} = \max\{X_{n,1}^{(1)}, X_{n,2}^{(1)}, \dots, X_{n,n}^{(1)}\}$, $S_{n1} = \sum_{s=1}^{n} X_{n,s}^{(1)}$, $\sigma_n^2 = Var(S_{n1})$. *Under the conditions of Theorem 1 and Theorem 2, we have*

$$\lim_{n\to\infty} \mathbb{P}\left\{M_{n1} \le u_n(x_1), \frac{S_{n1}}{\sigma_n} \le y\right\} = \exp(-\vartheta e^{-x_1})\Phi(y),$$

where ϑ *is given in* (6) *and* $\Phi(y)$ *is the cumulative distribution function of a standard normal random variable.*

3 An illustrative example

In this section, a numeric example is provided to illustrate our results. We compare the distributions of the point process of clusters $N_n^{(1)}(B)$ with the Poisson process N due to Corollary 1, and the empirical distribution with the asymptotic distribution of the joint probability of maximum and partial sum based on Corollary 2.

Similar to Hsing et al. (1996), for each n let $\left\{Z_i\right\}_{i=0}^{n}$ be independent and identically distributed standard normal random variables, $\xi_{0,n} = Z_0$, and for $1 \le i \le n$,

$$\xi_{i,n} = d_n \xi_{i-1,n} + \sqrt{1 - d_n^2} Z_i.$$

Then, the stationary AR(1) process $\{\xi_{i,n}, 1 \le i \le n\}$ forms a Gaussian triangular array with mean zero and variance one. Let

$$d_n = 1 - \frac{\zeta}{\log n} \quad \text{for some} \quad \zeta \in (0, \infty). \tag{14}$$

Then, the correlation function of $\{\xi_{i,n}, 1 \le i \le n\}$ is $\rho_{j,n} = \mathbb{E}\left(\xi_{0,n}\xi_{j,n}\right) = (1 - \zeta/\log n)^j$ and (2) holds with $\delta_{ii}(j) = \delta_j = j\zeta$. Let $\ell_n = (\log n)(\log\log n)^2$, then conditions (11) and (12) are satisfied. Hence, Corollary 1 holds with $r_n = \sqrt{n\ell_n}$, as the conditions in Remark 1 are satisfied. Now, we present some numerical analysis regarding the stationary process. Let $d = 0.7$, then $\widetilde{\zeta} = (1-d)\log n$ and

$$\widetilde{\vartheta} = \mathbb{P}\left(E/2 + \sqrt{\widetilde{\zeta}}\sum_{i=1}^{s} Z_i \le s\widetilde{\zeta} \quad \text{for all} \quad s \ge 1\right).$$

Here, $W_{s,1}^{(1)}$ in (6) is replaced by $s^{-1/2}\sum_{i=1}^{s} Z_i$, where E denotes a standard exponential random variable independent of Z_i and $\{Z_i\}_{i=1}^{n}$ are independent and identically distributed standard normal random variables as before. The distributions of the point process of clusters $N_n^{(1)}(B)$ and the Poisson process N for different sample size n when $x_1 = 0$ are displayed in Fig. 2. From the figure, we can see that distribution of $N_n^{(1)}(B)$ approximates that of N better as n becomes larger.

To show that the point process of clusters $N_n^{(1)}(B)$ and the partial sum S_{n1} are asymptotically independent in this example, it is sufficient to show (13) because of Theorem 2. From the definition of $\ell_n = (\log n)(\log\log n)^2$ and (11), we have

$$\begin{aligned}
&\frac{\log n}{n^2}\sum_{s=1}^{n}\sum_{\ell=1}^{n} |\rho_{s-\ell,n}| \\
&\le \frac{2}{n}\sum_{l=0}^{n} |\rho_{l,n}|\log n = \frac{2}{n}\sum_{l=0}^{\ell_n} |\rho_{l,n}|\log n + \frac{2}{n}\sum_{l=\ell_n+1}^{n} |\rho_{l,n}|\log n \\
&\le \frac{2\log n}{n}\left(\log n(\log\log n)^2 + 1\right) + \frac{2}{n}\sum_{l=\ell_n+1}^{n} |\rho_{l,n}|\log n \\
&\to 0 \ \ as \ \ n \to \infty,
\end{aligned}$$

which implies the condition (13).

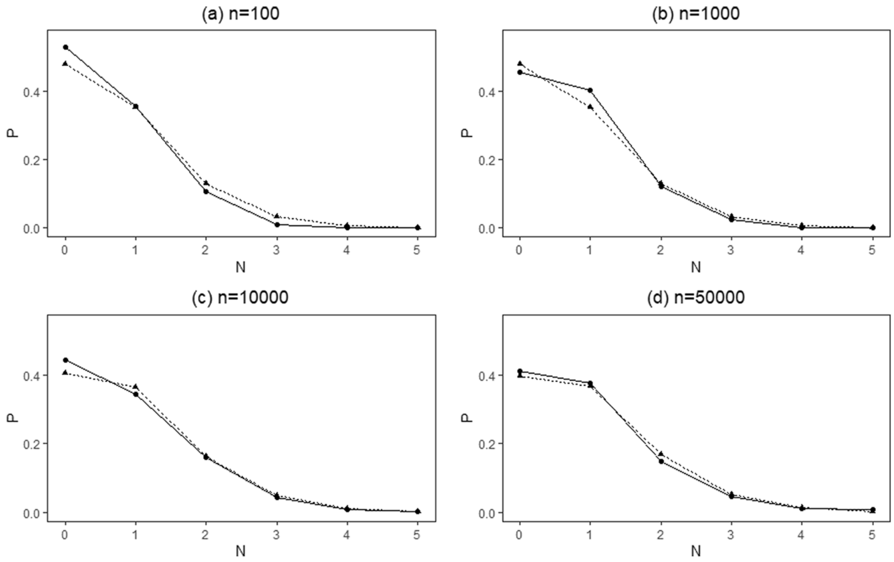

Fig. 2 Solid lines are for the distributions of $N_n^{(1)}(B)$ based on the observed frequencies; dotted lines are for the distribution of the Poisson process N; n represents different sample sizes

