Supplementary Material for "Nonparametric tests for multistate processes with clustered data"

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Asymptotic Theory Proofs

In this Supplementary Material we provide the proofs of the theorems stated in the main manuscript. The proofs rely on empirical process theory (van der Vaart & Wellner 1996; Kosorok 2008). Standard empirical process theory notation is used throughout this Supplementary Material. Specifically, for any measurable function $f : \mathcal{D} \mapsto \mathbb{R}$, where $\mathcal{D}$ is the sample space, we define

$$\mathbb{P}_n f = \frac{1}{n}\sum_{i=1}^{n} f(D_i),$$

and

$$Pf = \int_{\mathcal{D}} f dP,$$

where P is the underlying true probability measure on the Borel σ-algebra on $\mathcal{D}$.

Proof of Theorem 1

Under the null hypothesis $H_0 : P_{0,1hj}(s,\cdot) = P_{0,2hj}(s,\cdot)$, it follows that,

$$\begin{aligned}
\sqrt{\frac{n_1 n_2}{n_1+n_2}} Z_{n_1,n_2,hj}(s) &= \sqrt{\frac{n_1 n_2}{n_1+n_2}} \int_s^{\tau} \hat{W}_{hj}(t)\left[\hat{P}_{n_1,1hj}(s,t) - \hat{P}_{n_2,2hj}(s,t)\right] d\mu(t) \\
&\quad - \sqrt{\frac{n_1 n_2}{n_1+n_2}} \int_s^{\tau} \hat{W}_{hj}(t)\left[P_{0,1hj}(s,t) - P_{0,2hj}(s,t)\right] d\mu(t) \\
&= \sqrt{1-\frac{n_1}{n_1+n_2}} \int_s^{\tau} \left[\hat{W}_{hj}(t) - W_{hj}(t)\right]\sqrt{n_1}\left[\hat{P}_{n_1,1hj}(s,t) - P_{0,1hj}(s,t)\right] d\mu(t) \\
&\quad + \sqrt{1-\frac{n_1}{n_1+n_2}} \int_s^{\tau} W_{hj}(t)\sqrt{n_1}\left[\hat{P}_{n_1,1hj}(s,t) - P_{0,1hj}(s,t)\right] d\mu(t) \\
&\quad - \sqrt{\frac{n_1}{n_1+n_2}} \int_s^{\tau} \left[\hat{W}_{hj}(t) - W_{hj}(t)\right]\sqrt{n_2}\left[\hat{P}_{n_2,2hj}(s,t) - P_{0,2hj}(s,t)\right] d\mu(t) \\
&\quad - \sqrt{\frac{n_1}{n_1+n_2}} \int_s^{\tau} W_{hj}(t)\sqrt{n_2}\left[\hat{P}_{n_2,2hj}(s,t) - P_{0,2hj}(s,t)\right] d\mu(t), \qquad (1)
\end{aligned}$$

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for any $s \in [0, \tau)$. By conditions C1-C6 and Theorem 2 in (Bakoyannis 2021)

$$\sqrt{n_p}\left[\hat{P}_{n_p,phj}(s,t) - P_{0,phj}(s,t)\right] = \sqrt{n_p}\mathbb{P}_{n_p}\gamma_{phj}(s,t) + \epsilon(t), \qquad p = 1, 2, \tag{2}$$

where $\sup_{t\in[s,\tau]} |\epsilon(t)| = o_p(1)$, and the classes of functions

$$\mathcal{F}_p = \{\gamma_{phj}(s,t) : t \in [s,\tau]\}, \qquad p = 1, 2,$$

are P-Donsker. These facts along with condition C7 lead to the conclusion that

$$\sup_{t\in[s,\tau]} \left|\left[\hat{W}_{hj}(t) - W_{hj}(t)\right]\sqrt{n_p}\left[\hat{P}_{n_p,phj}(s,t) - P_{0,phj}(s,t)\right]\right| = o_p(1), \qquad p = 1, 2.$$

