Supplementary to "Inference of Random Effects for Linear Mixed-Effects Models with a Fixed Number of Clusters"

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Supplementary Material

The supplementary materials consist of three appendices that prove all the theoretical results except for Theorem 2, whose proof is straightforward and is hence omitted. Appendix A contains auxiliary lemmas that are required in the proofs. Appendix B provides proofs for Example 1 and Theorems 1 and 3–5. Appendix C gives proofs for all the lemmas.

A Auxiliary Lemmas

We start with the following matrix identities, which will be repeated applied:

$$\det(\boldsymbol{A}+\boldsymbol{c}\boldsymbol{d}') = \det(\boldsymbol{A})(1+\boldsymbol{d}'\boldsymbol{A}^{-1}\boldsymbol{c}), \tag{A.1}$$

$$(\boldsymbol{A}+\boldsymbol{c}\boldsymbol{d}')^{-1} = \boldsymbol{A}^{-1} - \frac{\boldsymbol{A}^{-1}\boldsymbol{c}\boldsymbol{d}'\boldsymbol{A}^{-1}}{1+\boldsymbol{d}'\boldsymbol{A}^{-1}\boldsymbol{c}}, \tag{A.2}$$

where $\boldsymbol{A}$ is an $n \times n$ nonsingular matrix, and $\boldsymbol{c}$ and $\boldsymbol{d}$ are $n \times 1$ column vectors. Note that (A.2) is applied iteratively to establish the decomposition of the precision matrix $\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})$, where

$$\boldsymbol{H}_i(\gamma,\boldsymbol{\theta}) \equiv \sum_{k\in\gamma} \theta_k \boldsymbol{z}_{i,k}\boldsymbol{z}_{i,k}' + \boldsymbol{I}_{n_i}. \tag{A.3}$$

Heuristically speaking, let $\boldsymbol{z}_{i,(s)}$; $s = 1,\dots,q(\gamma)$ be the s-th column of $\boldsymbol{Z}_i(\gamma)$ and

$$\boldsymbol{H}_{i,t}(\gamma,\boldsymbol{\theta}) = \sum_{s=1}^{t} \theta_{(s)}\boldsymbol{z}_{i,(s)}\boldsymbol{z}_{i,(s)}' + \boldsymbol{I}_{n_i}; \quad t = 1,\dots,q(\gamma), \tag{A.4}$$

where $\theta_{(s)}$ denotes the s-th element of $\boldsymbol{\theta}$; $s = 1,\dots,q(\gamma)$. Suppose that $q(\gamma) = q$. Then by (A.2),

$$\boldsymbol{H}_{i,q}^{-1}(\gamma,\boldsymbol{\theta}) = \boldsymbol{H}_{i,q-1}^{-1}(\gamma,\boldsymbol{\theta}) - \frac{\theta_q \boldsymbol{H}_{i,q-1}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,q}\boldsymbol{z}_{i,q}'\boldsymbol{H}_{i,q-1}^{-1}(\gamma,\boldsymbol{\theta})}{1+\theta_q\boldsymbol{z}_{i,q}'\boldsymbol{H}_{i,q-1}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,q}}. \tag{A.5}$$

Applying (A.2) iteratively, we obtain the decomposition

$$\boldsymbol{H}_{i,q}^{-1}(\gamma,\boldsymbol{\theta}) = \boldsymbol{I}_{n_i} - \sum_{k=1}^{q} \frac{\theta_k \boldsymbol{H}_{i,k-1}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,k}\boldsymbol{z}_{i,k}'\boldsymbol{H}_{i,k-1}^{-1}(\gamma,\boldsymbol{\theta})}{1+\theta_k\boldsymbol{z}_{i,k}'\boldsymbol{H}_{i,k-1}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,k}}; \tag{A.6}$$

note that $\boldsymbol{H}_{i,0}(\gamma,\boldsymbol{\theta}) = \boldsymbol{I}_{n_i}$. The proofs of Lemmas 2, 3, and 4 are then based on the induction and the decomposition of (A.6).

The proofs of theorems in Section 3 heavily rely on the asymptotic properties of the quadratic forms, $\boldsymbol{x}'_{i,j}\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{x}_{i,j^*}$, $\boldsymbol{z}'_{i,k}\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,k^*}$, $\boldsymbol{\epsilon}'_i\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{\epsilon}_i$, $\boldsymbol{x}'_{i,j}\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,k}$, $\boldsymbol{x}'_{i,j}\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{\epsilon}_i$, and $\boldsymbol{z}'_{i,k}\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{\epsilon}_i$, with $\boldsymbol{H}_i(\gamma,\boldsymbol{\theta})$ defined in (A.3), for $i = 1,\ldots,m$; $j, j^* = 1,\ldots,p$ and $k, k^* = 1,\ldots,q$. The following lemmas give their convergence rates.

Lemma 2 *Consider the linear mixed-effects model* (α,γ) *of* (2.4)*. Suppose that (A0)–(A3) hold. Then for* $\boldsymbol{H}_i(\gamma,\boldsymbol{\theta})$ *defined in (A.3), we have*

(i) For $i = 1,\ldots,m$ *and* $j, j^* = 1,\ldots,p$,

$$\sup_{\boldsymbol{\theta}\in[0,\infty)^{q(\gamma)}} \left|\boldsymbol{x}'_{i,j}\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{x}_{i,j^*}\right| = \begin{cases} d_{i,j}n_i^{\xi} + o(n_i^{\xi}); & \text{if } j = j^*, \\ o(n_i^{\xi-\tau}); & \text{if } j \neq j^*. \end{cases}$$

(ii) For $i = 1,\ldots,m$, $j = 1,\ldots,p$ *and* $k \notin \gamma$,

$$\sup_{\boldsymbol{\theta}\in[0,\infty)^{q(\gamma)}} \left|\boldsymbol{x}'_{i,j}\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,k}\right| = o(n_i^{(\xi+\ell)/2-\tau}).$$

(iii) For $i = 1,\ldots,m$, $j = 1,\ldots,p$ *and* $k \in \gamma$,

$$\sup_{\boldsymbol{\theta}\in[0,\infty)^{q(\gamma)}} \theta_k\left|\boldsymbol{x}'_{i,j}\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,k}\right| = o_p(n_i^{(\xi-\ell)/2-\tau}),$$

$$\sup_{\boldsymbol{\theta}\in[0,\infty)^{q(\gamma)}} \left|\boldsymbol{x}'_{i,j}\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,k}\right| = o(n_i^{(\xi+\ell)/2-\tau}).$$

Lemma 3 *Consider the linear mixed-effects model* (α,γ) *of* (2.4)*. Suppose that (A0) and (A2) hold. Then for* $\boldsymbol{H}_i(\gamma,\boldsymbol{\theta})$ *defined in (A.3), we have*

(i) For $i = 1,\ldots,m$ *and* $k, k^* \notin \gamma$,

$$\sup_{\boldsymbol{\theta}\in[0,\infty)^{q(\gamma)}} \left|\boldsymbol{z}'_{i,k}\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,k^*}\right| = \begin{cases} c_{i,k}n_i^{\ell} + o(n_i^{\ell}); & \text{if } k = k^*, \\ o(n_i^{\ell-\tau}); & \text{if } k \neq k^*. \end{cases}$$

(ii) For $i = 1,\ldots,m$ *and* $k \in \gamma$,

$$\sup_{\boldsymbol{\theta}\in[0,\infty)^{q(\gamma)}} \left|\theta_k^2\boldsymbol{z}'_{i,k}\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,k} - \theta_k\right| = O(n_i^{-\ell}),$$

$$\sup_{\boldsymbol{\theta}\in[0,\infty)^{q(\gamma)}} \left|\boldsymbol{z}'_{i,k}\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,k}\right| = O(n_i^{\ell}).$$

(iii) For $i = 1,\ldots,m$ *and* $k, k^* \in \gamma$ *with* $k \neq k^*$,

$$\sup_{\boldsymbol{\theta}\in[0,\infty)^{q(\gamma)}} \theta_k\theta_{k^*}\left|\boldsymbol{z}'_{i,k}\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,k^*}\right| = o(n_i^{-\ell-\tau}),$$

$$\sup_{\boldsymbol{\theta}\in[0,\infty)^{q(\gamma)}} \theta_k\left|\boldsymbol{z}'_{i,k}\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,k^*}\right| = o(n_i^{-\tau}),$$

$$\sup_{\boldsymbol{\theta}\in[0,\infty)^{q(\gamma)}} \left|\boldsymbol{z}'_{i,k}\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,k^*}\right| = o(n_i^{\ell-\tau}).$$

(iv) For $i = 1,\ldots,m$, $k \in \gamma$ *and* $k^* \notin \gamma$,

$$\sup_{\boldsymbol{\theta}\in[0,\infty)^{q(\gamma)}} \theta_k\left|\boldsymbol{z}'_{i,k}\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,k^*}\right| = o(n_i^{-\tau}),$$

$$\sup_{\boldsymbol{\theta}\in[0,\infty)^{q(\gamma)}} \left|\boldsymbol{z}'_{i,k}\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,k^*}\right| = o(n_i^{\ell-\tau}).$$

Lemma 4 *Consider the linear mixed-effects model* (α, γ) *of* (2.4)*. Suppose that (A0)–(A3) hold. Then for* $\boldsymbol{H}_i(\gamma, \boldsymbol{\theta})$ *defined in (A.3), we have*

(i) For $i = 1, \ldots, m$ *and* $k \in \gamma$,

$$\sup_{\boldsymbol{\theta} \in [0,\infty)^{q(\gamma)}} \theta_k \big| \boldsymbol{z}'_{i,k} \boldsymbol{H}_i^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{\epsilon}_i \big| = O_p(n_i^{-\ell/2}),$$
$$\sup_{\boldsymbol{\theta} \in [0,\infty)^{q(\gamma)}} \big| \boldsymbol{z}'_{i,k} \boldsymbol{H}_i^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{\epsilon}_i \big| = O_p(n_i^{\ell/2}).$$

(ii) For $i = 1, \ldots, m$ *and* $k \notin \gamma$,

$$\sup_{\boldsymbol{\theta} \in [0,\infty)^{q(\gamma)}} \big| \boldsymbol{z}'_{i,k} \boldsymbol{H}_i^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{\epsilon}_i \big| = O_p(n_i^{\ell/2}).$$

(iii) For $i = 1, \ldots, m$ *and* $j = 1, \ldots, p$,

$$\sup_{\boldsymbol{\theta} \in [0,\infty)^{q(\gamma)}} \big| \boldsymbol{x}'_{i,j} \boldsymbol{H}_i^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{\epsilon}_i \big| = O_p(n_i^{\xi/2}).$$

In addition,

$$\sup_{\boldsymbol{\theta} \in [0,\infty)^{q(\gamma)}} \Big| \sum_{i=1}^{m} \boldsymbol{x}'_{i,j} \boldsymbol{H}_i^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{\epsilon}_i \Big| = O_p\bigg(\Big(\sum_{i=1}^{m} n_i^{\xi} \Big)^{1/2} \bigg).$$

(iv) For $i = 1, \ldots, m$,

$$\sup_{\boldsymbol{\theta} \in [0,\infty)^{q(\gamma)}} \boldsymbol{\epsilon}'_i \boldsymbol{H}_i^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{\epsilon}_i = \boldsymbol{\epsilon}'_i \boldsymbol{\epsilon}_i + O_p(q).$$

Note that Lemma 2 (i) implies that, for $(\alpha, \gamma) \in \mathcal{A} \times \mathcal{G}$,

$$\begin{aligned} \sum_{i=1}^{m} \boldsymbol{X}_i(\alpha)' \boldsymbol{H}_i^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{X}_i(\alpha) &= \Big(\sum_{i=1}^{m} n_i^{\xi} \Big) \boldsymbol{T}(\alpha) + \Big\{ o\Big(\sum_{i=1}^{m} n_i^{\xi - \tau} \Big) \Big\}_{p(\alpha) \times p(\alpha)} \\ &= \Big(\sum_{i=1}^{m} n_i^{\xi} \Big) \boldsymbol{T}(\alpha) + \Big\{ o\Big(n_{\min}^{-\tau} \sum_{i=1}^{m} n_i^{\xi} \Big) \Big\}_{p(\alpha) \times p(\alpha)} \end{aligned}$$

uniformly over $\boldsymbol{\theta} \in [0, \infty)^{q(\gamma)}$, where $\{a\}_{k \times j}$ denotes a $k \times j$ matrix with elements equal to a and $\boldsymbol{T}(\alpha)$ is a diagonal matrix with diagonal elements bounded away from 0 and ∞. Hence by (A.2) with $p(\alpha)$-vectors $\boldsymbol{c} = \{o(n_{\min}^{-\tau/2})\}_{p(\alpha) \times 1}$ and $\boldsymbol{d} = \{o(n_{\min}^{-\tau/2})\}_{p(\alpha) \times 1}$, and a $p(\alpha) \times p(\alpha)$ diagonal matrix $\boldsymbol{A} = \boldsymbol{T}(\alpha)$, we have, for $(\alpha, \gamma) \in \mathcal{A} \times \mathcal{G}$,

$$\begin{aligned} \bigg(\frac{\sum_{i=1}^{m} \boldsymbol{X}_i(\alpha)' \boldsymbol{H}_i^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{X}_i(\alpha)}{\sum_{i=1}^{m} n_i^{\xi}} \bigg)^{-1} &= \Big(\boldsymbol{T}(\alpha) + \big\{ o\big(n_{\min}^{-\tau}\big) \big\}_{p(\alpha) \times p(\alpha)} \Big)^{-1} \\ &= \boldsymbol{T}^{-1}(\alpha) + \{ o(n_{\min}^{-\tau}) \}_{p(\alpha) \times p(\alpha)} \end{aligned} \tag{A.7}$$

uniformly over $\boldsymbol{\theta} \in [0, \infty)^{q(\gamma)}$, which plays a key role in proving lemmas for theorems.

The following lemma shows that $\hat{\theta}_k$ does not converge to 0 in probability for $k \in \gamma \cap \gamma_0$, which allows us to restrict the parameter space of $\boldsymbol{\theta}$ from $[0, \infty)^{q(\gamma)}$ to

$$\Theta_\gamma = \{ \boldsymbol{\theta} \in [0, \infty)^{q(\gamma)} : \boldsymbol{\theta}(\gamma \cap \gamma_0) \in (0, \infty)^{q(\gamma \cap \gamma_0)} \}. \tag{A.8}$$

Lemma 5 *Under the assumptions of Theorem 1, let* $\boldsymbol{\theta}_0^{\dagger}$ *be* $\boldsymbol{\theta}$ *except that* $\{\theta_k : k \in \gamma \cap \gamma_0\}$ *are replaced by* $\{\theta_{k,0} : k \in \gamma \cap \gamma_0\}$*. Then for any* $(\alpha, \gamma) \in \mathcal{A} \times \mathcal{G}$*,* $v^2 > 0$*, and* $\boldsymbol{\theta} \in [0, \infty)^{q(\gamma)}$ *with* $\theta_k \to 0$ *for some* $k \in \gamma \cap \gamma_0$*, we have*

$$-2 \log L(\boldsymbol{\theta}, v^2; \alpha, \gamma) - \{-2 \log L(\boldsymbol{\theta}_0^{\dagger}, v^2, \alpha, \gamma)\} \xrightarrow{p} \infty$$

as $N \to \infty$*, where* $-2 \log L(\boldsymbol{\theta}, v^2; \alpha, \gamma)$ *is given in (2.7).*

Based on Lemma 5, the following lemma is needed to develop the convergence rates of components of the likelihood equations given in (B.1) and (B.2), uniformly over Θ_γ defined in (A.8).

Lemma 6 *Consider a mixed-effects model* $(\alpha, \gamma) \in \mathcal{A} \times \mathcal{G}$ *with* $\boldsymbol{H}(\gamma, \boldsymbol{\theta})$ *defined in (2.5) and* Θ_γ *defined in (A.8). Suppose that (A0)–(A3) hold. Then*

(i) For $i, i^* = 1, \ldots, m$*,* $(\alpha, \gamma) \in \mathcal{A} \times \mathcal{G}$ *and* $k, k^* \in \gamma$*,*

$$\sup_{\boldsymbol{\theta} \in \Theta_\gamma} \theta_k \theta_{k^*} \left| \boldsymbol{h}'_{i,k} \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}) \boldsymbol{h}_{i^*,k^*} \right| = o\left(\frac{n_i^{(\xi-\ell)/2} n_{i^*}^{(\xi-\ell)/2-\tau}}{\sum_{i=1}^m n_i^\xi} \right),$$

$$\sup_{\boldsymbol{\theta} \in \Theta_\gamma} \theta_k \left| \boldsymbol{h}'_{i,k} \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}) \boldsymbol{h}_{i^*,k^*} \right| = o\left(\frac{n_i^{(\xi-\ell)/2} n_{i^*}^{(\xi+\ell)/2-\tau}}{\sum_{i=1}^m n_i^\xi} \right),$$

$$\sup_{\boldsymbol{\theta} \in \Theta_\gamma} \left| \boldsymbol{h}'_{i,k} \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}) \boldsymbol{h}_{i^*,k^*} \right| = o\left(\frac{n_i^{(\xi+\ell)/2} n_{i^*}^{(\xi+\ell)/2-\tau}}{\sum_{i=1}^m n_i^\xi} \right).$$

(ii) For $i, i^* = 1, \ldots, m$*,* $(\alpha, \gamma) \in \mathcal{A} \times \mathcal{G}$*,* $k \in \gamma$ *and* $k^* \notin \gamma$*,*

$$\sup_{\boldsymbol{\theta} \in \Theta_\gamma} \theta_k \left| \boldsymbol{h}'_{i,k} \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}) \boldsymbol{h}_{i^*,k^*} \right| = o\left(\frac{n_i^{(\xi-\ell)/2} n_{i^*}^{(\xi+\ell)/2-\tau}}{\sum_{i=1}^m n_i^\xi} \right),$$

$$\sup_{\boldsymbol{\theta} \in \Theta_\gamma} \left| \boldsymbol{h}'_{i,k} \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}) \boldsymbol{h}_{i^*,k^*} \right| = o\left(\frac{n_i^{(\xi+\ell)/2} n_{i^*}^{(\xi+\ell)/2-\tau}}{\sum_{i=1}^m n_i^\xi} \right).$$

(iii) For $i = 1, \ldots, m$*,* $(\alpha, \gamma) \in \mathcal{A} \times \mathcal{G}$ *and* $k \in \gamma$*,*

$$\sup_{\boldsymbol{\theta} \in \Theta_\gamma} \theta_k \left| \boldsymbol{h}'_{i,k} \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}) \boldsymbol{\epsilon} \right| = o_p(n_i^{-\ell/2}),$$

$$\sup_{\boldsymbol{\theta} \in \Theta_\gamma} \left| \boldsymbol{h}'_{i,k} \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}) \boldsymbol{\epsilon} \right| = o_p(n_i^{\ell/2}).$$

(iv) For $i = 1, \ldots, m$*,* $(\alpha, \gamma) \in (\mathcal{A} \setminus \mathcal{A}_0) \times \mathcal{G}$ *and* $k \in \gamma$*,*

$$\sup_{\boldsymbol{\theta} \in \Theta_\gamma} \theta_k \left| \boldsymbol{h}'_{i,k} \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}) \boldsymbol{X}(\alpha_0 \setminus \alpha) \boldsymbol{\beta}(\alpha_0 \setminus \alpha) \right| = o(n_i^{(\xi-\ell)/2-\tau}),$$

$$\sup_{\boldsymbol{\theta} \in \Theta_\gamma} \left| \boldsymbol{h}'_{i,k} \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}) \boldsymbol{X}(\alpha_0 \setminus \alpha) \boldsymbol{\beta}(\alpha_0 \setminus \alpha) \right| = o(n_i^{(\xi+\ell)/2-\tau}).$$

(v) For $i = 1, \ldots, m$ *and* $(\alpha, \gamma) \in \mathcal{A} \times \mathcal{G}$*,*

$$\sup_{\boldsymbol{\theta} \in \Theta_\gamma} \boldsymbol{\epsilon}' \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}) \boldsymbol{\epsilon} = O_p(p(\alpha)).$$

(vi) For $i = 1, \ldots, m$*,* $(\alpha, \gamma) \in \mathcal{A} \times \mathcal{G}$ *and* $k \notin \gamma$*,*

$$\sup_{\boldsymbol{\theta} \in \Theta_\gamma} \left| \boldsymbol{h}'_{i,k} \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}) \boldsymbol{\epsilon} \right| = o_p(n_i^{\ell/2}).$$

(vii) *For* $(\alpha,\gamma) \in (\mathcal{A} \setminus \mathcal{A}_0) \times \mathcal{G}$,

$$\sup_{\boldsymbol{\theta}\in\Theta_\gamma} \left|\boldsymbol{\epsilon}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\boldsymbol{X}(\alpha_0 \setminus \alpha)\boldsymbol{\beta}(\alpha_0 \setminus \alpha)\right| = o_p\left(\left(\sum_{i=1}^{m} n_i^{\xi}\right)^{1/2}\right).$$

(viii) *For* $i, i^* = 1,\dots,m$, $(\alpha,\gamma) \in \mathcal{A} \times \mathcal{G}$ *and* $k, k^* \notin \gamma$,

$$\sup_{\boldsymbol{\theta}\in\Theta_\gamma} \left|\boldsymbol{h}'_{i,k}\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\boldsymbol{h}_{i^*,k^*}\right| = o_p\left(\frac{n_i^{(\xi+\ell)/2} n_{i^*}^{(\xi+\ell)/2-\tau}}{\sum_{i=1}^{m} n_i^{\xi}}\right).$$

(ix) *For* $i = 1,\dots,m$, $(\alpha,\gamma) \in (\mathcal{A} \setminus \mathcal{A}_0) \times \mathcal{G}$ *and* $k \notin \gamma$,

$$\sup_{\boldsymbol{\theta}\in\Theta_\gamma} \left|\boldsymbol{h}'_{i,k}\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\boldsymbol{X}(\alpha_0 \setminus \alpha)\boldsymbol{\beta}(\alpha_0 \setminus \alpha)\right| = o(n_i^{(\xi+\ell)/2-\tau}).$$

(x) *For* $(\alpha,\gamma) \in (\mathcal{A} \setminus \mathcal{A}_0) \times \mathcal{G}$,

$$\begin{aligned}\sup_{\boldsymbol{\theta}\in\Theta_\gamma} &\left|\boldsymbol{\beta}(\alpha_0 \setminus \alpha)'\boldsymbol{X}(\alpha_0 \setminus \alpha)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\right.\\ &\quad\left.\times \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\boldsymbol{X}(\alpha_0 \setminus \alpha)\boldsymbol{\beta}(\alpha_0 \setminus \alpha)\right| = o\left(\sum_{i=1}^{m} n_i^{\xi-\tau}\right).\end{aligned}$$

B Theoretical Proofs

B.1 Proof of Theorem 1

We shall focus on the asymptotic properties of $\hat{v}^2(\alpha,\gamma)$ and $\{\hat{\theta}_k(\alpha,\gamma) : k \in \gamma\}$, and derive the asymptotic properties of $\{\hat{\sigma}_k^2(\alpha,\gamma) : k \in \gamma\}$ via $\hat{\sigma}_k^2(\alpha,\gamma) = \hat{v}^2(\alpha,\gamma)\hat{\theta}_k(\alpha,\gamma)$; $k \in \gamma$. If $\hat{v}^2(\alpha,\gamma) > 0$ and $\hat{\theta}_k(\alpha,\gamma) > 0$; $k \in \gamma$, then we can derive them using the likelihood equations. Differentiating the profile log-likelihood function of (2.7) with respect to v^2 and $\{\theta_k : k \in \gamma\}$, we obtain

$$\frac{\partial}{\partial v^2}\{-2\log L(\boldsymbol{\theta}, v^2;\alpha,\gamma)\} = \frac{N}{v^2} - \frac{\boldsymbol{y}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}))\boldsymbol{y}}{v^4} \tag{B.1}$$

and

$$\begin{aligned}\frac{\partial}{\partial \theta_k}\{-2\log L(\boldsymbol{\theta}, v^2;\alpha,\gamma)\} = \sum_{i=1}^{m}&\left\{\boldsymbol{z}'_{i,k}\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,k}\right.\\ &\left.- \frac{\{\boldsymbol{h}'_{i,k}\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}))\boldsymbol{y}\}^2}{v^2}\right\}.\end{aligned} \tag{B.2}$$

To derive $\hat{v}^2(\alpha,\gamma)$ and $\{\hat{\theta}_k(\alpha,\gamma) : k \in \gamma\}$, we must study the convergence rate of each term on the right-hand sides of both (B.1) and (B.2) by Lemmas 2–4 and Lemma 6.

We first prove (3.1) using (B.1). Consider the following decomposition of $\boldsymbol{y}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}))\boldsymbol{y}$ in (B.1):

$$\begin{aligned}&\boldsymbol{y}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}))\boldsymbol{y}\\ &\quad= \boldsymbol{\mu}_0'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}))\boldsymbol{\mu}_0\\ &\qquad+ 2\boldsymbol{\mu}_0'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}))(\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0) + \boldsymbol{\epsilon})\\ &\qquad+ (\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0) + \boldsymbol{\epsilon})'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0) + \boldsymbol{\epsilon})\\ &\qquad- (\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0) + \boldsymbol{\epsilon})'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})(\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0) + \boldsymbol{\epsilon}).\end{aligned} \tag{B.3}$$

The first two terms of (B.3) are zeros because

$$(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}))\boldsymbol{\mu}_0 = \boldsymbol{0}; \quad \alpha \in \mathcal{A}_0, \tag{B.4}$$

which is obtained by treating $\boldsymbol{\mu}_0 = \boldsymbol{X}(\alpha)\boldsymbol{\beta}(\alpha)$ for some $\boldsymbol{\beta}(\alpha) \in \mathbb{R}^{p(\alpha)}$ under $\alpha \in \mathcal{A}_0$, where note that by (2.9), $\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\boldsymbol{X}(\alpha) = \boldsymbol{X}(\alpha)$. By Lemma 3 (ii)–(iii), Lemma 4 (i), and Lemma 4 (iv), the third term of (B.3) can be written as

$$\begin{aligned}\sum_{i=1}^{m}(\boldsymbol{Z}_i(\gamma_0)\boldsymbol{b}_i(\gamma_0)+\boldsymbol{\epsilon}_i)'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{Z}_i(\gamma_0)\boldsymbol{b}_i(\gamma_0)+\boldsymbol{\epsilon}_i)&\\ = \sum_{i=1}^{m}\boldsymbol{\epsilon}_i'\boldsymbol{\epsilon}_i + O_p\Big(\sum_{k\in\gamma_0}\frac{m}{\theta_k}\Big) + o_p\Big(\sum_{k\in\gamma_0}\frac{m}{\theta_k^2}\Big) + O_p(mq)&\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in \Theta_\gamma$. Note that by the Cauchy–Schwarz inequality,

$$\Big(\sum_{i=1}^{m} n_i^{(\xi-\ell)/2}\Big)^2 = O\Big(\sum_{i=1}^{m} n_i^{\xi}\sum_{i^*=1}^{m} n_{i^*}^{-\ell}\Big). \tag{B.5}$$

Hence, by Lemma 6 (i), Lemma 6 (iii), and Lemma 6 (v), the last term of (B.3) can be written as

$$\begin{aligned}\Big\{\Big(\sum_{i=1}^{m}\sum_{k\in\gamma_0} b_{i,k}\boldsymbol{h}_{i,k}\Big)+\boldsymbol{\epsilon}\Big\}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\Big\{\Big(\sum_{i=1}^{m}\sum_{k\in\gamma_0} b_{i,k}\boldsymbol{h}_{i,k}\Big)+\boldsymbol{\epsilon}\Big\}&\\ = o_p\Big(\sum_{k,k^*\in\gamma_0}\frac{m}{\theta_k\theta_{k^*}}\Big) + o_p\Big(\sum_{k\in\gamma_0}\frac{m}{\theta_k}\Big) + O_p(p+mq)&\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in \Theta_\gamma$. Therefore, we can rewrite (B.3) as

$$\begin{aligned}\boldsymbol{y}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{I}_N-\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}))\boldsymbol{y}&\\ = \boldsymbol{\epsilon}'\boldsymbol{\epsilon} + o_p\Big(\sum_{k,k^*\in\gamma_0}\frac{m}{\theta_k\theta_{k^*}}\Big) + O_p\Big(\sum_{k\in\gamma_0}\frac{m}{\theta_k}\Big) + O_p(p+mq).&\end{aligned}$$

It follows from (B.1) that for $v^2 \in (0,\infty)$,

$$\begin{aligned}v^4\Big\{\frac{\partial}{\partial v^2}\{-2\log L(\boldsymbol{\theta},v^2;\alpha,\gamma)\}\Big\} = N\Big(v^2-\frac{\boldsymbol{\epsilon}'\boldsymbol{\epsilon}}{N}\Big) + o_p\Big(\sum_{k,k^*\in\gamma_0}\frac{m}{\theta_k\theta_{k^*}}\Big)&\\ + O_p\Big(\sum_{k\in\gamma_0}\frac{m}{\theta_k}\Big) + O_p(p+mq)&\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in \Theta_\gamma$. This and Lemma 5 imply that

$$\hat{v}^2(\alpha,\gamma) = \frac{\boldsymbol{\epsilon}'\boldsymbol{\epsilon}}{N} + O_p\Big(\frac{p+mq}{N}\Big). \tag{B.6}$$

Thus (3.1) follows by applying the law of large numbers to $\boldsymbol{\epsilon}'\boldsymbol{\epsilon}/N$. In addition, the asymptotic normality of $\hat{v}^2(\alpha,\gamma)$ follows by $p+mq = o(N^{1/2})$ and an application of the central limit theorem to $\boldsymbol{\epsilon}'\boldsymbol{\epsilon}/N$ in (B.6).

