Supplement to “Directed Hybrid Random Networks Mixing Preferential Attachment with Uniform Attachment Mechanisms”

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This supplement provides the omitted technical details to the proofs of Theorem 1 and Proposition 1, in the main manuscript. The setting, notation, equation reference numbers are retained from the main paper.

1 Proof of Theorem 1

Here we only present the details for the in-degree sequence, and those for out-degrees will follow from a similar reasoning. The derivations for out-degree sequence are similar, so omitted.

For $k, \ell \geq 0$, we have

$$\begin{aligned}
\mathbb{E}\left[M^{\text{in}}_{k+\ell} - M^{\text{in}}_{k}\right] &= \sum_{s=k+1}^{k+\ell} \mathbb{E}\left[M^{\text{in}}_{s} - M^{\text{in}}_{s-1}\right] \\
&= \sum_{s=k+1}^{k+\ell} \mathbb{E}\left[\frac{(1-p)(\alpha+\beta)/|V_{S_i+s}|}{\prod_{\ell=1}^{s}\left(1+\frac{p(\alpha+\beta)}{S_i+\ell+1+\delta_{\text{in}}|V_{S_i+\ell}|}\right)}\right] \\
&\leq \sum_{s=k+1}^{k+\ell} \mathbb{E}\left[\prod_{\ell=1}^{s}\left(1+\frac{p(\alpha+\beta)}{(1+\delta_{\text{in}})(S_i+\ell+1)}\right)\frac{(1-p)(\alpha+\beta)}{|V_{S_i+s}|}\right] \\
&\leq \sum_{s=k+1}^{k+\ell} \prod_{\ell=1}^{s}\left(1+\frac{p(\alpha+\beta)}{(1+\delta_{\text{in}})(\ell+1)}\right)\mathbb{E}\left[\frac{(1-p)(\alpha+\beta)}{|V_s|}\right],
\end{aligned}$$

where

$$\prod_{\ell=1}^{s}\left(1+\frac{p(\alpha+\beta)}{(1+\delta_{\text{in}})(\ell+1)}\right) = \frac{\Gamma\left(s+2+\frac{p(\alpha+\beta)}{1+\delta_{\text{in}}}\right)}{\Gamma(s+2)\Gamma\left(2+\frac{p(\alpha+\beta)}{1+\delta_{\text{in}}}\right)} \sim s^{-(p(\alpha+\beta))/(1+\delta_{\text{in}})},$$

for s large. For the expectation in the summand, we have

$$\mathbb{E}\left[\frac{(1-p)(\alpha+\beta)}{|V(s)|}\right] = (1-p)(\alpha+\beta)\left(\frac{1}{(1-\beta)s} + \mathbb{E}\left[\frac{1}{|V(s)|} - \frac{1}{(1-\beta)s}\right]\right).$$

Since for $n \geq 1$, $|V_n| - 1$ is a binomial random variable with success probability $1-\beta$, then we apply the Chernoff bound to get

$$\mathbb{P}\left(||V_n| - (1-\beta)n| \geq \sqrt{12(1-\beta)n\log n}\right) \leq \frac{2}{n^4}. \quad (1)$$

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Then by (1), we get

$$\mathbb{E}\left[\left|\frac{1}{|V(s)|}-\frac{1}{(1-\beta)s}\right|\right]=\mathbb{E}\left[\frac{\big||V(s)|-(1-\beta)s\big|}{(1-\beta)s|V(s)|}\right]$$
$$\leq \frac{\sqrt{12(1-\beta)s\log s}}{(1-\beta)s|(1-\beta)s-\sqrt{12(1-\beta)s\log s}|}$$
$$+\frac{(2-\beta)s}{(1-\beta)s\sqrt{12(1-\beta)s\log s}}\times\frac{2}{s^4}. \tag{2}$$