Thus, the point process of clusters $N_n^{(1)}(B)$ and the partial sum S_{n1} are indeed asymptotically independent. Combining with Theorem 1, the maximum M_{n1} and the partial sum S_{n1} are also asymptotically independent and their asymptotic joint distribution is provided by Corollary 2. We calculate the joint empirical distribution of M_{n1} and S_{n1} using 1000 groups of generated data as well as their asymptotic limiting function $\exp(-\widetilde{\vartheta})\Phi(y)$ ($\vartheta = \widetilde{\vartheta}$ and $x_1 = 0$).

Displayed in Fig. 3 are the empirical and asymptotic distributions based on different lengths of sequences. From the figure, we can see that the differences in the empirical distribution (red lines) and the asymptotic distributions (black lines) become smaller as lengths of the sequences n become larger. This confirms our asymptotic results, Theorem 2 and Corollary 2.

4 Auxiliary results

Let

$$\xi_k = \begin{cases} 1\,; \text{ when } \max\{X_{n,s}^{(1)}, s \in U_k\} > u_n(x_1); \\ 0; \text{ otherwise,} \end{cases}$$

and

$$\eta_k = \begin{cases} 1\,; \text{ when } \max\{X_{n,s}^{(2)}, s \in U_k\} > u_n(x_2); \\ 0; \text{ otherwise,} \end{cases}$$

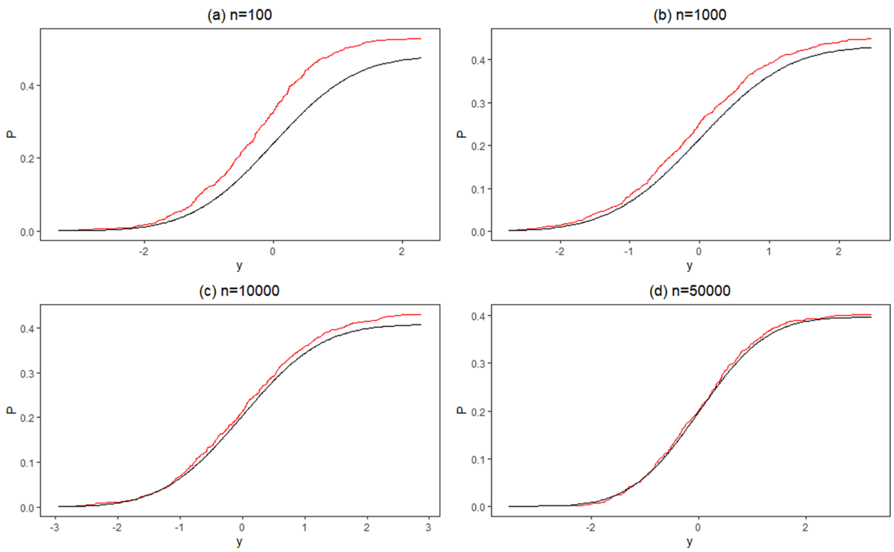

Fig. 3 Red lines are for the distributions calculated based on the generated data; black lines are the theoretical asymptotic distributions by Theorem 2.2; n represents different sequence lengths

where $U_k = \{(k-1)(r_n+\ell_n)+1, (k-1)(r_n+\ell_n)+2, \dots, (k-1)(r_n+\ell_n)+r_n\}$ and $k=1,2,\dots,q_n$. Note that the length of U_k is r_n. For simplicity, we use the same C to denote positive constants that may take different values at different places.

Lemma 1 *If y_k, y'_k are variables assuming only the values 0 and 1 for $k=1,2\dots,q_n$, and the condition (8) holds, we have*

$$\lim_{n\to\infty}\left|\mathbb{P}\left(\xi_k=y_k,\eta_k=y'_k,k=1,2,\dots,q_n\right)-\prod_{k=1}^{q_n}\mathbb{P}(\xi_k=y_k,\eta_k=y'_k)\right|=0. \tag{15}$$

Proof It follows from Lemma 8.1 of Berman (1971) that we need only to prove that (15) holds when $y_k=y'_k=0, k=1,2,\dots,q_n$. By the Normal Comparison Lemma (see e.g. Leadbetter, 1983), we have

$$\begin{aligned}
&\left|\mathbb{P}\left(\xi_k=0,\eta_k=0,k=1,2,\dots,q_n\right)-\prod_{k=1}^{q_n}\mathbb{P}(\xi_k=0,\eta_k=0)\right|\\
&\quad=\left|\mathbb{P}\left(\max_{s\in U_k}X^{(1)}_{n,s}\le u_n(x_1),\max_{s\in U_k}X^{(2)}_{n,s}\le u_n(x_2),k=1,2,\dots,q_n\right)\right.\\
&\qquad\left.-\prod_{k=1}^{q_n}\mathbb{P}\left(\max_{s\in U_k}X^{(1)}_{n,s}\le u_n(x_1),\max_{s\in U_k}X^{(2)}_{n,s}\le u_n(x_2)\right)\right|\\
&\quad\le C\sum\nolimits_{i,j=1}^{2}\sum\nolimits_{s\in U_1,|t-s|\ge\ell_n}|\rho_{ij}(|t-s|,n)|\exp\left(-\frac{u_n^2(x_1)+u_n^2(x_2)}{2(1+|\rho_{ij}(|t-s|,n)|)}\right)\\
&\qquad+C\sum\nolimits_{i,j=1}^{2}\sum\nolimits_{s\in U_2,|t-s|\ge\ell_n}|\rho_{ij}(|t-s|,n)|\exp\left(-\frac{u_n^2(x_1)+u_n^2(x_2)}{2(1+|\rho_{ij}(|t-s|,n)|)}\right)\\
&\qquad+\cdots+C\sum\nolimits_{i,j=1}^{2}\sum\nolimits_{s\in U_{q_n},|t-s|\ge\ell_n}|\rho_{ij}(|t-s|,n)|\exp\left(-\frac{u_n^2(x_1)+u_n^2(x_2)}{2(1+|\rho_{ij}(|t-s|,n)|)}\right).
\end{aligned} \tag{16}$$