Next, we prove (3.2), for $k \in \gamma\cap\gamma_0$, using (B.2). By Lemma 6 (i) and Lemma 6 (iii), we have, for $k \in \gamma\cap\gamma_0$,

$$\begin{aligned}&\theta_k\boldsymbol{h}_{i,k}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})(\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)+\boldsymbol{\epsilon})\\ &\quad= \theta_k\boldsymbol{h}_{i,k}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\Big\{\Big(\sum_{i^*=1}^{m}\sum_{k^*\in\gamma_0} b_{i^*,k^*}\boldsymbol{h}_{i^*,k^*}\Big)+\boldsymbol{\epsilon}\Big\}\\ &\quad= o_p\Big(\sum_{k^*\in\gamma_0}\frac{n_i^{(\xi-\ell)/2}\sum_{i^*=1}^{m} n_{i^*}^{(\xi-\ell)/2}}{\theta_{k^*}\sum_{i=1}^{m} n_i^{\xi}}\Big) + o_p(n_i^{-\ell/2})\end{aligned} \tag{B.7}$$

uniformly over $\boldsymbol{\theta} \in \Theta_\gamma$. This and (B.4) imply that for $k \in \gamma \cap \gamma_0$,

$$\begin{aligned}
&\theta_k \boldsymbol{h}'_{i,k} \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}))\boldsymbol{y} \\
&= \theta_k \boldsymbol{h}'_{i,k} \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}))(\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0) + \boldsymbol{\epsilon}) \\
&= \theta_k \boldsymbol{h}'_{i,k} \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta})(\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0) + \boldsymbol{\epsilon}) \\
&\quad + o_p\bigg(\sum_{k^* \in \gamma_0} \frac{n_i^{(\xi-\ell)/2} \sum_{i^*=1}^m n_{i^*}^{(\xi-\ell)/2}}{\theta_{k^*} \sum_{i=1}^m n_i^{\xi}} \bigg) + o_p(n_i^{-\ell/2}) \\
&= \theta_k \boldsymbol{z}'_{i,k} \boldsymbol{H}_i^{-1}(\gamma, \boldsymbol{\theta})(\boldsymbol{Z}_i(\gamma_0)\boldsymbol{b}_i(\gamma_0) + \boldsymbol{\epsilon}_i) \\
&\quad + o_p\bigg(\sum_{k^* \in \gamma_0} \frac{n_i^{(\xi-\ell)/2} \sum_{i^*=1}^m n_{i^*}^{(\xi-\ell)/2}}{\theta_{k^*} \sum_{i=1}^m n_i^{\xi}} \bigg) + o_p(n_i^{-\ell/2}) \\
&= b_{i,k} + O_p(n_i^{-\ell/2}) + o_p\bigg(\sum_{k^* \in \gamma_0} \frac{n_i^{(\xi-\ell)/2} \sum_{i^*=1}^m n_{i^*}^{(\xi-\ell)/2}}{\theta_{k^*} \sum_{i=1}^m n_i^{\xi}} \bigg)
\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in \Theta_\gamma$, where the last equality follows from Lemma 3 (ii)–(iii) and Lemma 4 (i). Hence, for $k \in \gamma \cap \gamma_0$,

$$\begin{aligned}
&\theta_k^2 \{\boldsymbol{h}'_{i,k} \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}))\boldsymbol{y}\}^2 \\
&= b_{i,k}^2 + O_p(n_i^{-\ell/2}) + o_p\bigg(\sum_{k^* \in \gamma_0} \frac{n_i^{(\xi-\ell)/2} \sum_{i^*=1}^m n_{i^*}^{(\xi-\ell)/2}}{\theta_{k^*} \sum_{i=1}^m n_i^{\xi}} \bigg)
\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in \Theta_\gamma$. This together with Lemma 3 (ii) and (B.2) imply that for $k \in \gamma \cap \gamma_0$,

$$\begin{aligned}
&\theta_k^2 \bigg\{ \frac{\partial}{\partial \theta_k} \{-2 \log L(\boldsymbol{\theta}, v^2; \alpha, \gamma)\} \bigg\} \\
&= m\bigg(\theta_k - \frac{1}{m} \sum_{i=1}^m \frac{b_{i,k}^2}{v^2} \bigg) + O_p\bigg(\sum_{i=1}^m n_i^{-\ell/2} \bigg) \\
&\quad + o_p\bigg(\sum_{k^* \in \gamma_0} \frac{\sum_{i=1}^m n_i^{(\xi-\ell)/2} \sum_{i^*=1}^m n_{i^*}^{(\xi-\ell)/2-\tau}}{\theta_{k^*} \sum_{i=1}^m n_i^{\xi}} \bigg)
\end{aligned} \tag{B.8}$$

uniformly over $\boldsymbol{\theta} \in \Theta_\gamma$. By (B.5), Lemma 5 and setting (B.8) to 0, we obtain

$$\hat{\theta}_k(\alpha, \gamma) = \frac{1}{m} \sum_{i=1}^m \frac{b_{i,k}^2}{\hat{v}^2(\alpha, \gamma)} + O_p\bigg(\frac{1}{m} \sum_{i=1}^m n_i^{-\ell/2} \bigg); \quad k \in \gamma \cap \gamma_0.$$

This proves (3.2), for $k \in \gamma \cap \gamma_0$.

It remains to prove (3.2), for $k \in \gamma \backslash \gamma_0$. We prove by showing that (B.2) is asymptotically nonnegative, for $\theta_k \in (n_{\max}^{-\ell}, \infty)$; $k \in \gamma \setminus \gamma_0$ using a recursive argument. Let $\boldsymbol{\theta}^\dagger$ be $\boldsymbol{\theta}$ except that $\{\theta_k : k \in \gamma \cap \gamma_0\}$ are replaced by $\{\hat{\theta}_k(\alpha, \gamma) : k \in \gamma \cap \gamma_0\}$. By Lemma 6 (i) and Lemma 6 (iii), we have, for $k \in \gamma \setminus \gamma_0$,

$$\begin{aligned}
&\theta_k \boldsymbol{h}'_{i,k} \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}^\dagger) \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}^\dagger)(\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0) + \boldsymbol{\epsilon}) \\
&= \theta_k \boldsymbol{h}'_{i,k} \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}^\dagger) \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}^\dagger) \bigg(\sum_{i^*=1}^m \sum_{k^* \in \gamma_0} b_{i^*,k^*} \boldsymbol{h}_{i^*,k^*} + \boldsymbol{\epsilon} \bigg) \\
&= o_p\bigg(\frac{n_i^{(\xi-\ell)/2} \sum_{i^*=1}^m n_{i^*}^{(\xi-\ell)/2}}{\sum_{i=1}^m n_i^{\xi}} \bigg) + o_p(n_i^{-\ell/2})
\end{aligned}$$

uniformly over $\boldsymbol{\theta}(\gamma \setminus \gamma_0) \in [0,\infty)^{q(\gamma\setminus\gamma_0)}$. This and (B.4) imply that for $k \in \gamma \setminus \gamma_0$,

$$\begin{aligned}
&\theta_k \boldsymbol{h}'_{i,k}\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}^\dagger)(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}^\dagger))\boldsymbol{y} \\
&= \theta_k \boldsymbol{h}'_{i,k}\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}^\dagger)(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}^\dagger))(\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)+\boldsymbol{\epsilon}) \\
&= \theta_k \boldsymbol{h}'_{i,k}\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}^\dagger)(\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)+\boldsymbol{\epsilon}) \\
&\quad + o_p\bigg(\frac{n_i^{(\xi-\ell)/2}\sum_{i^*=1}^m n_{i^*}^{(\xi-\ell)/2}}{\sum_{i=1}^m n_i^\xi}\bigg) + o_p(n_i^{-\ell/2}) \\
&= \theta_k \boldsymbol{z}'_{i,k}\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta}^\dagger)\bigg(\sum_{k^*\in\gamma_0}\boldsymbol{z}_{i,k^*}b_{i,k^*}+\boldsymbol{\epsilon}_i\bigg) \\
&\quad + o_p\bigg(\frac{n_i^{(\xi-\ell)/2}\sum_{i^*=1}^m n_{i^*}^{(\xi-\ell)/2}}{\sum_{i=1}^m n_i^\xi}\bigg) + o_p(n_i^{-\ell/2}) \\
&= O_p(n_i^{-\ell/2}) + o_p\bigg(\frac{n_i^{(\xi-\ell)/2}\sum_{i^*=1}^m n_{i^*}^{(\xi-\ell)/2}}{\sum_{i=1}^m n_i^\xi}\bigg)
\end{aligned} \tag{B.9}$$

uniformly over $\boldsymbol{\theta}(\gamma \setminus \gamma_0) \in [0,\infty)^{q(\gamma\setminus\gamma_0)}$, where the last equality follows from Lemma 3 (iii) and Lemma 4 (i). Hence by (B.5), Lemma 3 (ii), and (B.2), we have, for $\boldsymbol{\theta}(\gamma \setminus \gamma_0) \in [0,\infty)^{q(\gamma\setminus\gamma_0)}$ and $k \in \gamma \setminus \gamma_0$,

$$\begin{aligned}
&\theta_k^2\bigg\{\frac{\partial}{\partial\theta_k}\{-2\log L(\boldsymbol{\theta}^\dagger, v^2;\alpha,\gamma)\}\bigg\} \\
&= m\theta_k + O_p\bigg(\sum_{i=1}^m n_i^{-\ell}\bigg) + o_p\bigg(\frac{\sum_{i=1}^m n_i^{(\xi-\ell)/2}\sum_{i^*=1}^m n_{i^*}^{(\xi-\ell)/2-\tau}}{\sum_{i=1}^m n_i^\xi}\bigg) \\
&= m\theta_k + O_p\bigg(\sum_{i=1}^m n_i^{-\ell}\bigg) \\
&= m\theta_k + o_p(m\log(n_{\min})n_{\min}^{-\ell}).
\end{aligned}$$

This implies that $-2\log L(\boldsymbol{\theta}^\dagger, v^2;\alpha,\gamma)$ is an asymptotically nondecreasing function on $\theta_k \in (\log(n_{\min})n_{\min}^{-\ell},\infty)$, for $k \in \gamma \setminus \gamma_0$ given other $\boldsymbol{\theta}(\gamma \setminus \gamma_0) \in [0,\infty)^{q(\gamma\setminus\gamma_0)}$. It follows that $\hat{\theta}_k(\alpha,\gamma) \in \big[0, \log(n_{\min})n_{\min}^{-\ell}\big)$; $k \in \gamma \setminus \gamma_0$. The above convergence rate can be recursively improved. Without loss of generality, assume that $n_{\min} = n_1 \le n_2 \le \cdots \le n_m = n_{\max}$. We can restrict the parameter space of θ_k in the next step to

$$\varTheta_{\gamma,k,i} = \big\{\boldsymbol{\theta}(\gamma \setminus \gamma_0) \in [0,\infty)^{q(\gamma\setminus\gamma_0)} : \theta_k \le \log(n_{\min})n_i^{-\ell}\big\} \tag{B.10}$$

with $i = 1$. Then, by Lemma 6 (i) and Lemma 6 (iii), we have, for $k \in \gamma \setminus \gamma_0$,

$$\begin{aligned}
&\theta_k \boldsymbol{h}'_{1,k}\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}^\dagger)\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}^\dagger)(\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)+\boldsymbol{\epsilon}) \\
&= o_p\bigg(\frac{n_1^{(\xi-\ell)/2}\sum_{i^*=1}^m n_{i^*}^{(\xi-\ell)/2}}{\sum_{i=1}^m n_i^\xi}\bigg) + o_p(\theta_k n_1^{\ell/2})
\end{aligned}$$

uniformly over $\boldsymbol{\theta}(\gamma \setminus \gamma_0) \in \Theta_{\gamma,k,1}$. This and (B.4) imply that for $k \in \gamma \setminus \gamma_0$,

$$\begin{aligned}
&\theta_k \boldsymbol{h}'_{1,k} \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}^\dagger)(\boldsymbol{I}_N - \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}^\dagger))\boldsymbol{y} \\
&\quad = \theta_k \boldsymbol{h}'_{1,k} \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}^\dagger)(\boldsymbol{I}_N - \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}^\dagger))(\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0) + \boldsymbol{\epsilon}) \\
&\quad = \theta_k \boldsymbol{h}'_{1,k} \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}^\dagger)(\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0) + \boldsymbol{\epsilon}) \\
&\qquad + o_p\left(\frac{n_1^{(\xi-\ell)/2} \sum_{i^*=1}^m n_{i^*}^{(\xi-\ell)/2}}{\sum_{i=1}^m n_i^\xi}\right) + o_p(\theta_k n_1^{\ell/2}) \\
&\quad = \theta_k \boldsymbol{z}'_{1,k} \boldsymbol{H}_1^{-1}(\gamma, \boldsymbol{\theta}^\dagger)(\boldsymbol{Z}_1(\gamma_0)\boldsymbol{b}_1(\gamma_0) + \boldsymbol{\epsilon}_1) \\
&\qquad + o_p\left(\frac{n_1^{(\xi-\ell)/2} \sum_{i^*=1}^m n_{i^*}^{(\xi-\ell)/2}}{\sum_{i=1}^m n_i^\xi}\right) + o_p(\theta_k n_1^{\ell/2}) \\
&\quad = O_p(\theta_k n_1^{\ell/2}) + o_p\left(\frac{n_1^{(\xi-\ell)/2} \sum_{i^*=1}^m n_{i^*}^{(\xi-\ell)/2}}{\sum_{i=1}^m n_i^\xi}\right)
\end{aligned}$$

uniformly over $\boldsymbol{\theta}(\gamma \setminus \gamma_0) \in \Theta_{\gamma,k,1}$, where the last equality follows from Lemma 3 (iii) and Lemma 4 (i). Hence by (B.5), Lemma 3 (ii), (B.2), and (B.9), we have

$$\begin{aligned}
&\theta_k^2 \left\{ \frac{\partial}{\partial \theta_k} \{-2 \log L(\boldsymbol{\theta}^\dagger, v^2; \alpha, \gamma)\} \right\} \\
&= (m-1)\theta_k + O_p(\theta_k^2 n_1^\ell) + O_p\left(\sum_{i=2}^m n_i^{-\ell}\right) + o_p\left(\sum_{i=1}^m n_i^{-\ell}\right) \\
&= (m-1)\theta_k + O_p(\log(n_{\min})\theta_k) + O_p\left(\sum_{i=2}^m n_i^{-\ell}\right)
\end{aligned}$$

uniformly over $\boldsymbol{\theta}(\gamma \setminus \gamma_0) \in \Theta_{\gamma,k,1}$. Hence, setting the above equation equal to 0, we have

$$\hat{\theta}_k(\alpha, \gamma) = \frac{1}{m - 1 + O_p(\log(n_{\min}))} O_p\left(\sum_{i=2}^m n_i^{-\ell}\right) = O_p(n_2^{-\ell}).$$

Now we can further restrict the parameter space of θ_k to $\Theta_{\gamma,k,2}$ in (B.10). Continuing this procedure, we can recursively obtain $\hat{\theta}_k(\alpha, \gamma) = O_p(n_i^{-\ell})$; $k \in \gamma \setminus \gamma_0$, for $i = 3, \ldots, m$. This completes the proof of (3.2), for $k \in \gamma \setminus \gamma_0$. Hence the proof of Theorem 1 is complete.

B.2 Proof of Example 1

Note that for $q = 1$, $\boldsymbol{Z}_i = \boldsymbol{z}_{i,1}$ and $\boldsymbol{b}_i = b_{i,1}$. Note that by Lemma 5, we consider the sample space $(\sigma_1^2, v^2) \in (0, \infty)^2$. We first derive the explicit forms of the ML estimators $\hat{\theta}_1$ and $\hat{v}^2$.

By (B.2), we have

$$\begin{aligned}
\frac{\partial}{\partial\theta_1}\{-2\log L(\theta_1, v^2)\} &= \sum_{i=1}^m \frac{\boldsymbol{z}'_{i,1}\boldsymbol{z}_{i,1}}{1+\theta_1\boldsymbol{z}'_{i,1}\boldsymbol{z}_{i,1}} - \frac{1}{v^2}\sum_{i=1}^m \left\{\boldsymbol{z}'_{i,1}\left(\boldsymbol{I}_n - \frac{\theta_1\boldsymbol{z}_{i,1}\boldsymbol{z}'_{i,1}}{1+\theta_1\boldsymbol{z}'_{i,1}\boldsymbol{z}_{i,1}}\right)\boldsymbol{y}_i\right\}^2 \\
&= \sum_{i=1}^m \frac{\boldsymbol{z}'_{i,1}\boldsymbol{z}_{i,1}}{1+\theta_1\boldsymbol{z}'_{i,1}\boldsymbol{z}_{i,1}} - \frac{1}{v^2}\sum_{i=1}^m \left\{\frac{\boldsymbol{z}'_{i,1}\boldsymbol{z}_{i,1}b_{i,1}}{1+\theta_1\boldsymbol{z}'_{i,1}\boldsymbol{z}_{i,1}} + \frac{\boldsymbol{z}'_{i,1}\boldsymbol{\epsilon}_i}{1+\theta_1\boldsymbol{z}'_{i,1}\boldsymbol{z}_{i,1}}\right\}^2 \\
&= \sum_{i=1}^m \left(\frac{1}{\theta_1} - \frac{1}{\theta_1(1+\theta_1\boldsymbol{z}'_{i,1}\boldsymbol{z}_{i,1})}\right) \\
&\quad - \frac{1}{v^2}\sum_{i=1}^m \left\{\frac{b_{i,1}}{\theta_1} - \frac{b_{i,1}}{\theta_1(1+\theta_1\boldsymbol{z}'_{i,1}\boldsymbol{z}_{i,1})} + \frac{\boldsymbol{z}'_{i,1}\boldsymbol{\epsilon}_i}{1+\theta_1\boldsymbol{z}'_{i,1}\boldsymbol{z}_{i,1}}\right\}^2 \\
&= \frac{m}{\theta_1} - \frac{\sum_{i=1}^m b_{i,1}^2}{v^2\theta_1^2} + 2\sum_{i=1}^m \frac{b_{i,1}\boldsymbol{z}'_{i,1}\boldsymbol{\epsilon}_i}{v^2\theta_1(1+\theta_1\boldsymbol{z}'_{i,1}\boldsymbol{z}_{i,1})} + R(\sigma_1^2, v^2),
\end{aligned}$$

where $\sigma_1^2 = \theta_1 v^2$ and

$$\begin{aligned}
R(\sigma_1^2, v^2) = &-\sum_{i=1}^m \frac{1}{\theta_1(1+\theta_1\boldsymbol{z}'_{i,1}\boldsymbol{z}_{i,1})} - \sum_{i=1}^m \frac{(\boldsymbol{z}'_{i,1}\boldsymbol{\epsilon}_i)^2}{v^2\{1+\theta_1\boldsymbol{z}'_{i,1}\boldsymbol{z}_{i,1}\}^2} + \sum_{i=1}^m \frac{2b_{i,1}\boldsymbol{z}'_{i,1}\boldsymbol{\epsilon}_i}{v^2\theta_1\{1+\theta_1\boldsymbol{z}'_{i,1}\boldsymbol{z}_{i,1}\}^2} \\
&+ \sum_{i=1}^m \frac{2b_{i,1}^2}{v^2\theta_1^2(1+\theta_1\boldsymbol{z}'_{i,1}\boldsymbol{z}_{i,1})} - \sum_{i=1}^m \frac{b_{i,1}^2}{v^2\theta_1^2\{1+\theta_1\boldsymbol{z}'_{i,1}\boldsymbol{z}_{i,1}\}^2}.
\end{aligned} \tag{B.11}$$

Note that ML estimators $\hat{\sigma}_1^2 = \hat{\theta}_1\hat{v}^2$ and $\hat{v}^2$ satisfy

$$0 = \frac{m}{\hat{\theta}_1} - \frac{\sum_{i=1}^m b_{i,1}^2}{\hat{v}^2\hat{\theta}_1^2} + \sum_{i=1}^m \frac{2b_{i,1}\boldsymbol{z}'_{i,1}\boldsymbol{\epsilon}_i}{\hat{v}^2\hat{\theta}_1(1+\hat{\theta}_1(\boldsymbol{z}'_{i,1}\boldsymbol{z}_{i,1}))} + R(\hat{\sigma}_1^2, \hat{v}^2),$$

which implies that

$$\begin{aligned}
\hat{\sigma}_1^2 = \hat{\theta}_1\hat{v}^2 &= \frac{1}{m}\sum_{i=1}^m b_{i,1}^2 + \frac{1}{m}\sum_{i=1}^m \frac{2\hat{\theta}_1 b_{i,1}\boldsymbol{z}'_{i,1}\boldsymbol{\epsilon}_i}{1+\hat{\theta}_1\boldsymbol{z}'_{i,1}\boldsymbol{z}_{i,1}} + \frac{\hat{\theta}_1^2}{m}R(\hat{\sigma}_1^2, \hat{v}^2) \\
&= \frac{1}{m}\sum_{i=1}^m b_{i,1}^2 + \frac{1}{m}\sum_{i=1}^m \frac{2b_{i,1}\boldsymbol{z}'_{i,1}\boldsymbol{\epsilon}_i}{\boldsymbol{z}'_{i,1}\boldsymbol{z}_{i,1}} - \frac{1}{m}\sum_{i=1}^m \frac{2b_{i,1}\boldsymbol{z}'_{i,1}\boldsymbol{\epsilon}_i}{(1+\hat{\theta}_1\boldsymbol{z}'_{i,1}\boldsymbol{z}_{i,1})\boldsymbol{z}'_{i,1}\boldsymbol{z}_{i,1}} + \frac{\hat{\theta}_1^2}{m}R(\hat{\sigma}_1^2, \hat{v}^2) \\
&= \frac{1}{m}\sum_{i=1}^m b_{i,1}^2 + \frac{1}{m}\sum_{i=1}^m \frac{2b_{i,1}\boldsymbol{z}'_{i,1}\boldsymbol{\epsilon}_i}{\boldsymbol{z}'_{i,1}\boldsymbol{z}_{i,1}} + R^*(\hat{\sigma}_1^2, \hat{v^2}),
\end{aligned} \tag{B.12}$$

where

$$R^*(\hat{\sigma}_1^2, \hat{v}^2) = -\frac{1}{m}\sum_{i=1}^m \frac{2b_{i,1}\boldsymbol{z}'_{i,1}\boldsymbol{\epsilon}_i}{(1+\hat{\theta}_1\boldsymbol{z}'_{i,1}\boldsymbol{z}_{i,1})\boldsymbol{z}'_{i,1}\boldsymbol{z}_{i,1}} + \frac{\hat{\theta}_1^2}{m}R(\hat{\theta}_1, \hat{v}^2) \tag{B.13}$$

with $R(\sigma_1^2, v^2)$ defined in (B.11). By (B.12), we have

$$\begin{aligned}
\frac{\sum_{i=1}^m b_{i,1}^2}{\hat{\sigma}_1^2} &= O_p(1), \\
\frac{b_{i,1}^2}{\hat{\sigma}_1^2} &= O_p(1), \\
\frac{(b_{i,1}\boldsymbol{z}'_{i,1}\boldsymbol{\epsilon}_i)^2}{1+\hat{\theta}_1\boldsymbol{z}'_{i,1}\boldsymbol{z}_{i,1}} &= O_p(1).
\end{aligned} \tag{B.14}$$

By (B.13) and (B.14), we have

$$R^*(\hat{\sigma}_1^2, \hat{v}^2) = o_p(n^{-1}). \tag{B.15}$$

Similarly, by (B.1), we have

$$\begin{aligned}
\frac{\partial}{\partial v^2}\{-2\log L(\theta_1, v^2)\} &= \frac{N}{v^2} - \frac{1}{v^4}\sum_{i=1}^m \boldsymbol{y}_i'\Big(\boldsymbol{I}_n - \frac{\theta_1 \boldsymbol{z}_{i,1}\boldsymbol{z}_{i,1}'}{1+\theta_1\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}}\Big)\boldsymbol{y}_i \\
&= \frac{N}{v^2} - \frac{1}{v^4}\sum_{i=1}^m (\boldsymbol{z}_{i,1}b_{i,1} + \boldsymbol{\epsilon}_i)'\Big(\boldsymbol{I}_n - \frac{\theta_1 \boldsymbol{z}_{i,1}\boldsymbol{z}_{i,1}'}{1+\theta_1\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}}\Big)(\boldsymbol{z}_{i,1}b_{i,1} + \boldsymbol{\epsilon}_i) \\
&= \frac{N}{v^2} - \frac{1}{v^4}\sum_{i=1}^m \Big\{\frac{b_{i,1}^2\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}}{1+\theta_1\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}} + \frac{2b_{i,1}\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i}{1+\theta_1\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}} + \boldsymbol{\epsilon}_i'\boldsymbol{\epsilon}_i - \frac{\theta_1(\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i)^2}{1+\theta_1\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}}\Big\}.
\end{aligned}$$