Putting them together, we conclude that $\mathbb{E}\left|M^{\text{in}}_{k+\ell}-M^{\text{in}}_{k}\right|\to 0$ as $k\to\infty$, suggesting that $\left\{\mathbb{E}\left[M^{\text{in}}_n\right]\right\}_{n\geq 1}$ is a *cauchy* sequence in L_1 space. Applying the martingale convergence theorem (Durrett, 2006, Theorem 4.2.11) gives that there exists some finite random variable L_i such that as $n\to\infty$,

$$M^{\text{in}}_n \xrightarrow{a.s.} L_i. \tag{3}$$

Then it remains to show the almost sure convergence of

$$X_n := \prod_{k=0}^{n-1}\left(1+\frac{(\alpha+\beta)p}{S_i+k+1+\delta_{\text{in}}|V_{S_i+k}|}\right).$$

Consider $\log X_n$, and rewrite it as

$$\log X_n = \left[\sum_{k=0}^{n-1}\log\left(1+\frac{(\alpha+\beta)p}{S_i+k+1+\delta_{\text{in}}|V_{S_i+k}|}\right)-\sum_{k=0}^{n-1}\frac{(\alpha+\beta)p}{S_i+k+1+\delta_{\text{in}}|V_{S_i+k}|}\right]$$
$$+\left[\sum_{k=0}^{n-1}\frac{(\alpha+\beta)p}{S_i+k+1+\delta_{\text{in}}|V_{S_i+k}|}-\sum_{k=1}^{n}\frac{C_1}{S_i+k+1}\right]$$
$$+\left[\sum_{k=1}^{n}\frac{C_1}{S_i+k}-C_1\log n\right]$$
$$+C_1\log n =: P_1(n)+P_2(n)+P_3(n)+C_1\log n. \tag{4}$$

For $P_1(n)$, we first note that $\log(1+x)-x\leq 0$ for all $x\geq 0$. Then $P_1(n+1)-P_1(n)\leq 0$, i.e. $P_1(n)$ is decreasing in n, and it suffices to show

$$P_1(\infty) := \sum_{k=0}^{\infty}\left(\log\left(1+\frac{(\alpha+\beta)p}{S_i+k+1+\delta_{\text{in}}|V_{S_i+k}|}\right)-\frac{(\alpha+\beta)p}{S_i+k+1+\delta_{\text{in}}|V_{S_i+k}|}\right)$$

is finite almost surely. Note also that $|\log(1+x)-x|\leq x^2/2$, for all $x\geq 0$, then

$$\mathbb{E}\left|\sum_{k=0}^{\infty}\left(\log\left(1+\frac{(\alpha+\beta)p}{S_i+k+1+\delta_{\text{in}}|V_{S_i+k}|}\right)-\frac{(\alpha+\beta)p}{S_i+k+1+\delta_{\text{in}}|V_{S_i+k}|}\right)\right|$$
$$\leq \sum_{k=0}^{\infty}\mathbb{E}\left|\log\left(1+\frac{(\alpha+\beta)p}{S_i+k+1+\delta_{\text{in}}|V_{S_i+k}|}\right)-\frac{(\alpha+\beta)p}{S_i+k+1+\delta_{\text{in}}|V_{S_i+k}|}\right|$$
$$\leq \frac{(\alpha+\beta)^2p^2}{2}\sum_{k=0}^{\infty}\mathbb{E}\left(\frac{1}{S_i+k+1+\delta_{\text{in}}|V_{S_i+k}|}\right)^2\leq\sum_{k=1}^{\infty}\frac{1}{k^2}<\infty.$$

Hence, $P_1(n)\xrightarrow{a.s.}P_1(\infty)$.