Since

$$\frac{u_n^2(x_1)+u_n^2(x_2)}{2}=2\log n-\log(\log n)+O(1),$$

it follows from (8) that (16) can be bounded by

$$Cr_nq_n\sum\nolimits_{i,j=1}^{2}\sum\nolimits_{s=\ell_n}^{n}|\rho_{ij}(s,n)|\exp\left(-\frac{2\log n-\log\log n}{1+|\rho_{ij}(s,n)|}\right)\to 0 \quad \text{as} \quad n\to\infty.$$

This proves (15). □

For notational simplicity we define

$$M^{(i)}_{k,\ell}=\max_{k<s\le\ell}X^{(i)}_{n,s}, \qquad M^{(i)}_{\ell}=M^{(i)}_{0,\ell}=\max_{1\le s\le\ell}X^{(i)}_{n,s},$$

for $i = 1, 2$, $k = 1, 2, \ldots, \ell - 1$ and $\ell = 1, 2, \ldots, n$.

Lemma 2 *Under the conditions of Theorem 1, for any bounded index set $K \subseteq \{2, 3, \ldots, \}$, we have*

$$\lim_{n\to\infty} \mathbb{P}\left(X_{n,s}^{(1)} \le u_n(x_1), X_{n,s}^{(2)} \le u_n(x_2), s \in K \middle| X_{n,1}^{(1)} > u_n(x_1)\right) = \vartheta_1(x_1, x_2), \quad (17)$$

and

$$\lim_{n\to\infty} \mathbb{P}\left(X_{n,1}^{(1)} \le u_n(x_1), X_{n,s}^{(1)} \le u_n(x_1), X_{n,s}^{(2)} \le u_n(x_2), s \in K \middle| X_{n,1}^{(2)} > u_n(x_2)\right)$$
$$= \vartheta_2(x_1, x_2), \quad (18)$$

where $\vartheta_1(x_1, x_2)$ and $\vartheta_2(x_1, x_2)$ are given in (4) and (5), respectively.

Proof Our arguments are similar to those used in the proof of Lemma 3.3 in Hashorva et al. (2015). First, we have

$$\begin{aligned}
&\mathbb{P}\left(X_{n,1}^{(1)} \le u_n(x_1), X_{n,s}^{(1)} \le u_n(x_1), X_{n,s}^{(2)} \le u_n(x_2), s \in K \middle| X_{n,1}^{(2)} > u_n(x_2)\right)\\
&\sim \int_0^\infty \mathbb{P}\left(X_{n,1}^{(1)} \le u_n(x_1), X_{n,s}^{(1)} \le u_n(x_1), X_{n,s}^{(2)} \le u_n(x_2), s \in K \middle| X_{n,1}^{(2)} = u_n(x_2) + \frac{z}{u_n(x_2)}\right)\\
&\quad \times \exp\left(-z - \frac{z^2}{2u_n^2(x_2)}\right) \mathrm{d}z.
\end{aligned} \quad (19)$$

Let $\left\{\left(Y_{n,s,2}^{(1)}, Y_{n,s,2}^{(2)}\right), s \in \{1\} \cup K\right\}$ be a Gaussian triangular array with the same distribution as the conditional distribution of $\left\{\left(X_{n,s}^{(1)}, X_{n,s}^{(2)}\right), s \in \{1\} \cup K\right\}$ given $X_{n,1}^{(2)} = u_n(x_2) + \frac{z}{u_n(x_2)}$. Then,

$$\mathbb{E}(Y_{n,s,2}^{(i)}) = \rho_{i2}(s-1, n)\left(u_n(x_2) + \frac{z}{u_n(x_2)}\right),$$

and

$$\mathrm{Cov}(Y_{n,s,2}^{(i)}, Y_{n,\ell,2}^{(j)}) = \rho_{ij}(|s-\ell|, n) - \rho_{i2}(s-1, n)\rho_{j2}(\ell-1, n),$$

for $i, j = 1, 2$ and $s, \ell \in \{1\} \cup K$. Since

$$\frac{\rho_{ij}(|s-\ell|, n) - \rho_{i2}(s-1, n)\rho_{j2}(\ell-1, n)}{\sqrt{(1-\rho_{i2}^2(s-1, n))(1-\rho_{j2}^2(\ell-1, n))}} \to \frac{\delta_{i2}(s-1) + \delta_{j2}(\ell-1) - \delta_{ij}(|s-\ell|)}{2\sqrt{\delta_{i2}(s-1)\delta_{j2}(\ell-1)}}$$

for $i, j = 1, 2$ and $s, \ell \in \{1\} \cup K$ if $i \ne 2$ and $j \ne 2$, $s, \ell \in K$ if $i = 2$ or $j = 2$. Then, using $u_n^2(x) \sim 2\log n$ for $x \in R$, we have

$$
\begin{aligned}
&\lim_{n\to\infty} \mathbb{P}\Big(Y^{(1)}_{n,1,2} \le u_n(x_1), Y^{(1)}_{n,s,2} \le u_n(x_1), Y^{(2)}_{n,s,2} \le u_n(x_2), s \in K\Big) \\
&\quad = \mathbb{P}\Big(\frac{z}{2} + \sqrt{\delta_{12}(0)}W^{(1)}_{1,2} \le \delta_{12}(0) + \frac{x_1 - x_2}{2}, \delta_{12}(0) < \infty, \\
&\qquad \frac{z}{2} + \sqrt{\delta_{12}(s-1)}W^{(1)}_{s,2} \le \delta_{12}(s-1) + \frac{x_1 - x_2}{2}, \delta_{12}(s-1) < \infty, \\
&\qquad \frac{z}{2} + \sqrt{\delta_{22}(s-1)}W^{(2)}_{s,2} \le \delta_{22}(s-1), \quad \delta_{22}(s-1) < \infty \quad \text{for all } s \in K\Big).
\end{aligned} \tag{20}
$$

