

Bahadur efficiency of the maximum likelihood estimator and one-step estimator for quasi-arithmetic means of the Cauchy distribution

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Abstract

Some quasi-arithmetic means of random variables easily give unbiased strongly consistent closed-form estimators of the joint of the location and scale parameters of the Cauchy distribution. The one-step estimators of those quasi-arithmetic means of the Cauchy distribution are considered. We establish the Bahadur efficiency of the maximum likelihood estimator and the one-step estimators. We also show that the rate of the convergence of the mean-squared errors achieves the Cramér–Rao bound. Our results are also applicable to the circular Cauchy distribution .

Keywords Bahadur efficiency · Cauchy distribution · Maximum likelihood estimator · One-step estimator

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1 Introduction

In the parameter estimation of the location $\mu \in \mathbb{R}$ and the scale $\sigma > 0$ of the Cauchy distribution, it is difficult to balance efficiency with computational difficulty. So far, various approaches have been taken. The maximal likelihood estimation (MLE) has been considered by Haas et al. (1970); Copas (1975); Ferguson (1978); Gabrielsen (1982); Reeds (1985); Saleh et al. (1985); Bai and Fu (1987); Vaughan (1992); McCullagh (1992, 1993, 1996); Matsui and Takemura (2005). The order statistics, which includes analysis for central values and quantiles, is used in Ogawa (1962a, 1962b); Rothenberg et al. (1964); Barnett (1966); Bloch (1966); Chan (1970); Balmer et al. (1974); Cane (1974); Rublik (2001); Zhang (2009); Kravchuk and Pollett (2012). Other approaches are taken by Howlader and Weiss (1988); Higgins and Tichenor (1977, 1978); Boos (1981); Gürtler and Henze (2000); Besbeas and Morgan (2001); Onen et al. (2001); Kravchuk (2005); Cohen Freue (2007). Results obtained before 1994 are thoroughly surveyed in the book by Chapter 16 in Johnson et al. (1994).

In Akaoka et al. (2021b), the authors suggest new estimators of the parameters of the Cauchy distribution in the case that neither the location μ nor the scale σ is known by dealing with quasi-arithmetic means of independent and identically distributed (i.i.d.) random variables. For the parameter estimation of the Cauchy distribution, some of the quasi-arithmetic means of a sample have closed-forms, and are unbiased and strongly consistent, under McCullagh's parametrization McCullagh (1993, 1996). Those quasi-arithmetic means are easy to construct and analyze rigorously. Indeed, in Akaoka et al. (2021a), the authors construct confidence discs for $\mu + \sigma i$ without numerical analysis.

Considering the one-step estimators of \sqrt{n} -consistent estimators is a useful way to obtain efficient estimators. It is well known that under some conditions, they achieve the Cramér–Rao bound via the central limit theorem for the one-step estimators (See Section 7.3 in Lehmann (1999)). Our one-step estimators are simple and easy to calculate, according that the initial estimators, which are quasi-arithmetic means here, are simple and easy to calculate. We also show that our one-step estimators and the MLE are efficient in the Bahadur sense, which concerns large deviation estimates. Our proof of the case of the MLE depends on Arcones (2006), Shen (2001) and the explicit formula of the Kullback–Leibler divergence between the Cauchy distributions recently obtained by Chyzak and Nielsen (2019). We also show that the rate of the convergence of the mean-squared errors achieves the Cramér–Rao bound. Our results are also applicable to the circular Cauchy distribution, which are closely connected with the Cauchy distribution via the Möbius transformations.

This paper is organized as follows. In Sect. 2, we give some preliminary results for tail estimates of quasi-arithmetic means. In Sect. 3, we establish the Bahadur efficiency of the MLE. In Sect. 4, we establish the Bahadur efficiency and the rate of the convergence of the mean-squared errors of the one-step estimators of the quasi-arithmetic means, which are the main results of this paper. In Sect. 5, we give an application of parameter estimation for the circular Cauchy distribution. Section 6 is devoted to proofs of the results in Sects. 2, 3, 4 and 5. In Appendix, we give numerical computations for the mean-squared errors of the one-step estimators in Sect. 4.

1.1 Framework

Let \mathbb{H} be the upper-half plane and $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{R}$. Let *i* be the imaginary unit. We often denote $\theta := \mu + \sigma i$. For $\theta = \mu + \sigma i$, we let

$$P_{\theta}(dx) = p(x;\theta)dx, \ p(x;\theta) = p(x;(\mu,\sigma)) := \frac{\sigma}{\pi} \frac{1}{(x-\mu)^2 + \sigma^2}.$$

For $\alpha \in \mathbb{C}$, we denote its complex conjugate by $\overline{\alpha}$.

Let $(X_i)_i$ be an i.i.d. sequence of random variables with distribution $C(\mu, \sigma)$ on a probability space. We often denote the distribution following P_{θ} by $C(\theta)$.

We deal with the quasi-arithmetic means of the i.i.d. random variables. This has the form that

$$f^{-1}\left(\frac{1}{n}\sum_{j=1}^n f(X_j)\right),\,$$

where U is a domain containing $\overline{\mathbb{H}} \setminus \{\alpha\}$ for some $\alpha \in \{x + yi : y \le 0\}$ and $f : U \to \mathbb{C}$ is a continuous injective holomorphic function such that $f(\overline{\mathbb{H}} \setminus \{\alpha\})$ is convex, $\lim_{z\to\alpha} (z-\alpha)f(z) = 0$ and furthermore f has sublinear growth at infinity, specifically, $\lim_{|z|\to\infty} f(z)/|z| = 0$.