For $P_2(n)$, we apply (Athreya and Ney, 2004, Theorem 3.9.4) to conclude that there exists a finite r.v. Z such that

$$\sum_{k=1}^{\infty}\left(\frac{(\alpha+\beta)p}{k+\delta_{\text{in}}|V(k-1)|}-\frac{C_1}{k}\right)\xrightarrow{a.s.} Z,$$

then

$$P_2(n)\xrightarrow{a.s.} Z-\sum_{k=1}^{S_i}\left(\frac{(\alpha+\beta)p}{k+\delta_{\text{in}}|V(k-1)|}-\frac{C_1}{k}\right)=: P_2(\infty).$$

Meanwhile, since $\sum_{k=1}^{n} 1/k-\log n\to\tilde{c}$ as $n\to\infty$, where $\tilde{c}$ is Euler's constant, then for $i\geq 1$,

$$P_3(n)\xrightarrow{a.s.} C_1\left(\tilde{c}+\log(S_i+1)-\sum_{k=1}^{S_i}\frac{1}{k}\right)=: P_3(\infty).$$

Then we conclude from (4) that

$$\frac{X_n}{n^{C_1}}\xrightarrow{a.s.}\exp\left(P_1(\infty)+P_2(\infty)+P_3(\infty)\right)<\infty. \tag{5}$$

The results in the theorem follow by combining (5) with (3), where we set

$$\zeta_i := L_i\exp\left(-(P_1(\infty)+P_2(\infty)+P_3(\infty))\right).$$

□

2 Proof of Proposition 1

We explicitly demonstrate the derivations of Proposition 1 for the in-degree distribution, and the methodology is also applicable to out-degrees. Let $\mathcal{F}_n$ denote the σ-field generated by the network evolution up to n steps. Set τ to be an $(\mathcal{F}_n)_{n\geq 0}$-stopping time, then

$$\mathcal{F}_\tau=\{F : F\cap\{\tau=n\}\in\mathcal{F}_n\}.$$

For $i\geq 1$, let S_i be the time when node i is created, i.e.,

$$S_i=\inf\{n\geq 0 : |V_n|=i\}.$$

Then S_i is an $(\mathcal{F}_n)_{n\geq 0}$-stopping time. Also, for $n\geq k\geq 0$, we have

$$\{S_i+k=n\}=\{S_i=n-k\}\in\mathcal{F}_{n-k}\subset\mathcal{F}_n,$$

so S_i+k, $k\geq 0$, is a stopping time with respect to $(\mathcal{F}_n)_{n\geq 0}$.

Since the in-degree of i is increased at most by 1 at each evolutionary step, then under $\mathbb{P}^{\mathcal{F}_{S_i+n}}(\cdot) := \mathbb{P}(\cdot|\mathcal{F}_{S_i+n})$, we have

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_{S_i+n}}\left[\left(D_i^{\text{in}}(S_i+n+1)+\delta_{\text{in}}\right)^2\right]\\
&=\left(D_i^{\text{in}}(S_i+n)+\delta_{\text{in}}\right)^2+(\alpha+\beta)\left[\frac{p(D_i^{\text{in}}(S_i+n)+\delta_{\text{in}})}{\sum_{k\in V_{S_i+n}}(D_k^{\text{in}}(S_i+n)+\delta_{\text{in}})}+\frac{(1-p)}{|V_{S_i+n}|}\right]\\
&\quad+2(\alpha+\beta)\left(D_i^{\text{in}}(S_i+n)+\delta_{\text{in}}\right)\left[\frac{p(D_i^{\text{in}}(S_i+n)+\delta_{\text{in}})}{\sum_{k\in V_{S_i+n}}(D_k^{\text{in}}(S_i+n)+\delta_{\text{in}})}+\frac{(1-p)}{|V_{S_i+n}|}\right]\\
&=\left(D_i^{\text{in}}(S_i+n)+\delta_{\text{in}}\right)^2\left[1+\frac{2(\alpha+\beta)p}{S_i+n+1+\delta_{\text{in}}|V_{S_i+n}|}\right]\\
&\quad\ +\left(D_i^{\text{in}}(S_i+n)+\delta_{\text{in}}\right)\left[\frac{(\alpha+\beta)p}{S_i+n+1+\delta_{\text{in}}|V_{S_i+n}|}+\frac{2(\alpha+\beta)(1-p)}{|V_{S_i+n}|}\right]\\
&\quad\ +(\alpha+\beta)\frac{1-p}{|V_{S_i+n}|}.
\end{aligned}\tag{6}$$