Therefore, by the assumption that

$$\frac{n_1}{n_1+n_2} \to \lambda \in (0,1),$$

as $n_1 \wedge n_2 \to \infty$, it follows that

$$\sqrt{1 - \frac{n_1}{n_1+n_2}}\int_s^\tau \left[\hat{W}_{hj}(t) - W_{hj}(t)\right]\sqrt{n_1}\left[\hat{P}_{n_1,1hj}(s,t) - P_{0,1hj}(s,t)\right]d\mu(t) = o_p(1)$$

and

$$\sqrt{\frac{n_1}{n_1+n_2}}\int_s^\tau \left[\hat{W}_{hj}(t) - W_{hj}(t)\right]\sqrt{n_2}\left[\hat{P}_{n_2,2hj}(s,t) - P_{0,2hj}(s,t)\right]d\mu(t) = o_p(1).$$

Next, the map $\phi : D[s,\tau] \mapsto \mathbb{R}$, defined as

$$\phi(\theta) = \int_s^\tau \theta(t)d\mu(t), \qquad \theta \in D[s,\tau],$$

satisfies

$$\left|\frac{\phi(\theta + t_n h_n) - \phi(\theta)}{t_n} - \int_s^\tau h(t)d\mu(t)\right| \leq \sup_{t\in[s,\tau]} |h_n(t) - h(t)|(\tau - s) \to 0$$

for all sequences $t_n \to 0$ and $\sup_{t\in[s,\tau]} |h_n(t) - h(t)| \to 0$ with $\theta + t_n h_n \in D[s,\tau]$ for all n and, thus, this map is Hadamard-differentiable at $\theta \in D[s,\tau]$ with derivative

$$\phi'_\theta(h) = \int_s^\tau h(t)d\mu(t).$$

Therefore, by (2), condition C7, the continuous mapping theorem, and the stronger assertion of the functional delta method (van der Vaart & Wellner 1996), it follows that

$$\sqrt{1 - \frac{n_1}{n_1+n_2}}\int_s^\tau W_{hj}(t)\sqrt{n_1}\left[\hat{P}_{n_1,1hj}(s,t) - P_{0,1hj}(s,t)\right]d\mu(t) =$$

$$\sqrt{1-\lambda}\sqrt{n_1}\mathbb{P}_{n_1}\int_s^\tau W_{hj}(t)\gamma_{1hj}(s,t)d\mu(t) + o_p(1),$$

and

$$\sqrt{\frac{n_1}{n_1+n_2}}\int_s^\tau W_{hj}(t)\sqrt{n_2}\left[\hat{P}_{n_2,2hj}(s,t) - P_{0,2hj}(s,t)\right]d\mu(t) = \sqrt{\lambda}\sqrt{n_2}\mathbb{P}_{n_2}\int_s^\tau W_{hj}(t)\gamma_{2hj}(s,t)d\mu(t) + o_p(1).$$

Thus, by (1) it follows that

$$\begin{aligned}\sqrt{\frac{n_1 n_2}{n_1+n_2}}Z_{n_1,n_2,hj}(s) = {} & \sqrt{1-\lambda}\sqrt{n_1}\mathbb{P}_{n_1}\int_s^\tau W_{hj}(t)\gamma_{1hj}(s,t)d\mu(t) \\ & -\sqrt{\lambda}\sqrt{n_2}\mathbb{P}_{n_2}\int_s^\tau W_{hj}(t)\gamma_{2hj}(s,t)d\mu(t) + o_p(1).\end{aligned}$$