We remark that f is not only in the C^{∞} class but also holomorphic on a domain containing $\overline{\mathbb{H}} \setminus \{\alpha\}$. By the residue theorem,

$$E[f(X_1)] = f(\theta). \tag{1}$$

If $\operatorname{Im}(\alpha) < 0$, then, $\overline{\mathbb{H}} \setminus \{\alpha\} = \overline{\mathbb{H}}$. On the other hand, if $\operatorname{Im}(\alpha) = 0$, then, $\overline{\mathbb{H}} \setminus \{\alpha\} \subseteq \overline{\mathbb{H}}$. See Akaoka et al. (2021b) for more details.

In this paper we additionally assume that there exists a constant $\lambda > 0$ such that

$$E\left[\exp\left(\lambda \left| f(X_1) \right| \right)\right] < +\infty.$$
⁽²⁾

Akaoka et al. (2021b) deals with quasi-arithmetic means with its generators of the form

$$f(x) = \log(x + \alpha), \ \alpha \in \mathbb{H},$$

or

$$f(x) = \frac{1}{x + \alpha}, \quad \alpha \in \mathbb{H},$$

each of which corresponds to the geometric mean and a modification of the harmonic mean. We remark that $\alpha \in \mathbb{R}$ is allowed when we deal with the case that $f(x) = \log(x + \alpha)$. (1) and (2) hold for the case that $f(x) = \log(x + \alpha)$, $\alpha \in \overline{\mathbb{H}}$ and that $f(x) = 1/(x + \alpha)$, $\alpha \in \mathbb{H}$. We remark that the quasi-arithmetic means with generator $f(x) = 1/(x + \alpha)$ are identical with the quasi-arithmetic means with generator

$$f(x) = \frac{x + \overline{\alpha}}{x + \alpha} = \frac{x - \beta}{x - \overline{\beta}}, \ \beta = -\overline{\alpha}.$$

See Example 1.2 (ii) in Akaoka et al. (2021b) for more details.

2 Large deviations for quasi-arithmetic means

We first give a decay rate of quasi-arithmetic means. We assume that (1) and (2) hold.

Theorem 1 *There exist positive constants* c_1 *and* c_2 *such that for every* $\epsilon \in (0, \sigma)$ *and every* $n \ge 2$,

$$P\left(\left|f^{-1}\left(\frac{1}{n}\sum_{j=1}^{n}f(X_{j})\right)-\theta\right|>\epsilon\right)\leq c_{1}\exp\left(-c_{2}n\epsilon^{2}\right).$$

This also plays a crucial role in the proof of the Bahadur efficiency of the onestep estimator with initial estimator.

By the contraction principle, we have the following:

Theorem 2 The quasi-arithmetic means $f^{-1}\left(\frac{1}{n}\sum_{j=1}^{n}f(X_j)\right)$ satisfy the large deviation principle with rate function

$$I(z) = \sup_{\lambda_1, \lambda_2 \in \mathbb{R}} \left(\lambda_1 \operatorname{Re}(f(z)) + \lambda_2 \operatorname{Im}(f(z)) - \log E \left[\exp \left(\lambda_1 \operatorname{Re}(f(z)) + \lambda_2 \operatorname{Im}(f(z)) \right) \right] \right),$$

where $z \in \mathbb{H}$.

See Dembo and Zeitouni (2010) for the basic terminologies and results of the theory of large deviations.

Under the consideration of Eq.(1.5) in Akaoka et al. (2021b), we conjecture that

$$\lim_{\epsilon \to +0} \frac{1}{\epsilon^2} \lim_{n \to \infty} \frac{1}{n} P\left(\left| f^{-1}\left(\frac{1}{n} \sum_{j=1}^n f(X_j)\right) - \theta \right| > \epsilon \right) = -\frac{|f'(\theta)|^2}{\operatorname{Var}(f(X_1))}.$$

The right-hand side is called an *inaccuracy rate*. The quasi-arithmetic mean $f^{-1}\left(\frac{1}{n}\sum_{j=1}^{n}f(X_j)\right)$ can be regarded as an M-estimator. Let $\phi(x,\theta) := f(x) - f(\theta)$. Then, $\theta = f^{-1}\left(\frac{1}{n}\sum_{j=1}^{n}f(X_j)\right)$ if and only if $\sum_{j=1}^{n}\phi(X_j,\theta) = 0$.

The inaccuracy rates for one-dimensional M-estimators are considered by Jurečková and Kallenberg (1987). Multidimensional large deviation principles for M-estimators are considered by Arcones (2006). Theorem 3.5 in Arcones (2006) is not applicable to the case the both of the location and the scale are unknown. We do not pursue further properties for large deviations for complex-valued M-estimators here.

3 Bahadur efficiency of the maximum likelihood estimator

Let $\hat{\theta}_n$ be the maximal likelihood estimator (MLE) of $\theta = \mu + \sigma i$. Copas (1975) showed that the joint likelihood function for the location and scale parameters of the Cauchy distribution is unimodal. Let

$$h(x,t) := \frac{x-t}{x-\overline{t}}, x \in \overline{\mathbb{H}}, t \in \mathbb{H}.$$

The MLE $\hat{\theta}_n = \hat{\theta}_n(X_1, \dots, X_n)$ is a unique solution of the following equation for α :

$$\sum_{j=1}^n h(X_j, \alpha) = \sum_{j=1}^n \frac{X_j - \alpha}{X_j - \overline{\alpha}} = 0.$$

This is the likelihood equation in the complex form. See Corollary 2.8 in Okamura and Otobe (2021) for more details and Proposition 2.2 in Okamura and Otobe (2021) for another expression for the likelihood equation in the complex form. For $n \le 4$, the solution of the likelihood equation has a closed form (Ferguson (1978)). However, for $n \ge 5$, we do not have any algebraic closed-form formula (Okamura and Otobe (2021)).