The ML estimators $\hat{\theta}_1$ and $\hat{v}^2$ satisfy

$$0 = \frac{N}{\hat{v}^2} - \frac{N}{\hat{v}^4}\sum_{i=1}^m \Big\{\frac{b_{i,1}^2\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}}{1+\hat{\theta}_1\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}} + \frac{2b_{i,1}\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i}{1+\hat{\theta}_1\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}} + \boldsymbol{\epsilon}_i'\boldsymbol{\epsilon}_i - \frac{\hat{\theta}_1(\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i)^2}{1+\hat{\theta}_1\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}}\Big\},$$

which implies that

$$\hat{v}^2 = \frac{1}{N}\sum_{i=1}^m \boldsymbol{\epsilon}_i'\boldsymbol{\epsilon}_i + R^\dagger(\hat{\sigma}_1^2, \hat{v}^2), \tag{B.16}$$

with

$$R^\dagger(\hat{\sigma}_1^2, \hat{v}^2) = \frac{1}{N}\sum_{i=1}^m \Big\{\frac{b_{i,1}^2\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}}{1+\hat{\theta}_1\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}} + \frac{2b_{i,1}\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i}{1+\hat{\theta}_1\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}} - \frac{\hat{\theta}_1(\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i)^2}{1+\hat{\theta}_1\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}}\Big\}.$$

This together with (B.14) yields

$$R^\dagger(\hat{\sigma}_1^2, \hat{v}^2) = O_p(n^{-1}). \tag{B.17}$$

We are now ready to compare the asymptotic behaviors between the LS predictors and the empirical BLUPs. Note that for $i = 1, \dots, m$, we have

$$\begin{aligned}
\tilde{b}_{i,1} &= (\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})^{-1}\boldsymbol{z}_{i,1}'\boldsymbol{y}_i, \\
\hat{b}_{i,1}(\sigma_1^2, v^2) &= \sigma_1^2\boldsymbol{z}_{i,1}'(\sigma_1^2\boldsymbol{z}_{i,1}\boldsymbol{z}_{i,1}' + v^2\boldsymbol{I}_n)^{-1}\boldsymbol{y}_i.
\end{aligned}$$

Hence

$$\boldsymbol{z}_{i,1}\big(\tilde{b}_{i,1} - b_{i,1}\big) = \frac{\boldsymbol{z}_{i,1}\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i}{\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}}, \tag{B.18}$$

and

$$\begin{aligned}
\hat{b}_{i,1}(\hat{\sigma}_1^2, \hat{v}^2) - b_{i,1} &= \hat{\sigma}_1^2\boldsymbol{z}_{i,1}'(\hat{\sigma}_1^2\boldsymbol{z}_{i,1}\boldsymbol{z}_{i,1}' + \hat{v}^2\boldsymbol{I}_n)^{-1}(\boldsymbol{z}_{i,1}b_{i,1} + \boldsymbol{\epsilon}_i) - b_{i,1} \\
&= \big\{\hat{\sigma}_1^2\boldsymbol{z}_{i,1}'(\hat{\sigma}_1^2\boldsymbol{z}_{i,1}\boldsymbol{z}_{i,1}' + \hat{v}^2\boldsymbol{I}_n)^{-1}\boldsymbol{z}_{i,1} - 1\big\}b_{i,1} + \hat{\sigma}_1^2\boldsymbol{z}_{i,1}'(\hat{\sigma}_1^2\boldsymbol{z}_{i,1}\boldsymbol{z}_{i,1}' + \hat{v}^2\boldsymbol{I}_n)^{-1}\boldsymbol{\epsilon}_i \\
&= \Big\{(\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}'\Big(\boldsymbol{I}_n - \frac{(\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}\boldsymbol{z}_{i,1}'}{1+(\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}}\Big)\boldsymbol{z}_{i,1} - 1\Big\}b_i \\
&\quad + (\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}'\Big(\boldsymbol{I}_n - \frac{(\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}\boldsymbol{z}_{i,1}'}{1+(\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}}\Big)\boldsymbol{\epsilon}_i \\
&= \frac{(\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i - b_i}{1+(\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}},
\end{aligned}$$

which implies that

$$\boldsymbol{z}_{i,1}\big(\hat{b}_{i,1}(\hat{\sigma}_1^2,\hat{v}^2)-b_{i,1}\big)=\frac{\boldsymbol{z}_{i,1}\{(\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i-b_i\}}{1+(\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}}. \tag{B.19}$$

Note that by (B.18),

$$\sum_{i=1}^m\big\|\boldsymbol{z}_{i,1}\big(\tilde{b}_{i,1}-b_{i,1}\big)\big\|^2=\sum_{i=1}^m(\tilde{b}_{i,1}-b_{i,1})^2\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}=\sum_{i=1}^m\frac{(\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i)^2}{\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}},$$

and by (B.19),

$$\sum_{i=1}^m\big\|\boldsymbol{z}_{i,1}\big(\hat{b}_{i,1}(\hat{\sigma}_1^2,\hat{v}^2)-b_{i,1}\big)\big\|^2=\sum_{i=1}^m\frac{\{(\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i-b_{i,1}\}^2\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}}{\{1+(\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}\}^2},$$

which implies that

$$\begin{aligned}D(\hat{\sigma}_1^2,\hat{v}^2)&=\sum_{i=1}^m\bigg(\frac{(\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i)^2}{\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}}-\frac{\{(\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i-b_{i,1}\}^2\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}}{\{1+(\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}\}^2}\bigg)\\&=\sum_{i=1}^m\frac{(\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i)^2\{1+(\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}\}^2-\{(\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i-b_{i,1}\}^2(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})^2}{\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}\{1+(\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}\}^2}\\&=\sum_{i=1}^m\frac{(\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i)^2+2(\hat{\sigma}_1^2/\hat{v}^2)(\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i)^2\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}+2(\hat{\sigma}_1^2/\hat{v}^2)b_{i,1}\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})^2-b_{i,1}^2(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})^2}{\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}\{1+(\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}\}^2}.\end{aligned} \tag{B.20}$$

Note that by (B.20) and

$$\begin{aligned}\frac{2(\hat{\sigma}_1^2/\hat{v}^2)(\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i)^2}{\{1+(\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}\}^2}&=\frac{2(\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i)^2}{(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})^2(\hat{\sigma}_1^2/\hat{v}^2)}-\frac{2(\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i)^2+4(\hat{\sigma}_1^2/\hat{v}^2)(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})(\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i)^2}{\{1+(\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}\}^2(\hat{\sigma}_1^2/\hat{v}^2)(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})^2},\\\frac{2(\hat{\sigma}_1^2/\hat{v}^2)b_{i,1}\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})}{\{1+(\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}\}^2}&=\frac{2b_{i,1}\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i}{(\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}}-\frac{2b_{i,1}\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i+4b_{i,1}\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i(\hat{\sigma}_1^2/\hat{v}^2)(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})}{\{1+(\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}\}^2(\hat{\sigma}_1^2/\hat{v}^2)(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})},\\\frac{b_{i,1}^2(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})}{\{1+(\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}\}^2}&=\frac{b_{i,1}^2}{(\hat{\sigma}_1^2/\hat{v}^2)^2(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})}-\frac{b_{i,1}^2+2(\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}}{\{1+(\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}\}^2(\hat{\sigma}_1^2/\hat{v}^2)^2(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})},\end{aligned}$$

we have

$$D(\hat{\sigma}_1^2,\hat{v}^2)=\sum_{i=1}^m\bigg(\frac{2(\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i)^2}{(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})^2(\hat{\sigma}_1^2/\hat{v}^2)}+\frac{2b_{i,1}\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i}{(\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}}-\frac{b_{i,1}^2}{(\hat{\sigma}_1^2/\hat{v}^2)^2(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})}\bigg)+R^{\ddagger}(\hat{\sigma}_1^2,\hat{v}^2) \tag{B.21}$$

with

$$\begin{aligned}R^{\ddagger}(\hat{\sigma}_1^2,\hat{v}^2)=\sum_{i=1}^m\bigg(&\frac{(\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i)^2}{\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}\{1+(\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}\}^2}-\frac{2(\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i)^2+4(\hat{\sigma}_1^2/\hat{v}^2)(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})(\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i)^2}{\{1+(\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}\}^2(\hat{\sigma}_1^2/\hat{v}^2)(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})^2}\\&-\frac{2b_{i,1}\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i+4b_{i,1}\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i(\hat{\sigma}_1^2/\hat{v}^2)(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})}{\{1+(\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}\}^2(\hat{\sigma}_1^2/\hat{v}^2)(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})}+\frac{b_{i,1}^2+2(\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}}{\{1+(\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}\}^2(\hat{\sigma}_1^2/\hat{v}^2)^2(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})}\bigg).\end{aligned}$$

Note that by (B.14),

$$R^{\ddagger}(\hat{\sigma}_1^2,\hat{v}^2)=O_p(n^{-3/2}). \tag{B.22}$$

Further, by (B.12) and (B.16), we have

$$\begin{aligned}\frac{2(\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i)^2}{(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})^2(\hat{\sigma}_1^2/\hat{v}^2)} &= \frac{2(\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i)^2(\sum_{k=1}^m \boldsymbol{\epsilon}_k'\boldsymbol{\epsilon}_k/N)}{(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})^2(\sum_{k=1}^m b_{k,1}^2/m)} + \frac{2(\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i)^2}{(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})^2\hat{\sigma}_1^2(\sum_{k=1}^m b_{k,1}^2/m)} \\ &\quad \times \left\{R^\dagger(\hat{\sigma}_1^2,\hat{v}^2)\frac{\sum_{k=1}^m b_{k,1}^2}{m} - \left(\frac{\sum_{k=1}^m \boldsymbol{\epsilon}_k'\boldsymbol{\epsilon}_k}{N}\right)\left(\sum_{i=1}^m \frac{2b_{i,1}\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i}{m\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}}\right) - R^*(\hat{\sigma}_1^2,\hat{v}^2)\frac{\sum_{k=1}^m \boldsymbol{\epsilon}_k'\boldsymbol{\epsilon}_k}{N}\right\} \\ &\equiv \frac{2(\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i)^2\sum_{k=1}^m \boldsymbol{\epsilon}_k'\boldsymbol{\epsilon}_k}{n(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})^2\sum_{k=1}^m b_{k,1}^2} + R_{i,1}(\hat{\sigma}_1^2,\hat{v}^2),\end{aligned} \tag{B.23}$$

with

$$\begin{aligned}R_{i,1}(\hat{\sigma}_1^2,\hat{v}^2) &= \frac{2(\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i)^2}{(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})^2\hat{\sigma}_1^2(\sum_{k=1}^m b_{k,1}^2/m)}\left\{R^\dagger(\hat{\sigma}_1^2,\hat{v}^2)\frac{\sum_{k=1}^m b_{k,1}^2}{m}\right. \\ &\quad \left. - \left(\frac{\sum_{k=1}^m \boldsymbol{\epsilon}_k'\boldsymbol{\epsilon}_k}{N}\right)\left(\sum_{i=1}^m \frac{2b_{i,1}\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i}{m\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}}\right) - R^*(\hat{\sigma}_1^2,\hat{v}^2)\frac{\sum_{k=1}^m \boldsymbol{\epsilon}_k'\boldsymbol{\epsilon}_k}{N}\right\}.\end{aligned}$$

Similarly,

$$\frac{2b_{i,1}\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i}{(\hat{\sigma}_1^2/\hat{v}^2)\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}} = \frac{2b_{i,1}\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i\sum_{k=1}^m \boldsymbol{\epsilon}_k'\boldsymbol{\epsilon}_k}{n\{\sum_{k=1}^m b_{k,1}^2 + 2\sum_{k=1}^m b_{k,1}\boldsymbol{z}_{k,1}'\boldsymbol{\epsilon}_k/(\boldsymbol{z}_{k,1}'\boldsymbol{z}_{k,1})\}(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})} + R_{i,2}(\hat{\sigma}_1^2,\hat{v}^2), \tag{B.24}$$

$$\frac{b_{i,1}^2}{(\hat{\sigma}_1^2/\hat{v}^2)^2(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})} = \frac{b_{i,1}^2(\sum_{k=1}^m \boldsymbol{\epsilon}_k'\boldsymbol{\epsilon}_k)^2}{n^2(\sum_{k=1}^m b_{k,1}^2)^2\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}} + R_{i,3}(\hat{\sigma}_1^2,\hat{v}^2), \tag{B.25}$$

with

$$\begin{aligned}R_{i,2}(\hat{\sigma}_1^2,\hat{v}^2) &= \frac{2b_{i,1}\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i}{\hat{\sigma}_1^2\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}\{\sum_{k=1}^m b_{k,1}^2/m + 2\sum_{k=1}^m b_{k,1}\boldsymbol{z}_{k,1}'\boldsymbol{\epsilon}_k/(m\boldsymbol{z}_{k,1}'\boldsymbol{z}_{k,1})\}} \\ &\quad \times \left\{R^\dagger(\hat{\sigma}_1^2,\hat{v}^2)\left(\frac{\sum_{k=1}^m b_{k,1}^2}{m} + \sum_{k=1}^m \frac{2b_{k,1}\boldsymbol{z}_{k,1}'\boldsymbol{\epsilon}_k}{m\boldsymbol{z}_{k,1}'\boldsymbol{z}_{k,1}}\right) - R^*(\hat{\sigma}_1^2,\hat{v}^2)\sum_{k=1}^m \frac{\boldsymbol{\epsilon}_k'\boldsymbol{\epsilon}_k}{N}\right\}, \\ R_{i,3}(\hat{\sigma}_1^2,\hat{v}^2) &= \frac{b_{i,1}^2}{\hat{\sigma}_1^4(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})\{\sum_{k=1}^m b_{k,1}^2/m\}^2}\left(\hat{v}^2\sum_{k=1}^m \frac{b_{k,1}^2}{m} + \hat{\sigma}_1^2\sum_{k=1}^m \frac{\boldsymbol{\epsilon}_k'\boldsymbol{\epsilon}_k}{N}\right) \\ &\quad \times \left\{R^\dagger(\hat{\sigma}_1^2,\hat{v}^2)\frac{\sum_{k=1}^m b_{k,1}^2}{m} - \left(\frac{\sum_{k=1}^m \boldsymbol{\epsilon}_k'\boldsymbol{\epsilon}_k}{N}\right)\left(\sum_{i=1}^m \frac{2b_{i,1}\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i}{m\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}}\right) - R^*(\hat{\sigma}_1^2,\hat{v}^2)\frac{\sum_{k=1}^m \boldsymbol{\epsilon}_k'\boldsymbol{\epsilon}_k}{N}\right\}.\end{aligned}$$

Hence by (B.14), (B.15), and (B.17), we have

$$R_{i,1}(\hat{\sigma}_1^2,\hat{v}^2) = O_p(n^{-3/2}), \quad i = 1,2,3. \tag{B.26}$$

Furthermore, we have

$$\begin{aligned}
&\frac{1}{n}\frac{2b_{i,1}\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i\sum_{k=1}^m\boldsymbol{\epsilon}_k'\boldsymbol{\epsilon}_k}{\{\sum_{k=1}^m b_{k,1}^2+2\sum_{k=1}^m b_{k,1}\boldsymbol{z}_{k,1}'\boldsymbol{\epsilon}_k/(m\boldsymbol{z}_{k,1}'\boldsymbol{z}_{k,1})\}(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})}\\
&=\frac{1}{n(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})}\bigg\{\frac{2b_{i,1}\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i\sum_{k=1}^m\boldsymbol{\epsilon}_k'\boldsymbol{\epsilon}_k}{\sum_{k=1}^m b_{k,1}^2}-\frac{4b_{i,1}\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i\{\sum_{k=1}^m\boldsymbol{\epsilon}_k'\boldsymbol{\epsilon}_k\}\{\sum_{k=1}^m b_{k,1}\boldsymbol{z}_{k,1}'\boldsymbol{\epsilon}_k/(m\boldsymbol{z}_{k,1}'\boldsymbol{z}_{k,1})\}}{\{\sum_{k=1}^m b_{k,1}^2\}^2}\\
&\quad+\frac{8b_{i,1}\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i\sum_{k=1}^m\boldsymbol{\epsilon}_k'\boldsymbol{\epsilon}_k\{\sum_{k=1}^m b_{k,1}\boldsymbol{z}_{k,1}'\boldsymbol{\epsilon}_k/(m\boldsymbol{z}_{k,1}'\boldsymbol{z}_{k,1})\}^2}{\{\sum_{k=1}^m b_{k,1}^2+2\sum_{k=1}^m b_{k,1}\boldsymbol{z}_{k,1}'\boldsymbol{\epsilon}_k/(m\boldsymbol{z}_{k,1}'\boldsymbol{z}_{k,1})\}\{\sum_{k=1}^m b_{k,1}^2\}^2}\bigg\}\\
&\equiv\bigg\{\frac{2b_{i,1}\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i\sum_{k=1}^m\boldsymbol{\epsilon}_k'\boldsymbol{\epsilon}_k}{n(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})\sum_{k=1}^m b_{k,1}^2}-\frac{4b_{i,1}\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i\{\sum_{k=1}^m\boldsymbol{\epsilon}_k'\boldsymbol{\epsilon}_k\}\{\sum_{k=1}^m b_{k,1}\boldsymbol{z}_{k,1}'\boldsymbol{\epsilon}_k/(m\boldsymbol{z}_{k,1}'\boldsymbol{z}_{k,1})\}}{n(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})\{\sum_{k=1}^m b_{k,1}^2\}^2}\bigg\}\\
&\quad+R_{i,4},
\end{aligned}\tag{B.27}$$

with

$$R_{i,4}=\frac{8b_{i,1}\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i\sum_{k=1}^m\boldsymbol{\epsilon}_k'\boldsymbol{\epsilon}_k\{\sum_{k=1}^m b_{k,1}\boldsymbol{z}_{k,1}'\boldsymbol{\epsilon}_k/(m\boldsymbol{z}_{k,1}'\boldsymbol{z}_{k,1})\}^2}{n(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})\{\sum_{k=1}^m b_{k,1}^2+2\sum_{k=1}^m b_{k,1}\boldsymbol{z}_{k,1}'\boldsymbol{\epsilon}_k/(m\boldsymbol{z}_{k,1}'\boldsymbol{z}_{k,1})\}\{\sum_{k=1}^m b_{k,1}^2\}^2}.$$

Note that

$$R_{i,4}=O_p(n^{-3/2}).\tag{B.28}$$

By (B.21), (B.23), (B.24), (B.25), and (B.27), we have

$$\begin{aligned}
nD(\hat\sigma_1^2,\hat v^2)&=A_{n,m}+nR^{\ddagger}(\hat\sigma_1^2,\hat v^2)+n\sum_{i=1}^m\Big\{R_{i,1}(\hat\sigma_1^2,\hat v^2)+R_{i,2}(\hat\sigma_1^2,\hat v^2)-R_{i,3}(\hat\sigma_1^2,\hat v^2)+R_{i,4}\Big\}\\
&\equiv A_{n,m}+O_p(n^{-1/2})
\end{aligned}$$

with

$$\begin{aligned}
A_{n,m}=\sum_{i=1}^m\bigg\{&\frac{2(\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i)^2\sum_{k=1}^m\boldsymbol{\epsilon}_k'\boldsymbol{\epsilon}_k}{(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})^2\sum_{k=1}^m b_{k,1}^2}-\frac{b_{i,1}^2(\sum_{k=1}^m\boldsymbol{\epsilon}_k'\boldsymbol{\epsilon}_k)^2}{n(\sum_{k=1}^m b_{k,1}^2)^2\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}}\frac{2b_{i,1}\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i\sum_{k=1}^m\boldsymbol{\epsilon}_k'\boldsymbol{\epsilon}_k}{(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})\sum_{k=1}^m b_{k,1}^2}\\
&-\frac{4b_{i,1}\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i\{\sum_{k=1}^m\boldsymbol{\epsilon}_k'\boldsymbol{\epsilon}_k\}\{\sum_{k=1}^m b_{k,1}\boldsymbol{z}_{k,1}'\boldsymbol{\epsilon}_k/(\boldsymbol{z}_{k,1}'\boldsymbol{z}_{k,1})\}}{(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})\{\sum_{k=1}^m b_{k,1}^2\}^2}\bigg\},
\end{aligned}$$

where the last equality follows from (B.22), (B.26), and (B.28). Note that $\big(\sum_{k=1}^m b_{i,k}^2/\sigma_{1,0}^2\big)^{-1}$ follows the inverse-chi-squared distribution with m degrees of freedom. We have

$$\begin{aligned}
\mathrm{E}\bigg(\frac{1}{\sum_{i=1}^m b_{i,1}^2}\bigg)&=\frac{1}{(m-2)\sigma_{1,0}^2},\quad\text{provided } m>2,\\
\mathrm{E}\bigg(\frac{b_{i,1}^2}{\{\sum_{k=1}^m b_{k,1}^2\}^2}\bigg)&=\frac{1}{m(m-2)\sigma_{1,0}^2}\quad\text{provided } m>4.
\end{aligned}\tag{B.29}$$

By (B.29) and

$$\begin{aligned}
\mathrm{E}\bigg(\bigg\{\sum_{i=1}^m\boldsymbol{\epsilon}_i'\boldsymbol{\epsilon}_i\bigg\}^2\bigg)&=(2mn+m^2n^2)v_0^4,\\
\mathrm{E}\big(\boldsymbol{\epsilon}_i'\boldsymbol{\epsilon}_i(\boldsymbol{z}_{i,1}\boldsymbol{\epsilon}_i)\big)&=0,\\
\mathrm{E}\big(\boldsymbol{\epsilon}_i'\boldsymbol{\epsilon}_i(\boldsymbol{z}_{i,1}\boldsymbol{\epsilon}_i)^2\big)&=n^2v_0^4+o(n^2),
\end{aligned}$$

we have that for $m > 4$,

$$
\begin{aligned}
\mathrm{E}(A_{n,m}) &= \mathrm{E}\sum_{i=1}^{m}\bigg\{\frac{2(\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i)^2\sum_{k=1}^{m}\boldsymbol{\epsilon}_k'\boldsymbol{\epsilon}_k}{(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})^2\sum_{k=1}^{m}b_{k,1}^2} - \frac{b_{i,1}^2(\sum_{k=1}^{m}\boldsymbol{\epsilon}_k'\boldsymbol{\epsilon}_k)^2}{n(\sum_{k=1}^{m}b_{k,1}^2)^2\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1}} + \frac{2b_{i,1}\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i\sum_{k=1}^{m}\boldsymbol{\epsilon}_k'\boldsymbol{\epsilon}_k}{(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})\sum_{k=1}^{m}b_{k,1}^2} \\
&\quad - \frac{4b_{i,1}\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i\{\sum_{k=1}^{m}\boldsymbol{\epsilon}_k'\boldsymbol{\epsilon}_k\}\{\sum_{k=1}^{m}b_{k,1}\boldsymbol{z}_{k,1}'\boldsymbol{\epsilon}_k/(\boldsymbol{z}_{k,1}'\boldsymbol{z}_{k,1})\}}{(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})\{\sum_{k=1}^{m}b_{k,1}^2\}^2}\bigg\} \\
&= \frac{2m^2v_0^4}{(m-2)\sigma_{1,0}^2} - \frac{m^2v_0^4}{(m-2)\sigma_{1,0}^2} + o(1) \\
&\quad - \mathrm{E}\bigg(\mathrm{E}\bigg(\sum_{i=1}^{m}\frac{4b_{i,1}\boldsymbol{z}_{i,1}'\boldsymbol{\epsilon}_i\{\sum_{k=1}^{m}\boldsymbol{\epsilon}_k'\boldsymbol{\epsilon}_k\}\{\sum_{k=1}^{m}b_{k,1}\boldsymbol{z}_{k,1}'\boldsymbol{\epsilon}_k/(\boldsymbol{z}_{k,1}'\boldsymbol{z}_{k,1})\}}{(\boldsymbol{z}_{i,1}'\boldsymbol{z}_{i,1})\{\sum_{k=1}^{m}b_{k,1}^2\}^2}\bigg| b_{1,1},\ldots,b_{m,1}\bigg)\bigg) \\
&= \frac{2m^2v_0^4}{(m-2)\sigma_{1,0}^2} - \frac{m^2v_0^4}{(m-2)\sigma_{1,0}^2} - \mathrm{E}\bigg(\sum_{i=1}^{m}\frac{4mv_0^4b_{i,1}^2}{\{\sum_{k=1}^{m}b_{k,1}^2\}^2}\bigg) + o(1) \\
&= \frac{2m^2v_0^4}{(m-2)\sigma_{1,0}^2} - \frac{m^2v_0^4}{(m-2)\sigma_{1,0}^2} - \frac{4mv_0^4}{(m-2)\sigma_{1,0}^2} + o(1) \\
&= \frac{m(m-4)v_0^4}{(m-2)\sigma_{1,0}^2} + o(1).
\end{aligned}
$$

This completes the proofs.

B.3 Proof of Theorem 5

In this section, we first prove Theorem 5 to simplify the proofs of Theorems 3 and 4. As with the proof of Theorem 1, we shall focus on the asymptotic properties of $\hat{v}^2(\alpha,\gamma)$ and $\{\hat{\theta}_k(\alpha,\gamma) : k \in \gamma\}$, and derive them by solving the likelihood equations directly.