By conditions C2, C3, and C5-C7, the Donsker property of the classes $\mathcal{F}_p$, $p = 1, 2$, and corollary 9.32 in Kosorok (2008), it follows that the classes

$$\{W_{hj}(t)\gamma_{phj}(s,t) : t \in [s,\tau]\}, \quad p = 1, 2,$$

are P-Donsker. By this and Lemma 15.10 in Kosorok (2008) it follows that the classes

$$\left\{\int_s^t W_{hj}(t)\gamma_{phj}(s,t)d\mu(t) : t \in [s,\tau]\right\}, \quad p = 1, 2,$$

are P-Donsker and this implies that

$$\sqrt{n_p}\mathbb{P}_{n_p}\int_s^\tau W_{hj}(t)\gamma_{phj}(s,t)d\mu(t) \rightsquigarrow G_{hj}^{(p)}(s), \quad p = 1, 2,$$

where $G_{hj}^{(p)}(s)$ follows a normal distribution with mean zero and variance

$$P\left[\int_s^\tau W_{hj}(t)\gamma_{phj}(s,t)d\mu(t)\right]^2.$$

Finally, the independence between the two groups and Slutsky's theorem complete the proof of the assertion (i) in Theorem 1. The proof of assertion (ii) follows from similar arguments.

Proof of Theorem 2

Using similar arguments to those used in the proof of Theorem 1 and under the null $H_0 : P_{0,1hj}(s,\cdot) = P_{0,2hj}(s,\cdot)$, it follows that

$$\begin{aligned}\sqrt{\frac{n_1n_2}{n_1+n_2}}\hat{W}_{hj}(t)\hat{\Delta}_{n_1,n_2,hj}(s,t) = \sqrt{1-\lambda}\sqrt{n_1}\mathbb{P}_{n_1}W_{hj}(t)\gamma_{1hj}(s,t)\\ -\sqrt{\lambda}\sqrt{n_2}\mathbb{P}_{n_2}W_{hj}(t)\gamma_{2hj}(s,t) + o_p(1),\end{aligned}$$

as $n_1 \wedge n_2 \to \infty$. Now, the first result in assertion (i) follows from the Donsker property of the classes $\{W_{hj}(t)\gamma_{1hj}(s,t) : t \in [s,\tau]\}$ and $\{W_{hj}(t)\gamma_{phj}(s,t) : t \in [s,\tau]\}$, as argued in the proof of Theorem 1, and the independence between the two groups.

For the second result in assertion (i), define the multiplier process

$$\begin{aligned}\tilde{C}_{n_1,n_2,hj}(s,t) = \sqrt{1-\lambda}W_{hj}(t)\sqrt{n_1}\mathbb{P}_{n_1}\gamma_{1hj}(s,t)\xi_1\\ -\sqrt{\lambda}W_{hj}(t)\sqrt{n_2}\mathbb{P}_{n_2}\gamma_{2hj}(s,t)\xi_2, \quad t \in [s,\tau],\end{aligned}$$

and define $\mathbb{G}_{hj}(s,\cdot) \equiv \sqrt{1-\lambda}\mathbb{G}_{1hj}(s,\cdot) - \sqrt{\lambda}\mathbb{G}_{2hj}(s,\cdot)$, for simplicity. Also, let BL_1 denote the set of all Lipschitz functionals $h : D[s,\tau] \mapsto [0,1]$ with Lipschitz norm bounded by 1 and let E_ξ denote conditional expectation over the random variable ξ conditionally on the data. Now,

$$\begin{aligned}\sup_{h\in BL_1}\left|E_\xi h[\hat{C}_{n_1,n_2,hj}(s,\cdot)] - Eh[\mathbb{G}_{hj}(s,\cdot)]\right| \le \sup_{h\in BL_1}\left|E_\xi h[\hat{C}_{n_1,n_2,hj}(s,\cdot)] - E_\xi h[\tilde{C}_{n_1,n_2,hj}(s,\cdot)]\right|\\ + \sup_{h\in BL_1}\left|E_\xi h[\tilde{C}_{n_1,n_2,hj}(s,\cdot)] - Eh[\mathbb{G}_{hj}(s,\cdot)]\right|. \qquad (3)\end{aligned}$$