Let the Kullback-Leibler divergence be

$$K(P_{\theta'}|P_{\theta}) := \int_{\mathbb{R}} p(x;\theta') \log \frac{p(x;\theta')}{p(x;\theta)} dx, \ \theta \in \mathbb{H}.$$

Proposition 3 (Chyzak and Nielsen (2019)) For $\theta, \theta' \in \mathbb{H}$,

$$K(P_{\theta'}|P_{\theta}) = \log\left(1 + \frac{|\theta - \theta'|^2}{4\mathrm{Im}(\theta)\mathrm{Im}(\theta')}\right).$$
(3)

We remark that a version of (3) already appears in Eq.(18) in McCullagh (1996). See also the discussions around (8) and (9) below. Let

$$SL(2,\mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}.$$

The quantity $\frac{|\theta - \theta'|^2}{4 \operatorname{Im}(\theta) \operatorname{Im}(\theta')}$ in (3) is the maximal invariant for the action of $SL(2, \mathbb{R})$ to $\mathbb{H} \times \mathbb{H}$ defined by

$$A \cdot (z, w) = \left(\frac{az+b}{cz+d}, \frac{aw+b}{cw+d}\right), \ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}), \ z, w \in \mathbb{H}.$$

See also McCullagh (1993, 1996). In particular we see that

$$K(P_{A\cdot\theta'}|P_{A\cdot\theta}) = K(P_{\theta'}|P_{\theta}).$$

Let

$$b(\epsilon,\theta) := \inf \left\{ K(P_{\theta'}|P_{\theta}) : \theta' \in \mathbb{H}, |\theta' - \theta| > \epsilon \right\}.$$

Then, by (3),

$$b(\epsilon, \theta) = \log\left(1 + \frac{\epsilon^2}{4\mathrm{Im}(\theta)(\mathrm{Im}(\theta) + \epsilon)}\right). \tag{4}$$

This does not depend on the location μ . In particular,

$$\lim_{\epsilon \to +0} \frac{b(\epsilon, \theta)}{\epsilon^2} = \frac{1}{4 \text{Im}(\theta)^2}.$$
(5)

Bahadur et al. (1980) showed that for every consistent estimator T_n for each θ ,

$$\liminf_{n\to\infty} \frac{\log P(|T_n-\theta|>\epsilon)}{n} \ge -b(\epsilon,\theta).$$

See Eq. (4) in Arcones (2006) or Proposition 1 in Shen (2001). It is a natural question whether we can show that

$$\liminf_{n \to \infty} \frac{\log P(|T_n - \theta| > \epsilon)}{n} = -b(\epsilon, \theta), \tag{6}$$

in the case that $(T_n)_n$ is the MLE. Kester and Kallenberg (1986) gave an example of the MLE which does not satisfy (6).

The following theorem along with the above estimate indicates that the MLE is best in the sense of Bahadur efficiency.

Theorem 4 (Bahadur efficiency of the MLE)

$$\limsup_{\epsilon \to +0} \limsup_{n \to \infty} \frac{\log P(|\hat{\theta}_n - \theta| > \epsilon)}{nb(\epsilon, \theta)} = -1.$$

Bai and Fu (1987) considered the MLE of location μ in the case that the scale parameter σ is known. In that case the likelihood equation is given by

$$\sum_{j=1}^{n} \frac{X_j - \mu}{(X_j - \mu)^2 + \sigma^2} = 0$$
(7)

and this equation for μ has multiple solutions (Reeds (1985)), so from this equation itself, we cannot see which μ of the solutions gives the maximum of the log-likelihood. Eq. (2.12) and (2.13) in Bai and Fu (1987) shows that

$$\lim_{n \to \infty} \frac{\log P(|\hat{\theta}_n - \mu| > \epsilon)}{n} = -\beta(\epsilon),$$

where

$$\beta(\epsilon) = \frac{\epsilon^2}{4\sigma^2} \Big(1 + O(\sqrt{\epsilon}) \Big).$$

However, by Theorem 3.2 in Arcones (2006), it holds that $\beta(\epsilon) \neq b(\epsilon, \sigma i)$.

Our strategy of the proof of Theorem 4 is to firstly establish that $P(|\hat{\theta}_n - \theta| > \epsilon)$ decays exponentially for every $\epsilon > 0$, and then to adopt Theorem 3 in Shen (2001). McCullagh (1996) obtains for an asymptotic pointwise lower bound for the density function $p_n(\chi)$ of the MLE $\hat{\theta}_n$ with respect to an invariant measure on \mathbb{H} for the action of the special linear group $SL_2(\mathbb{R})$, under the assumption of existence of the continuous density function. However it might not lead any estimates for the upper bound of $P(|\hat{\theta}_n - \theta| > \epsilon)$.

We still conjecture that the rate function in Theorem 3.8 in Arcones (2006) is correct, although it is not applicable to the Cauchy distribution in its form. The rate function is defined by

$$I_{\theta}(\gamma) := -\inf_{\lambda_1, \lambda_2 \in \mathbb{R}} \log E^{\theta} \Big[\exp \big(\lambda_1 \operatorname{Re}(h(x, \gamma)) + \lambda_2 \operatorname{Im}(h(x, \gamma)) \big) \Big],$$

where $\gamma, \theta \in \mathbb{H}$ and E^{θ} denotes the expectation with respect to P_{θ} . Let $\gamma = \gamma_1 + i\gamma_2$. This might be the limit of $-\log p_n(\chi)/n$, where $\chi := \frac{(\gamma_1 - \mu)^2 + (\gamma_2 - \sigma)^2}{4\gamma_2\sigma}$.