Now (18) follows immediately from (19) and (20). By similar arguments, (17) can be established. □

Lemma 3 *Under the conditions of Theorem 1, we have*

$$
\lim_{n\to\infty} \mathbb{P}\Big(M^{(1)}_{1,r_n} \le u_n(x_1), M^{(2)}_{1,r_n} \le u_n(x_2) \big| X^{(1)}_{n,1} > u_n(x_1)\Big) = \vartheta_1(x_1, x_2), \tag{21}
$$

and

$$
\lim_{n\to\infty} \mathbb{P}\Big(M^{(1)}_{r_n} \le u_n(x_1), M^{(2)}_{1,r_n} \le u_n(x_2) \big| X^{(2)}_{n,1} > u_n(x_2)\Big) = \vartheta_2(x_1, x_2), \tag{22}
$$

where $\vartheta_1(x_1, x_2)$ and $\vartheta_2(x_1, x_2)$ are given in (4) and (5), respectively.

Proof To prove (21) and (22), it suffices to show that for each $j \in \{1, 2\}$

$$
\lim_{m\to\infty} \limsup_{n\to\infty} \mathbb{P}\left(\bigcup_{s=m}^{r_n} X^{(j)}_{n,s} > u_n(x_j) \big| X^{(i)}_{n,1} > u_n(x_i)\right) = 0, \tag{23}
$$

for $i = 1, 2$. Using similar arguments as those used in Lemma 2, we have

$$
\begin{aligned}
&\mathbb{P}\left(\bigcup_{s=m}^{r_n} X^{(j)}_{n,s} > u_n(x_j) \big| X^{(i)}_{n,1} > u_n(x_i)\right) \\
&\quad \sim \int_0^\infty \mathbb{P}\left(\bigcup_{s=m}^{r_n} \rho_{ij}(s-1, n)\left(u_n(x_i) + \frac{z}{u_n(x_i)}\right)\right. \\
&\qquad + \frac{Y^{(j)}_{n,s,i} - \rho_{ij}(s-1, n)\left(u_n(x_i) + \frac{z}{u_n(x_i)}\right)}{\sqrt{1 - \rho^2_{ij}(s-1, n)}} \\
&\qquad \left. \times \sqrt{1 - \rho^2_{ij}(s-1, n)} > u_n(x_j)\right) \exp\left(-z - \frac{z^2}{2u_n^2(x_i)}\right) \mathrm{d}z.
\end{aligned}
$$

Noting that condition (9) implies

$$
\lim_{m\to\infty} \limsup_{n\to\infty} \max_{m \le s \le r_n} \big((1 - \rho_{ij}(s-1, n)) \log n\big)^{-1} = 0. \tag{24}
$$

Thus, for large enough n and $s \in [m, r_n]$, we have

$$\theta_{ns} := \frac{u_n(x_j) - u_n(x_i)\rho_{ij}(s-1,n)}{\sqrt{1-\rho_{ij}^2(s-1,n)}} - \frac{z\rho_{ij}(s-1,n)}{u_n(x_i)\sqrt{1-\rho_{ij}^2(s-1,n)}} > 0.$$

By Mill's inequality we have

$$\mathbb{P}\left(\frac{Y_{n,s,j}^{(j)} - \rho_{ij}(s-1,n)\left(u_n(x_i) + \frac{z}{u_n(x_i)}\right)}{\sqrt{1-\rho_{ij}^2(s-1,n)}} > \theta_{ns}\right) \le \frac{1}{\theta_{ns}\sqrt{2\pi}} \exp\left(-\frac{1}{2}\theta_{ns}^2\right). \quad (25)$$

Since, for large enough n,

$$\theta_{ns}^2 \ge C + \frac{1-\rho_{ij}(s-1,n)}{1+\rho_{ij}(s-1,n)} b_n^2 \ge C + \frac{1-\rho_{ij}(s-1,n)}{1+\rho_{ij}(s-1,n)}(2\log n - \log(\log n)),$$

it follows that the left side of (25) can be bounded by

$$C\frac{1}{\sqrt{1-\rho_{ij}^2(s-1)}} n^{-\frac{1-\rho_{ij}(s-1,n)}{1+\rho_{ij}(s-1,n)}} (\log n)^{-\frac{\rho_{ij}(s-1,n)}{1+\rho_{ij}(s-1,n)}}. \quad (26)$$

Thus, for each fixes $z_0 > 0$ we have

$$\begin{aligned}
&\lim_{m\to\infty} \limsup_{n\to\infty} \int_0^{z_0} \mathbb{P} \\
&\left(\bigcup_{s=m}^{r_n} \rho_{ij}(s-1,n)\left(u_n(x_i) + \frac{z}{u_n(x_i)}\right)\right. \\
&+ \frac{Y_{n,s,i}^{(j)} - \rho_{ij}(s-1,n)\left(u_n(x_i) + \frac{z}{u_n(x_i)}\right)}{\sqrt{1-\rho_{ij}^2(s-1,n)}} \\
&\left.\times\sqrt{1-\rho_{ij}^2(s-1,n)} > u_n(x_j)\right) \exp\left(-z - \frac{z^2}{2u_n^2(x_i)}\right) \mathrm{d}z = 0.
\end{aligned}$$

Hence, (23) holds. By Lemma 2, the claim of this lemma follows. □

Lemma 4 *Under conditions of Theorem 1, we have*

$$\limsup_{n\to\infty} q_n \mathbb{P}(\xi_1 = \eta_1 = 1) = \vartheta e^{-x_1} - \vartheta_1(x_1,x_2)e^{-x_1} + \vartheta e^{-x_2} - \vartheta_2(x_1,x_2)e^{-x_2},$$

where $\vartheta_1(x_1,x_2)$, $\vartheta_2(x_1,x_2)$ and ϑ are given in (4), (5) and (6), respectively.