We first prove (3.11) using (B.1). For $(\alpha,\gamma) \in (\mathcal{A}\setminus\mathcal{A}_0)\times\mathcal{G}$, we have

$$(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}))\boldsymbol{\mu}_0 = (\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}))\boldsymbol{X}(\alpha_0\setminus\alpha)\boldsymbol{\beta}_0(\alpha_0\setminus\alpha), \tag{B.30}$$

where $\boldsymbol{\beta}_0(\alpha_0\setminus\alpha)$ denotes the sub-vector of $\boldsymbol{\beta}_0$ corresponding to $\alpha_0\setminus\alpha$. Note that by the Cauchy–Schwarz inequality, we have

$$\bigg(\sum_{i=1}^{m} n_i^{(\xi+\ell)/2}\bigg)^2 = O\bigg(\sum_{i=1}^{m} n_i^{\xi}\sum_{i^*=1}^{m} n_i^{\ell}\bigg). \tag{B.31}$$

Hence by (B.31) and Lemma 6, we have

$$
\begin{aligned}
&\big(\boldsymbol{X}(\alpha_0\setminus\alpha)\boldsymbol{\beta}_0(\alpha_0\setminus\alpha)+\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)+\boldsymbol{\epsilon}\big)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\\
&\quad\times\big(\boldsymbol{X}(\alpha_0\setminus\alpha)\boldsymbol{\beta}_0(\alpha_0\setminus\alpha)+\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)+\boldsymbol{\epsilon}\big)\\
&=\boldsymbol{\beta}_0(\alpha_0\setminus\alpha)'\big(\boldsymbol{X}(\alpha_0\setminus\alpha)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\boldsymbol{X}(\alpha_0\setminus\alpha)\boldsymbol{\beta}_0(\alpha_0\setminus\alpha)\\
&\quad+\boldsymbol{b}(\gamma_0)'\boldsymbol{Z}(\gamma_0)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)\\
&\quad+\boldsymbol{\epsilon}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\boldsymbol{\epsilon}\\
&\quad+2\boldsymbol{b}(\gamma_0)'\boldsymbol{Z}(\gamma_0)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\boldsymbol{X}(\alpha_0\setminus\alpha)\boldsymbol{\beta}_0(\alpha_0\setminus\alpha)\\
&\quad+2\boldsymbol{\epsilon}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\boldsymbol{X}(\alpha_0\setminus\alpha)\boldsymbol{\beta}_0(\alpha_0\setminus\alpha)\\
&\quad+2\boldsymbol{b}(\gamma_0)'\boldsymbol{Z}(\gamma_0)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\boldsymbol{\epsilon}\\
&=\bigg(\sum_{i=1}^m\sum_{k\in\gamma_0}b_{i,k}\boldsymbol{h}_{i,k}\bigg)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\bigg(\sum_{i=1}^m\sum_{k\in\gamma_0}b_{i,k}\boldsymbol{h}_{i,k}\bigg)\\
&\quad+2\bigg(\sum_{i=1}^m\sum_{k\in\gamma_0}b_{i,k}\boldsymbol{h}_{i,k}\bigg)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\boldsymbol{X}(\alpha_0\setminus\alpha)\boldsymbol{\beta}_0(\alpha_0\setminus\alpha)\\
&\quad+2\bigg(\sum_{i=1}^m\sum_{k\in\gamma_0}b_{i,k}\boldsymbol{h}_{i,k}\bigg)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\boldsymbol{\epsilon}+o_p\bigg(\sum_{i=1}^m n_i^{\xi-\tau}\bigg)\\
&\quad+O_p(p)\\
&=o_p\bigg(\sum_{i=1}^m n_i^{\ell-\tau}\bigg)+o_p\bigg(\sum_{i=1}^m n_i^{(\xi+\ell)/2-\tau}\bigg)+o_p\bigg(\sum_{i=1}^m n_i^{\xi-\tau}\bigg)+O_p(p)\\
&=o_p\bigg(\sum_{i=1}^m n_i^{\xi}\bigg)+o_p\bigg(\sum_{i=1}^m n_i^{\ell}\bigg)+O_p(p)
\end{aligned}
$$

uniformly over $\boldsymbol{\theta}\in\Theta_\gamma$. This and (B.30) imply

$$
\begin{aligned}
&\boldsymbol{y}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{I}_N-\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}))\boldsymbol{y}\\
&\quad=\big(\boldsymbol{X}(\alpha_0\setminus\alpha)\boldsymbol{\beta}_0(\alpha_0\setminus\alpha)+\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)+\boldsymbol{\epsilon}\big)'\\
&\qquad\times\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{I}_N-\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}))\\
&\qquad\times\big(\boldsymbol{X}(\alpha_0\setminus\alpha)\boldsymbol{\beta}_0(\alpha_0\setminus\alpha)+\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)+\boldsymbol{\epsilon}\big)\\
&\quad=\big(\boldsymbol{X}(\alpha_0\setminus\alpha)\boldsymbol{\beta}_0(\alpha_0\setminus\alpha)+\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)+\boldsymbol{\epsilon}\big)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\\
&\qquad\times\big(\boldsymbol{X}(\alpha_0\setminus\alpha)\boldsymbol{\beta}_0(\alpha_0\setminus\alpha)+\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)+\boldsymbol{\epsilon}\big)\\
&\qquad+o_p\bigg(\sum_{i=1}^m n_i^{\xi}\bigg)+o_p\bigg(\sum_{i=1}^m n_i^{\ell}\bigg)+O_p(p)\\
&\quad=\sum_{i=1}^m\boldsymbol{\beta}_0(\alpha_0\setminus\alpha)'\boldsymbol{X}_i(\alpha_0\setminus\alpha)'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}_i(\alpha_0\setminus\alpha)\boldsymbol{\beta}_0(\alpha_0\setminus\alpha)\\
&\qquad+2\sum_{i=1}^m\boldsymbol{\beta}_0(\alpha_0\setminus\alpha)'\boldsymbol{X}_i(\alpha_0\setminus\alpha)'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{Z}_i(\gamma_0)\boldsymbol{b}_i(\gamma_0)+\boldsymbol{\epsilon}_i)\\
&\qquad+\sum_{i=1}^m\bigg(\sum_{k\in\gamma_0}\boldsymbol{z}_{i,k}b_{i,k}+\boldsymbol{\epsilon}_i\bigg)'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\bigg(\sum_{k\in\gamma_0}\boldsymbol{z}_{i,k}b_{i,k}+\boldsymbol{\epsilon}_i\bigg)\\
&\qquad+o_p\bigg(\sum_{i=1}^m n_i^{\xi}\bigg)+o_p\bigg(\sum_{i=1}^m n_i^{\ell}\bigg)+O_p(p)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} \boldsymbol{\epsilon}_i' \boldsymbol{\epsilon}_i + \sum_{i=1}^{m} \sum_{j \in \alpha_0 \setminus \alpha} \beta_{j,0}^2 d_{i,j} n_i^{\xi} + \sum_{i=1}^{m} \sum_{k \in \gamma_0 \setminus \gamma} b_{i,k}^2 c_{i,k} n_i^{\ell} \\
&\quad + o_p\biggl(\sum_{k,k^* \in \gamma \cap \gamma_0} \frac{m}{\theta_k \theta_{k^*}} \biggr) + O_p\biggl(\sum_{k \in \gamma \cap \gamma_0} \frac{m}{\theta_k} \biggr) \\
&\quad + o_p\biggl(\sum_{i=1}^{m} n_i^{\xi} \biggr) + o_p\biggl(\sum_{i=1}^{m} n_i^{\ell} \biggr) + O_p(p + mq)
\end{aligned}
$$

uniformly over $\boldsymbol{\theta} \in \Theta_\gamma$, where the last equality follows from (B.5) and Lemmas 2–4. Hence by (B.1), we have, for $v^2 \in (0, \infty)$,

$$
\begin{aligned}
&v^4 \biggl\{ \frac{\partial}{\partial v^2} \{ -2 \log L(\boldsymbol{\theta}, v^2; \alpha, \gamma) \} \biggr\} \\
&= N\biggl(v^2 - \frac{\boldsymbol{\epsilon}'\boldsymbol{\epsilon}}{N} + \frac{1}{N} \sum_{i=1}^{m} \sum_{j \in \alpha_0 \setminus \alpha} \beta_{j,0}^2 d_{i,j} n_i^{\xi} + \frac{1}{N} \sum_{i=1}^{m} \sum_{k \in \gamma_0 \setminus \gamma} b_{i,k}^2 c_{i,k} n_i^{\ell} \biggr) \\
&\quad + o_p\biggl(\sum_{i=1}^{m} n_i^{\xi} \biggr) + o_p\biggl(\sum_{i=1}^{m} n_i^{\ell} \biggr) + O_p\biggl(\sum_{k,k^* \in \gamma \cap \gamma_0} \frac{m}{\theta_k \theta_{k^*}} \biggr) \\
&\quad + O_p\biggl(\sum_{k \in \gamma \cap \gamma_0} \frac{m}{\theta_k} \biggr) + O_p(p + mq)
\end{aligned}
$$

uniformly over $\boldsymbol{\theta} \in \Theta_\gamma$. This and Lemma 5 imply that

$$
\begin{aligned}
\hat{v}^2(\alpha, \gamma) &= \frac{\boldsymbol{\epsilon}'\boldsymbol{\epsilon}}{N} + \frac{1}{N} \sum_{i=1}^{m} \sum_{j \in \alpha_0 \setminus \alpha} \beta_{j,0}^2 d_{i,j} n_i^{\xi} + \frac{1}{N} \sum_{i=1}^{m} \sum_{k \in \gamma_0 \setminus \gamma} b_{i,k}^2 c_{i,k} n_i^{\ell} \\
&\quad + o_p\biggl(\frac{1}{N} \sum_{i=1}^{m} n_i^{\xi} \biggr) + o_p\biggl(\frac{1}{N} \sum_{i=1}^{m} n_i^{\ell} \biggr) + O_p\Bigl(\frac{p + mq}{N} \Bigr).
\end{aligned}
\tag{B.32}
$$

Thus (3.11) follows by applying the law of large numbers to $\boldsymbol{\epsilon}'\boldsymbol{\epsilon}/N$. In addition, if $(\xi, \ell) \in (0, 1/2) \times (0, 1/2)$, the asymptotic normality of $\hat{v}^2(\alpha, \gamma)$ follows by $p + mq = o(N^{1/2})$ and an application of the central limit theorem to $\boldsymbol{\epsilon}'\boldsymbol{\epsilon}/N$ in (B.32).

Next, we prove (3.12), for $k \in \gamma \cap \gamma_0$, using (B.2). By (B.31) and Lemma 6 (i)–(iv), we have, for $k \in \gamma \cap \gamma_0$,

$$
\begin{aligned}
&\theta_k \boldsymbol{h}_{i,k}' \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}) \bigl(\boldsymbol{X}(\alpha_0 \setminus \alpha) \boldsymbol{\beta}_0(\alpha_0 \setminus \alpha) + \boldsymbol{Z}(\gamma_0) \boldsymbol{b}(\gamma_0) + \boldsymbol{\epsilon} \bigr) \\
&= \theta_k \boldsymbol{h}_{i,k}' \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}) \\
&\quad \times \biggl(\boldsymbol{X}(\alpha_0 \setminus \alpha) \boldsymbol{\beta}_0(\alpha_0 \setminus \alpha) + \sum_{i^*=1}^{m} \sum_{k^* \in \gamma_0} b_{i^*,k^*} \boldsymbol{h}_{i^*,k^*} + \boldsymbol{\epsilon} \biggr) \\
&= o_p\biggl(\frac{n_i^{(\xi-\ell)/2} \sum_{i^*=1}^{m} n_{i^*}^{(\xi+\ell)/2-\tau}}{\sum_{i=1}^{m} n_i^{\xi}} \biggr) + o_p(n_i^{(\xi-\ell)/2}) + o_p(n_i^{-\ell/2}) \\
&= o_p\biggl(n_i^{(\xi-\ell)/2} \biggl(\frac{\sum_{i=1}^{m} n_i^{\ell}}{\sum_{i=1}^{m} n_i^{\xi}} \biggr)^{1/2} \biggr) + o_p(n_i^{(\xi-\ell)/2}) + o_p(1)
\end{aligned}
$$

uniformly over $\boldsymbol{\theta} \in \Theta_\gamma$. This and (B.30) imply that for $k \in \gamma \cap \gamma_0$,

$$\begin{aligned}
&\theta_k \boldsymbol{h}'_{i,k}\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}))\boldsymbol{y}\\
&= \theta_k \boldsymbol{h}'_{i,k}\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}))\\
&\quad \times \big(\boldsymbol{X}(\alpha_0 \setminus \alpha)\boldsymbol{\beta}_0(\alpha_0 \setminus \alpha) + \boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0) + \boldsymbol{\epsilon}\big)\\
&= \theta_k \boldsymbol{h}'_{i,k}\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\big(\boldsymbol{X}(\alpha_0 \setminus \alpha)\boldsymbol{\beta}_0(\alpha_0 \setminus \alpha) + \boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0) + \boldsymbol{\epsilon}\big)\\
&\quad + o_p\bigg(n_i^{(\xi-\ell)/2}\bigg(\frac{\sum_{i=1}^m n_i^\ell}{\sum_{i=1}^m n_i^\xi}\bigg)^{1/2}\bigg) + o_p(n_i^{(\xi-\ell)/2}) + o_p(1)\\
&= \theta_k \boldsymbol{z}'_{i,k}\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\bigg(\boldsymbol{X}_i(\alpha_0 \setminus \alpha)\boldsymbol{\beta}_0(\alpha_0 \setminus \alpha) + \sum_{k^* \in \gamma_0} \boldsymbol{z}_{i,k^*} b_{i,k^*} + \boldsymbol{\epsilon}_i\bigg)\\
&\quad + o_p\bigg(n_i^{(\xi-\ell)/2}\bigg(\frac{\sum_{i=1}^m n_i^\ell}{\sum_{i=1}^m n_i^\xi}\bigg)^{1/2}\bigg) + o_p(n_i^{(\xi-\ell)/2}) + o_p(1)\\
&= b_{i,k} + o_p\bigg(n_i^{(\xi-\ell)/2}\bigg(\frac{\sum_{i=1}^m n_i^\ell}{\sum_{i=1}^m n_i^\xi}\bigg)^{1/2}\bigg) + o_p(n_i^{(\xi-\ell)/2}) + o_p(1)
\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in \Theta_\gamma$, where the last equality follows from Lemma 2 (iii), Lemma 3 (ii)–(iv), and Lemma 4 (i). It follows that for $k \in \gamma \cap \gamma_0$,

$$\begin{aligned}
&\theta_k^2 \{\boldsymbol{h}'_{i,k}\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}))\boldsymbol{y}\}^2\\
&= b_{i,k}^2 + o_p\bigg(n_i^{\xi-\ell}\bigg(\frac{\sum_{i=1}^m n_i^\ell}{\sum_{i=1}^m n_i^\xi}\bigg)\bigg) + o_p(n_i^{\xi-\ell}) + o_p(1)
\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in \Theta_\gamma$. Hence by Lemma 3 (ii) and (B.2), we have, for $k \in \gamma \cap \gamma_0$,

$$\begin{aligned}
&\theta_k^2 \bigg\{\frac{\partial}{\partial \theta_k}\{-2\log L(\boldsymbol{\theta}, v^2; \alpha, \gamma)\}\bigg\}\\
&= m\bigg(\theta_k - \frac{1}{m}\sum_{i=1}^m \frac{b_{i,k}^2}{v^2}\bigg) + o_p\bigg(\sum_{i=1}^m n_i^{\xi-\ell}\bigg(1 + \frac{\sum_{i=1}^m n_i^\ell}{\sum_{i=1}^m n_i^\xi}\bigg)\bigg) + o_p(m)
\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in \Theta_\gamma$. This implies that for $k \in \gamma \cap \gamma_0$,

$$\hat{\theta}_k(\alpha,\gamma) = \frac{1}{m}\sum_{i=1}^m \frac{b_{i,k}^2}{\hat{v}^2(\alpha,\gamma)} + o_p\bigg(\frac{1}{m}\sum_{i=1}^m n_i^{\xi-\ell}\bigg(1 + \frac{\sum_{i=1}^m n_i^\ell}{\sum_{i=1}^m n_i^\xi}\bigg)\bigg) + o_p(1).$$

This proves (3.12), for $k \in \gamma \cap \gamma_0$.

It remains to prove (3.12), for $k \in \gamma \setminus \gamma_0$. Let $\boldsymbol{\theta}^\dagger$ be $\boldsymbol{\theta}$ except that $\{\theta_k : k \in \gamma \cap \gamma_0\}$ are replaced by $\{\hat{\theta}_k(\alpha,\gamma) : k \in \gamma \cap \gamma_0\}$. By (B.31) and Lemma 6 (i)–(iv), we have, for $k \in \gamma \setminus \gamma_0$,

$$\begin{aligned}
&\theta_k \boldsymbol{h}'_{i,k}\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}^\dagger)\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}^\dagger)\big(\boldsymbol{X}(\alpha_0 \setminus \alpha)\boldsymbol{\beta}_0(\alpha_0 \setminus \alpha) + \boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0) + \boldsymbol{\epsilon}\big)\\
&= \theta_k \boldsymbol{h}'_{i,k}\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}^\dagger)\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}^\dagger)\\
&\quad \times \bigg(\boldsymbol{X}(\alpha_0 \setminus \alpha)\boldsymbol{\beta}_0(\alpha_0 \setminus \alpha) + \sum_{i^*=1}^m \sum_{k^* \in \gamma_0} b_{i^*,k^*}\boldsymbol{h}_{i^*,k^*} + \boldsymbol{\epsilon}\bigg)\\
&= o_p\bigg(n_i^{(\xi-\ell)/2}\bigg(\frac{\sum_{i=1}^m n_i^\ell}{\sum_{i=1}^m n_i^\xi}\bigg)^{1/2}\bigg) + o_p(n_i^{(\xi-\ell)/2-\tau}) + o_p(n_i^{-\ell/2})\\
&= o_p\bigg(n_i^{(\xi-\ell)/2}\bigg(\frac{\sum_{i=1}^m n_i^\ell}{\sum_{i=1}^m n_i^\xi}\bigg)^{1/2}\bigg) + o_p(n_i^{(\xi-\ell)/2}) + o_p(1)
\end{aligned}$$

uniformly over $\boldsymbol{\theta}(\gamma \setminus \gamma_0) \in [0,\infty)^{q(\gamma\setminus\gamma_0)}$. This and (B.30) imply that for $k \in \gamma \setminus \gamma_0$,

$$
\begin{aligned}
&\theta_k \boldsymbol{h}_{i,k}' \boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}^\dagger)(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}^\dagger))\boldsymbol{y} \\
&= \theta_k \boldsymbol{h}_{i,k}' \boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}^\dagger)(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}^\dagger))\big(\boldsymbol{X}(\alpha_0\setminus\alpha)\boldsymbol{\beta}_0(\alpha_0\setminus\alpha) \\
&\quad + \boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0) + \boldsymbol{\epsilon}\big) \\
&= \theta_k \boldsymbol{h}_{i,k}' \boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}^\dagger)\big(\boldsymbol{X}(\alpha_0\setminus\alpha)\boldsymbol{\beta}_0(\alpha_0\setminus\alpha) + \boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0) + \boldsymbol{\epsilon}\big) \\
&\quad + o_p(n_i^{(\xi-\ell)/2} n_{\max}^{(\ell-\xi)/2}) + o_p(n_i^{(\xi-\ell)/2}) + o_p(1) \\
&= \theta_k \boldsymbol{z}_{i,k}' \boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta}^\dagger)\bigg(\boldsymbol{X}_i(\alpha_0\setminus\alpha)\boldsymbol{\beta}_0(\alpha_0\setminus\alpha) + \sum_{i^*=1}^m \sum_{k^*\in\gamma_0} b_{i^*,k^*}\boldsymbol{h}_{i^*,k^*} + \boldsymbol{\epsilon}_i\bigg) \\
&\quad + o_p\bigg(n_i^{(\xi-\ell)/2}\bigg(\frac{\sum_{i=1}^m n_i^\ell}{\sum_{i=1}^m n_i^\xi}\bigg)^{1/2}\bigg) + o_p(n_i^{(\xi-\ell)/2}) + o_p(1) \\
&= o_p\bigg(n_i^{(\xi-\ell)/2}\bigg(\frac{\sum_{i=1}^m n_i^\ell}{\sum_{i=1}^m n_i^\xi}\bigg)^{1/2}\bigg) + o_p(n_i^{(\xi-\ell)/2}) + o_p(1)
\end{aligned}
$$

uniformly over $\boldsymbol{\theta}(\gamma\setminus\gamma_0) \in [0,\infty)^{q(\gamma\setminus\gamma_0)}$, where the last equality follows from Lemma 2 (iii), Lemma 3 (iii)–(iv), and Lemma 4 (i). Therefore,

$$
\begin{aligned}
&\theta_k^2 \{\boldsymbol{h}_{i,k}' \boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}^\dagger)(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}^\dagger))\boldsymbol{y}\}^2 \\
&= o_p\bigg(n_i^{\xi-\ell}\bigg(\frac{\sum_{i=1}^m n_i^\ell}{\sum_{i=1}^m n_i^\xi}\bigg)\bigg) + o_p(n_i^{\xi-\ell}) + o_p(1)
\end{aligned}
$$

uniformly over $\boldsymbol{\theta}(\gamma \setminus \gamma_0) \in [0,\infty)^{q(\gamma\setminus\gamma_0)}$. Hence by Lemma 3 (ii) and (B.2), we have for $k \in \gamma \setminus \gamma_0$,

$$
\begin{aligned}
&\theta_k^2 \bigg\{\frac{\partial}{\partial\theta_k}\{-2\log L(\boldsymbol{\theta}^\dagger, v^2;\alpha,\gamma)\}\bigg\} \\
&= m\theta_k + o_p\bigg(\sum_{i=1}^m n_i^{\xi-\ell}\bigg(1 + \frac{\sum_{i=1}^m n_i^\ell}{\sum_{i=1}^m n_i^\xi}\bigg)\bigg) + o_p(m)
\end{aligned}
$$

uniformly over $\boldsymbol{\theta}(\gamma \setminus \gamma_0) \in [0,\infty)^{q(\gamma\setminus\gamma_0)}$. This implies that for $k \in \gamma \setminus \gamma_0$,

$$
\hat{\theta}_k(\alpha,\gamma) = o_p\bigg(\frac{1}{m}\sum_{i=1}^m n_i^{\xi-\ell}\bigg(1 + \frac{\sum_{i=1}^m n_i^\ell}{\sum_{i=1}^m n_i^\xi}\bigg)\bigg) + o_p(1).
$$

This completes the proof of (3.12). Thus the proof of Theorem 5 is complete.

B.4 Proof of Theorem 3

As with the proof of Theorem 1, we shall focus on the asymptotic properties of $\hat{v}^2(\alpha,\gamma)$ and $\{\hat{\theta}_k(\alpha,\gamma) : k \in \gamma\}$, and derive them by solving the likelihood equations directly.

We first prove (3.7) using (B.1). Hence by (B.31), Lemma 6 (i)–(iii), Lemma 6 (v)–(vi), and Lemma 6 (viii), we have

$$
\begin{aligned}
&(\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)+\boldsymbol{\epsilon})'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})(\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)+\boldsymbol{\epsilon})\\
&\quad=\bigg(\sum_{i=1}^{m}\sum_{k\in\gamma_0}b_{i,k}\boldsymbol{h}_{i,k}+\boldsymbol{\epsilon}\bigg)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\bigg(\sum_{i=1}^{m}\sum_{k\in\gamma_0}b_{i,k}\boldsymbol{h}_{i,k}+\boldsymbol{\epsilon}\bigg)\\
&\quad=o_p\bigg(\sum_{i=1}^{m}n_i^{\ell-\tau}\bigg)+o_p\bigg(\sum_{i=1}^{m}n_i^{\ell/2}\bigg)+O_p(p)\\
&\quad=o_p\bigg(\sum_{i=1}^{m}n_i^{\ell}\bigg)+O_p(p)
\end{aligned}
$$

uniformly over $\boldsymbol{\theta}\in\Theta_\gamma$. This and (B.4) imply

$$
\begin{aligned}
&\boldsymbol{y}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{I}_N-\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}))\boldsymbol{y}\\
&=(\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)+\boldsymbol{\epsilon})'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{I}_N-\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}))(\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)+\boldsymbol{\epsilon})\\
&=(\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)+\boldsymbol{\epsilon})'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)+\boldsymbol{\epsilon})+o_p\bigg(\sum_{i=1}^{m}n_i^{\ell}\bigg)+O_p(p)\\
&=\sum_{i=1}^{m}(\boldsymbol{Z}_i(\gamma_0)\boldsymbol{b}_i(\gamma_0)+\boldsymbol{\epsilon}_i)'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{Z}_i(\gamma_0)\boldsymbol{b}_i(\gamma_0)+\boldsymbol{\epsilon}_i)+o_p\bigg(\sum_{i=1}^{m}n_i^{\ell}\bigg)\\
&\quad+O_p(p)\\
&=\sum_{i=1}^{m}\boldsymbol{\epsilon}_i'\boldsymbol{\epsilon}_i+\sum_{i=1}^{m}\sum_{k\in\gamma_0\setminus\gamma}b_{i,k}^2c_{i,k}n_i^{\ell}+o_p\bigg(\sum_{i=1}^{m}n_i^{\ell}\bigg)+O_p\bigg(\sum_{k\in\gamma\cap\gamma_0}\frac{m}{\theta_k}\bigg)\\
&\quad+o_p\bigg(\sum_{k,k^*\in\gamma\cap\gamma_0}\frac{m}{\theta_k\theta_{k^*}}\bigg)+O_p(p+mq)
\end{aligned}
$$

uniformly over $\boldsymbol{\theta}\in\Theta_\gamma$, where the last equality follows from Lemma 3, Lemma 4 (i)–(ii), and Lemma 4 (iv). Hence by (B.1), we have, for $v^2\in(0,\infty)$,

$$
\begin{aligned}
&v^4\ \bigg\{\frac{\partial}{\partial v^2}\{-2\log L(\boldsymbol{\theta},v^2;\alpha,\gamma)\}\bigg\}\\
&=N\bigg(v^2-\frac{\boldsymbol{\epsilon}'\boldsymbol{\epsilon}}{N}+\frac{1}{N}\sum_{k\in\gamma_0\setminus\gamma}b_{i,k}^2c_{i,k}n_i^{\ell}\bigg)+o_p\bigg(\sum_{i=1}^{m}n_i^{\ell}\bigg)\\
&\quad+O_p\bigg(\sum_{k\in\gamma\cap\gamma_0}\frac{m}{\theta_k}\bigg)+o_p\bigg(\sum_{k,k^*\in\gamma\cap\gamma_0}\frac{m}{\theta_k\theta_{k^*}}\bigg)+O_p(p+mq)
\end{aligned}
$$

uniformly over $\boldsymbol{\theta}\in\Theta_\gamma$. This and Lemma 5 imply that for $(\xi,\ell)\in(0,1]\times(0,1]$,

$$
\begin{aligned}
\hat{v}^2(\alpha,\gamma)=&\ \frac{\boldsymbol{\epsilon}'\boldsymbol{\epsilon}}{N}+\frac{1}{N}\sum_{i=1}^{m}\sum_{k\in\gamma_0\setminus\gamma}b_{i,k}^2c_{i,k}n_i^{\ell}\\
&+o_p\bigg(\frac{1}{N}\sum_{i=1}^{m}n_i^{\ell}\bigg)+O_p\bigg(\frac{p+mq}{N}\bigg).
\end{aligned}
\tag{B.33}
$$

Thus (3.7) follows by applying the law of large numbers to $\boldsymbol{\epsilon}'\boldsymbol{\epsilon}/N$. In addition, if $\ell\in(0,1/2)$, the asymptotic normality of $\hat{v}^2(\alpha,\gamma)$ follows by $p+mq=o(N^{1/2})$ and an application of the central limit theorem to $\boldsymbol{\epsilon}'\boldsymbol{\epsilon}/N$ in (B.33).

Next, we prove (3.8), for $k \in \gamma \cap \gamma_0$, using (B.2). By (B.31) and Lemma 6 (i)–(iii), we have, for $k \in \gamma \cap \gamma_0$,

$$\begin{aligned}
&\theta_k \boldsymbol{h}'_{i,k} \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta})(\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0) + \boldsymbol{\epsilon}) \\
&\quad = \theta_k \boldsymbol{h}'_{i,k} \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}) \bigg(\sum_{i^*=1}^{m} \sum_{k^* \in \gamma_0} b_{i^*,k^*} \boldsymbol{h}_{i^*,k^*} + \boldsymbol{\epsilon} \bigg) \\
&\quad = o_p\bigg(n_i^{(\xi-\ell)/2} \bigg(\frac{\sum_{i=1}^m n_i^\ell}{\sum_{i=1}^m n_i^\xi} \bigg)^{1/2} \bigg) + o_p(n_i^{-\ell/2}) \\
&\quad = o_p\bigg(n_i^{(\xi-\ell)/2} \bigg(\frac{\sum_{i=1}^m n_i^\ell}{\sum_{i=1}^m n_i^\xi} \bigg)^{1/2} \bigg) + o_p(1)
\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in \Theta_\gamma$. This and (B.4) imply that for $k \in \gamma \cap \gamma_0$,

$$\begin{aligned}
&\theta_k \boldsymbol{h}'_{i,k} \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}))\boldsymbol{y} \\
&= \theta_k \boldsymbol{h}'_{i,k} \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}))(\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0) + \boldsymbol{\epsilon}) \\
&= \theta_k \boldsymbol{h}'_{i,k} \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta})(\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0) + \boldsymbol{\epsilon}) + o_p\bigg(n_i^{(\xi-\ell)/2} \bigg(\frac{\sum_{i=1}^m n_i^\ell}{\sum_{i=1}^m n_i^\xi} \bigg)^{1/2} \bigg) + o_p(1) \\
&= \theta_k \boldsymbol{z}'_{i,k} \boldsymbol{H}_i^{-1}(\gamma, \boldsymbol{\theta}) \bigg(\sum_{k^* \in \gamma_0} \boldsymbol{z}_{i,k^*} b_{i,k^*} + \boldsymbol{\epsilon}_i \bigg) \\
&\quad + o_p\bigg(n_i^{(\xi-\ell)/2} \bigg(\frac{\sum_{i=1}^m n_i^\ell}{\sum_{i=1}^m n_i^\xi} \bigg)^{1/2} \bigg) + o_p(1) \\
&= b_{i,k} + o_p\bigg(n_i^{(\xi-\ell)/2} \bigg(\frac{\sum_{i=1}^m n_i^\ell}{\sum_{i=1}^m n_i^\xi} \bigg)^{1/2} \bigg) + o_p(1)
\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in \Theta_\gamma$, where the last equality follows from Lemma 3 (ii)–(iv) and Lemma 4 (i). Hence, for $k \in \gamma \cap \gamma_0$,

$$\begin{aligned}
&\theta_k^2 \{\boldsymbol{h}'_{i,k} \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}))\boldsymbol{y}\}^2 \\
&\quad = b_{i,k}^2 + o_p\bigg(n_i^{\xi-\ell} \bigg(\frac{\sum_{i=1}^m n_i^\ell}{\sum_{i=1}^m n_i^\xi} \bigg) \bigg) + o_p(1)
\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in \Theta_\gamma$. Hence by Lemma 3 (ii) and (B.2), we have, for $k \in \gamma \cap \gamma_0$,

$$\begin{aligned}
&\theta_k^2 \bigg\{ \frac{\partial}{\partial \theta_k} \{-2 \log L(\boldsymbol{\theta}, v^2; \alpha, \gamma)\} \bigg\} \\
&\quad = m\bigg(\theta_k - \frac{1}{m} \sum_{i=1}^m \frac{b_{i,k}^2}{v^2} \bigg) + o_p\bigg(\sum_{i=1}^m n_i^{\xi-\ell} \bigg(\frac{\sum_{i=1}^m n_i^\ell}{\sum_{i=1}^m n_i^\xi} \bigg) \bigg) + o_p(m)
\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in \Theta_\gamma$. Hence we have, for $k \in \gamma \cap \gamma_0$,

$$\hat{\theta}_k(\alpha, \gamma) = \frac{1}{m} \sum_{i=1}^m \frac{b_{i,k}^2}{\hat{v}^2(\alpha, \gamma)} + o_p\bigg(\frac{1}{m} \sum_{i=1}^m n_i^{\xi-\ell} \bigg(\frac{\sum_{i=1}^m n_i^\ell}{\sum_{i=1}^m n_i^\xi} \bigg) \bigg) + o_p(1).$$

This completes the proof of (3.8), for $k \in \gamma \cap \gamma_0$.