In p.801 in McCullagh (1996), it is stated that

$$\liminf_{n \to \infty} \frac{\log p_n(\chi)}{n} \ge -\log\left(1 + \frac{(\gamma_1 - \mu)^2 + (\gamma_2 - \sigma)^2}{4\gamma_2\sigma}\right) = -K(P_{\gamma}|P_{\theta}).$$
(8)

Since it holds that for Borel measurable set A,

$$P_{\theta}(\hat{\theta}_{n} \in A) = \iint_{A} \frac{1}{4\pi t_{2}^{2}} p_{n} \left(\frac{(t_{1} - \mu)^{2} + (t_{2} - \sigma)^{2}}{4t_{2}\sigma} \right) dt_{1} dt_{2},$$

we would naturally expect that for sufficiently small $\epsilon > 0$,

$$\frac{\log p_n(\chi)}{n} \approx \frac{\log P_{\theta}(\hat{\theta}_n \in B(\gamma, \epsilon))}{n} \approx -I_{\theta}(\gamma), \ n \to \infty.$$
(9)

By Eq.(33) in Arcones (2006), we see that $K(P_{\gamma}|P_{\theta}) \ge I_{\theta}(\gamma)$, which is consistent with (8) and (9). If the conjecture Eq.(18) in McCullagh (1996) is true, then, $K(P_{\gamma}|P_{\theta}) = I_{\theta}(\gamma)$. Furthermore, we can show that $I_{A\cdot\theta}(A \cdot \gamma) = I_{\theta}(\gamma)$ for every

 $A \in SL(2, \mathbb{R})$ by the same technique as in the proof of Lemma 2.3 in Okamura (2020).

4 Bahadur efficiency of the one-step estimator of quasi-arithmetic means

For ease of notation, we let

$$Y_n := f^{-1} \left(\frac{1}{n} \sum_{i=1}^n f(X_i) \right).$$
(10)

Let

$$\Psi(x;\theta) = \Psi(x;(\mu,\sigma)) := \begin{pmatrix} \frac{\partial}{\partial\mu} \log p(x;(\mu,\sigma)) \\ \frac{\partial}{\partial\sigma} \log p(x;(\mu,\sigma)) \end{pmatrix}$$

Let $I_n(\theta) = I_n(\mu, \sigma) := \frac{n}{2\sigma^2}I_2$, that is, the Fisher information matrix for the Cauchy sample of size *n*. Now we regard Y_n as an \mathbb{R}^2 -valued random variable. Then, a version of the one-step estimator of Y_n is given by

$$Z_n := Y_n - I_n (Y_n)^{-1} \sum_{j=1}^n \Psi(X_j; Y_n).$$

Now we rewrite this in terms of the complex parametrization. By recalling the definition of h,

$$\Psi(x;\theta) = \frac{1}{\mathrm{Im}(\theta)|x-\theta|^2} \begin{pmatrix} \mathrm{Im}((x-\theta)^2) \\ \mathrm{Re}((x-\theta)^2) \end{pmatrix} = \frac{1}{\mathrm{Im}(\theta)} \begin{pmatrix} \mathrm{Im}(h(x,\theta)) \\ \mathrm{Re}(h(x,\theta)) \end{pmatrix}$$

Now we see that

$$Z_n = Y_n + \frac{2\mathrm{Im}(Y_n)i}{n} \sum_{j=1}^n h(X_j, Y_n) = Y_n + \frac{2\mathrm{Im}(Y_n)i}{n} \sum_{j=1}^n \frac{X_j - Y_n}{X_j - \overline{Y_n}}.$$
 (11)

The main feature is that the initial estimators $(Y_n)_n$ and the one-step estimators $(Z_n)_n$ have closed-form. Therefore it is easy to compute and do not need numerical approximations, which is contrary to the MLE $(\hat{\theta}_n)_n$. If $\operatorname{Var}(f(X_1)) < +\infty$, then, by the central limit theorem for (Y_n) , which is stated in Theorem 1.5 in Akaoka et al. (2021b), $(Y_n)_n$ is a \sqrt{n} -consistent estimator. Then it holds that

$$\sqrt{n}(Z_n - \theta) \Rightarrow N\left(0, \frac{1}{2\sigma^2}I_2\right), \ n \to \infty,$$
(12)

where \Rightarrow means the convergence in distribution, I_2 denotes the identity matrix of degree 2, and, $N(\cdot, \cdot)$ is the 2-dimensional normal distribution. See Section 7.3 in Lehmann (1999) for details.

The one-step estimators are easily obtained by efficient estimators. In general, the difficulty arising from multiple roots of the likelihood equations, as in (7) above, is overcome by a one-step estimator which only requires a \sqrt{n} -consistent estimator in the initial point. The following is the main result of this paper.

Theorem 5 (Bahadur efficiency of the one-step estimator) If(2) holds, then,

$$\limsup_{\epsilon \to +0} \limsup_{n \to \infty} \frac{\log P(|Z_n - \theta| > \epsilon)}{nb(\epsilon, \theta)} \le -1.$$

The proof is done by estimating $Z_n - \hat{\theta}_n$. This is different from the consideration of one-step estimators by Janssen et al. (1985) in the one-dimensional case. We might be able to consider it for not only the Cauchy distribution but also other distributions, if the tail of the starting point estimator decays exponentially fast.