With $C_1 = p(\alpha+\beta)/(1+\delta_{\text{in}}(1-\beta))$, we have

$$
\begin{aligned}
&\mathbb{E}\left[\left(D_i^{\text{in}}(S_i+n+1)+\delta_{\text{in}}\right)^2\right] \\
&\quad = \mathbb{E}\left[\left(D_i^{\text{in}}(S_i+n)+\delta_{\text{in}}\right)^2\left(1+\frac{2(\alpha+\beta)p}{S_i+n+1+\delta_{\text{in}}|V_{S_i+n}|}\right)\right] \\
&\qquad + \mathbb{E}\left[\left(D_i^{\text{in}}(S_i+n)+\delta_{\text{in}}\right)\left(\frac{(\alpha+\beta)p}{S_i+n+1+\delta_{\text{in}}|V_{S_i+n}|}+\frac{2(\alpha+\beta)(1-p)}{|V_{S_i+n}|}\right)\right] \\
&\qquad + (\alpha+\beta)\mathbb{E}\left(\frac{1-p}{|V_{S_i+n}|}\right) \\
&\quad =: R_i^{(1)}(n)+R_i^{(2)}(n)+R_i^{(3)}(n).
\end{aligned} \tag{7}
$$

First, we rewrite $R_i^{(1)}(n)$ to obtain

$$
\begin{aligned}
R_i^{(1)}(n) &= \mathbb{E}\left[\left(D_i^{\text{in}}(S_i+n)+\delta_{\text{in}}\right)^2\right]\left(1+\frac{2C_1}{n}\right) \\
&\quad + \mathbb{E}\left[\left(D_i^{\text{in}}(S_i+n)+\delta_{\text{in}}\right)^2\left(\frac{2(\alpha+\beta)p}{S_i+n+1+\delta_{\text{in}}|V_{S_i+n}|}-\frac{2C_1}{n}\right)\right] \\
&\le \mathbb{E}\left[\left(D_i^{\text{in}}(S_i+n)+\delta_{\text{in}}\right)^2\right]\left(1+\frac{2C_1}{n}\right) \\
&\quad + \mathbb{E}\left[\left(D_i^{\text{in}}(S_i+n)+\delta_{\text{in}}\right)^2\frac{2(\alpha+\beta)p\delta_{\text{in}}\big||V_n|-(1-\beta)n\big|}{(n+\delta_{\text{in}}|V_n|)(1+\delta_{\text{in}}(1-\beta))n}\right] \\
&\le \mathbb{E}\left[\left(D_i^{\text{in}}(S_i+n)+\delta_{\text{in}}\right)^2\right]\left(1+\frac{2C_1}{n}\right) \\
&\quad + \mathbb{E}\left[\left(D_i^{\text{in}}(S_i+n)+\delta_{\text{in}}\right)^2\frac{2\delta_{\text{in}}\big||V_n|-(1-\beta)n\big|}{n^2}\right].
\end{aligned}
$$

Note also that $\big||V_n|-(1-\beta)n\big| \le n+1$, and $\left(D_i^{\text{in}}(S_i+n)+\delta_{\text{in}}\right)^2 \le (n+1+\delta_{\text{in}})^2$, then

$$
\begin{aligned}
&\mathbb{E}\left[\left(D_i^{\text{in}}(S_i+n)+\delta_{\text{in}}\right)^2\frac{2\delta_{\text{in}}\big||V_n|-(1-\beta)n\big|}{n^2}\right] \\
&\le \mathbb{E}\left[\left(D_i^{\text{in}}(S_i+n)+\delta_{\text{in}}\right)^2\right]\frac{2\delta_{\text{in}}\sqrt{12n\log n}}{n^2}+\frac{2\delta_{\text{in}}(n+1+\delta_{\text{in}})^2(n+1)}{n^2}\times\frac{2}{n^4}.
\end{aligned}
$$