Proof Note that

$$
\begin{aligned}
\mathbb{P}(\xi_1 = \eta_1 = 1) &= \mathbb{P}\Big(M_{r_n}^{(1)} > u_n(x_1), M_{r_n}^{(2)} > u_n(x_2)\Big) \\
&= \mathbb{P}\Big(M_{r_n}^{(2)} > u_n(x_2)\Big) - \mathbb{P}\Big(M_{r_n}^{(2)} > u_n(x_2), M_{r_n}^{(1)} \le u_n(x_1)\Big) \\
&= \mathbb{P}\Big(M_{r_n}^{(2)} > u_n(x_2)\Big) \\
&\quad - \sum\nolimits_{k=1}^{r_n} \mathbb{P}\Big(X_{n,k}^{(2)} > u_n(x_2), M_{k,r_n}^{(2)} \le u_n(x_2), M_{k-1,r_n}^{(1)} \le u_n(x_1)\Big) \\
&+ \sum\nolimits_{k=1}^{r_n} \mathbb{P} \\
&\quad \Big(X_{n,k}^{(2)} > u_n(x_2), M_{k,r_n}^{(2)} \le u_n(x_2), M_{k-1,r_n}^{(1)} \le u_n(x_1), M_{k-1}^{(1)} > u_n(x_1)\Big) \\
&= \mathbb{P}\Big(M_{r_n}^{(2)} > u_n(x_2)\Big) \\
&\quad - \sum\nolimits_{k=1}^{r_n} \mathbb{P}\Big(X_{n,k}^{(2)} > u_n(x_2), M_{k,r_n}^{(2)} \le u_n(x_2), M_{k-1,r_n}^{(1)} \le u_n(x_1)\Big) \\
&+ \sum\nolimits_{\ell=1}^{r_n-1} \mathbb{P}\Big(M_{\ell,r_n}^{(2)} > u_n(x_2), X_{n,\ell}^{(1)} > u_n(x_1), M_{\ell,r_n}^{(1)} \le u_n(x_1)\Big).
\end{aligned} \tag{27}
$$

By Lemma 3, we have

$$
\mathbb{P}\Big(X_{n,1}^{(2)} > u_n(x_2), M_{1,r_n}^{(2)} \le u_n(x_2), M_{r_n}^{(1)} \le u_n(x_1)\Big) \sim \frac{1}{n}\vartheta_2(x_1, x_2)e^{-x_2}, \tag{28}
$$

and

$$
\begin{aligned}
&\mathbb{P}\Big(M_{1,r_n}^{(2)} > u_n(x_2), X_{n,1}^{(1)} > u_n(x_1), M_{1,r_n}^{(1)} \le u_n(x_1)\Big) \\
&= \mathbb{P}\Big(X_{n,1}^{(1)} > u_n(x_1), M_{1,r_n}^{(1)} \le u_n(x_1)\Big) \\
&\quad - \mathbb{P}\Big(M_{1,r_n}^{(1)} \le u_n(x_1), M_{1,r_n}^{(2)} \le u_n(x_2), X_{n_1}^{(1)} > u_n(x_2)\Big) \\
&\sim \frac{1}{n}\vartheta e^{-x_1} - \frac{1}{n}\vartheta_1(x_1, x_2)e^{-x_1}.
\end{aligned} \tag{29}
$$

By Hsing et al. (1996) and O'Brien (1987),

$$
\lim_{n\to\infty} q_n \mathbb{P}\Big(M_{r_n}^{(2)} > u_n(x_2)\Big) = \vartheta e^{-x_2}. \tag{30}
$$

Combing (27–30), we can get the assertion of this lemma. □

5 Proofs

5.1 Proof of Theorem 1

By the fact that $\lim_{n\to\infty} n(1 - \Phi(u_n(x))) = e^{-x}$ and (7), we have

$$
\begin{aligned}
&\mathbb{E}\left|\omega^{N_n^{(1)}} z^{N_n^{(2)}} - \omega^{\sum_{k=1}^{q_n}\xi_k} z^{\sum_{k=1}^{q_n}\eta_k}\right| \\
&\quad \le \mathbb{P}\left(\cup_{s\in V_k} X_{n,s}^{(1)} > u_n(x_1), k=1,2,\dots,q_n\right) \\
&\quad + \mathbb{P}\left(\cup_{s\in V_k} X_{n,s}^{(2)} > u_n(x_2), k=1,2,\dots,q_n\right) \\
&\quad \le \ell_n q_n\left[(1-\Phi(u_n(x_1)) + (1-\Phi(u_n(x_2))\right] \\
&\quad \le \frac{\ell_n}{r_n+\ell_n}\left[n(1-\Phi(u_n(x_1)) + n(1-\Phi(u_n(x_2))\right] \to 0
\end{aligned}
\tag{31}
$$

as $n\to\infty$, where $V_k = \{(k-1)(r_n+\ell_n)+r_n+1, (k-1)(r_n+\ell_n)+r_n+2, \dots, k(r_n+\ell_n)\}$ and $k=1,2,\dots,q_n$. Since

$$
\begin{aligned}
\mathbb{E}(\omega^{\xi_1} z^{\eta_1}) &= \mathbb{P}(\xi_1=\eta_1=0) + \omega\mathbb{P}(\xi_1=1,\eta_1=0) \\
&\quad + z\mathbb{P}(\xi_1=0,\eta_1=1) + \omega z\mathbb{P}(\xi_1=\eta_1=1) \\
&= 1-(1-\omega)\mathbb{P}(\xi_1=1) - (1-z)\mathbb{P}(\eta_1=1) + (1-\omega)(1-z)\mathbb{P}(\xi_1=\eta_1=1).
\end{aligned}
$$