To complete the proof of the second result in assertion (i) of Theorem 2 it suffices to show that the right side of (3) converges to 0 in probability (Kosorok 2008). For the first term in the right side of (3) we have

$$\begin{aligned}\sup_{h\in BL_1}\left|E_\xi h[\hat{C}_{n_1,n_2,hj}(s,\cdot)] - E_\xi h[\tilde{C}_{n_1,n_2,hj}(s,\cdot)]\right| \le \sup_{h\in BL_1}E_\xi\left|h[\hat{C}_{n_1,n_2,hj}(s,\cdot)] - h[\tilde{C}_{n_1,n_2,hj}(s,\cdot)]\right|\\ \le E_\xi\left\{\sup_{h\in BL_1}\left|h[\hat{C}_{n_1,n_2,hj}(s,\cdot)] - h[\tilde{C}_{n_1,n_2,hj}(s,\cdot)]\right|\right\}^\star, \qquad (4)\end{aligned}$$

where the first inequality follows from Jensen's inequality and the notation $Y^\star$ is used to denote the minimal measurable majorant of the (possibly non-measurable) variable Y (van der Vaart & Wellner 1996; Kosorok 2008). Note that,

$$\begin{aligned}\left\{\sup_{h\in BL_1}\left|h[\hat{C}_{n_1,n_2,hj}(s,\cdot)]-h[\tilde{C}_{n_1,n_2,hj}(s,\cdot)]\right|\right\}^\star &\leq \left[\sup_{t\in[s,\tau]}\left|\hat{C}_{n_1,n_2,hj}(s,t)-\tilde{C}_{n_1,n_2,hj}(s,t)\right|\right]^\star \\ &\leq \sum_{p=1}^{2}\left\{\sup_{t\in[s,\tau]}\left|\hat{W}_{hj}(t)-W_{hj}(t)\right|\right. \\ &\quad\left.\times\sup_{t\in[s,\tau]}\left|\sqrt{n_p}\mathbb{P}_{n_p}\left[\hat{\gamma}_{phj}(s,t)-\gamma_{phj}(s,t)\right]\xi_p\right|\right\}^\star \\ &\quad+c_1\sum_{p=1}^{2}\left\{\sup_{t\in[s,\tau]}\left|\sqrt{n_p}\mathbb{P}_{n_p}\left[\hat{\gamma}_{phj}(s,t)-\gamma_{phj}(s,t)\right]\xi_p\right|\right\}^\star \\ &\quad+\sum_{p=1}^{2}\left\{\sup_{t\in[s,\tau]}\left|\left[\hat{W}_{hj}(t)-W_{hj}(t)\right]\sqrt{n_p}\mathbb{P}_{n_p}\gamma_{phj}(s,t)\xi_p\right|\right\}^\star \\ &\equiv I^{(1)}_{n_1,n_2}+I^{(2)}_{n_1,n_2}+I^{(3)}_{n_1,n_2}, \end{aligned} \tag{5}$$

where c_1 is a constant satisfying $\sup_{t\in[0,\tau]}|W_{hj}(t)|\leq c_1$, in light of condition C7. By the Donsker property of the classes $\mathcal{F}_p$, $p=1,2$, the unconditional multiplier central limit theorem (Kosorok 2008), condition C7, and recognizing that, in empirical process theory, convergence in probability is defined based on the minimal measurable majorant of a metric (Kosorok 2008), it follows that $I^{(3)}_{n_1,n_2}\xrightarrow{p}0$. Next, using similar calculations to those used in the proof of Theorem 2 in Bakoyannis (2021), it follows that

$$\sup_{t\in[s,\tau]}\left|\sqrt{n_p}\mathbb{P}_{n_p}\left[\hat{\gamma}_{phj}(s,t)-\gamma_{phj}(s,t)\right]\xi_p\right|\xrightarrow{p}0,\qquad p=1,2,$$

and, thus, $I^{(2)}_{n_1,n_2}\xrightarrow{p}0$. Similar arguments and condition C7 lead to the conclusion that $I^{(1)}_{n_1,n_2}\xrightarrow{p}0$ and, therefore, by (5)

$$\left\{\sup_{h\in BL_1}\left|h[\hat{C}_{n_1,n_2,hj}(s,\cdot)]-h[\tilde{C}_{n_1,n_2,hj}(s,\cdot)]\right|\right\}^\star\xrightarrow{p}0.$$