Let $(T_n)_n$ be a sequence of complex-valued *unbiased* estimators of $\mu + \sigma i$, where T_n is an unbiased estimator for samples of size *n*. Then, Re (T_n) and Im (T_n) are unbiased estimators of μ and σ , respectively. By the Cramér–Rao inequality,

$$n\operatorname{Var}(T_n) \ge 4\operatorname{Im}(\theta)^2.$$
 (13)

The following theorem shows $(Z_n)_n$ achieves the lower bound of (13), although it may not be unbiased.

Theorem 6 (Variance asymptotics for the one-step estimator) If

$$E\Big[\big|f(X_1)\big|^2\Big] < +\infty$$

and

$$\lim_{n \to \infty} n \operatorname{Var}(Y_n) = \frac{\operatorname{Var}(f(X_1))}{\left| f'(\theta) \right|^2},\tag{14}$$

then,

$$\lim_{n \to \infty} nE\left[\left|Z_n - \theta\right|^2\right] = 4 \operatorname{Im}(\theta)^2,$$

in particular,

$$\lim_{n\to\infty} n \operatorname{Var}(Z_n) \le 4 \operatorname{Im}(\theta)^2.$$

(14) is identical with Eq.(1.5) in Akaoka et al. (2021b) and is verified for the case that $f(x) = \log(x + \alpha)$, $\alpha \in \overline{\mathbb{H}}$ and $f(x) = \frac{x - \alpha}{x - \overline{\alpha}}$, $\alpha \in \mathbb{H}$ in Theorems 4.2 and 4.4 in Akaoka et al. (2021b). The proof is done by establishing the uniform integrability

for $\left\{\sqrt{n}(Z_n - \theta)\right\}_n$. Numerical computations for $nE\left[\left|Z_n - \theta\right|^2\right]/\text{Im}(\theta)^2$ and related discussions are given in Appendix.

5 Parameter estimation for the circular Cauchy distribution

In this section, we apply the results in Sect. 4 to parameter estimations for the circular Cauchy distribution. The circular Cauchy distribution, also known as the wrapped Cauchy distribution, appears in the area of directional statistics. It is regarded as a distribution on the unit circle. It is connected with the Cauchy distribution via the Möbius transformations. Such connection is considered by McCullagh (1992, 1996). Recently, in Kato and McCullagh (2020), an extension to the high-dimensional sphere is investigated in terms of the Möbius transformations. The maximum likelihood estimation of the circular Cauchy distribution is attributed to that of the Cauchy distribution. Due to the connection, we can apply our results in Sect. 4 to the circular Cauchy distribution in a simple manner.

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. The circular Cauchy distribution P_w^{cc} with parameter $w \in \mathbb{D}$ is the continuous distribution on $[0, 2\pi)$ with density function

$$p^{\rm cc}(x;w) := \frac{1}{2\pi} \frac{1 - |w|^2}{|\exp(ix) - w|^2}$$

where we have used McCullagh (1996). We estimate the parameter w.

In this section we let ϕ_{α} : $\mathbb{H} \to \mathbb{D}$ be the function defined by

$$\phi_{\alpha}(z) := h(z, \alpha) = \frac{z - \alpha}{z - \overline{\alpha}}.$$

Then ϕ_{α} is a bijection and its inverse is given by

$$\phi_{\alpha}^{-1}(w) = \frac{\alpha - \overline{\alpha}w}{1 - w}, \ w \in \mathbb{D}.$$

Let $\widetilde{X}_n, n \ge 1$, be i.i.d. random variables following the circular Cauchy distribution P_w^{cc} . Let $X_n := \phi_a^{-1}(\exp(i\widetilde{X}_n))$. They are i.i.d. random variables following the Cauchy distribution with parameter $\phi_a^{-1}(w)$. Let Y_n be the quasi-arithmetic mean of X_n with a generator *f* defined by (10). Let Z_n be the one-step estimator of Y_n defined by (11).

Let $W_n := \phi_{\alpha}(Z_n)$. Then, by straightforward computations,

Lemma 7 For $w_1, w_2 \in \mathbb{D}$ and $\alpha \in \mathbb{H}$,

$$K(P_{w_1}^{cc}|P_{w_2}^{cc}) = K\left(P_{\phi_{\alpha}^{-1}(w_1)}|P_{\phi_{\alpha}^{-1}(w_2)}\right) = \log\left(1 + \frac{|w_1 - w_2|^2}{(1 - |w_1|^2)(1 - |w_2|^2)}\right).$$

The following are main results in this section, which correspond to Theorems 5 and 6, respectively.

$$\limsup_{\epsilon \to +0} \limsup_{n \to \infty} \frac{\log P(|W_n - w| > \epsilon)}{n\widetilde{b}(\epsilon, w)} \le -1,$$

Theorem 8

where we let

$$\widetilde{b}(\epsilon, w) := \inf \left\{ K(P_w^{\rm cc} | P_{w'}^{\rm cc}) : |w - w'| > \epsilon \right\}.$$

$$\limsup_{n \to \infty} n \operatorname{Var}(W_n) \le \lim_{n \to \infty} n E \left[|W_n - w|^2 \right] = \left(1 - |w|^2 \right)^2.$$

Theorem 9

The estimator W_n has a simple closed-form, contrary to the MLE for the circular Cauchy distribution. These assertions are somewhat easy consequences of Theorems 5 and 6.

Remark 10 Section 4.3 in Kato and McCullagh (2020) deals with a one-step estimator different from ours, which is applicable to the higher dimensional case.