It follows from Lemma 4 and (30) that

$$
\begin{aligned}
\lim_{n\to\infty}\mathbb{E}\left(\omega^{\xi_1} z^{\eta_1}\right)^{q_n} &= \exp\{-(1-\omega)\vartheta e^{-x_1} - (1-z)\vartheta e^{-x_2} + (1-\omega)(1-z) \\
&\qquad \left[\vartheta e^{-x_1} - \vartheta_1(x_1,x_2)e^{-x_1} + \vartheta e^{-x_2} - \vartheta_2(x_1,x_2)e^{-x_2}\right]\}.
\end{aligned}
\tag{32}
$$

Therefore, by combing Lemma 1 and (31)-(32), the assertion of this theorem can be obtained. □

5.2 Proof of Theorem 2

The proof is similar to those of McCormick and Qi (2000) and Hu et al. (2009). Under the condition (13), we can find a sequence of integer $m(n)$ satisfying

$$
\lim_{n\to\infty} m(n) = \infty, \tag{33}
$$

and

$$
\lim_{n\to\infty}\frac{m(n)\log n}{n^2}\sum_{s=1}^{n}\sum_{\ell=1}^{n}|\rho_{ij}(|s-\ell|,n)| = 0, \tag{34}
$$

for $i,j=1,2$. Recall that $I_n=\{1,2,\dots,n\}$ and denote

$$
\delta_{ni}(s) = \mathbb{E}\left(X_{n,s}^{(i)} S_{n1}\right) = \sum_{\ell=1}^{n}\rho_{1i}(|s-\ell|,n) \quad \text{for} \quad s\in I_n, \quad i=1,2.
$$

Let $W_{ni}^{+} = \sum_{s=1}^{n}\delta_{ni}^{+}(s)$ and $W_{ni}^{-} = \sum_{s=1}^{n}\delta_{ni}^{-}(s)$, where $\delta_{ni}^{+}(s)$ and $\delta_{ni}^{-}(s)$ are the positive and negative part of $\delta_{ni}(s)$, respectively. Define

$$J_n = \left\{ s : \delta_{ni}(s) \geq 0, \frac{\delta_{ni}(s)}{W_{ni}^+} \leq \frac{\log m(n)}{n} \right\}$$
$$\cup \left\{ s : \delta_{ni}(s) < 0, \frac{\delta_{ni}(s)}{W_{ni}^-} \leq \frac{\log m(n)}{n} \right\} \quad i = 1, 2,$$

and $S_{ni}^+ = \sum_{\delta_{ni}(s) \geq 0} X_{n,s}^{(i)}$, $S_{ni}^- = \sum_{\delta_{ni}(s) < 0} X_{n,s}^{(i)}$ for $i = 1, 2$.

Next, we construct an array $\{Y_{n,s}^{(i)}, s \in J_n, n \geq 1\}$ that is independent of $S_{n1} = \sum_{s=1}^n X_{n,s}^{(1)}$ by

$$Y_{n,s}^{(i)} = \begin{cases} X_{n,s}^{(i)} - \frac{\delta_{ni}(s)}{W_{ni}^+} S_{ni}^+; & \text{if } \delta_{ni}(s) \geq 0 \\ X_{n,s}^{(i)} - \frac{\delta_{ni}(s)}{W_{ni}^-} S_{ni}^-; & \text{if } \delta_{ni}(s) < 0 \end{cases}$$

for $i = 1, 2$.

Note that

$$\mathrm{Var}(S_{ni}^{\pm}) \leq \sum_{s=1}^n \sum_{\ell=1}^n \left| \rho_{ii}(|s - \ell|, n) \right|.$$

It follows from (13) that

$$\lim_{n \to \infty} \frac{m(n) \log n}{n^2} \mathrm{Var}(S_{ni}^{\pm}) = 0,$$

which implies $\frac{\sqrt{m(n) \log n}}{n} S_{ni}^{\pm} \to 0$ in probability.

Let $\rho'_{ij}(s, \ell, n) = \mathbb{E}\left(Y_{n,s}^{(i)} Y_{n,\ell}^{(j)} \right)$ for $i, j = 1, 2$ and $s, \ell \in J_n$. It follows from James et al. (2007) that

$$\rho'_{ij}(s, \ell, n) = \rho_{ij}(|s - \ell|, n) + o\left(\frac{\log^2 m(n)}{m(n) \log n} \right),$$

which yields

$$\lim_{n \to \infty} \frac{m(n) \log n}{n^2 \log^2 m(n)} \sum_{s=1}^n \sum_{\ell=1}^n |\rho'_{ij}(s, \ell, n)| = 0, \quad i, j = 1, 2. \tag{35}$$

Similarly, we construct a new array $\{Z_{n,s}^{(i)}\}$ that is independent of $S_{n2} = \sum_{s=1}^n X_{n,s}^{(2)}$ which maintains that independent of S_{n1} based on $Y_{n,s}^{(i)}$, $s \in J_n, i = 1, 2$.