Now, an application of the dominated convergence theorem and inequality (4) lead to the conclusion that

$$\sup_{h\in BL_1}\left|E_\xi h[\hat{C}_{n_1,n_2,hj}(s,\cdot)]-E_\xi h[\tilde{C}_{n_1,n_2,hj}(s,\cdot)]\right|\xrightarrow{p}0.$$

For the second term in the right side of (3), condition C7, the independence between the two groups, the Donsker property of the classes $\mathcal{F}_p$, $p=1,2$, as argued in the proof of Theorem 1, and Theorem 2.9.6 in van der Vaart & Wellner (1996) imply that

$$\sup_{h\in BL_1}\left|E_\xi h[\tilde{C}_{n_1,n_2,hj}(s,\cdot)]-Eh[\mathbb{G}_{hj}(s,\cdot)]\right|\xrightarrow{p}0$$

and, thus, by (3) the proof of the second result in assertion (i) of Theorem 2 is complete.

For the conditional weak convergence of the cluster bootstrap (third result in assertion (i)), note that a cluster bootstrap version of the estimator $\hat{\mathbf{P}}_{n_p,p}(s,t)$ is

$$\hat{\mathbf{P}}^*_{n_p,p}(s,t)=\prod_{(s,t]}\left[\mathbf{I}_S+d\hat{\mathbf{A}}^*_{n_p,p}(u)\right],\qquad p=1,2,\quad 0\leq s\leq t\leq\tau,$$

where $\hat{\mathbf{A}}^*_{n_p,p}(t)$ consists of the elements

$$\hat{A}^*_{n_p,phj}(t)=\int_0^t\frac{\sum_{i=1}^{n_p}O_{n_p,i}\sum_{m=1}^{M_{ip}}dN_{ipm,hj}(u)}{\sum_{i=1}^{n_p}O_{n_p,i}\sum_{m=1}^{M_{ip}}Y_{ipm,h}(u)},\qquad h\neq j,$$

where $(O_{n_p,1},\dots,O_{n_p,n_p})$ is a random vector following the multinomial distribution with n_p trials and probabilities of success $1/n_p$ for each trial, and $\hat{A}^*_{n_p,phh}(t) = -\sum_{j\neq h}\hat{A}^*_{n_p,phj}(t)$. The cluster bootstrap versions $\hat{\mathbf{P}}'^*_{n_p,p}(s,t)$, $\hat{P}^*_{n_p,pj}(s)$, and $\hat{P}'^*_{n_p,pj}(s)$, are defined similarly. Now, denoting conditional expectation over the multinomial weights O conditionally on the observed data by E_O, we have