Therefore, we conclude that

$$
\begin{aligned}
R_i^{(1)}(n) &\le \mathbb{E}\left[\left(D_i^{\text{in}}(S_i+n)+\delta_{\text{in}}\right)^2\right]\left(1+\frac{2C_1}{n}+\frac{2\delta_{\text{in}}\sqrt{12n\log n}}{n^2}\right) \\
&\quad + \frac{2\delta_{\text{in}}(n+1+\delta_{\text{in}})^2(n+1)}{n^2}\times\frac{2}{n^4}.
\end{aligned} \tag{8}
$$

For $R_i^{(3)}(n)$, we apply the Cauchy-Schwartz inequality to obtain

$$
R_i^{(3)}(n) \le \frac{1-p}{n(1-\beta)}\left(\mathbb{E}\left[\left((1-\beta)(S_i+n+1)-|V_{S_i+n}|\right)^2\right]\right)^{1/2}\left(\mathbb{E}\left[|V_n|^{-2}\right]\right)^{1/2}.
$$

By Theorem 3.9.4 in Athreya and Ney (2004), we have as $n\to\infty$,

$$
n^2\mathbb{E}\left[|V_n|^{-2}\right] \to (1-\beta)^{-2}.
$$

Meanwhile,

$$
\mathbb{E}\left[\left((1-\beta)(S_i+n+1)-|V_{S_i+n}|\right)^2\right]
$$

$$= \mathrm{Var}\left(|V_{S_i+n}|\right) + \mathrm{Var}\left((1-\beta)(S_i+n+1)\right)$$
$$= \beta(1-\beta)\left(\mathbb{E}\left(S_i\right)+n\right) + 2(1-\beta)^2\mathrm{Var}(S_i).$$

Hence, there exists some constant $A_i > 0$ such that

$$R_i^{(3)}(n) \le A_i n^{-3/2}. \tag{9}$$

For $R_i^{(2)}(n)$, we now claim that

$$\sup_{i\ge1} \frac{\mathbb{E}(D_i^{\mathrm{in}}(n)+\delta_{\mathrm{in}})}{n^{C_1}} < \infty, \tag{10}$$

then for some constant $K > 0$, $R_i^{(2)}(n)$ is bounded by

$$Kn^{C_1}\mathbb{E}\left(\frac{(\alpha+\beta)p}{S_i+n+1+\delta_{\mathrm{in}}|V_{S_i+n}|} + \frac{2(\alpha+\beta)(1-p)}{|V_{S_i+n}|}\right),$$

and by (1) and (9) we see that

$$\le Kn^{C_1}\left(\frac{C_1}{n} + \frac{2\delta_{\mathrm{in}}\sqrt{12n\log n}}{n^2} + \frac{4\delta_{\mathrm{in}}(n+1+\delta_{\mathrm{in}})^2(n+1)}{n^6} + 2A_i n^{-3/2}\right). \tag{11}$$

Then combining (8), (9) and (11) gives that there exists some constant $A_0 > 0$ such that

$$\begin{aligned}
&\mathbb{E}\left[\left(D_i^{\mathrm{in}}(S_i+n+1)+\delta_{\mathrm{in}}\right)^2\right]\\
&\le \mathbb{E}\left[\left(D_i^{\mathrm{in}}(S_i+n)+\delta_{\mathrm{in}}\right)^2\right]\left(1+\frac{2C_1}{n}+\frac{2\delta_{\mathrm{in}}\sqrt{12n\log n}}{n^2}\right) + A_0 n^{C_1-1}\\
&\le \cdots \le \left(\alpha\delta_{\mathrm{in}}^2+\gamma(1+\delta_{\mathrm{in}})^2\right)\prod_{k=1}^{n}\left(1+\frac{2C_1}{k}+\frac{2\delta_{\mathrm{in}}\sqrt{12k\log k}}{k^2}\right)\\
&\qquad + A_0\sum_{k=1}^{n}k^{C_1-1}\prod_{l=k+1}^{n}\left(1+\frac{2C_1}{l}+\frac{2\delta_{\mathrm{in}}\sqrt{12l\log l}}{l^2}\right).
\end{aligned}$$