Define

$$\sigma_{ni}(s) = \mathbb{E}(Y_{n,s}^{(i)} S_{n2}), \quad s \in J_n, i = 1, 2.$$

Let $V_{ni}^+ = \sum_{s \in J_n} \sigma_{ni}^+(s)$ and $V_{ni}^- = \sum_{s \in J_n} \sigma_{ni}^-(s)$, where $\sigma_{ni}^+(s)$ and $\sigma_{ni}^-(s)$ are the positive and negative parts of $\sigma_{ni}(s)$, respectively.

Similarly, put

$$R_n = \left\{ s \in J_n; \sigma_{ni}(s) \geq 0, \frac{\sigma_{ni}(s)}{V^+_{ni}} \leq \frac{\log m(n)}{n} \right\}$$
$$\cup \left\{ s \in J_n; \sigma_{ni}(s) < 0, \frac{\sigma_{ni}(s)}{V^-_{ni}} \leq \frac{\log m(n)}{n} \right\}$$

and let $T^+_{ni} = \sum_{s \in J_n; \sigma_{ni}(s) \geq 0} Y^{(i)}_{n,s}$ and $T^-_{ni} = \sum_{s \in J_n; \sigma_{ni}(s) < 0} Y^{(i)}_{n,s}$ for $i = 1, 2$.

For $s \in R_n$ and $n \geq 1$, let

$$Z^{(i)}_{n,s} = \begin{cases} Y^{(i)}_{n,s} - \frac{\delta_{ni}(s)}{V^+_{ni}} T^+_{ni}; & \text{if } \sigma_{ni}(s) \geq 0, \\ Y^{(i)}_{n,s} - \frac{\delta_{ni}(s)}{V^-_{ni}} T^-_{ni}; & \text{if } \sigma_{ni}(s) < 0. \end{cases}$$

Then, we have $\mathbb{E}(Z^{(i)}_{n,s} S_{n2}) = 0$, which implies $\{Z^{(i)}_{n,s}, s \in R_n, n \geq 1\}$, $i = 1, 2$ is independent of $\mathbf{S_n} = (S_{n1}, S_{n2})$. By (35), we have

$$\lim_{n\to\infty} \frac{m(n) \log n}{n^2 \log^2 m(n)} \mathrm{Var}(T^{\pm}_{ni}) = 0,$$

which implies

$$\frac{\sqrt{m(n) \log n}}{n \log m(n)} T^{\pm}_{ni} \xrightarrow{P} 0.$$

Next, define

$$\mathbf{N}^*_n(\mathbf{B}, \mathbf{x}) = \sum_{i=1}^{2} \sum_{k=1}^{q_n} I\Big(\cup_{s \in Q_k} Y^{(i)}_{n,s} > u_n(x_i), s \in J_n, \frac{s}{n} \in B_i \Big),$$

and

$$\mathbf{N}'_n(\mathbf{B}, \mathbf{x}) = \sum_{i=1}^{2} \sum_{k=1}^{q_n} I\Big(\cup_{s \in Q_k} Z^{(i)}_{n,s} > u_n(x_i), s \in R_n, \frac{s}{n} \in B_i \Big),$$

for $\mathbf{x} \in R^2$, $\mathbf{B} = \cup_{i=1}^2 (B_i \times \{i\})$, where B_i are Borel sets on (0, 1].

It suffices to prove that

$$\mathbf{N}_n(\mathbf{B}, \mathbf{x}) - \mathbf{N}'_n(\mathbf{B}, \mathbf{x}) = o_p(1).$$

Note that

$$\begin{aligned}\mathbf{N}_n(\mathbf{B},\mathbf{x}) - \mathbf{N}_n^{'}(\mathbf{B},\mathbf{x}) &= (\mathbf{N}_n(\mathbf{B},\mathbf{x}) - \mathbf{N}_n^{*}(\mathbf{B},\mathbf{x})) + (\mathbf{N}_n^{*}(\mathbf{B},\mathbf{x}) - \mathbf{N}_n^{'}(\mathbf{B},\mathbf{x}))\\
&= \sum_{i=1}^{2}\sum_{k=1}^{q_n} I\Big(\cup_{s\in Q_k} X_{n,s}^{(i)} > u_n(x_i), s\notin J_n\Big)\\
&\quad + \sum_{i=1}^{2}\sum_{k=1}^{q_n}\Big[I\Big(\cup_{s\in Q_k} X_{n,s}^{(i)} > u_n(x_i), s\in J_n\Big)\\
&\qquad - I\Big(\cup_{s\in Q_k} Y_{n,s}^{(i)} > u_n(x_i), s\in J_n\Big)\Big]\\
&\quad + \sum_{i=1}^{2}\sum_{k=1}^{q_n} I\Big(\cup_{s\in Q_k} Y_{n,s}^{(i)} > u_n(x_i), s\notin R_n, s\in J_n\Big)\\
&\quad + \sum_{i=1}^{2}\sum_{k=1}^{q_n}\Big[I\Big(\cup_{s\in Q_k} Y_{n,s}^{(i)} > u_n(x_i), s\in R_n\Big)\\
&\qquad - I\Big(\cup_{s\in Q_k} Z_{n,s}^{(i)} > u_n(x_i), s\in R_n\Big)\Big]\\
&=: \sum_{i=1}^{2}\Big(A_1^{(i)} + A_2^{(i)} + A_3^{(i)} + A_4^{(i)}\Big).\end{aligned}$$

By the definition of J_n and R_n, we have

$$\sharp(I_n \setminus J_n) \le \frac{2n}{\log m(n)}, \quad n - \frac{2n}{\log m(n)} \le \sharp(J_n) \le n,$$

and

$$\sharp(I_n \setminus R_n) \le \frac{4n}{\log m(n)}, \quad n - \frac{4n}{\log m(n)} \le \sharp(R_n) \le n,$$

where $\sharp(\cdot)$ denotes the cardinality of any set.