6 Proofs

6.1 Proofs of assertions in Section 2

Proof of Theorem 1 Since f^{-1} is injective and holomorphic on an open neighborhood of $f(\theta)$, we see that there exists a constant $c_3 = c_3(\mu, \sigma) > 0$ such that for every $\epsilon \in (0, \sigma)$ and every $n \ge 2$,

$$\begin{split} & P\left(\left|f^{-1}\left(\frac{1}{n}\sum_{j=1}^{n}f(X_{j})\right)-\theta\right|>\epsilon\right)\leq P\left(\left|\frac{1}{n}\sum_{i=1}^{n}f(X_{i})-f(\theta)\right|>c_{3}\epsilon\right)\\ & \leq P\left(\left|\frac{1}{n}\sum_{i=1}^{n}\operatorname{Re}(f(X_{i}))-\operatorname{Re}(f(\theta))\right|>\frac{c_{3}\epsilon}{2}\right)\\ & + P\left(\left|\frac{1}{n}\sum_{i=1}^{n}\operatorname{Im}(f(X_{i}))-\operatorname{Im}(f(\theta))\right|>\frac{c_{3}\epsilon}{2}\right). \end{split}$$

By (1) and (2), we can apply the Cramér–Chernoff method and have that there exist positive constants c_4 and c_5 such that for every $\epsilon \in (0, \sigma)$ and every $n \ge 2$,

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n}\operatorname{Re}(f(X_{i}))-\operatorname{Re}(f(\theta))\right|>\frac{c_{3}\epsilon}{2}\right)\leq c_{4}\exp(-c_{5}n\epsilon^{2}),$$

and

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n}\operatorname{Im}(f(X_{i})) - \operatorname{Im}(f(\theta))\right| > \frac{c_{3}\epsilon}{2}\right) \le c_{4}\exp(-c_{5}n\epsilon^{2})$$

Now we have the assertion.

6.2 Proofs of assertions in Section 3

Let H(t) := E[h(X, t)] for X following $C(\theta)$. Then, by the Cauchy integral formula,

$$H(t) = \frac{\theta - t}{\theta - \overline{t}} = h(\theta, t).$$

It is continuous on \overline{H} . H(t) = 0 if and only if $t = \theta$.

Lemma 11 There exists a compact set K of \mathbb{H} and a constant $\alpha \in (0, 1)$ such that $\theta \in K$ and

$$P(\hat{\theta}_n \notin K) = O(\alpha^n).$$

Proof Let $F_{\theta}(z) := h(z, \theta)$ and

$$K_m := F_{\theta}^{-1} \left(\left\{ z \in \mathbb{C} : |z|^2 \le \frac{m}{m+1} \right\} \right), \ m \ge 1.$$

We see that each K_m is a compact subset of \mathbb{H} and $\theta \in K_1$. By the definition of H, we see that

$$\inf_{t \notin K_1} |H(t)| > 0.$$

Let

$$R_m^{(k)}(x_1, \dots, x_k) := \sup_{t \notin K_m} \frac{1}{k} \left| \sum_{j=1}^k \left(\frac{h(x_j, t)}{H(t)} - 1 \right) \right|, \ x_1, \dots, x_k \in \mathbb{R}.$$

This is decreasing with respect to m. Let

$$R^{(k)}(x_1,\ldots,x_k) := \lim_{m \to \infty} R^{(k)}_m(x_1,\ldots,x_k).$$

Since $|h(x_i, t)| = 1$, we see that

$$|R_m^{(k)}(x_1, \dots, x_k)| \le 1 + \frac{1}{\inf_{t \notin K_1} |H(t)|}$$

Hence,

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$$|R^{(k)}(x_1, \dots, x_k)| \le 1 + \frac{1}{\inf_{t \notin K_1} |H(t)|}.$$

Since

$$\frac{h(x_j,t)}{H(t)} - 1 = \frac{2\mathrm{Im}(t)(x_j - \theta)}{(x_j - \bar{t})(\theta - t)},$$

we see that if $a \in \mathbb{R} \setminus \{x_i\}$, then,

$$\lim_{t \to a} \frac{h(x_j, t)}{H(t)} - 1 = 0,$$

and furthermore,

$$\lim_{|t|\to\infty;t\in\mathbb{H}}\frac{h(x_j,t)}{H(t)}-1=0.$$

Therefore,

$$\lim_{t \to a} \sum_{j=1}^k \left(\frac{h(x_j, t)}{H(t)} - 1 \right) = 0, \ a \in \mathbb{R} \setminus \{x_1, \dots, x_k\},$$

and,

$$\lim_{t\to\infty}\sum_{j=1}^k\left(\frac{h(x_j,t)}{H(t)}-1\right)=0.$$

Furthermore, if x_1, \ldots, x_k are distinctive, then,

$$\limsup_{t \to a} \left| \sum_{j=1}^{k} \left(\frac{h(x_j, t)}{H(t)} - 1 \right) \right| \le 1 + \frac{1}{\inf_{t \notin K_1} |H(t)|}, \ a \in \{x_1, \dots, x_k\}.$$

Therefore we see that if x_1, \ldots, x_k are distinctive, then,

$$|R^{(k)}(x_1, \dots, x_k)| \le \frac{1}{k} \left(1 + \frac{1}{\inf_{t \notin K_1} |H(t)|} \right).$$

Let k_0 be an integer such that

$$k_0 > 2 + \frac{1}{\inf_{t \notin K_1} |H(t)|}$$

Now by following the argument in the proof of Theorem 3.8 in Arcones (2006), we see that for every $\epsilon \in (0, 1)$,

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where $\lfloor n/k_0 \rfloor$ denotes the integer part of n/k_0 .