$$\begin{aligned}
&\sup_{h\in BL_1}\left|E_O h\left\{\sqrt{\frac{n_1n_2}{n_1+n_2}}\hat{W}_{hj}(\cdot)[\hat{\Delta}^*_{n_1,n_2,hj}(s,\cdot)-\hat{\Delta}_{n_1,n_2,hj}(s,\cdot)]\right\}-Eh[\mathbb{G}_{hj}(s,\cdot)]\right| \\
&\le \sup_{h\in BL_1}\left|E_O h\left\{\sqrt{\frac{n_1n_2}{n_1+n_2}}\hat{W}_{hj}(\cdot)[\hat{\Delta}^*_{n_1,n_2,hj}(s,\cdot)-\hat{\Delta}_{n_1,n_2,hj}(s,\cdot)]\right\}\right. \\
&\quad\left.-E_O h\left\{\sqrt{\frac{n_1n_2}{n_1+n_2}}W_{hj}(\cdot)[\hat{\Delta}^*_{n_1,n_2,hj}(s,\cdot)-\hat{\Delta}_{n_1,n_2,hj}(s,\cdot)]\right\}\right| \\
&\quad+\sup_{h\in BL_1}\left|E_O h\left\{\sqrt{\frac{n_1n_2}{n_1+n_2}}W_{hj}(\cdot)[\hat{\Delta}^*_{n_1,n_2,hj}(s,\cdot)-\hat{\Delta}_{n_1,n_2,hj}(s,\cdot)]\right\}-Eh[\mathbb{G}_{hj}(s,\cdot)]\right|. \qquad (6)
\end{aligned}$$

For the first term in the right side of (6) we have

$$\begin{aligned}
&\sup_{h\in BL_1}\left|E_O h\left\{\sqrt{\frac{n_1n_2}{n_1+n_2}}\hat{W}_{hj}(\cdot)[\hat{\Delta}^*_{n_1,n_2,hj}(s,\cdot)-\hat{\Delta}_{n_1,n_2,hj}(s,\cdot)]\right\}\right. \\
&\quad\left.-E_O h\left\{\sqrt{\frac{n_1n_2}{n_1+n_2}}W_{hj}(\cdot)[\hat{\Delta}^*_{n_1,n_2,hj}(s,\cdot)-\hat{\Delta}_{n_1,n_2,hj}(s,\cdot)]\right\}\right| \\
&\le \sup_{h\in BL_1}E_O\left|h\left\{\sqrt{\frac{n_1n_2}{n_1+n_2}}\hat{W}_{hj}(\cdot)[\hat{\Delta}^*_{n_1,n_2,hj}(s,\cdot)-\hat{\Delta}_{n_1,n_2,hj}(s,\cdot)]\right\}\right. \\
&\quad\left.-h\left\{\sqrt{\frac{n_1n_2}{n_1+n_2}}W_{hj}(\cdot)[\hat{\Delta}^*_{n_1,n_2,hj}(s,\cdot)-\hat{\Delta}_{n_1,n_2,hj}(s,\cdot)]\right\}\right| \\
&\le E_O\left(\sup_{h\in BL_1}\left|h\left\{\sqrt{\frac{n_1n_2}{n_1+n_2}}\hat{W}_{hj}(\cdot)[\hat{\Delta}^*_{n_1,n_2,hj}(s,\cdot)-\hat{\Delta}_{n_1,n_2,hj}(s,\cdot)]\right\}\right.\right. \\
&\quad\left.\left.-h\left\{\sqrt{\frac{n_1n_2}{n_1+n_2}}W_{hj}(\cdot)[\hat{\Delta}^*_{n_1,n_2,hj}(s,\cdot)-\hat{\Delta}_{n_1,n_2,hj}(s,\cdot)]\right\}\right|\right)^\star, \qquad (7)
\end{aligned}$$

where the first inequality follows from Jensen's inequality. Now,

$$\begin{aligned}
&\left(\sup_{h\in BL_1}\left|h\left\{\sqrt{\frac{n_1n_2}{n_1+n_2}}\hat{W}_{hj}(\cdot)[\hat{\Delta}^*_{n_1,n_2,hj}(s,\cdot)-\hat{\Delta}_{n_1,n_2,hj}(s,\cdot)]\right\}\right.\right. \\
&\quad\left.\left.-h\left\{\sqrt{\frac{n_1n_2}{n_1+n_2}}W_{hj}(\cdot)[\hat{\Delta}^*_{n_1,n_2,hj}(s,\cdot)-\hat{\Delta}_{n_1,n_2,hj}(s,\cdot)]\right\}\right|\right)^\star \\
&\le \left\{\sup_{t\in[s,\tau]}\left|[\hat{W}_{hj}(t)-W_{hj}(t)]\sqrt{\frac{n_1n_2}{n_1+n_2}}[\hat{\Delta}^*_{n_1,n_2,hj}(s,\cdot)-\hat{\Delta}_{n_1,n_2,hj}(s,\cdot)]\right|\right\}^\star. \qquad (8)
\end{aligned}$$