It remains to prove (3.8), for $k \in \gamma \setminus \gamma_0$. Let $\boldsymbol{\theta}^\dagger$ be $\boldsymbol{\theta}$ except that $\{\theta_k : k \in \gamma \cap \gamma_0\}$ are replaced by $\{\hat{\theta}_k(\alpha,\gamma) : k \in \gamma \cap \gamma_0\}$. By (B.31) and Lemma 6 (i)–(iii), we have, for $k \in \gamma \setminus \gamma_0$,

$$\begin{aligned}
&\theta_k \boldsymbol{h}'_{i,k}\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}^\dagger)\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}^\dagger)(\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)+\boldsymbol{\epsilon})\\
&= \theta_k \boldsymbol{h}'_{i,k}\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}^\dagger)\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}^\dagger)\bigg(\sum_{i^*=1}^{m}\sum_{k^*\in\gamma_0} b_{i^*,k^*}\boldsymbol{h}_{i^*,k^*}+\boldsymbol{\epsilon}\bigg)\\
&= o_p\bigg(n_i^{(\xi-\ell)/2}\bigg(\frac{\sum_{i=1}^m n_i^\ell}{\sum_{i=1}^m n_i^\xi}\bigg)^{1/2}\bigg)+o_p(1)
\end{aligned}$$

uniformly over $\boldsymbol{\theta}(\gamma\setminus\gamma_0) \in [0,\infty)^{q(\gamma\setminus\gamma_0)}$. This and (B.4) imply that for $k \in \gamma \setminus \gamma_0$,

$$\begin{aligned}
&\theta_k \boldsymbol{h}'_{i,k}\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}^\dagger)(\boldsymbol{I}_N-\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}^\dagger))\boldsymbol{y}\\
&= \theta_k \boldsymbol{h}'_{i,k}\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}^\dagger)(\boldsymbol{I}_N-\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}^\dagger))(\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)+\boldsymbol{\epsilon})\\
&= \theta_k \boldsymbol{h}'_{i,k}\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}^\dagger)(\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)+\boldsymbol{\epsilon})+o_p\bigg(n_i^{(\xi-\ell)/2}\bigg(\frac{\sum_{i=1}^m n_i^\ell}{\sum_{i=1}^m n_i^\xi}\bigg)^{1/2}\bigg)+o_p(1)\\
&= \theta_k \boldsymbol{z}'_{i,k}\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta}^\dagger)\bigg(\sum_{k^*\in\gamma_0}\boldsymbol{z}_{i,k^*}b_{i,k^*}+\boldsymbol{\epsilon}_i\bigg)\\
&\quad + o_p\bigg(n_i^{(\xi-\ell)/2}\bigg(\frac{\sum_{i=1}^m n_i^\ell}{\sum_{i=1}^m n_i^\xi}\bigg)^{1/2}\bigg)+o_p(1)\\
&= o_p\bigg(n_i^{(\xi-\ell)/2}\bigg(\frac{\sum_{i=1}^m n_i^\ell}{\sum_{i=1}^m n_i^\xi}\bigg)^{1/2}\bigg)+o_p(1)
\end{aligned}$$

uniformly over $\boldsymbol{\theta}(\gamma\setminus\gamma_0) \in [0,\infty)^{q(\gamma\setminus\gamma_0)}$, where the last equality follows from Lemma 3 (iii)–(iv) and Lemma 4 (i). Therefore,

$$\theta_k^2\{\boldsymbol{h}'_{i,k}\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}^\dagger)(\boldsymbol{I}_N-\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}^\dagger))\boldsymbol{y}\}^2 = o_p\bigg(n_i^{\xi-\ell}\bigg(\frac{\sum_{i=1}^m n_i^\ell}{\sum_{i=1}^m n_i^\xi}\bigg)\bigg)+o_p(1)$$

uniformly over $\boldsymbol{\theta}(\gamma\setminus\gamma_0) \in [0,\infty)^{q(\gamma\setminus\gamma_0)}$. Hence by Lemma 3 (ii) and (B.2), we have, for $k \in \gamma \setminus \gamma_0$,

$$\theta_k^2\bigg\{\frac{\partial}{\partial\theta_k}\{-2\log L(\boldsymbol{\theta}^\dagger,v^2;\alpha,\gamma)\}\bigg\} = m\theta_k + o_p\bigg(\sum_{i=1}^m n_i^{\xi-\ell}\bigg(\frac{\sum_{i=1}^m n_i^\ell}{\sum_{i=1}^m n_i^\xi}\bigg)\bigg)+o_p(m)$$

uniformly over $\boldsymbol{\theta}(\gamma\setminus\gamma_0) \in [0,\infty)^{q(\gamma\setminus\gamma_0)}$. This implies that, for $k \in \gamma \setminus \gamma_0$,

$$\hat{\theta}_k(\alpha,\gamma) = o_p\bigg(\frac{1}{m}\sum_{i=1}^m n_i^{\xi-\ell}\bigg(\frac{\sum_{i=1}^m n_i^\ell}{\sum_{i=1}^m n_i^\xi}\bigg)\bigg)+o_p(1).$$

This completes the proof of (3.8). Hence the proof of Theorem 3 is complete.

B.5 Proof of Theorem 4

As with the proof of Theorem 1, we shall focus on the asymptotic properties of $\hat{v}^2(\alpha,\gamma)$ and $\{\hat{\theta}_k(\alpha,\gamma) : k \in \gamma\}$, and derive them by solving the likelihood equations directly.

We first prove (3.9) using (B.1). By Lemma 6 (i), Lemma 6 (iii)–(v), Lemma 6 (vii), and Lemma 6 (x), we have

$$
\begin{aligned}
&\big(\boldsymbol{X}(\alpha_0\setminus\alpha)\boldsymbol{\beta}_0(\alpha_0\setminus\alpha)+\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)+\boldsymbol{\epsilon}\big)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\\
&\quad\times\big(\boldsymbol{X}(\alpha_0\setminus\alpha)\boldsymbol{\beta}_0(\alpha_0\setminus\alpha)+\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)+\boldsymbol{\epsilon}\big)\\
&=\bigg(\boldsymbol{X}(\alpha_0\setminus\alpha)\boldsymbol{\beta}_0(\alpha_0\setminus\alpha)+\sum_{i=1}^m\sum_{k\in\gamma_0}b_{i,k}\boldsymbol{h}_{i,k}+\boldsymbol{\epsilon}\bigg)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\\
&\quad\times\bigg(\boldsymbol{X}(\alpha_0\setminus\alpha)\boldsymbol{\beta}_0(\alpha_0\setminus\alpha)+\sum_{i=1}^m\sum_{k\in\gamma_0}b_{i,k}\boldsymbol{h}_{i,k}+\boldsymbol{\epsilon}\bigg)\\
&=o\bigg(\sum_{i=1}^n n_i^{\xi}\bigg)+o_p\bigg(\sum_{k,k^*\in\gamma_0}\frac{m}{\theta_k\theta_{k^*}}\bigg)+o_p\bigg(\sum_{k\in\gamma_0}\frac{m}{\theta_k}\bigg)+O_p(p)
\end{aligned}
$$

uniformly over $\boldsymbol{\theta}\in\Theta_\gamma$. This and (B.30) imply

$$
\begin{aligned}
&\boldsymbol{y}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{I}_N-\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}))\boldsymbol{y}\\
&\quad=\big(\boldsymbol{X}(\alpha_0\setminus\alpha)\boldsymbol{\beta}_0(\alpha_0\setminus\alpha)+\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)+\boldsymbol{\epsilon}\big)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\\
&\qquad\times(\boldsymbol{I}_N-\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}))\big(\boldsymbol{X}(\alpha_0\setminus\alpha)\boldsymbol{\beta}_0(\alpha_0\setminus\alpha)+\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)+\boldsymbol{\epsilon}\big)\\
&=\big(\boldsymbol{X}(\alpha_0\setminus\alpha)\boldsymbol{\beta}_0(\alpha_0\setminus\alpha)+\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)+\boldsymbol{\epsilon}\big)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\\
&\quad\times\big(\boldsymbol{X}(\alpha_0\setminus\alpha)\boldsymbol{\beta}_0(\alpha_0\setminus\alpha)+\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)+\boldsymbol{\epsilon}\big)\\
&\quad+o\bigg(\sum_{i=1}^n n_i^{\xi}\bigg)+o_p\bigg(\sum_{k,k^*\in\gamma_0}\frac{m}{\theta_k\theta_{k^*}}\bigg)+o_p\bigg(\sum_{k\in\gamma_0}\frac{m}{\theta_k}\bigg)+O_p(p)\\
&=\sum_{i=1}^m\boldsymbol{\beta}_0(\alpha_0\setminus\alpha)'\boldsymbol{X}_i(\alpha_0\setminus\alpha)'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}_i(\alpha_0\setminus\alpha)\boldsymbol{\beta}_0(\alpha_0\setminus\alpha)\\
&\quad+2\sum_{i=1}^m\boldsymbol{\beta}_0(\alpha_0\setminus\alpha)'\boldsymbol{X}_i(\alpha_0\setminus\alpha)'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{Z}_i(\gamma_0)\boldsymbol{b}_i(\gamma_0)+\boldsymbol{\epsilon}_i)\\
&\quad+\sum_{i=1}^m(\boldsymbol{Z}_i(\gamma_0)\boldsymbol{b}_i(\gamma_0)+\boldsymbol{\epsilon}_i)'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{Z}_i(\gamma_0)\boldsymbol{b}_i(\gamma_0)+\boldsymbol{\epsilon}_i)\\
&\quad+o\bigg(\sum_{i=1}^n n_i^{\xi}\bigg)+o_p\bigg(\sum_{k,k^*\in\gamma_0}\frac{m}{\theta_k\theta_{k^*}}\bigg)+o_p\bigg(\sum_{k\in\gamma_0}\frac{m}{\theta_k}\bigg)+O_p(p)\\
&=\sum_{i=1}^m\boldsymbol{\epsilon}_i'\boldsymbol{\epsilon}_i+\sum_{i=1}^m\sum_{j\in\alpha_0\setminus\alpha}\beta_{j,0}^2d_{i,j}n_i^{\xi}+o_p\bigg(\sum_{i=1}^m n_i^{\xi}\bigg)+o_p\bigg(\sum_{k,k^*\in\gamma_0}\frac{m}{\theta_k\theta_{k^*}}\bigg)\\
&\quad+O_p\bigg(\sum_{k\in\gamma_0}\frac{m}{\theta_k}\bigg)+O_p(p+mq)
\end{aligned}
$$

uniformly over $\boldsymbol{\theta}\in\Theta_\gamma$, where the last equality follows from Lemma 3 (ii)–(iv) and Lemma 4. Hence by (B.1), we have, for $v^2\in(0,\infty)$,

$$
\begin{aligned}
&v^4\bigg\{\frac{\partial}{\partial v^2}\{-2\log L(\boldsymbol{\theta},v^2;\alpha,\gamma)\}\bigg\}\\
&\quad=N\bigg(v^2-\frac{\boldsymbol{\epsilon}'\boldsymbol{\epsilon}}{N}+\frac{1}{N}\sum_{i=1}^m\sum_{j\in\alpha_0\setminus\alpha}\beta_{j,0}^2d_{i,j}n_i^{\xi}\bigg)+o_p\bigg(\sum_{i=1}^m n_i^{\xi}\bigg)\\
&\qquad+o_p\bigg(\sum_{k,k^*\in\gamma_0}\frac{m}{\theta_k\theta_{k^*}}\bigg)+O_p\bigg(\sum_{k\in\gamma_0}\frac{m}{\theta_k}\bigg)+O_p(p+mq)
\end{aligned}
$$

uniformly over $\boldsymbol{\theta} \in \Theta_\gamma$. This and Lemma 5 imply that for $(\xi, \ell) \in (0,1] \times (0,1]$,

$$\begin{aligned} \hat{v}^2(\alpha,\gamma) &= \frac{\boldsymbol{\epsilon}'\boldsymbol{\epsilon}}{N} + \frac{1}{N}\sum_{i=1}^m \sum_{j\in\alpha_0\setminus\alpha} \beta_{j,0}^2 d_{i,j} n_i^{\xi} \\ &\quad + o_p\bigg(\frac{1}{N}\sum_{i=1}^m n_i^{\xi}\bigg) + O_p\bigg(\frac{p+mq}{N}\bigg). \end{aligned} \tag{B.34}$$

Thus (3.9) follows by applying the law of large numbers to $\boldsymbol{\epsilon}'\boldsymbol{\epsilon}/N$. In addition, if $\xi \in (0, 1/2)$, the asymptotic normality of $\hat{v}^2(\alpha,\gamma)$ follows by $p + mq = o(N^{1/2})$ and an application of the central limit theorem to $\boldsymbol{\epsilon}'\boldsymbol{\epsilon}/N$ in (B.34).

Next, we prove (3.10), for $k \in \gamma \cap \gamma_0$, using (B.2). By Lemma 6 (i) and Lemma 6 (iii)–(iv), we have, for $k \in \gamma \cap \gamma_0$,

$$\begin{aligned} &\theta_k \boldsymbol{h}_{i,k}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\big(\boldsymbol{X}(\alpha_0\setminus\alpha)\boldsymbol{\beta}_0(\alpha_0\setminus\alpha) + \boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0) + \boldsymbol{\epsilon}\big) \\ &= \theta_k \boldsymbol{h}_{i,k}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}) \\ &\quad \times \bigg(\boldsymbol{X}(\alpha_0\setminus\alpha)\boldsymbol{\beta}_0(\alpha_0\setminus\alpha) + \sum_{i^*=1}^m \sum_{k^*\in\gamma_0} b_{i^*,k^*}\boldsymbol{h}_{i^*,k^*} + \boldsymbol{\epsilon}\bigg) \\ &= o_p\bigg(\sum_{k^*\in\gamma_0}\frac{n_i^{(\xi-\ell)/2}}{\theta_{k^*}}\bigg) + o_p(n_i^{(\xi-\ell)/2-\tau}) + o_p(n_i^{-\ell/2}) \\ &= o_p\bigg(\sum_{k^*\in\gamma_0}\frac{n_i^{(\xi-\ell)/2}}{\theta_{k^*}}\bigg) + o_p(1) \end{aligned}$$

uniformly over $\boldsymbol{\theta} \in \Theta_\gamma$. This and (B.30) imply that for $k \in \gamma \cap \gamma_0$,

$$\begin{aligned} &\theta_k \boldsymbol{h}_{i,k}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}))\boldsymbol{y} \\ &\quad = \theta_k \boldsymbol{h}_{i,k}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})) \\ &\qquad \times \big(\boldsymbol{X}(\alpha_0\setminus\alpha)\boldsymbol{\beta}_0(\alpha_0\setminus\alpha) + \boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0) + \boldsymbol{\epsilon}\big) \\ &\quad = \theta_k \boldsymbol{h}_{i,k}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\big(\boldsymbol{X}(\alpha_0\setminus\alpha)\boldsymbol{\beta}_0(\alpha_0\setminus\alpha) + \boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0) + \boldsymbol{\epsilon}\big) \\ &\qquad + o_p\bigg(\sum_{k^*\in\gamma_0}\frac{n_i^{(\xi-\ell)/2}}{\theta_{k^*}}\bigg) + o_p(1) \\ &\quad = \theta_k \boldsymbol{z}_{i,k}'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\bigg(\boldsymbol{X}_i(\alpha_0\setminus\alpha)\boldsymbol{\beta}_0(\alpha_0\setminus\alpha) + \sum_{k^*\in\gamma_0}\boldsymbol{z}_{i,k^*}b_{i,k^*} + \boldsymbol{\epsilon}_i\bigg) \\ &\qquad + o_p\bigg(\sum_{k^*\in\gamma_0}\frac{n_i^{(\xi-\ell)/2}}{\theta_{k^*}}\bigg) + o_p(1) \\ &\quad = b_{i,k} + o_p\bigg(\sum_{k^*\in\gamma_0}\frac{n_i^{(\xi-\ell)/2}}{\theta_{k^*}}\bigg) + o_p(1) \end{aligned}$$

uniformly over $\boldsymbol{\theta} \in \Theta_\gamma$, where the last equality follows from Lemma 2 (iii), Lemma 3 (ii)–(iii), and Lemma 4 (i). Hence, for $k \in \gamma \cap \gamma_0$,

$$\theta_k^2\{\boldsymbol{h}_{i,k}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}))\boldsymbol{y}\}^2 = b_{i,k}^2 + o_p\bigg(\sum_{k,k^*\in\gamma_0}\frac{n_i^{\xi-\ell}}{\theta_k\theta_{k^*}}\bigg) + o_p(1)$$

uniformly over $\boldsymbol{\theta} \in \Theta_\gamma$. Hence by Lemma 3 (ii) and (B.2), we have, for $k \in \gamma \cap \gamma_0$,

$$\begin{aligned}
&\theta_k^2 \left\{ \frac{\partial}{\partial \theta_k} \{-2 \log L(\boldsymbol{\theta}, v^2; \alpha, \gamma)\} \right\} \\
&= m \left(\theta_k - \frac{1}{m} \sum_{i=1}^m \frac{b_{i,k}^2}{v^2} \right) + o_p \left(\sum_{i=1}^m \sum_{k,k^* \in \gamma_0} \frac{n_i^{\xi-\ell}}{\theta_k \theta_{k^*}} \right) + o_p(1)
\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in \Theta_\gamma$. This and Lemma 5 imply that for $k \in \gamma \cap \gamma_0$,

$$\hat{\theta}_k(\alpha, \gamma) = \frac{1}{m} \sum_{i=1}^m \frac{b_{i,k}^2}{\hat{v}^2(\alpha, \gamma)} + o_p \left(\frac{1}{m} \sum_{i=1}^m n_i^{\xi-\ell} \right) + o_p(1).$$

This completes the proof of (3.10) when $k \in \gamma \cap \gamma_0$.

It remains to prove (3.10), for $k \in \gamma \setminus \gamma_0$. Let $\boldsymbol{\theta}^\dagger$ be $\boldsymbol{\theta}$ except that $\{\theta_k : k \in \gamma \cap \gamma_0\}$ are replaced by $\{\hat{\theta}_k(\alpha, \gamma) : k \in \gamma \cap \gamma_0\}$. By Lemma 6 (i) and Lemma 6 (iii)–(iv), we have, for $k \in \gamma \setminus \gamma_0$,

$$\begin{aligned}
&\theta_k \boldsymbol{h}_{i,k}' \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}^\dagger) \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}^\dagger) \big(\boldsymbol{X}(\alpha_0 \setminus \alpha) \boldsymbol{\beta}_0(\alpha_0 \setminus \alpha) + \boldsymbol{Z}(\gamma_0) \boldsymbol{b}(\gamma_0) + \boldsymbol{\epsilon} \big) \\
&= \theta_k \boldsymbol{h}_{i,k}' \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}^\dagger) \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}^\dagger) \\
&\quad \times \left(\boldsymbol{X}(\alpha_0 \setminus \alpha) \boldsymbol{\beta}_0(\alpha_0 \setminus \alpha) + \sum_{i^*=1}^m \sum_{k^* \in \gamma_0} b_{i^*,k^*} \boldsymbol{h}_{i^*,k^*} + \boldsymbol{\epsilon} \right) \\
&= o_p(n_i^{(\xi-\ell)/2-\tau}) + o_p(n_i^{-\ell/2}) \\
&= o_p(n_i^{(\xi-\ell)/2}) + o_p(1)
\end{aligned}$$

uniformly over $\boldsymbol{\theta}(\gamma \setminus \gamma_0) \in [0, \infty)^{q(\gamma \setminus \gamma_0)}$. This and (B.30) imply that for $k \in \gamma \setminus \gamma_0$,

$$\begin{aligned}
&\theta_k \boldsymbol{h}_{i,k}' \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}^\dagger)(\boldsymbol{I}_N - \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}^\dagger)) \boldsymbol{y} \\
&= \theta_k \boldsymbol{h}_{i,k}' \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}^\dagger)(\boldsymbol{I}_N - \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}^\dagger)) \\
&\quad \times \big(\boldsymbol{X}(\alpha_0 \setminus \alpha) \boldsymbol{\beta}_0(\alpha_0 \setminus \alpha) + \boldsymbol{Z}(\gamma_0) \boldsymbol{b}(\gamma_0) + \boldsymbol{\epsilon} \big) \\
&= \theta_k \boldsymbol{h}_{i,k}' \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}^\dagger) \big(\boldsymbol{X}(\alpha_0 \setminus \alpha) \boldsymbol{\beta}_0(\alpha_0 \setminus \alpha) + \boldsymbol{Z}(\gamma_0) \boldsymbol{b}(\gamma_0) + \boldsymbol{\epsilon} \big) \\
&\quad + o_p(n_i^{(\xi-\ell)/2}) + o_p(1) \\
&= \theta_k \boldsymbol{z}_{i,k}' \boldsymbol{H}_i^{-1}(\gamma, \boldsymbol{\theta}^\dagger) \left(\boldsymbol{X}_i(\alpha_0 \setminus \alpha) \boldsymbol{\beta}_0(\alpha_0 \setminus \alpha) + \sum_{k^* \in \gamma_0} \boldsymbol{z}_{i,k^*} b_{i,k^*} + \boldsymbol{\epsilon}_i \right) \\
&\quad + o_p(n_i^{(\xi-\ell)/2}) + o_p(1) \\
&= o_p(n_i^{(\xi-\ell)/2}) + o_p(1)
\end{aligned}$$

uniformly over $\boldsymbol{\theta}(\gamma \setminus \gamma_0) \in [0, \infty)^{q(\gamma \setminus \gamma_0)}$, where the last equality follows from Lemma 2 (iii), Lemma 3 (iii), and Lemma 4 (i). Therefore,

$$\theta_k^2 \{ \boldsymbol{h}_{i,k}' \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}^\dagger)(\boldsymbol{I}_N - \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}^\dagger)) \boldsymbol{y} \}^2 = o_p(n_i^{\xi-\ell}) + o_p(1)$$

uniformly over $\boldsymbol{\theta}(\gamma \setminus \gamma_0) \in [0, \infty)^{q(\gamma \setminus \gamma_0)}$. Hence by Lemma 3 (ii) and (B.2), we have, for $k \in \gamma \setminus \gamma_0$,

$$\theta_k^2 \left\{ \frac{\partial}{\partial \theta_k} \{-2 \log L(\boldsymbol{\theta}^\dagger, v^2; \alpha, \gamma)\} \right\} = m \theta_k + o_p \left(\sum_{i=1}^m n_i^{\xi-\ell} \right) + o_p(m)$$

uniformly over $\boldsymbol{\theta}(\gamma \setminus \gamma_0) \in [0,\infty)^{q(\gamma\setminus\gamma_0)}$. This and Lemma 5 imply that for $k \in \gamma \setminus \gamma_0$,

$$\hat{\theta}_k(\alpha,\gamma) = o_p\left(\frac{1}{m}\sum_{i=1}^{m} n_i^{\xi-\ell}\right) + o_p(1).$$

This completes the proof of (3.10), for $k \in \gamma \setminus \gamma_0$. Hence the proof of Theorem 4 is complete.