Remark 12 Although we have used the techniques in the proof of Theorem 3.8 in Arcones (2006), Condition (v) in Theorem 3.8 in Arcones (2006) does not hold in our case. Our main idea is introducing $R_m^{(k)}(X_1, \ldots, X_k)$ for general k.

Let
$$h_1(x,t) := \operatorname{Re}(h(x,t))$$
 and $h_2(x,t) := \operatorname{Im}(h(x,t))$. As a function of $t \in K$,
 $\frac{1}{n} \sum_{j=1}^n h_i(X_j, t)$ is a $\ell_{\infty}(K)$ -valued random variable.
Let

Now we have the assertion for the case that *n* is a multiple of
$$k_0$$
. Even if *n* is not a multiple of k_0 , with remark the fact that $\frac{h(x, t)}{H(t)}$ is uniformly bounded, we see that for a positive constant *C*,

$$P(\hat{\theta}_n \notin K_m) \leq C \Big(\exp(-(1 - \epsilon)) E \Big[\exp\left(R_m^{(k_0)} (X_1, \dots, X_{k_0}) \right) \Big] \Big)^{\lfloor n/k_0 \rfloor},$$

$$\exp(-(1-\epsilon))E\left[\exp\left(R_m^{(k_0)}(X_1,\ldots,X_{k_0})\right)\right]<1.$$

$$E\left[\exp\left(R^{(k_0)}\left(X_1,\ldots,X_{k_0}\right)\right)\right] \le \exp\left(\frac{1}{k_0}\left(1+\frac{1}{\inf_{t\notin K_1}|H(t)|}\right)\right).$$

By recalling the definition of
$$k_0$$
, if we take sufficiently small ϵ and sufficiently large

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$$k_0$$
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By recalling the definition of
$$k_0$$
, if we take sufficiently small ϵ and sufficiently large m ,

$$\exp(-(1-\epsilon))E\left[\exp\left(R_m^{(k_0)}(X_1,\ldots,X_{k_0})\right)\right] < 1.$$

By the bounded convergence theorem,

$$\lim_{m \to \infty} E\left[\exp\left(R_m^{(k_0)}(X_1, \dots, X_{k_0})\right)\right] = E\left[\exp\left(R^{(k_0)}(X_1, \dots, X_{k_0})\right)\right].$$

 $= \left(\exp(-(1-\epsilon))E\left[\exp\left(R_m^{(k_0)}(X_1,\ldots,X_k)\right)\right]\right)^n.$

 $\left\{\sum_{i=1}^n R_m^{(k_0)} \left(X_{(j-1)k_0+1}, \dots, X_{jk_0}\right) \le n(1-\epsilon)\right\} \subset \left\{\hat{\theta}_{k_0n} \in K_m\right\}.$

 $P(\hat{\theta}_{k_0n} \notin K_m) \le P\left(\sum_{i=1}^n R_m^{(k_0)} (X_{(j-1)k_0+1}, \dots, X_{jk_0}) > n(1-\epsilon)\right)$

 $\leq \exp(-n(1-\epsilon))E\left[\exp\left(\sum_{i=1}^{n} R_m^{(k_0)}\left(X_{(j-1)k_0+1}, \dots, X_{jk_0}\right)\right)\right]$

Since
$$X_1, \ldots, X_{k_0}$$
 are distinctive almost surely,

By this and the fact that $\{(X_{(j-1)k_0+1}, \dots, X_{jk_0})\}_j$ are independent,

$$\mathcal{L} := \left\{ f : \mathbb{R} \to \mathbb{R} \mid \exists \lambda < +\infty \text{ such that } E\left[\exp(\lambda |f(X_1)|)\right] < +\infty \right\}.$$

Let \mathcal{L}^* be the dual space of \mathcal{L} . Let

$$J(l) := \sup_{f \in \mathcal{L}} \left(l(f) - \log E[\exp(f(X_1))] \right), \ l \in \mathcal{L}^*.$$

Let $\ell_{\infty}(K)$ be the set of bounded continuous functions on *K*. Let P_* and P^* be the inner and outer measures of a probability measure *P*, respectively.

Lemma 13 Assume that i = 1 or 2. For each non-empty compact subset K of \mathbb{H} , $\left\{\frac{1}{n}\sum_{j=1}^{n}h_{i}(X_{j},t)\right\}_{n}$ follows the large deviation principle in $\ell_{\infty}(K)$. Specifically, there exists a good rate function I on $\ell_{\infty}(K)$ such that (i) $\{I \leq c\}$ is compact in $\ell_{\infty}(K)$ for every c > 0,

(ii) for every open subset U of $\ell_{\infty}(K)$,

$$\liminf_{n \to \infty} \frac{1}{n} \log P_* \left(\frac{1}{n} \sum_{j=1}^n h_i(X_j, t) \in U \right) \ge -\inf_{z \in U} I(z),$$

and, (iii) for every closed subset F of $\ell_{\infty}(K)$,

$$\liminf_{n \to \infty} \frac{1}{n} \log P^* \left(\frac{1}{n} \sum_{j=1}^n h_i(X_j, t) \in F \right) \le -\inf_{z \in F} I(z).$$

Furthermore,

$$I_i(z) = \inf \left\{ J(l) : l \in \mathcal{L}^*, \ l(h_i(\cdot, t)) = z(t) \text{ for every } t \in K \right\}, \ z \in \mathscr{E}_{\infty}(K).$$

Proof It suffices to check the conditions in Theorem 2.5 in Arcones (2006). We remark that for every $x \in \mathbb{R}$ and every $t \in \mathbb{H}$,

$$|h_i(x,t)| \le |h(x,t)| = 1.$$

This implies conditions (i) and (ii) in Theorem 2.5 in Arcones (2006).