By the unconditional multiplier central limit theorem (Kosorok 2008), the Hadamard differentiability of the Nelson–Aalen integral (Kosorok 2008) and the product integral (Andersen et al. 2012), and a double application of the functional delta method (Kosorok 2008), it follows that

$$\sqrt{\frac{n_1n_2}{n_1+n_2}}[\hat{\Delta}^*_{n_1,n_2,hj}(s,\cdot)-\hat{\Delta}_{n_1,n_2,hj}(s,\cdot)] \rightsquigarrow \sqrt{1-\lambda}\tilde{\mathbb{G}}_{1hj}(s,\cdot)-\sqrt{\lambda}\tilde{\mathbb{G}}_{2hj}(s,\cdot) \ \text{ in } \ D[s,\tau],$$

(unconditionally) where $\tilde{\mathbb{G}}_{phj}$, $p=1,2$, is a tight zero-mean Gaussian process with covariance function

$$P[\gamma_{phj}(s,t_1)\gamma_{phj}(s,t_2)], \quad \text{for} \quad t_1,t_2\in[s,\tau].$$

This and condition C7 lead to the conclusion that

$$\left\{\sup_{t\in[s,\tau]}\left|[\hat{W}_{hj}(t)-W_{hj}(t)]\sqrt{\frac{n_1n_2}{n_1+n_2}}[\hat{\Delta}^*_{n_1,n_2,hj}(s,\cdot)-\hat{\Delta}_{n_1,n_2,hj}(s,\cdot)]\right|\right\}^\star \xrightarrow{p} 0.$$

Now, by (8), the dominated convergence theorem, and (7), it follows that

$$\sup_{h\in BL_1}\left|E_O h\left\{\sqrt{\frac{n_1n_2}{n_1+n_2}}\hat{W}_{hj}(\cdot)[\hat{\Delta}^*_{n_1,n_2,hj}(s,\cdot)-\hat{\Delta}_{n_1,n_2,hj}(s,\cdot)]\right\}\right.$$
$$\left.-E_O h\left\{\sqrt{\frac{n_1n_2}{n_1+n_2}}W_{hj}(\cdot)[\hat{\Delta}^*_{n_1,n_2,hj}(s,\cdot)-\hat{\Delta}_{n_1,n_2,hj}(s,\cdot)]\right\}\right| \xrightarrow{p} 0.$$

For second term in the right side of (6), the assumption that

$$\frac{n_1}{n_1+n_2}\to\lambda\in(0,1),$$

as $n_1\wedge n_2\to\infty$, the independence between the two groups, a double application of Theorem 2 in Bakoyannis (2021), and condition C7 along with the bootstrap continuous mapping theorem (Theorem 10.8 in Kosorok 2008), lead to the conclusion that

$$\sup_{h\in BL_1}\left|E_O h\left\{\sqrt{\frac{n_1n_2}{n_1+n_2}}W_{hj}(\cdot)[\hat{\Delta}^*_{n_1,n_2,hj}(s,\cdot)-\hat{\Delta}_{n_1,n_2,hj}(s,\cdot)]\right\}-Eh[\mathbb{G}_{hj}(s,\cdot)]\right|\xrightarrow{p} 0.$$

Therefore, by (6), the proof of the third result in assertion (i) of Theorem 2 is complete. Assertion (ii) can be proven similarly using the same arguments.