C Proofs of Auxiliary Lemmas

C.1 Proof of Lemma 2

Let $\boldsymbol{z}_{i,(s)}$; $s = 1,\dots,q(\gamma)$ be the s-th column of $\boldsymbol{Z}_i(\gamma)$ and $\boldsymbol{H}_{i,t}(\gamma,\boldsymbol{\theta})$ defined in (A.4). For Lemma 2 (i)–(ii) to hold, it suffices to prove that for $k \notin \gamma$ and $j, j^* = 1,\dots,p$,

$$\boldsymbol{x}_{i,j}'\boldsymbol{H}_{i,t}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{x}_{i,j} = d_{i,j}n_i^{\xi} + o(n_i^{\xi}) + o(tn_i^{\xi-2\tau}), \tag{C.1}$$

$$\boldsymbol{x}_{i,j}'\boldsymbol{H}_{i,t}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{x}_{i,j^*} = o(n_i^{\xi-\tau}) + o(tn_i^{\xi-2\tau}), \tag{C.2}$$

$$\boldsymbol{x}_{i,j}'\boldsymbol{H}_{i,t}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,k} = o(n_i^{(\xi+\ell)/2-\tau}) + o(tn_i^{(\xi+\ell)/2-2\tau}) \tag{C.3}$$

uniformly over $\boldsymbol{\theta} \in [0,\infty)^{q(\gamma)}$. We prove (C.1)–(C.3) by induction. For $j = 1,\dots,p$ and $t = 1$, by (A.2) and (A1)–(A3), we have

$$\begin{aligned}\boldsymbol{x}_{i,j}'\boldsymbol{H}_{i,1}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{x}_{i,j} &= \boldsymbol{x}_{i,j}'\boldsymbol{x}_{i,j} - \frac{\theta_{(1)}\boldsymbol{x}_{i,j}'\boldsymbol{z}_{i,(1)}\boldsymbol{z}_{i,(1)}'\boldsymbol{x}_{i,j}}{1+\theta_{(1)}\boldsymbol{z}_{i,(1)}'\boldsymbol{z}_{i,(1)}} \\ &= d_{i,j}n_i^{\xi} + o(n_i^{\xi}) + o(n_i^{\xi-2\tau})\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in [0,\infty)^{q(\gamma)}$. For $j, j^* = 1,\dots,p$, $j \neq j^*$ and $t = 1$, by (A.2) and (A1)–(A3), we have

$$\begin{aligned}\boldsymbol{x}_{i,j}'\boldsymbol{H}_{i,1}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{x}_{i,j^*} &= \boldsymbol{x}_{i,j}'\boldsymbol{x}_{i,j^*} - \frac{\theta_{(1)}\boldsymbol{x}_{i,j}'\boldsymbol{z}_{i,(1)}\boldsymbol{z}_{i,(1)}'\boldsymbol{x}_{i,j^*}}{1+\theta_{(1)}\boldsymbol{z}_{i,(1)}'\boldsymbol{z}_{i,(1)}} \\ &= o(n_i^{\xi-\tau}) + o(n_i^{\xi-2\tau})\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in [0,\infty)^{q(\gamma)}$. For $j = 1,\dots,p$, $k \notin \gamma$ and $t = 1$, by (A.2) and (A1)–(A3), we have

$$\begin{aligned}\boldsymbol{x}_{i,j}'\boldsymbol{H}_{i,1}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,k} &= \boldsymbol{x}_{i,j}'\boldsymbol{z}_{i,k} - \frac{\theta_{(1)}\boldsymbol{x}_{i,j}'\boldsymbol{z}_{i,(1)}\boldsymbol{z}_{i,(1)}'\boldsymbol{z}_{i,k}}{1+\theta_{(1)}\boldsymbol{z}_{i,(1)}'\boldsymbol{z}_{i,(1)}} \\ &= o(n_i^{(\xi+\ell)/2-\tau}) + o(n_i^{(\xi+\ell)/2-2\tau})\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in [0,\infty)^{q(\gamma)}$. Now suppose that (C.1)–(C.3) hold for $t = r$. Then for $j = 1,\dots,p$ and $t = r+1$, by (A.2) and (C.1)–(C.3) with $t = r$, and Lemma 3 (i), we have

$$\begin{aligned}&\boldsymbol{x}_{i,j}'\boldsymbol{H}_{i,r+1}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{x}_{i,j} \\ &= \boldsymbol{x}_{i,j}'\boldsymbol{H}_{i,r}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{x}_{i,j} - \frac{\theta_{(r+1)}\boldsymbol{x}_{i,j}'\boldsymbol{H}_{i,r}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(r+1)}\boldsymbol{z}_{i,(r+1)}'\boldsymbol{H}_{i,r}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{x}_{i,j}}{1+\theta_{(r+1)}\boldsymbol{z}_{i,(r+1)}'\boldsymbol{H}_{i,r}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(r+1)}} \\ &= d_{i,j}n_i^{\xi} + o(n_i^{\xi}) + o(\{r+1\}n_i^{\xi-2\tau})\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in [0,\infty)^{q(\gamma)}$. For $j, j^* = 1,\ldots,p$, $j \neq j^*$, and $t = r+1$, by (A.2) and (C.1)–(C.3) with $t = r$, and Lemma 3 (i), we have

$$\begin{aligned}
&\boldsymbol{x}'_{i,j}\boldsymbol{H}^{-1}_{i,r+1}(\gamma,\boldsymbol{\theta})\boldsymbol{x}_{i,j^*} \\
&\quad = \boldsymbol{x}'_{i,j}\boldsymbol{H}^{-1}_{i,r}(\gamma,\boldsymbol{\theta})\boldsymbol{x}_{i,j^*} - \frac{\theta_{(r+1)}\boldsymbol{x}'_{i,j}\boldsymbol{H}^{-1}_{i,r}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(r+1)}\boldsymbol{z}'_{i,(r+1)}\boldsymbol{H}^{-1}_{i,r}(\gamma,\boldsymbol{\theta})\boldsymbol{x}_{i,j^*}}{1+\theta_{(r+1)}\boldsymbol{z}'_{i,(r+1)}\boldsymbol{H}^{-1}_{i,r}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(r+1)}} \\
&\quad = o(n_i^{\xi-\tau}) + o(\{r+1\}n_i^{\xi-2\tau})
\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in [0,\infty)^{q(\gamma)}$. For $j, j^* = 1,\ldots,p$, $k \notin \gamma$, and $t = r+1$, by (A.2) and (C.1)–(C.3) with $t = r$, and Lemma 3 (i), we have

$$\begin{aligned}
&\boldsymbol{x}'_{i,j}\boldsymbol{H}^{-1}_{i,r+1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,k} \\
&\quad = \boldsymbol{x}'_{i,j}\boldsymbol{H}^{-1}_{i,r}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,k} - \frac{\theta_{(r+1)}\boldsymbol{x}'_{i,j}\boldsymbol{H}^{-1}_{i,r}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(r+1)}\boldsymbol{z}'_{i,(r+1)}\boldsymbol{H}^{-1}_{i,r}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,k}}{1+\theta_{(r+1)}\boldsymbol{z}'_{i,(r+1)}\boldsymbol{H}^{-1}_{i,r}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(r+1)}} \\
&\quad = o(n_i^{(\xi+\ell)/2-\tau}) + o(\{r+1\}n_i^{(\xi+\ell)/2-2\tau})
\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in [0,\infty)^{q(\gamma)}$. This completes the proofs of (C.1)–(C.3). Hence the proofs of Lemma 2 (i)–(ii) are complete.

We finally prove Lemma 2 (iii). Without loss of generality, we assume that $q(\gamma) = q$, $t = q$, and $k = (q)$. Then by (A.2),

$$\begin{aligned}
\theta_{(q)}\boldsymbol{x}'_{i,j}\boldsymbol{H}^{-1}_{i,q}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q)} &= \theta_{(q)}\Bigg\{\boldsymbol{x}'_{i,j}\boldsymbol{H}^{-1}_{i,q-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q)} \\
&\qquad - \frac{\theta_{(q)}\boldsymbol{x}'_{i,j}\boldsymbol{H}^{-1}_{i,q-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q)}\boldsymbol{z}'_{i,(q)}\boldsymbol{H}^{-1}_{i,q-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,q}}{1+\theta_{(q)}\boldsymbol{z}'_{i,(q)}\boldsymbol{H}^{-1}_{i,q-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q)}}\Bigg\} \\
&= \frac{\theta_{(q)}\boldsymbol{x}'_{i,j}\boldsymbol{H}^{-1}_{i,q-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q)}}{1+\theta_{(q)}\boldsymbol{z}'_{i,(q)}\boldsymbol{H}^{-1}_{i,q-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q)}},
\end{aligned}$$

where we note that $\theta_{(q)}$ can be arbitrarily small and the dominant term of the denominator of the last equation can be equal to (i) $\theta_{(q)}\boldsymbol{z}'_{i,(q)}\boldsymbol{H}^{-1}_{i,q-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q)}$ or (ii) 1. For the case of (i), $\theta_{(q)}n_i^{\ell} \to \infty$ by Lemma 3 (i); hence, using Lemma 2 (ii) and Lemma 3 (i), we have

$$\theta_{(q)}\boldsymbol{x}'_{i,j}\boldsymbol{H}^{-1}_{i,q}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q)} = o(n_i^{(\xi-\ell)/2-\tau}),$$

and thus

$$\boldsymbol{x}'_{i,j}\boldsymbol{H}^{-1}_{i,q}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q)} = o(n_i^{(\xi+\ell)/2-\tau}).$$

For the case of (ii), $\theta_{(q)} = O(n_i^{-\ell})$ by Lemma 3 (i); hence, using Lemma 3 (i), we have

$$\theta_{(q)}\boldsymbol{x}'_{i,j}\boldsymbol{H}^{-1}_{i,q}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q)} = o(\theta_{(q)}n_i^{(\xi+\ell)/2-\tau}),$$

which also gives the following two results:

$$\begin{aligned}
\boldsymbol{x}'_{i,j}\boldsymbol{H}^{-1}_{i,q}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q)} &= o(n_i^{(\xi+\ell)/2-\tau}), \\
\theta_{(q)}\boldsymbol{x}'_{i,j}\boldsymbol{H}^{-1}_{i,q}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q)} &= o(n_i^{(\xi-\ell)/2-\tau}).
\end{aligned}$$

In conclusion, we have

$$\begin{aligned}
\theta_{(q)}\boldsymbol{x}'_{i,j}\boldsymbol{H}^{-1}_{i,q}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q)} &= o(n_i^{(\xi-\ell)/2-\tau}), \\
\boldsymbol{x}'_{i,j}\boldsymbol{H}^{-1}_{i,q}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q)} &= o(n_i^{(\xi+\ell)/2-\tau})
\end{aligned} \tag{C.4}$$

uniformly over $\boldsymbol{\theta} \in [0,\infty)^{q}$. This completes the proof.

C.2 Proof of Lemma 3

Let $\boldsymbol{z}_{i,(s)}$; $s = 1, \ldots, q(\gamma)$ be the s-th column of $\boldsymbol{Z}_i(\gamma)$ and $\boldsymbol{H}_{i,t}(\gamma, \boldsymbol{\theta})$ defined in (A.4). We first prove Lemma 3 (i). By (A.4), it suffices to prove that for $k \notin \gamma$,

$$\boldsymbol{z}'_{i,k}\boldsymbol{H}^{-1}_{i,t}(\gamma, \boldsymbol{\theta})\boldsymbol{z}_{i,k} = c_{i,k}n_i^{\ell} + o(n_i^{\ell}) + o(tn_i^{\ell-2\tau}), \tag{C.5}$$

and for $k, k^* \notin \gamma$ and $k \neq k^*$,

$$\boldsymbol{z}'_{i,k}\boldsymbol{H}^{-1}_{i,t}(\gamma, \boldsymbol{\theta})\boldsymbol{z}_{i,k^*} = o(n_i^{\ell-\tau}) + o(tn_i^{\ell-2\tau}) \tag{C.6}$$

uniformly over $\boldsymbol{\theta} \in [0, \infty)^{q(\gamma)}$ by induction. For $t = 1$ and $k \notin \gamma$, by (A.2) and (A2), we have

$$\begin{aligned}
\boldsymbol{z}'_{i,k}\boldsymbol{H}^{-1}_{i,1}(\gamma, \boldsymbol{\theta})\boldsymbol{z}_{i,k} &= \boldsymbol{z}'_{i,k}\left(\boldsymbol{I}_{n_i} - \frac{\theta_{(1)}\boldsymbol{z}_{i,(1)}\boldsymbol{z}'_{i,(1)}}{1 + \theta_{(1)}\boldsymbol{z}'_{i,(1)}\boldsymbol{z}_{i,(1)}}\right)\boldsymbol{z}_{i,k} \\
&= \boldsymbol{z}'_{i,k}\boldsymbol{z}_{i,k} - \frac{\theta_{(1)}\boldsymbol{z}'_{i,k}\boldsymbol{z}_{i,(1)}\boldsymbol{z}'_{i,(1)}\boldsymbol{z}_{i,k}}{1 + \theta_{(1)}\boldsymbol{z}'_{i,(1)}\boldsymbol{z}_{i,(1)}} \\
&= c_{i,k}n_i^{\ell} + o(n_i^{\ell}) + o(n_i^{\ell-2\tau})
\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in [0, \infty)^{q(\gamma)}$. For $k, k^* \notin \gamma$ and $k \neq k^*$, by (A.2) and (A2), we have

$$\begin{aligned}
\boldsymbol{z}'_{i,k}\boldsymbol{H}^{-1}_{i,1}(\gamma, \boldsymbol{\theta})\boldsymbol{z}_{i,k^*} &= \boldsymbol{z}'_{i,k}\boldsymbol{z}_{i,k^*} - \frac{\theta_{(1)}\boldsymbol{z}'_{i,k}\boldsymbol{z}_{i,(1)}\boldsymbol{z}'_{i,(1)}\boldsymbol{z}_{i,k^*}}{1 + \theta_{(1)}\boldsymbol{z}'_{i,(1)}\boldsymbol{z}_{i,(1)}} \\
&= o(n_i^{\ell-\tau}) + o(n_i^{\ell-2\tau})
\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in [0, \infty)^{q(\gamma)}$. Now suppose that (C.5) and (C.6) hold for $t = r$. Then for $k \notin \gamma$ and $t = r + 1$, by (A.2), and (C.5) and (C.6) with $t = r$, we have

$$\begin{aligned}
\boldsymbol{z}'_{i,k}\boldsymbol{H}^{-1}_{i,r+1}(\gamma, \boldsymbol{\theta})\boldsymbol{z}_{i,k} &= \boldsymbol{z}'_{i,k}\boldsymbol{H}^{-1}_{i,r}(\gamma, \boldsymbol{\theta})\boldsymbol{z}_{i,k} \\
&\quad - \frac{\theta_{(r+1)}\boldsymbol{z}'_{i,k}\boldsymbol{H}^{-1}_{i,r}(\gamma, \boldsymbol{\theta})\boldsymbol{z}_{i,(r+1)}\boldsymbol{z}'_{i,(r+1)}\boldsymbol{H}^{-1}_{i,r}(\gamma, \boldsymbol{\theta})\boldsymbol{z}_{i,k}}{1 + \theta_{(r+1)}\boldsymbol{z}'_{i,(r+1)}\boldsymbol{H}^{-1}_{i,r}(\gamma, \boldsymbol{\theta})\boldsymbol{z}_{i,(r+1)}} \\
&= c_{i,k}n_i^{\ell} + o(n_i^{\ell}) + o(\{r+1\}n_i^{\ell-2\tau})
\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in [0, \infty)^{q(\gamma)}$. For $k, k^* \notin \gamma$ and $t = r + 1$, by (A.2), and (C.5) and (C.6) with $t = r$, we have

$$\begin{aligned}
\boldsymbol{z}'_{i,k}\boldsymbol{H}^{-1}_{i,r+1}(\gamma, \boldsymbol{\theta})\boldsymbol{z}_{i,k^*} &= \boldsymbol{z}'_{i,k}\boldsymbol{H}^{-1}_{i,r}(\gamma, \boldsymbol{\theta})\boldsymbol{z}_{i,k^*} \\
&\quad - \frac{\theta_{(r+1)}\boldsymbol{z}'_{i,k}\boldsymbol{H}^{-1}_{i,r}(\gamma, \boldsymbol{\theta})\boldsymbol{z}_{i,(r+1)}\boldsymbol{z}'_{i,(r+1)}\boldsymbol{H}^{-1}_{i,r}(\gamma, \boldsymbol{\theta})\boldsymbol{z}_{i,k^*}}{1 + \theta_{(r+1)}\boldsymbol{z}'_{i,(r+1)}\boldsymbol{H}^{-1}_{i,r}(\gamma, \boldsymbol{\theta})\boldsymbol{z}_{i,(r+1)}} \\
&= o(n_i^{\ell-\tau}) + o(\{r+1\}n_i^{\ell-2\tau})
\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in [0, \infty)^{q(\gamma)}$. This completes the proof of (C.5) and (C.6). Hence Lemma 3 (i) follows from (C.5), (C.6) with $t = q(\gamma)$ and $q = o(n_{\min}^{\tau})$. This completes the proof of Lemma 3 (i).

We now prove Lemma 3 (ii). Without loss of generality, we assume that $q(\gamma) = q$ and $k = (q)$. Then by Lemma 3 (i) and (A.2),

$$\begin{aligned}\theta_{(q)}^2 \boldsymbol{z}_{i,(q)}' \boldsymbol{H}_{i,q}^{-1}(\gamma,\boldsymbol{\theta}) \boldsymbol{z}_{i,(q)} &= \theta_{(q)}^2 \left\{ \boldsymbol{z}_{i,(q)}' \boldsymbol{H}_{i,q-1}^{-1}(\gamma,\boldsymbol{\theta}) \boldsymbol{z}_{i,(q)} \right. \\ &\qquad \left. - \frac{\theta_{(q)} (\boldsymbol{z}_{i,(q)}' \boldsymbol{H}_{i,q-1}^{-1}(\gamma,\boldsymbol{\theta}) \boldsymbol{z}_{i,(q)})^2}{1 + \theta_{(q)} \boldsymbol{z}_{i,(q)}' \boldsymbol{H}_{i,q-1}^{-1}(\gamma,\boldsymbol{\theta}) \boldsymbol{z}_{i,(q)}} \right\} \\ &= \frac{\theta_{(q)}^2 \boldsymbol{z}_{i,(q)}' \boldsymbol{H}_{i,q-1}^{-1}(\gamma,\boldsymbol{\theta}) \boldsymbol{z}_{i,(q)}}{1 + \theta_{(q)} \boldsymbol{z}_{i,(q)}' \boldsymbol{H}_{i,q-1}^{-1}(\gamma,\boldsymbol{\theta}) \boldsymbol{z}_{i,(q)}} = O(\theta_{(q)}^2 n_i^{\ell})\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in [0,\infty)^q$. Again, by Lemma 3 (i), we have

$$\begin{aligned}\theta_{(q)}^2 \boldsymbol{z}_{i,(q)}' \boldsymbol{H}_{i,q}^{-1}(\gamma,\boldsymbol{\theta}) \boldsymbol{z}_{i,(q)} &= \frac{\theta_{(q)}^2 \boldsymbol{z}_{i,(q)}' \boldsymbol{H}_{i,q-1}^{-1}(\gamma,\boldsymbol{\theta}) \boldsymbol{z}_{i,(q)}}{1 + \theta_{(q)} \boldsymbol{z}_{i,(q)}' \boldsymbol{H}_{i,q-1}^{-1}(\gamma,\boldsymbol{\theta}) \boldsymbol{z}_{i,(q)}} \\ &= \theta_{(q)} - \frac{\theta_{(q)}}{1 + \theta_{(q)} \boldsymbol{z}_{i,(q)}' \boldsymbol{H}_{i,q-1}^{-1}(\gamma,\boldsymbol{\theta}) \boldsymbol{z}_{i,(q)}} \\ &= \theta_{(q)} + O(n_i^{-\ell})\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in [0,\infty)^q$. This completes the proof of Lemma 3 (ii).

We now prove Lemma 3 (iii). Without loss of generality, we assume that $q(\gamma) = q$, $k = (q)$, and $k^* = (q-1)$. Then by (A.2),

$$\begin{aligned}&\theta_{(q)}\theta_{(q-1)} \boldsymbol{z}_{i,(q)}' \boldsymbol{H}_{i,q}^{-1}(\gamma,\boldsymbol{\theta}) \boldsymbol{z}_{i,(q-1)} \\ &\quad = \theta_{(q)}\theta_{(q-1)} \left\{ \boldsymbol{z}_{i,(q)}' \boldsymbol{H}_{i,q-1}^{-1}(\gamma,\boldsymbol{\theta}) \boldsymbol{z}_{i,(q-1)} \right. \\ &\qquad \left. - \frac{\theta_{(q)} \boldsymbol{z}_{i,(q)}' \boldsymbol{H}_{i,q-1}^{-1}(\gamma,\boldsymbol{\theta}) \boldsymbol{z}_{i,(q)} \boldsymbol{z}_{i,(q)}' \boldsymbol{H}_{i,q-1}^{-1}(\gamma,\boldsymbol{\theta}) \boldsymbol{z}_{i,(q-1)}}{1 + \theta_{(q)} \boldsymbol{z}_{i,(q)}' \boldsymbol{H}_{i,q-1}^{-1}(\gamma,\boldsymbol{\theta}) \boldsymbol{z}_{i,(q)}} \right\} \\ &\quad = \frac{\theta_{(q)}\theta_{(q-1)} \boldsymbol{z}_{i,(q)}' \boldsymbol{H}_{i,q-1}^{-1}(\gamma,\boldsymbol{\theta}) \boldsymbol{z}_{i,(q-1)}}{1 + \theta_{(q)} \boldsymbol{z}_{i,(q)}' \boldsymbol{H}_{i,q-1}^{-1}(\gamma,\boldsymbol{\theta}) \boldsymbol{z}_{i,(q)}} \\ &\quad = \frac{\theta_{(q)}\theta_{(q-1)} \boldsymbol{z}_{i,(q)}' \boldsymbol{H}_{i,q-2}^{-1}(\gamma,\boldsymbol{\theta}) \boldsymbol{z}_{i,(q-1)}}{(1 + \theta_{(q)} \boldsymbol{z}_{i,(q)}' \boldsymbol{H}_{i,q-1}^{-1}(\gamma,\boldsymbol{\theta}) \boldsymbol{z}_{i,(q)})(1 + \theta_{(q-1)} \boldsymbol{z}_{i,(q-1)}' \boldsymbol{H}_{i,q-2}^{-1}(\gamma,\boldsymbol{\theta}) \boldsymbol{z}_{i,(q-1)})},\end{aligned}$$

where we note that $\theta_{(q)}$ and $\theta_{(q-1)}$ can be arbitrarily small and the dominant term of the denominator of the last equation can be equal to

(i) $\theta_{(q)}\theta_{(q-1)} \boldsymbol{z}_{i,(q)}' \boldsymbol{H}_{i,q-1}^{-1}(\gamma,\boldsymbol{\theta}) \boldsymbol{z}_{i,(q)} \boldsymbol{z}_{i,(q-1)}' \boldsymbol{H}_{i,q-2}^{-1}(\gamma,\boldsymbol{\theta}) \boldsymbol{z}_{i,(q-1)}$;

(ii) $\theta_{(q)} \boldsymbol{z}_{i,(q)}' \boldsymbol{H}_{i,q-1}^{-1}(\gamma,\boldsymbol{\theta}) \boldsymbol{z}_{i,(q)} + \theta_{(q-1)} \boldsymbol{z}_{i,(q-1)}' \boldsymbol{H}_{i,q-2}^{-1}(\gamma,\boldsymbol{\theta}) \boldsymbol{z}_{i,(q-1)}$;

(iii) 1.

For the case of (i), $\theta_{(q)} n_i^{\ell} \to \infty$ and $\theta_{(q-1)} n_i^{\ell} \to \infty$ by Lemma 3 (i); hence, using Lemma 3 (i), we have

$$\theta_{(q)}\theta_{(q-1)} \boldsymbol{z}_{i,(q)}' \boldsymbol{H}_{i,q}^{-1}(\gamma,\boldsymbol{\theta}) \boldsymbol{z}_{i,(q-1)} = o_P(n_i^{-\ell-\tau}),$$

which also gives the following two results:

$$\begin{aligned}\theta_{(q)} \boldsymbol{z}_{i,(q)}' \boldsymbol{H}_{i,q}^{-1}(\gamma,\boldsymbol{\theta}) \boldsymbol{z}_{i,(q-1)} &= o_P(n_i^{-\tau}), \\ \boldsymbol{z}_{i,(q)}' \boldsymbol{H}_{i,q}^{-1}(\gamma,\boldsymbol{\theta}) \boldsymbol{z}_{i,(q-1)} &= o_P(n_i^{\ell-\tau}).\end{aligned}$$

For the case of (ii), $\theta_{(q)} n_i^{\ell} \to \infty$ and $\theta_{(q)} = O(n_i^{-\ell})$ (or vice versa) by Lemma 3 (i); hence, using Lemma 3 (i), we have

$$\theta_{(q)}\theta_{(q-1)}\boldsymbol{z}'_{i,(q)}\boldsymbol{H}_{i,q}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q-1)} = o_P(\theta_{(q-1)}n_i^{-\tau}),$$

which gives the following three results:

$$\begin{aligned}\theta_{(q)}\theta_{(q-1)}\boldsymbol{z}'_{i,(q)}\boldsymbol{H}_{i,q}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q-1)} &= o_P(n_i^{-\ell-\tau}),\\ \theta_{(q)}\boldsymbol{z}'_{i,(q)}\boldsymbol{H}_{i,q}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q-1)} &= o_P(n_i^{-\tau}),\\ \boldsymbol{z}'_{i,(q)}\boldsymbol{H}_{i,q}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q-1)} &= o_P(n_i^{\ell-\tau}).\end{aligned}$$

For the case of (iii), $\theta_{(q)} = O(n_i^{-\ell})$ and $\theta_{(q)} = O(n_i^{-\ell})$ by Lemma 3 (i); hence, using Lemma 3 (i), we have

$$\theta_{(q)}\theta_{(q-1)}\boldsymbol{z}'_{i,(q)}\boldsymbol{H}_{i,q}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q-1)} = o_P(\theta_{(q)}\theta_{(q-1)}n_i^{\ell-\tau}),$$

which also gives the following three results:

$$\begin{aligned}\theta_{(q)}\theta_{(q-1)}\boldsymbol{z}'_{i,(q)}\boldsymbol{H}_{i,q}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q-1)} &= o_P(n_i^{-\ell-\tau}),\\ \theta_{(q)}\boldsymbol{z}'_{i,(q)}\boldsymbol{H}_{i,q}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q-1)} &= o_P(n_i^{-\tau}),\\ \boldsymbol{z}'_{i,(q)}\boldsymbol{H}_{i,q}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q-1)} &= o_P(n_i^{\ell-\tau}).\end{aligned}$$

In conclusion, we have

$$\begin{aligned}\theta_{(q)}\theta_{(q-1)}\boldsymbol{z}'_{i,(q)}\boldsymbol{H}_{i,q}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q-1)} &= o_P(n_i^{-\ell-\tau}),\\ \theta_{(q)}\boldsymbol{z}'_{i,(q)}\boldsymbol{H}_{i,q}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q-1)} &= o_P(n_i^{-\tau}),\\ \boldsymbol{z}'_{i,(q)}\boldsymbol{H}_{i,q}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q-1)} &= o_P(n_i^{\ell-\tau})\end{aligned} \tag{C.7}$$

uniformly over $\boldsymbol{\theta} \in [0,\infty)^q$. This completes the proof of Lemma 3 (iii).

We finally prove Lemma 3 (iv). Without loss of generality, it suffices to prove Lemma 3 (iv) by replacing $\boldsymbol{H}_i(\gamma,\boldsymbol{\theta})$ with $\boldsymbol{H}_{i,q-1}(\gamma,\boldsymbol{\theta})$ with $q(\gamma) = q$, $k = (q-1)$, and $k^* = (q)$. Then by (A.2),

$$\begin{aligned}&\theta_{(q-1)}\boldsymbol{z}'_{i,(q-1)}\boldsymbol{H}_{i,q-1}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q)}\\ &\quad= \theta_{(q-1)}\Bigg\{\boldsymbol{z}'_{i,(q-1)}\boldsymbol{H}_{i,q-2}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q)}\\ &\qquad - \frac{\theta_{(q-1)}\boldsymbol{z}'_{i,(q-1)}\boldsymbol{H}_{i,q-2}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q-1)}\boldsymbol{z}'_{i,(q-1)}\boldsymbol{H}_{i,q-2}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q)}}{1+\theta_{(q-1)}\boldsymbol{z}'_{i,(q-1)}\boldsymbol{H}_{i,q-2}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q-1)}}\Bigg\}\\ &\quad= \frac{\theta_{(q-1)}\boldsymbol{z}'_{i,(q-1)}\boldsymbol{H}_{i,q-2}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q)}}{1+\theta_{(q-1)}\boldsymbol{z}'_{i,(q-1)}\boldsymbol{H}_{i,q-2}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q-1)}}.\end{aligned}$$

Hence, Lemma 3 (iv) follows from Lemma 3 (i) and arguments similar to the proof of (C.4). This completes the proof.

C.3 Proof of Lemma 4

Note that for $k = 1,\ldots,q$ and $j = 1,\ldots,p$,

$$\begin{aligned}\boldsymbol{\epsilon}'_i\boldsymbol{z}_{i,k} &= O_p(n_i^{\ell/2}),\\ \boldsymbol{\epsilon}'_i\boldsymbol{x}_{i,j} &= O_p(n_i^{\xi/2}).\end{aligned}$$

Lemma 4 (ii)–(iii) then follow arguments similarly from the induction and the proofs of Lemma 2 (iii) are hence omitted.