Note that

$$\begin{aligned}
\prod_{l=k+1}^{n}\left(1+\frac{2C_1}{l}+\frac{2\delta_{\mathrm{in}}\sqrt{12l\log l}}{l^2}\right) &\le \exp\left\{\sum_{l=k+1}^{n}\left(\frac{2C_1}{l}+\frac{2\delta_{\mathrm{in}}\sqrt{12l\log l}}{l^2}\right)\right\}\\
&\le (n/k)^{2C_1}\exp\left\{\sum_{l=k+1}^{\infty}\frac{2\delta_{\mathrm{in}}\sqrt{12l\log l}}{l^2}\right\}.
\end{aligned} \tag{12}$$

Since $\sum_{l=k+1}^{\infty}\frac{2\delta_{\mathrm{in}}\sqrt{12l\log l}}{l^2} < \infty$, then we conclude that

$$\sup_{i\ge1}\frac{\mathbb{E}\left[\left(D_i^{\mathrm{in}}(n)\right)^2\right]}{n^{2C_1}} < \infty.$$

Hence, we are left with verifying (10). Note that

$$\begin{aligned}
&\mathbb{E}\left(D_i^{\mathrm{in}}(S_i+n+1)+\delta_{\mathrm{in}}\right)\\
&\quad= \mathbb{E}\left[\left(D_i^{\mathrm{in}}(S_i+n)+\delta_{\mathrm{in}}\right)\left(1+\frac{p(\alpha+\beta)}{S_i+n+1+\delta_{\mathrm{in}}|V_{S_i+n}|}\right)\right] + \mathbb{E}\left[\frac{1-p}{|V_{S_i+n}|}\right]
\end{aligned}$$

$$\le \mathbb{E}\left[(D_i^{\text{in}}(S_i+n)+\delta_{\text{in}}) \left(1+\frac{C_1}{n}\right)\right]$$
$$+\mathbb{E}\left[(D_i^{\text{in}}(S_i+n)+\delta_{\text{in}}) \left|\frac{p(\alpha+\beta)}{n+1+\delta_{\text{in}}|V_n|}-\frac{C_1}{n}\right|\right] + \mathbb{E}\left[\frac{1-p}{|V_{S_i+n}|}\right],$$

and applying (1) and (9) gives

$$\le \mathbb{E}\left(D_i^{\text{in}}(S_i+n)+\delta_{\text{in}}\right)\left(1+\frac{C_1}{n}+\frac{\delta_{\text{in}}\sqrt{12n\log n}}{n^2}\right)$$
$$+\frac{2\delta_{\text{in}}(n+1+\delta_{\text{in}})(n+1)}{n^6}+A_i n^{-3/2}. \tag{13}$$

Iterating the inequality in (13) backwards for n times gives

$$\mathbb{E}\big[D_i^{\text{in}}(S_i+n+1)+\delta_{\text{in}}\big]$$
$$\le \big(\delta_{\text{in}}\alpha+(1+\delta_{\text{in}})\gamma\big)\prod_{r=1}^{n}\left(1+\frac{C_1}{r}+\frac{1+\sqrt{12r\log r}}{r^2}\right)$$
$$+\sum_{r=1}^{n}\left(\frac{2\delta_{\text{in}}(r+1+\delta_{\text{in}})(r+1)}{r^6}+A_i r^{-3/2}\right)\prod_{s=r+1}^{n}\left(1+\frac{C_1}{s}+\frac{1+\sqrt{12s\log s}}{s^2}\right). \tag{14}$$

By a similar argument as in (12), we see that the bound in (14) implies (10), which completes the proof of Proposition 1. □

References

Athreya, K. and P. Ney (2004). *Branching processes. Reprint of the 1972 original.* Springer.

Durrett, R. T. (2006). *Random Graph Dynamics.* Cambridge, U.K.: Cambridge University Press.