Proof of Theorem 3

Under the null hypothesis $H_0: P_{0,1hj}(s,\cdot)=P_{0,2hj}(s,\cdot)$, it follows that,

$$\begin{aligned}
\sqrt{n}Z_{n,hj}(s) &= \sqrt{n}\int_s^\tau \hat{W}_{hj}(t)\left[\hat{P}_{n,1hj}(s,t)-\hat{P}_{n,2hj}(s,t)\right]d\mu(t)\\
&\quad -\sqrt{n}\int_s^\tau \hat{W}_{hj}(t)\left[P_{0,1hj}(s,t)-P_{0,2hj}(s,t)\right]d\mu(t)\\
&= \int_s^\tau\left[\hat{W}_{hj}(t)-W_{hj}(t)\right]\sqrt{n}\left[\hat{P}_{n,1hj}(s,t)-P_{0,1hj}(s,t)\right]d\mu(t)\\
&\quad +\int_s^\tau W_{hj}(t)\sqrt{n}\left[\hat{P}_{n,1hj}(s,t)-P_{0,1hj}(s,t)\right]d\mu(t)\\
&\quad -\int_s^\tau\left[\hat{W}_{hj}(t)-W_{hj}(t)\right]\sqrt{n}\left[\hat{P}_{n,2hj}(s,t)-P_{0,2hj}(s,t)\right]d\mu(t)\\
&\quad -\int_s^\tau W_{hj}(t)\sqrt{n}\left[\hat{P}_{n,2hj}(s,t)-P_{0,2hj}(s,t)\right]d\mu(t),
\end{aligned}\tag{9}$$

for any $s\in[0,\tau)$. By (2), the Donsker property of the classes $\mathcal{F}_p$, $p=1,2$, and condition C7 , it follows that

$$\sup_{t\in[s,\tau]}\left|\left[\hat{W}_{hj}(t)-W_{hj}(t)\right]\sqrt{n}\left[\hat{P}_{n,phj}(s,t)-P_{0,phj}(s,t)\right]\right|=o_p(1),\qquad p=1,2,$$

and, therefore,

$$\int_s^\tau\left[\hat{W}_{hj}(t)-W_{hj}(t)\right]\sqrt{n}\left[\hat{P}_{n,phj}(s,t)-P_{0,phj}(s,t)\right]d\mu(t)=o_p(1),\qquad p=1,2.$$

Similarly to the arguments in the proof of Theorem 1, utilizing the stronger assertion of the functional delta method (van der Vaart & Wellner 1996), conditions C1-C7, and Theorem 2 in Bakoyannis (2021), it follows that

$$\sqrt{n}Z_{n,hj}(s)=\sqrt{n}\mathbb{P}_n\int_s^\tau W_{hj}(t)[\gamma_{1hj}(s,t)-\gamma_{2hj}(s,t)]d\mu(t)+o_p(1),$$

by (9). By conditions C2, C3, and C5-C7, the Donsker property of the classes $\mathcal{F}_p$, $p=1,2$, and corollary 9.32 in Kosorok (2008), it follows that the class

$$\{W_{hj}(t)[\gamma_{1hj}(s,t)-\gamma_{2hj}(s,t)]:t\in[s,\tau]\},$$

is P-Donsker. By this and Lemma 15.10 in Kosorok (2008), it follows that the class

$$\left\{\int_s^t W_{hj}(t)[\gamma_{1hj}(s,t)-\gamma_{2hj}(s,t)]d\mu(t):t\in[s,\tau]\right\},$$

is P-Donsker. This, (9), and Slutsky's theorem imply that

$$\sqrt{n}Z_{n,hj}(s)\rightsquigarrow Z_{hj}(s),$$

where $Z_{hj}^{(p)}(s)$ follows a normal distribution with mean zero and variance

$$P\left[\int_s^\tau W_{hj}(t)[\gamma_{1hj}(s,t)-\gamma_{2hj}(s,t)]d\mu(t)\right]^2,$$

which completes the proof of assertion (i) in Theorem 3. The proof of assertion (ii) follows from similar arguments.

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