We next prove Lemma 4 (iv). Let $\boldsymbol{z}_{i,(s)}$ be the s-th column of $\boldsymbol{Z}_i(\gamma)$ and $\boldsymbol{H}_{i,t}(\gamma,\boldsymbol{\theta})$ be defined in (A.4). Without loss of generality, we assume $q(\gamma)=q$. Hence by (A.6), Lemma 3 (i), and Lemma 4 (ii), we have

$$\begin{aligned}\boldsymbol{\epsilon}_i'\boldsymbol{H}_{i,q}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{\epsilon}_i &= \boldsymbol{\epsilon}_i'\boldsymbol{\epsilon}_i - \sum_{k=1}^{q}\frac{\theta_{(k)}\boldsymbol{\epsilon}_i'\boldsymbol{H}_{i,k-1}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(k)}\boldsymbol{z}_{i,(k)}'\boldsymbol{H}_{i,k-1}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{\epsilon}_i}{1+\theta_{(k)}\boldsymbol{z}_{i,(k)}'\boldsymbol{H}_{i,k-1}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(k)}}\\ &= \boldsymbol{\epsilon}_i'\boldsymbol{\epsilon}_i + O_p(q)\end{aligned}$$

uniformly over $\boldsymbol{\theta}\in[0,\infty)^q$. This completes the proof of Lemma 4 (iv).

It remains to prove Lemma 4 (i). Again, without loss of generality, it suffices to prove Lemma 4 (i) for $q(\gamma)=q$ and $k=(q)$. Then by (A.2),

$$\theta_{(q)}\boldsymbol{\epsilon}_i'\boldsymbol{H}_{i,q}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q)} = \frac{\theta_{(q)}\boldsymbol{\epsilon}_i'\boldsymbol{H}_{i,q-1}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q)}}{1+\theta_{(q)}\boldsymbol{z}_{i,(q)}'\boldsymbol{H}_{i,q-1}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q)}}.$$

Hence, Lemma 4 (i) follows from Lemma 3 (i), Lemma 4 (ii), and arguments similar to the proof of (C.4). This completes the proof.

C.4 Proof of Lemma 5

We show the lemma for $(\alpha,\gamma)\in\mathcal{A}_0\times\mathcal{G}_0$, where the proofs with respect to the remaining models are similar and are hence omitted.

Let $\boldsymbol{z}_{i,(s)}$ be the s-th column of $\boldsymbol{Z}_i(\gamma)$ and $\boldsymbol{H}_{i,t}(\gamma,\boldsymbol{\theta})$ be defined in (A.4). Without loss of generality, we assume that $q(\gamma)=q$ and $\boldsymbol{Z}_i(\gamma_0)\boldsymbol{b}_i(\gamma_0)=\sum_{s=q-q_0+1}^{q}\boldsymbol{z}_{i,(s)}b_{i,(s)}$. It then suffices to prove that for $(\alpha,\gamma)\in\mathcal{A}_0\times\mathcal{G}_0$ and $v^2>0$

$$-2\log L(\boldsymbol{\theta},v^2;\alpha,\gamma)-\{-2\log L(\boldsymbol{\theta}_0^{\dagger},v^2;\alpha,\gamma)\}\xrightarrow{p}\infty, \tag{C.8}$$

as both $N\to\infty$ and $\theta_{(k)}\to 0$ for some $k\in\{q-q_0+1,\ldots,q\}$, where $\boldsymbol{\theta}_0^{\dagger}\equiv(0,\ldots,0,\theta_{(q-q_0+1),0},\ldots,\theta_{(q),0})'$, and $\theta_{(s),0}$ being the true value of $\theta_{(s)}$; $s=q-q_0+1,\ldots,q$. Note that by (A.3) and (A.1), we have

$$\begin{aligned}\det(\boldsymbol{H}_i(\gamma,\boldsymbol{\theta})) &= \det\left(\boldsymbol{I}_{n_i}+\sum_{s=1}^{q}\theta_{(s)}\boldsymbol{z}_{i,(s)}\boldsymbol{z}_{i,(s)}'\right)\\ &= \det(\boldsymbol{H}_{i,q-1}(\gamma,\boldsymbol{\theta})+\theta_{(q)}\boldsymbol{z}_{i,(q)}\boldsymbol{z}_{i,(q)}')\\ &= \det(\boldsymbol{H}_{i,q-1}(\gamma,\boldsymbol{\theta}))(1+\theta_{(q)}\boldsymbol{z}_{i,(q)}'\boldsymbol{H}_{i,q-1}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q)}).\end{aligned}$$

Continuously expanding the above equation by (A.1) yields

$$\begin{aligned}\log\det(\boldsymbol{H}_i(\gamma,\boldsymbol{\theta})) &= \log\left\{\prod_{s=1}^{q}(1+\theta_{(s)}\boldsymbol{z}_{i,(s)}'\boldsymbol{H}_{i,s-1}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(s)})\right\}\\ &= \sum_{s=1}^{q}\log(1+\theta_{(s)}\boldsymbol{z}_{i,(s)}'\boldsymbol{H}_{i,s-1}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(s)}),\end{aligned}$$

where $\boldsymbol{H}_{i,0}(\gamma,\boldsymbol{\theta}) = \boldsymbol{I}_{n_i}$. This together with (2.7) and (B.4) yields for $(\alpha,\gamma) \in \mathcal{A}_0 \times \mathcal{G}_0$ and fixed $v^2 > 0$,

$$\begin{aligned}
&-2\log L(\boldsymbol{\theta}, v^2;\alpha,\gamma)\\
&= N\log(2\pi) + N\log(v^2) + \log\det(\boldsymbol{H}(\gamma,\boldsymbol{\theta})) + \frac{\boldsymbol{y}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{A}(\alpha,\gamma;\boldsymbol{\theta})\boldsymbol{y}}{v^2}\\
&= N\log(2\pi) + N\log(v^2) + \sum_{i=1}^{m}\sum_{s=1}^{q}\log(1+\theta_{(s)}\boldsymbol{z}_{i,(s)}'\boldsymbol{H}_{i,s-1}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(s)})\\
&\quad + \frac{(\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)+\boldsymbol{\epsilon})'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}))(\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)+\boldsymbol{\epsilon})}{v^2}.
\end{aligned}$$

Hence, we have, for $(\alpha,\gamma) \in \mathcal{A}_0 \times \mathcal{G}_0$,

$$\begin{aligned}
&-2\log L(\boldsymbol{\theta}, v^2;\alpha,\gamma) - \{-2\log L(\boldsymbol{\theta}_0^{\dagger}, v^2;\alpha,\gamma)\}\\
&= \sum_{i=1}^{m}\left\{\sum_{s=q-q_0+1}^{q}\log\left(\frac{1+\theta_{(s)}\boldsymbol{z}_{i,(s)}'\boldsymbol{H}_{i,s-1}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(s)}}{1+\theta_{(s),0}\boldsymbol{z}_{i,(s)}'\boldsymbol{H}_{i,s-1}^{-1}(\gamma,\boldsymbol{\theta}_0^{\dagger})\boldsymbol{z}_{i,(s)}}\right)\right\}\\
&\quad + \frac{1}{v^2}(\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)+\boldsymbol{\epsilon})'\{\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}))\\
&\quad - \boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^{\dagger})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}_0^{\dagger}))\}(\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)+\boldsymbol{\epsilon}),
\end{aligned}$$

where

$$\begin{aligned}
&(\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)+\boldsymbol{\epsilon})'\{\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}))\\
&\qquad - \boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^{\dagger})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}_0^{\dagger}))\}(\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)+\boldsymbol{\epsilon})\\
&\quad = \boldsymbol{b}(\gamma_0)'\boldsymbol{Z}(\gamma_0)'\{\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}))\\
&\qquad - \boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^{\dagger})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}_0^{\dagger}))\}\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)\\
&\qquad + 2\boldsymbol{b}(\gamma_0)'\boldsymbol{Z}(\gamma_0)'\{\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}))\\
&\qquad - \boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^{\dagger})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}_0^{\dagger}))\}\boldsymbol{\epsilon}\\
&\qquad + \boldsymbol{\epsilon}'\{\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}))\\
&\qquad - \boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^{\dagger})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}_0^{\dagger}))\}\boldsymbol{\epsilon}.
\end{aligned}$$

Hence, for (C.8) to hold, it suffices to prove

$$\begin{aligned}
&\boldsymbol{\epsilon}'\{\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}))\\
&\quad - \boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^{\dagger})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}_0^{\dagger}))\}\boldsymbol{\epsilon} = O_p(m)
\end{aligned} \tag{C.9}$$

uniformly over $\boldsymbol{\theta} \in [0,\infty)^q$ and

$$\begin{aligned}
&\sum_{i=1}^{m}\left\{\sum_{s=q-q_0+1}^{q}\log\left(\frac{1+\theta_{(s)}\boldsymbol{z}_{i,(s)}'\boldsymbol{H}_{i,s-1}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(s)}}{1+\theta_{(s),0}\boldsymbol{z}_{i,(s)}'\boldsymbol{H}_{i,s-1}^{-1}(\gamma,\boldsymbol{\theta}_0^{\dagger})\boldsymbol{z}_{i,(s)}}\right)\right\}\\
&\quad + \frac{1}{v^2}\Big(\boldsymbol{b}(\gamma_0)'\boldsymbol{Z}(\gamma_0)'\{\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}))\\
&\quad - \boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^{\dagger})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}_0^{\dagger}))\}\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)\Big) + O_p(m) \xrightarrow{p} \infty,
\end{aligned} \tag{C.10}$$

as both $N \to \infty$ and $\theta_{(k)} \to 0$ for some $k \in \{q-q_0+1,\ldots,q\}$. Before proving (C.9) and (C.10), we prove the following equations, for $\boldsymbol{h}_{i,k}$ being defined in (2.5) and $k =$

$q - q_0 + 1, \ldots, q$:

$$\boldsymbol{\epsilon}'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\boldsymbol{h}_{i,(k)}\boldsymbol{h}'_{i,(k)}\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta})\boldsymbol{\epsilon} = O_P(1), \tag{C.11}$$

$$\boldsymbol{\epsilon}'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta})\boldsymbol{h}_{i,(k)}\boldsymbol{h}'_{i,(k)}\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta})\boldsymbol{\epsilon} = o_P(1), \tag{C.12}$$

and

$$\begin{aligned} &\boldsymbol{\epsilon}'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\boldsymbol{X}(\alpha)(\boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta})\boldsymbol{X}(\alpha))^{-1} \\ &\quad \times \boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\boldsymbol{h}_{i,(k)}\boldsymbol{h}'_{i,(k)}\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta})\boldsymbol{\epsilon} = o_P(1), \end{aligned} \tag{C.13}$$

$$\begin{aligned} &\boldsymbol{\epsilon}'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\boldsymbol{X}(\alpha)(\boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\boldsymbol{X}(\alpha))^{-1} \\ &\quad \times \boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta})\boldsymbol{h}_{i,(k)}\boldsymbol{h}'_{i,(k)}\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\boldsymbol{X}(\alpha) \\ &\quad \times (\boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta})\boldsymbol{X}(\alpha))^{-1}\boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\boldsymbol{\epsilon} = o_P(1) \end{aligned} \tag{C.14}$$

uniformly over $\boldsymbol{\theta} \in [0, \infty)^q$. It suffices to prove (C.11)–(C.14) for $k = q$. For (C.11) with $k = q$, we have

$$\begin{aligned} &\boldsymbol{\epsilon}'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\boldsymbol{h}_{i,(q)}\boldsymbol{h}'_{i,(q)}\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta})\boldsymbol{\epsilon} \\ &= \{\boldsymbol{\epsilon}_i'\boldsymbol{H}_{i,q}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\boldsymbol{z}_{i,(q)}\}\{\boldsymbol{z}'_{i,(q)}\boldsymbol{H}_{i,q}^{-1}(\gamma, \boldsymbol{\theta})\boldsymbol{\epsilon}_i\} \\ &= \{\boldsymbol{\epsilon}_i'\boldsymbol{H}_{i,q}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\boldsymbol{z}_{i,(q)}\}\left(\frac{\boldsymbol{z}'_{i,(q)}\boldsymbol{H}_{i,q-1}^{-1}(\gamma, \boldsymbol{\theta})\boldsymbol{\epsilon}_i}{1 + \theta_{(q)}\boldsymbol{z}'_{i,(q)}\boldsymbol{H}_{i,q-1}^{-1}(\gamma, \boldsymbol{\theta})\boldsymbol{z}_{i,(q)}}\right) \\ &= \{O_P(n_i^{-\ell/2})\}_{1\times 1}\{O_P(n_i^{\ell/2})\}_{1\times 1} \\ &= O_P(1) \end{aligned}$$

uniformly over $\boldsymbol{\theta} \in [0, \infty)^q$, where the second last equality follows from Lemma 3 (i) and Lemma 4 (i)–(ii). For (C.12) with $k = q$, we have

$$\begin{aligned} &\boldsymbol{\epsilon}'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta})\boldsymbol{h}_{i,(q)}\boldsymbol{h}'_{i,(q)}\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta})\boldsymbol{\epsilon} \\ &= \{\boldsymbol{\epsilon}_i'\boldsymbol{H}_{i,q}^{-1}(\gamma, \boldsymbol{\theta})\boldsymbol{z}_{i,(q)}\}\boldsymbol{h}'_{i,(q)}\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta})\boldsymbol{\epsilon} \\ &= \left(\frac{\boldsymbol{\epsilon}_i'\boldsymbol{H}_{i,q-1}^{-1}(\gamma, \boldsymbol{\theta})\boldsymbol{z}_{i,(q)}}{1 + \theta_{(q)}\boldsymbol{z}'_{i,(q)}\boldsymbol{H}_{i,q-1}^{-1}(\gamma, \boldsymbol{\theta})\boldsymbol{z}_{i,(q)}}\right)\left(\frac{\boldsymbol{z}'_{i,(q)}\boldsymbol{H}_{i,q}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\boldsymbol{X}_i(\alpha)}{(\sum_{i=1}^m n_i^\xi)^{1/2}}\right) \\ &\quad \times \left(\frac{\sum_{i=1}^m \boldsymbol{X}_i(\alpha)'\boldsymbol{H}_{i,q}^{-1}(\gamma, \boldsymbol{\theta})\boldsymbol{X}_i(\alpha)}{\sum_{i=1}^m n_i^\xi}\right)^{-1}\left(\frac{\sum_{i=1}^m \boldsymbol{X}_i(\alpha)'\boldsymbol{H}_{i,q}^{-1}(\gamma, \boldsymbol{\theta})\boldsymbol{\epsilon}_i}{(\sum_{i=1}^m n_i^\xi)^{1/2}}\right) \\ &= \{O_P(n_i^{\ell/2})\}_{1\times 1}\{o(n_i^{-\ell/2-\tau})\}_{1\times p(\alpha)}\{\boldsymbol{T}^{-1}(\alpha) + \{o(n_{\min}^{-\tau})\}_{p(\alpha)\times p(\alpha)}\} \\ &\quad \times \{O_P(1)\}_{p(\alpha)\times 1} \\ &= o_P(1) \end{aligned}$$

uniformly over $\boldsymbol{\theta} \in [0, \infty)^q$, where the second equality follows from (2.9) and (A.5) and the third equality follows from (A.7), Lemma 2 (iii), and Lemma 4 (ii)–(iii). For (C.13) with

$k=q$, we have

$$
\begin{aligned}
&\boldsymbol{\epsilon}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)\boldsymbol{X}(\alpha)(\boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}(\alpha))^{-1}\\
&\quad\times\boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)\boldsymbol{h}_{i,(q)}\boldsymbol{h}_{i,(q)}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{\epsilon}\\
&=\bigg(\frac{\sum_{i=1}^m\boldsymbol{\epsilon}_i'\boldsymbol{H}_{i,q}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)\boldsymbol{X}_i(\alpha)}{(\sum_{i=1}^m n_i^\xi)^{1/2}}\bigg)\bigg(\frac{\sum_{i=1}^m\boldsymbol{X}_i(\alpha)'\boldsymbol{H}_{i,q}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}_i(\alpha)}{\sum_{i=1}^m n_i^\xi}\bigg)^{-1}\\
&\quad\times\bigg(\frac{\boldsymbol{X}_i(\alpha)'\boldsymbol{H}_{i,q}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)\boldsymbol{z}_{i,(q)}}{(\sum_{i=1}^m n_i^\xi)^{1/2}}\bigg)\bigg(\frac{\boldsymbol{z}_{i,(q)}'\boldsymbol{H}_{i,q-1}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{\epsilon}_i}{1+\theta_{(q)}\boldsymbol{z}_{i,(q)}'\boldsymbol{H}_{i,q-1}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q)}}\bigg)\\
&=\{O_p(1)\}_{1\times p(\alpha)}\{\boldsymbol{T}^{-1}(\alpha)+\{o(n_{\min}^{-\tau})\}_{p(\alpha)\times p(\alpha)}\}\{o(n_i^{-\ell/2-\tau})\}_{p(\alpha)\times 1}\\
&\quad\times\{O_p(n_i^{\ell/2})\}_{1\times 1}\\
&=o_p(1)
\end{aligned}
$$

uniformly over $\boldsymbol{\theta}\in[0,\infty)^q$, where the second equality follows from (A.7), Lemma 2 (iii), and Lemma 4 (ii)–(iii). For (C.14) with $k=q$,

$$
\begin{aligned}
&\boldsymbol{\epsilon}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)\boldsymbol{X}(\alpha)(\boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)\boldsymbol{X}(\alpha))^{-1}\boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{h}_{i,(q)}\\
&\quad\times\boldsymbol{h}_{i,(q)}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)\boldsymbol{X}(\alpha)(\boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}(\alpha))^{-1}\boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)\boldsymbol{\epsilon}\\
&=\bigg(\frac{\sum_{i=1}^m\boldsymbol{\epsilon}_i'\boldsymbol{H}_{i,q}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)\boldsymbol{X}_i(\alpha)}{(\sum_{i=1}^m n_i^\xi)^{1/2}}\bigg)\bigg(\frac{\sum_{i=1}^m\boldsymbol{X}_i(\alpha)'\boldsymbol{H}_{i,q}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)\boldsymbol{X}_i(\alpha)}{\sum_{i=1}^m n_i^\xi}\bigg)^{-1}\\
&\quad\times\bigg(\frac{\boldsymbol{X}_i(\alpha)'\boldsymbol{H}_{i,q-1}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q)}}{(\sum_{i=1}^m n_i^\xi)^{1/2}(1+\theta_{(q)}\boldsymbol{z}_{i,(q)}'\boldsymbol{H}_{i,q-1}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q)})}\bigg)\bigg(\frac{\boldsymbol{z}_{i,(q)}'\boldsymbol{H}_{i,q}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)\boldsymbol{X}_i(\alpha)}{(\sum_{i=1}^m n_i^\xi)^{1/2}}\bigg)\\
&\quad\times\bigg(\frac{\sum_{i=1}^m\boldsymbol{X}_i(\alpha)'\boldsymbol{H}_{i,q}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}_i(\alpha)}{\sum_{i=1}^m n_i^\xi}\bigg)^{-1}\bigg(\frac{\sum_{i=1}^m\boldsymbol{X}_i(\alpha)'\boldsymbol{H}_{i,q}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)\boldsymbol{\epsilon}_i}{(\sum_{i=1}^m n_i^\xi)^{1/2}}\bigg)\\
&=\{O_p(1)\}_{1\times p(\alpha)}\{\boldsymbol{T}^{-1}(\alpha)+\{o(n_{\min}^{-\tau})\}_{p(\alpha)\times p(\alpha)}\}\{o(n_i^{\ell/2-\tau})\}_{p(\alpha)\times 1}\\
&\quad\times\{o(n_{\min}^{-\tau})\}_{1\times p(\alpha)}\{\boldsymbol{T}^{-1}(\alpha)+\{o(n_{\min}^{-\tau})\}_{p(\alpha)\times p(\alpha)}\}\{O_p(1)\}_{p(\alpha)\times 1}\\
&=o_p(1)
\end{aligned}
$$

uniformly over $\boldsymbol{\theta}\in[0,\infty)^q$, where the second equality follows from (A.7), Lemma 2 (ii)–(iii), and Lemma 4 (iii). This completes the proofs of (C.11)–(C.14). We now prove (C.9). Note that

$$
\begin{aligned}
&\boldsymbol{\epsilon}'\{\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}_0^\dagger)-\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\}\boldsymbol{\epsilon}\\
&=\boldsymbol{\epsilon}'\{\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}_0^\dagger)-\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\\
&\quad+\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})-\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\}\boldsymbol{\epsilon}\\
&=\boldsymbol{\epsilon}'\{\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}_0^\dagger)-\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\}\boldsymbol{\epsilon}+o_p(m)\\
&=\boldsymbol{\epsilon}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)\{\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}_0^\dagger)\\
&\quad-\boldsymbol{X}(\alpha)(\boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}(\alpha))^{-1}\boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)\\
&\quad+\boldsymbol{X}(\alpha)(\boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}(\alpha))^{-1}\boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)\\
&\quad-\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\}\boldsymbol{\epsilon}+o_p(m)\\
&=\boldsymbol{\epsilon}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)\{\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta}_0^\dagger)\\
&\quad-\boldsymbol{X}(\alpha)(\boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}(\alpha))^{-1}\boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)\}\boldsymbol{\epsilon}+o_p(m)\\
&=o_p(m)
\end{aligned}
\tag{C.15}
$$

uniformly over $\boldsymbol{\theta}\in[0,\infty)^q$, where the second equality follows from (C.12) that

$$
\begin{aligned}
&\boldsymbol{\epsilon}'\{\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)-\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\}\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\boldsymbol{\epsilon}\\
&=\boldsymbol{\epsilon}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\{\boldsymbol{H}(\gamma,\boldsymbol{\theta})-\boldsymbol{H}(\gamma,\boldsymbol{\theta}_0^\dagger)\}\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\boldsymbol{\epsilon}\\
&=\sum_{i=1}^m\sum_{k=q-q_0+1}^q(\theta_{(k)}-\theta_{(k),0})\boldsymbol{\epsilon}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{h}_{i,(k)}\boldsymbol{h}_{i,(k)}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\boldsymbol{\epsilon}\\
&=o_p(m)
\end{aligned}
$$

uniformly over $\boldsymbol{\theta}\in[0,\infty)^q$, the second last equality follows from (C.13) that

$$
\begin{aligned}
&\boldsymbol{\epsilon}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)\{\boldsymbol{X}(\alpha)(\boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}(\alpha))^{-1}\boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)\\
&\quad-\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\}\boldsymbol{\epsilon}\\
&=\boldsymbol{\epsilon}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)\boldsymbol{X}(\alpha)(\boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}(\alpha))^{-1}\boldsymbol{X}(\alpha)'\\
&\quad\times\{\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)-\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\}\boldsymbol{\epsilon}\\
&=\sum_{i=1}^m\sum_{k=q-q_0+1}^q(\theta_{(k)}-\theta_{(k),0})\boldsymbol{\epsilon}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)\boldsymbol{X}(\alpha)(\boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)\boldsymbol{X}(\alpha))^{-1}\\
&\quad\times\boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{h}_{i,(k)}\boldsymbol{h}_{i,(k)}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^\dagger)\boldsymbol{X}(\alpha)\\
&=o_p(m)
\end{aligned}
$$

uniformly over $\boldsymbol{\theta} \in [0, \infty)^q$, and the last equality follows from (C.14) that

$$\begin{aligned}
&\boldsymbol{\epsilon}' \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\{\boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}_0^\dagger) \\
&\quad - \boldsymbol{X}(\alpha)(\boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta})\boldsymbol{X}(\alpha))^{-1}\boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\}\boldsymbol{\epsilon} \\
&= \boldsymbol{\epsilon}'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\boldsymbol{X}(\alpha)\{(\boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\boldsymbol{X}(\alpha))^{-1} \\
&\quad - (\boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta})\boldsymbol{X}(\alpha))^{-1}\}\boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\boldsymbol{\epsilon} \\
&= \boldsymbol{\epsilon}'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\boldsymbol{X}(\alpha)(\boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\boldsymbol{X}(\alpha))^{-1}\boldsymbol{X}(\alpha)'\{\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}) \\
&\quad - \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\}\boldsymbol{X}(\alpha)(\boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta})\boldsymbol{X}(\alpha))^{-1}\boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\boldsymbol{\epsilon} \\
&= \sum_{i=1}^{m} \sum_{k=q-q_0+1}^{q} (\theta_{(k),0} - \theta_{(k)})\boldsymbol{\epsilon}'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\boldsymbol{X}(\alpha)(\boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\boldsymbol{X}(\alpha))^{-1} \\
&\quad \times \boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta})\boldsymbol{h}_{i,(k)}\boldsymbol{h}_{i,(k)}'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\boldsymbol{X}(\alpha)(\boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta})\boldsymbol{X}(\alpha))^{-1} \\
&\quad \times \boldsymbol{X}(\alpha)'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\boldsymbol{\epsilon} \\
&= o_p(m)
\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in [0, \infty)^q$. Also, by (C.11),

$$\begin{aligned}
&\boldsymbol{\epsilon}'\{\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}) - \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\}\boldsymbol{\epsilon} \\
&= \boldsymbol{\epsilon}'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\{\boldsymbol{H}(\gamma, \boldsymbol{\theta}_0^\dagger) - \boldsymbol{H}(\gamma, \boldsymbol{\theta})\}\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta})\boldsymbol{\epsilon} \\
&= \sum_{i=1}^{m} \sum_{k=q-q_0+1}^{q} \{\theta_{(k),0} - \theta_{(k)}\}\boldsymbol{\epsilon}'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\boldsymbol{h}_{i,(k)}\boldsymbol{h}_{i,(k)}'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta})\boldsymbol{\epsilon} \\
&= O_p(m)
\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in [0, \infty)^q$. This together with (C.15) gives (C.9). We now prove (C.10). As with the proof of (C.15), we have

$$\begin{aligned}
&\boldsymbol{b}(\gamma_0)'\boldsymbol{Z}(\gamma_0)'\{\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}_0^\dagger) \\
&\qquad - \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta})\boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta})\}\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0) = o_p(m)
\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in [0, \infty)^q$. Hence

$$\begin{aligned}
&\boldsymbol{b}(\gamma_0)'\boldsymbol{Z}(\gamma_0)'\{\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta})(\boldsymbol{I}_N - \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta})) \\
&\quad - \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)(\boldsymbol{I}_N - \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}_0^\dagger))\}\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0) \\
&= \boldsymbol{b}(\gamma_0)'\boldsymbol{Z}(\gamma_0)'\{\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}) - \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\}\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0) + o_p(m) \\
&= \sum_{i=1}^{m} \sum_{s=q-q_0+1}^{q} (\theta_{(s),0} - \theta_{(s)})\boldsymbol{b}(\gamma_0)'\boldsymbol{Z}(\gamma_0)'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta})\boldsymbol{h}_{i,(s)} \\
&\quad \times \boldsymbol{h}_{i,(s)}'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0) + o_p(m)
\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in [0, \infty)^q$. Hence, for (C.10) to hold, it suffices to prove that for $k = q - q_0 + 1, \ldots, q$ and $i = 1, \ldots, m$,

$$\begin{aligned}
&\log\left(\frac{1 + \theta_{(k)}\boldsymbol{z}_{i,(k)}'\boldsymbol{H}_{i,k-1}^{-1}(\gamma, \boldsymbol{\theta})\boldsymbol{z}_{i,(k)}}{1 + \theta_{(k),0}\boldsymbol{z}_{i,(k)}'\boldsymbol{H}_{i,k-1}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\boldsymbol{z}_{i,(k)}}\right) \\
&\quad = o_p\Big(\boldsymbol{b}(\gamma_0)'\boldsymbol{Z}(\gamma_0)'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta})\boldsymbol{h}_{i,(k)}\boldsymbol{h}_{i,(k)}'\boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}_0^\dagger)\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)\Big),
\end{aligned} \tag{C.16}$$

as both $N \to \infty$ and $\theta_{(k)} \to 0$ for some $k \in \{q - q_0 + 1, \ldots, q\}$. It suffices to prove (C.16) for $k = q$. By Lemma 3 (ii)–(iii), we have

$$\begin{aligned}
&\boldsymbol{b}(\gamma_0)'\boldsymbol{Z}(\gamma_0)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{h}_{i,(q)}\boldsymbol{h}'_{i,(q)}\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta}_0^{\dagger})\boldsymbol{Z}(\gamma_0)\boldsymbol{b}(\gamma_0)\\
&\quad= \left(\frac{b_{i,(q)}\boldsymbol{z}'_{i,(q)}\boldsymbol{H}^{-1}_{i,q-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q)}}{\{1+\theta_{(q)}\boldsymbol{z}'_{i,(q)}\boldsymbol{H}^{-1}_{i,q-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q)}\}}\right)\{\boldsymbol{z}'_{i,(q)}\boldsymbol{H}^{-1}_{i,q}(\gamma,\boldsymbol{\theta}_0^{\dagger})\boldsymbol{Z}_i(\gamma_0)\boldsymbol{b}_i(\gamma_0)\}\\
&\quad= \left(\frac{b_{i,(q)}\boldsymbol{z}'_{i,(q)}\boldsymbol{H}^{-1}_{i,q-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q)}}{1+\theta_{(q)}\boldsymbol{z}'_{i,(q)}\boldsymbol{H}^{-1}_{i,q-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q)}}\right)\left(\frac{b_{i,(q)}}{\theta_{(q),0}} + o_P(n_i^{-\ell-\tau})\right).
\end{aligned}$$

Hence, for (C.16) with $k = q$ to hold, it suffices to prove that

$$\begin{aligned}
&\log\left(\frac{1+\theta_{(q)}\boldsymbol{z}'_{i,(q)}\boldsymbol{H}^{-1}_{i,q-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q)}}{1+\theta_{(q),0}\boldsymbol{z}'_{i,(q)}\boldsymbol{H}^{-1}_{i,q-1}(\gamma,\boldsymbol{\theta}_0^{\dagger})\boldsymbol{z}_{i,(q)}}\right)\left(\frac{\boldsymbol{z}'_{i,(q)}\boldsymbol{H}^{-1}_{i,q-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q)}}{1+\theta_{(q)}\boldsymbol{z}'_{i,(q)}\boldsymbol{H}^{-1}_{i,q-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i,(q)}}\right)^{-1}\\
&\quad\to 0,
\end{aligned}$$

as both $N \to \infty$ and $\theta_{(q)} \to 0$, which follows from Lemma 3 (i) and L'Hospital's rule. This completes the proof of (C.16). This completes the proof.