By the fact $n(1-\Phi(u_n(x))) \to e^{-x}$ as $n\to\infty$, we have

$$\mathbb{E}(A_1^{(i)}) \le \frac{2n}{\log m(n)}\big(1-\Phi(u_n(x))\big) \to 0 \quad \text{as} \quad n\to\infty.$$

Let $\zeta_n = \max_{s\in J_n}\left\{\left|\frac{\delta_{ni}(s)}{W_{ni}^{+}}\right|, \left|\frac{\delta_{ni}(s)}{W_{ni}^{-}}\right|\right\}\left(\left|S_{ni}^{+}\right| + \left|S_{ni}^{-}\right|\right)$, then by the Cauchy-Schwarz inequality and (13), we have

$$\varepsilon_n^2 := a_n\mathbb{E}(\zeta_n) \le \sqrt{2\log n}\frac{2\log m(n)}{n}\sqrt{\sum_{k=1}^{n}\sum_{\ell=1}^{n}\rho_{ii}(|k-\ell|, n)} \to 0 \quad (36)$$

as $n\to\infty$. So $\mathbb{P}(a_n\zeta_n > \varepsilon_n) \le \varepsilon_n$ as $n\to\infty$ by Chebyshev's inequality, which implies

$$\begin{aligned}&\mathbb{E}\left|A_2^{(i)} I\{\zeta_n \le a_n^{-1}\varepsilon_n\}\right|\\
&\quad\le \sum_{s\in J_n}\Big[\mathbb{P}\Big(X_{n,s}^{(i)} > u_n(x_i-\varepsilon_n)\Big) - \mathbb{P}\Big(X_{n,s}^{(i)} > u_n(x_i+\varepsilon_n)\Big)\Big] \to 0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

and by Hu et al. (2009), we have

$$\max_{1\le i\le 2}\max_{s\in J_n}\left|1-\rho_{ii}'(s,s,n)\right| = o\left(\frac{1}{\log n}\right),$$

which implies

$$\mathbb{E}(A_3^{(i)}) \le \frac{4n}{\log m(n)} \mathbb{P}\left(Y_{n,s}^{(i)} > u_n(x_i)\right) \to 0 \quad \text{as} \quad n \to \infty.$$

Similarly, let $\eta_n = \max_{s\in R_n}\left\{\left|\frac{\sigma_{ni}(s)}{V_{ni}^+}\right|, \left|\frac{\sigma_{ni}(s)}{V_{ni}^-}\right|\right\}\left(\left|T_{ni}^+\right| + \left|T_{ni}^-\right|\right)$, and by similar arguments as those used in (36), let

$$(\varepsilon_n')^2 := a_n\mathbb{E}(\eta_n) \le \sqrt{2\log n}\frac{2\log m(n)}{n}\sqrt{\sum_{k=1}^n \sum_{\ell=1}^n \rho_{ii}(|k-\ell|, n)} \to 0 \;\; as \;\; n \to \infty,$$

we have

$$\mathbb{P}(a_n\eta_n > \varepsilon_n') \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\left|A_4^{(i)} I\left(\{\eta_n \le a_n^{-1}\varepsilon_n'\} \cap \{\zeta_n \le a_n^{-1}\varepsilon_n\}\right)\right| \\
&\quad \le \sum_{s\in R_n}\left[\mathbb{P}\left(X_{n,s}^{(i)} > u_n(x_i - \varepsilon_n - \varepsilon_n')\right) - \mathbb{P}\left(X_{n,s}^{(i)} > u_n(x_i + \varepsilon_n + \varepsilon_n')\right)\right] \to 0
\end{aligned}$$

as $n \to \infty$. Hence, for an arbitrary $\varepsilon > 0$, we have

$$\begin{aligned}
&\mathbb{P}\left(\left|\mathbf{N}_n(\mathbf{B}, \mathbf{x}) - \mathbf{N}_n'(\mathbf{B}, \mathbf{x})\right| > \varepsilon\right) \\
&\quad \le \varepsilon_n + \varepsilon_n' \\
&\qquad + \frac{2}{\varepsilon}\Big[\mathbb{E}\Big(A_1^{(i)} + \left|A_2^{(i)}\right| I(\zeta_n \le a_n^{-1}\varepsilon_n) \\
&\qquad\qquad + A_3^{(i)} + \left|A_4^{(i)}\right| I\left(\{\eta_n \le a_n^{-1}\varepsilon_n'\} \cap \{\zeta_n \le a_n^{-1}\varepsilon_n\}\right)\Big)\Big] \\
&\quad \to 0.
\end{aligned}$$

This completes the proof of Theorem 2. □

5.3 Proof of Corollary 1

Under the conditions of Theorem 1, by (10) and let $x_2 \to \infty$, we have

$$\lim_{n\to\infty} \mathbb{E}\left(\omega^{N_n^{(1)}}\right) = \exp\{-(1-\omega)\vartheta e^{-x_1}\},$$

which implies that the number of clusters of $u_n(x_1)$ by $\left\{X_{n,s}^{(1)}, 1 \le s \le n\right\}$ in the set of I_n have an asymptotic Poisson distribution with intensity ϑe^{-x_1}. Hence, the point process $N_n^{(1)}(B)$ converges weakly to a Poisson process with intensity ϑe^{-x_1}. □

Acknowledgements The authors would like to thank the referees for careful reading and for their perceptive and useful comments which greatly improved the paper. We appreciate Professor Wan Tang for his fruitful discussions on this topic. Jinhui Guo gratefully acknowledges the financial support by the

National Natural Science Foundation of China (grant no. 71991472). Yingyin Lu was supported by Nanchong Municipal Government-Universities Scientific Cooperation Project SXHZ045.

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