C.5 Proof of Lemma 6

We first prove Lemma 6 (i). For $i, i^* = 1, \ldots, m$, $(\alpha, \gamma) \in \mathcal{A} \times \mathcal{G}$ and $k, k^* \in \gamma$, we have

$$\begin{aligned}
&\theta_k\theta_{k^*}\boldsymbol{h}'_{i,k}\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\boldsymbol{h}_{i^*,k^*}\\
&\quad= \left(\theta_k\boldsymbol{z}'_{i,k}\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}_i(\alpha)\right)\left(\frac{\sum_{i=1}^m \boldsymbol{X}_i(\alpha)'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}_i(\alpha)}{\sum_{i=1}^m n_i^{\xi}}\right)^{-1}\\
&\qquad\times\left(\frac{\theta_{k^*}\boldsymbol{X}_{i^*}(\alpha)'\boldsymbol{H}_{i^*}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i^*,k^*}}{\sum_{i=1}^m n_i^{\xi}}\right)\\
&\quad= \{o(n_i^{(\xi-\ell)/2-\tau})\}_{1\times p(\alpha)}\left\{\boldsymbol{T}^{-1}(\alpha) + \{o(n_{\min}^{-\tau})\}_{p(\alpha)\times p(\alpha)}\right\}\\
&\qquad\times\left\{o\left(\frac{n_{i^*}^{(\xi-\ell)/2-\tau}}{\sum_{i=1}^m n_i^{\xi}}\right)\right\}_{p(\alpha)\times 1}\\
&\quad= o\left(\frac{n_i^{(\xi-\ell)/2}n_{i^*}^{(\xi-\ell)/2-\tau}}{\sum_{i=1}^m n_i^{\xi}}\right)
\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in \Theta_\gamma$, where the second equality follows from (A.7) and Lemma 2 (iii). Similarly, by (A.7) and Lemma 2 (iii), we have

$$
\begin{aligned}
&\theta_k \boldsymbol{h}_{i,k}' \boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\boldsymbol{h}_{i^*,k^*} \\
&\quad = \big(\theta_k \boldsymbol{z}_{i,k}'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}_i(\alpha)\big)\bigg(\frac{\sum_{i=1}^m \boldsymbol{X}_i(\alpha)'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}_i(\alpha)}{\sum_{i=1}^m n_i^\xi}\bigg)^{-1} \\
&\qquad \times \bigg(\frac{\boldsymbol{X}_{i^*}(\alpha)'\boldsymbol{H}_{i^*}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i^*,k^*}}{\sum_{i=1}^m n_i^\xi}\bigg) \\
&\quad = \{o(n_i^{(\xi-\ell)/2-\tau})\}_{1\times p(\alpha)}\Big\{\boldsymbol{T}^{-1}(\alpha) + \{o(n_{\min}^{-\tau})\}_{p(\alpha)\times p(\alpha)}\Big\} \\
&\qquad \times \bigg\{o\bigg(\frac{n_{i^*}^{(\xi+\ell)/2-\tau}}{\sum_{i=1}^m n_i^\xi}\bigg)\bigg\}_{p(\alpha)\times 1} \\
&\quad = o\bigg(\frac{n_i^{(\xi-\ell)/2} n_{i^*}^{(\xi+\ell)/2-\tau}}{\sum_{i=1}^m n_i^\xi}\bigg)
\end{aligned}
$$

uniformly over $\boldsymbol{\theta} \in \Theta_\gamma$. Further, by (A.7) and Lemma 2 (iii), we have

$$
\begin{aligned}
&\boldsymbol{h}_{i,k}' \boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\boldsymbol{h}_{i^*,k^*} \\
&\quad = \big(\theta_k \boldsymbol{z}_{i,k}'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}_i(\alpha)\big)\bigg(\frac{\sum_{i=1}^m \boldsymbol{X}_i(\alpha)'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}_i(\alpha)}{\sum_{i=1}^m n_i^\xi}\bigg)^{-1} \\
&\qquad \times \bigg(\frac{\boldsymbol{X}_{i^*}(\alpha)'\boldsymbol{H}_{i^*}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i^*,k^*}}{\sum_{i=1}^m n_i^\xi}\bigg) \\
&\quad = \{o(n_i^{(\xi+\ell)/2-\tau})\}_{1\times p(\alpha)}\Big\{\boldsymbol{T}^{-1}(\alpha) + \{o(n_{\min}^{-\tau})\}_{p(\alpha)\times p(\alpha)}\Big\} \\
&\qquad \times \bigg\{o\bigg(\frac{n_{i^*}^{(\xi+\ell)/2-\tau}}{\sum_{i=1}^m n_i^\xi}\bigg)\bigg\}_{p(\alpha)\times 1} \\
&\quad = o\bigg(\frac{n_i^{(\xi+\ell)/2} n_{i^*}^{(\xi+\ell)/2-\tau}}{\sum_{i=1}^m n_i^\xi}\bigg)
\end{aligned}
$$

uniformly over $\boldsymbol{\theta} \in \Theta_\gamma$. This completes the proof of Lemma 6 (i).

We now prove Lemma 6 (ii). For $i, i^* = 1, \ldots, m$, $(\alpha,\gamma) \in \mathcal{A}\times\mathcal{G}$, $k \in \gamma$ and $k^* \notin \gamma$,

$$
\begin{aligned}
&\theta_k \boldsymbol{h}_{i,k}' \boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\boldsymbol{h}_{i^*,k^*} \\
&\quad = \big(\theta_k \boldsymbol{z}_{i,k}'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}_i(\alpha)\big)\bigg(\frac{\sum_{i=1}^m \boldsymbol{X}_i(\alpha)'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}_i(\alpha)}{\sum_{i=1}^m n_i^\xi}\bigg)^{-1} \\
&\qquad \times \bigg(\frac{\boldsymbol{X}_{i^*}(\alpha)'\boldsymbol{H}_{i^*}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i^*,k^*}}{\sum_{i=1}^m n_i^\xi}\bigg) \\
&\quad = \{o(n_i^{(\xi-\ell)/2-\tau})\}_{1\times p(\alpha)}\Big\{\boldsymbol{T}^{-1}(\alpha) + \{o(n_{\min}^{-\tau})\}_{p(\alpha)\times p(\alpha)}\Big\} \\
&\qquad \times \bigg\{o\bigg(\frac{n_i^{(\xi+\ell)/2-\tau}}{\sum_{i=1}^m n_i^\xi}\bigg)\bigg\}_{1\times p(\alpha)} \\
&\quad = o\bigg(\frac{n_i^{(\xi-\ell)/2} n_{i^*}^{(\xi+\ell)/2-\tau}}{\sum_{i=1}^m n_i^\xi}\bigg)
\end{aligned}
$$

uniformly over $\boldsymbol{\theta} \in \Theta_\gamma$, where the second equality follows from Lemma 2 (ii)–(iii) and (A.7). Similarly, by (A.7) and Lemma 2 (ii)–(iii), we have

$$\begin{aligned}
&\boldsymbol{h}_{i,k}' \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}) \boldsymbol{h}_{i^*,k^*} \\
&= \big(\theta_k \boldsymbol{z}_{i,k}' \boldsymbol{H}_i^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{X}_i(\alpha)\big) \left(\frac{\sum_{i=1}^m \boldsymbol{X}_i(\alpha)' \boldsymbol{H}_i^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{X}_i(\alpha)}{\sum_{i=1}^m n_i^\xi} \right)^{-1} \\
&\quad \times \left(\frac{\boldsymbol{X}_{i^*}(\alpha)' \boldsymbol{H}_{i^*}^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{z}_{i^*,k^*}}{\sum_{i=1}^m n_i^\xi} \right) \\
&= \{o(n_i^{(\xi+\ell)/2-\tau})\}_{1\times p(\alpha)} \left\{ \boldsymbol{T}^{-1}(\alpha) + \{o(n_{\min}^{-\tau})\}_{p(\alpha)\times p(\alpha)} \right\} \\
&\quad \times \left\{ o\left(\frac{n_{i^*}^{(\xi+\ell)/2-\tau}}{\sum_{i=1}^m n_i^\xi} \right) \right\}_{p(\alpha)\times 1} \\
&= o\left(\frac{n_i^{(\xi+\ell)/2} n_{i^*}^{(\xi+\ell)/2-\tau}}{\sum_{i=1}^m n_i^\xi} \right)
\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in \Theta_\gamma$. This completes the proof of Lemma 6 (ii).

We now prove Lemma 6 (iii). For $(\alpha, \gamma) \in \mathcal{A} \times \mathcal{G}$ and $k \in \gamma$,

$$\begin{aligned}
&\theta_k \boldsymbol{h}_{i,k}' \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}) \boldsymbol{\epsilon} \\
&= \left(\frac{\theta_k \boldsymbol{z}_{i,k}' \boldsymbol{H}_i^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{X}_i(\alpha)}{(\sum_{i=1}^m n_i^\xi)^{1/2}} \right) \left(\frac{\sum_{i=1}^m \boldsymbol{X}_i(\alpha)' \boldsymbol{H}_i^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{X}_i(\alpha)}{\sum_{i=1}^m n_i^\xi} \right)^{-1} \\
&\quad \times \left(\frac{\sum_{i=1}^m \boldsymbol{X}_i(\alpha)' \boldsymbol{H}_i^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{\epsilon}_i}{(\sum_{i=1}^m n_i^\xi)^{1/2}} \right) \\
&= \{o(n_i^{-\ell/2-\tau})\}_{1\times p(\alpha)} \left\{ \boldsymbol{T}^{-1}(\alpha) + \{o(n_{\min}^{-\tau})\}_{p(\alpha)\times p(\alpha)} \right\} \{O_P(1)\}_{p(\alpha)\times 1} \\
&= o_P(n_i^{-\ell/2})
\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in \Theta_\gamma$, where the second equality follows from (A.7), Lemma 2 (iii), and Lemma 4 (iii). Similarly, by (A.7), Lemma 2 (iii), and Lemma 4 (iii), we have

$$\begin{aligned}
&\boldsymbol{h}_{i,k}' \boldsymbol{H}^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{M}(\alpha, \gamma; \boldsymbol{\theta}) \boldsymbol{\epsilon} \\
&= \left(\frac{\boldsymbol{z}_{i,k}' \boldsymbol{H}_i^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{X}_i(\alpha)}{(\sum_{i=1}^m n_i^\xi)^{1/2}} \right) \left(\frac{\sum_{i=1}^m \boldsymbol{X}_i(\alpha)' \boldsymbol{H}_i^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{X}_i(\alpha)}{\sum_{i=1}^m n_i^\xi} \right)^{-1} \\
&\quad \times \left(\frac{\sum_{i=1}^m \boldsymbol{X}_i(\alpha)' \boldsymbol{H}_i^{-1}(\gamma, \boldsymbol{\theta}) \boldsymbol{\epsilon}_i}{(\sum_{i=1}^m n_i^\xi)^{1/2}} \right) \\
&= \{o(n_i^{\ell/2-\tau})\}_{1\times p(\alpha)} \left\{ \boldsymbol{T}^{-1}(\alpha) + \{o(n_{\min}^{-\tau})\}_{p(\alpha)\times p(\alpha)} \right\} \{O_P(1)\}_{p(\alpha)\times 1} \\
&= o_P(n_i^{\ell/2})
\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in \Theta_\gamma$. This completes the proof of Lemma 6 (iii).

We now prove Lemma 6 (iv). For $(\alpha,\gamma)\in(\mathcal{A}\setminus\mathcal{A}_0)\times\mathcal{G}$, $k\in\gamma$,

$$\begin{aligned}
&\theta_k\boldsymbol{h}_{i,k}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\boldsymbol{X}(\alpha_0\setminus\alpha)\boldsymbol{\beta}(\alpha_0\setminus\alpha)\\
&=\big(\theta_k\boldsymbol{z}_{i,k}'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}_i(\alpha)\big)\bigg(\frac{\sum_{i=1}^m\boldsymbol{X}_i(\alpha)'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}_i(\alpha)}{\sum_{i=1}^m n_i^{\xi}}\bigg)^{-1}\\
&\quad\times\bigg(\frac{\sum_{i=1}^m\sum_{j\in\gamma_0\setminus\alpha}\boldsymbol{X}_i(\alpha)'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{x}_{i,j}\beta_{j,0}}{\sum_{i=1}^m n_i^{\xi}}\bigg)\\
&=\{o(n_i^{(\xi-\ell)/2-\tau})\}_{1\times p(\alpha)}\Big\{\boldsymbol{T}^{-1}(\alpha)+\{o(n_{\min}^{-\tau})\}_{p(\alpha)\times p(\alpha)}\Big\}\bigg\{o\bigg(\frac{\sum_{i=1}^m n_i^{\xi-\tau}}{\sum_{i=1}^m n_i^{\xi}}\bigg)\bigg\}_{p(\alpha)\times 1}\\
&=\{o(n_i^{(\xi-\ell)/2-\tau})\}_{1\times p(\alpha)}\Big\{\boldsymbol{T}^{-1}(\alpha)+\{o(n_{\min}^{-\tau})\}_{p(\alpha)\times p(\alpha)}\Big\}\{o(n_{\min}^{-\tau})\}_{p(\alpha)\times 1}\\
&=o(n_i^{(\xi-\ell)/2-\tau})
\end{aligned}$$

uniformly over $\boldsymbol{\theta}\in\Theta_\gamma$, where the second equality follows from (A.7), Lemma 2 (i), and Lemma 2 (iii). Similarly, by (A.7) and Lemma 2 (i) and (iii), we have

$$\begin{aligned}
&\boldsymbol{h}_{i,k}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\boldsymbol{X}(\alpha_0\setminus\alpha)\boldsymbol{\beta}(\alpha_0\setminus\alpha)\\
&\quad=\big(\theta_k\boldsymbol{z}_{i,k}'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}_i(\alpha)\big)\bigg(\frac{\sum_{i=1}^m\boldsymbol{X}_i(\alpha)'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}_i(\alpha)}{\sum_{i=1}^m n_i^{\xi}}\bigg)^{-1}\\
&\qquad\times\bigg(\frac{\sum_{i=1}^m\sum_{j\in\alpha_0\setminus\alpha}\boldsymbol{X}_i(\alpha)'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{x}_{i,j}\beta_{j,0}}{\sum_{i=1}^m n_i^{\xi}}\bigg)\\
&\quad=\{o(n_i^{(\xi+\ell)/2-\tau})\}_{1\times p(\alpha)}\Big\{\boldsymbol{T}^{-1}(\alpha)+\{o(n_{\min}^{-\tau})\}_{p(\alpha)\times p(\alpha)}\Big\}\{o(n_{\min}^{-\tau})\}_{p(\alpha)\times 1}\\
&\quad=o_p(n_i^{(\xi+\ell)/2-\tau})
\end{aligned}$$

uniformly over $\boldsymbol{\theta}\in\Theta_\gamma$. This completes the proof of Lemma 6 (iv).

We now prove Lemma 6 (v). For $(\alpha,\gamma)\in\mathcal{A}\times\mathcal{G}$, we have

$$\begin{aligned}
&\boldsymbol{\epsilon}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\boldsymbol{\epsilon}\\
&=\bigg(\frac{\sum_{i=1}^m\boldsymbol{\epsilon}_i'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}_i(\alpha)}{(\sum_{i=1}^m n_i^{\xi})^{1/2}}\bigg)\bigg(\frac{\sum_{i=1}^m\boldsymbol{X}_i(\alpha)'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}_i(\alpha)}{\sum_{i=1}^m n_i^{\xi}}\bigg)^{-1}\\
&\quad\times\bigg(\frac{\sum_{i=1}^m\boldsymbol{X}_i(\alpha)'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{\epsilon}_i}{(\sum_{i=1}^m n_i^{\xi})^{1/2}}\bigg)\\
&=\{O_p(1)\}_{1\times p(\alpha)}\Big\{\boldsymbol{T}^{-1}(\alpha)+\{o(n_{\min}^{-\tau})\}_{p(\alpha)\times p(\alpha)}\Big\}\{O_p(1)\}_{p(\alpha)\times 1}\\
&=O_p(p(\alpha))
\end{aligned}$$

uniformly over $\boldsymbol{\theta}\in\Theta_\gamma$, where the second equality follows from (A.7) and Lemma 4 (iii). This completes the proof of Lemma 6 (v).

We now prove Lemma 6 (vi). For $(\alpha,\gamma) \in \mathcal{A} \times \mathcal{G}$ and $k \notin \gamma$, we have

$$\begin{aligned}
&\boldsymbol{h}_{i,k}' \boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\boldsymbol{\epsilon} \\
&= \left(\frac{\boldsymbol{z}_{i,k}'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}_i(\alpha)}{(\sum_{i=1}^m n_i^{\xi})^{1/2}}\right)\left(\frac{\sum_{i=1}^m \boldsymbol{X}_i(\alpha)'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}_i(\alpha)}{\sum_{i=1}^m n_i^{\xi}}\right)^{-1} \\
&\quad \times \left(\frac{\sum_{i=1}^m \boldsymbol{X}_i(\alpha)'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{\epsilon}_i}{(\sum_{i=1}^m n_i^{\xi})^{1/2}}\right) \\
&= \{o(n_i^{\ell/2-\tau})\}_{1\times p(\alpha)}\left\{\boldsymbol{T}^{-1}(\alpha) + \{o(n_{\min}^{-\tau})\}_{p(\alpha)\times p(\alpha)}\right\}\{O_p(1)\}_{p(\alpha)\times 1} \\
&= o_p(n_i^{\ell/2})
\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in \Theta_\gamma$, where the second equality follows from (A.7), Lemma 2 (ii), and Lemma 4 (iii). This completes the proof of Lemma 6 (vi).

We now prove Lemma 6 (vii). For $(\alpha,\gamma) \in (\mathcal{A} \setminus \mathcal{A}_0) \times \mathcal{G}$, we have

$$\begin{aligned}
&\boldsymbol{\epsilon}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\boldsymbol{X}(\alpha_0\setminus\alpha)\boldsymbol{\beta}(\alpha_0\setminus\alpha) \\
&= \left(\frac{\sum_{i=1}^m \boldsymbol{\epsilon}_i'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}_i(\alpha)}{(\sum_{i=1}^m n_i^{\xi})^{1/2}}\right)\left(\frac{\sum_{i=1}^m \boldsymbol{X}_i(\alpha)'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}_i(\alpha)}{\sum_{i=1}^m n_i^{\xi}}\right)^{-1} \\
&\quad \times \left(\frac{\sum_{i=1}^m \sum_{j\in\alpha_0\setminus\alpha} \boldsymbol{X}_i(\alpha)'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{x}_{i,j}\beta_{j,0}}{(\sum_{i=1}^m n_i^{\xi})^{1/2}}\right) \\
&= \{O_p(1)\}_{1\times p(\alpha)}\left\{\boldsymbol{T}^{-1}(\alpha) + \{o(n_{\min}^{-\tau})\}_{p(\alpha)\times p(\alpha)}\right\} \\
&\quad \times \left\{o\left(\left(\sum_{i=1}^m n_i^{\xi}\right)^{1/2} n_{\min}^{-\tau}\right)\right\}_{p(\alpha)\times 1} \\
&= o_p\left(\left(\sum_{i=1}^m n_i^{\xi}\right)^{1/2}\right)
\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in \Theta_\gamma$, where the second equality follows from (A.7), Lemma 2 (i), and Lemma 4 (iii). This completes the proof of Lemma 6 (vii).

We now prove Lemma 6 (viii). For $i, i^* = 1,\ldots,m$, $(\alpha,\gamma) \in \mathcal{A} \times \mathcal{G}$ and $k, k^* \notin \gamma$, we have

$$\begin{aligned}
&\boldsymbol{h}_{i,k}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\boldsymbol{h}_{i^*,k^*} \\
&= \left(\boldsymbol{z}_{i,k}'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}_i(\alpha)\right)\left(\frac{\sum_{i=1}^m \boldsymbol{X}_i(\alpha)'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}_i(\alpha)}{\sum_{i=1}^m n_i^{\xi}}\right)^{-1} \\
&\quad \times \left(\frac{\boldsymbol{X}_{i^*}(\alpha)'\boldsymbol{H}_{i^*}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{z}_{i^*,k^*}}{\sum_{i=1}^m n_i^{\xi}}\right) \\
&= \{o(n_i^{(\xi+\ell)/2-\tau})\}_{1\times p(\alpha)}\left\{\boldsymbol{T}^{-1}(\alpha) + \{o(n_{\min}^{-\tau})\}_{p(\alpha)\times p(\alpha)}\right\} \\
&\quad \times \left\{o_p\left(\frac{n_{i^*}^{(\xi+\ell)/2-\tau}}{\sum_{i=1}^m n_i^{\xi}}\right)\right\}_{p(\alpha)\times 1} \\
&= \text{`}o_p\left(\frac{n_i^{(\xi+\ell)/2} n_{i^*}^{(\xi+\ell)/2-\tau}}{\sum_{i=1}^m n_i^{\xi}}\right)
\end{aligned}$$

uniformly over $\boldsymbol{\theta} \in \Theta_\gamma$, where the second equality follows from (A.7) and Lemma 2 (ii). This completes the proof of Lemma 6 (viii).

We now prove Lemma 6 (ix). For $(\alpha,\gamma)\in(\mathcal{A}\setminus\mathcal{A}_0)\times\mathcal{G}$, $k\notin\gamma$, we have

$$
\begin{aligned}
&\boldsymbol{h}_{i,k}'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\boldsymbol{X}(\alpha_0\setminus\alpha)\boldsymbol{\beta}(\alpha_0\setminus\alpha)\\
&\quad=\big(\boldsymbol{z}_{i,k}'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}_i(\alpha)\big)\bigg(\frac{\sum_{i=1}^m\boldsymbol{X}_i(\alpha)'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}_i(\alpha)}{\sum_{i=1}^m n_i^{\xi}}\bigg)^{-1}\\
&\qquad\times\bigg(\frac{\sum_{i=1}^m\sum_{j\in\alpha_0\setminus\alpha}\boldsymbol{X}_i(\alpha)'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{x}_{i,j}\beta_{j,0}}{\sum_{i=1}^m n_i^{\xi}}\bigg)\\
&\quad=\{o(n_i^{(\xi+\ell)/2-\tau})\}_{1\times p(\alpha)}\Big\{\boldsymbol{T}^{-1}(\alpha)+\{o(n_{\min}^{-\tau})\}_{p(\alpha)\times p(\alpha)}\Big\}\\
&\qquad\times\{o(n_{\min}^{-\tau})\}_{p(\alpha)\times 1}\\
&\quad=o(n_i^{(\xi+\ell)/2-\tau})
\end{aligned}
$$

uniformly over $\boldsymbol{\theta}\in\Theta_\gamma$, where the second equality follows from (A.7) and Lemma 2 (i)–(ii). This completes the proof of Lemma 6 (ix).

We finally prove Lemma 6 (x). For $(\alpha,\gamma)\in(\mathcal{A}\setminus\mathcal{A}_0)\times\mathcal{G}$, we have

$$
\begin{aligned}
&\boldsymbol{\beta}(\alpha_0\setminus\alpha)'\boldsymbol{X}(\alpha_0\setminus\alpha)'\boldsymbol{H}^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{M}(\alpha,\gamma;\boldsymbol{\theta})\boldsymbol{X}(\alpha_0\setminus\alpha)\boldsymbol{\beta}(\alpha_0\setminus\alpha)\\
&=\bigg(\sum_{i=1}^m\sum_{j\in\alpha_0\setminus\alpha}\beta_{j,0}\boldsymbol{x}_{i,j}'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}_i(\alpha)\bigg)\bigg(\frac{\sum_{i=1}^m\boldsymbol{X}_i(\alpha)'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}_i(\alpha)}{\sum_{i=1}^m n_i^{\xi}}\bigg)^{-1}\\
&\quad\times\bigg(\frac{\sum_{i=1}^m\sum_{j\in\alpha_0\setminus\alpha}\boldsymbol{X}_i(\alpha)'\boldsymbol{H}_i^{-1}(\gamma,\boldsymbol{\theta})\boldsymbol{X}_i(\alpha)\boldsymbol{x}_{i,j}}{\sum_{i=1}^m n_i^{\xi}}\bigg)\\
&=\bigg\{o\bigg(\sum_{i=1}^m n_i^{\xi-\tau}\bigg)\bigg\}_{1\times p(\alpha)}\Big\{\boldsymbol{T}^{-1}(\alpha)+\{o(n_{\min}^{-\tau})\}_{p(\alpha)\times p(\alpha)}\Big\}\{o(n_{\min}^{-\tau})\}_{p(\alpha)\times 1}\\
&=o\bigg(\sum_{i=1}^m n_i^{\xi-\tau}\bigg)
\end{aligned}
$$

uniformly over $\boldsymbol{\theta}\in\Theta_\gamma$, where the second equality follows from (A.7) and Lemma 2 (i). This completes the proof.