Let $s, t \in \mathbb{H}$ and $|s - t| \le |t|/4$. Since

$$h(x,s) - h(x,t) = \frac{x-s}{(x-\overline{s})(x-\overline{t})}(\overline{s}-\overline{t}) + \frac{t-s}{x-\overline{t}}, \ x \in \mathbb{R},$$
(15)

we see that

$$|h(x,s) - h(x,t)| \le 2\frac{|s-t|}{\operatorname{Im}(t)}.$$
(16)

Hence condition (iii) in Theorem 2.5 Arcones (2006) holds.

Proposition 14 For every $\epsilon > 0$, there exists a constant $\beta \in (0, 1)$ such that

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$$P\left(\left|\hat{\theta}_n - \theta\right| \ge \epsilon\right) = O(\beta^n).$$

Proof By Lemma 11, there exists a compact set *K* of \mathbb{H} and $\alpha \in (0, 1)$ such that

.

$$P(\hat{\theta}_n \notin K) = O(\alpha^n).$$

Let

$$F := \{t \in \mathbb{H} : |t - \theta| \ge \epsilon\}.$$

Then,

$$P\Big(\Big|\hat{\theta}_n - \theta\Big| \ge \epsilon\Big) \le P\Big(\hat{\theta}_n \notin K\Big) + P\Big(\hat{\theta}_n \in K \cap F\Big).$$

Hence it suffices to show that there exists a constant $\beta \in (0, 1)$ such that

$$P(\hat{\theta}_n \in K \cap F) = O(\beta^n).$$

Let $H_1(t)$ and $H_2(t)$ be the real and imaginary parts of H(t), respectively. Since

$$\epsilon_0 := \inf_{t \in F} |H(t)| > 0,$$

we see that

$$\left\{t \in F : \left|H_1(t)\right| \le \epsilon_0/4\right\} \subset \left\{t \in F : \left|H_2(t)\right| \ge \epsilon_0/4\right\}.$$

Let

$$F_1 := \{t \in F : |H_1(t)| \ge \epsilon_0/4\}, \text{ and } F_2 := \{t \in F : |H_1(t)| \le \epsilon_0/4\}.$$

Then $F = F_1 \cup F_2$. Therefore,

$$\begin{split} P(\hat{\theta}_n \in K \cap F) &\leq P^* \left(\text{ there exists } t \in F \cap K \text{ such that } \frac{1}{n} \sum_{j=1}^n h(X_j, t) = 0 \right) \\ &\leq P^* \left(\text{ there exists } t \in F_1 \cap K \text{ such that } \frac{1}{n} \sum_{j=1}^n h_1(X_j, t) = 0 \right) \\ &+ P^* \left(\text{ there exists } t \in F_2 \cap K \text{ such that } \frac{1}{n} \sum_{j=1}^n h_2(X_j, t) = 0 \right). \end{split}$$

Let

$$C_i := \left\{ z \in \mathscr{C}_{\infty}(F_i \cap K) : \inf_{t \in F_i \cap K} |z(t)| = 0 \right\}, \ i = 1, 2.$$

These are closed subsets of $\ell_{\infty}(F_i \cap K)$, i = 1, 2. We see that

$$P^*\left(\text{there exists } t \in F_i \cap K \text{ such that } \frac{1}{n} \sum_{j=1}^n h_i(X_j, t) = 0\right)$$
$$= P^*\left(\frac{1}{n} \sum_{j=1}^n h_i(X_j, t) \in C_i\right).$$

Then, by Lemma 13, we see that for i = 1, 2,

$$\limsup_{n \to \infty} \frac{1}{n} \log P^* \left(\frac{1}{n} \sum_{j=1}^n h_i(X_j, t) \in C_i \right)$$

$$\leq -\inf_{z \in C_i} \inf \left\{ J(l) : l \in \mathcal{L}^*, \ l(h_i(\cdot, t)) = z(t) \text{ for every } t \in F_i \cap K \right\}$$
(17)

$$= -\inf \left\{ J(l) : l \in \mathcal{L}^*, \ \inf_{t \in F_i \cap K} \left| l(h_i(\cdot, t)) \right| = 0 \right\}.$$

By Eq. (31) in Arcones (2006), we see that

$$-\inf\left\{J(l): l \in \mathcal{L}^*, \inf_{t \in F_i \cap K} \left|l(h_i(\cdot, t))\right| = 0\right\}$$
$$= -\inf_{t \in F_i \cap K} \inf\left\{J(l): l \in \mathcal{L}^*, l(h_i(\cdot, t)) = 0\right\}$$
$$= \sup_{t \in F_i \cap K} \inf_{\lambda \in \mathbb{R}} \log E\left[\exp\left(\lambda h_i(X_1, t)\right)\right].$$
(18)

Since h_i is bounded, it holds that

$$\frac{d}{d\lambda}\bigg|_{\lambda=0}\log E\Big[\exp\big(\lambda h_i(X_1,t)\big)\Big] = H_i(t) \neq 0, \ t \in F_i \cap K.$$

By this and the fact that $\log E\left[\exp\left(\lambda h_i(X_1, t)\right)\right] = 0$ if $\lambda = 0$, we see that

$$\inf_{\lambda \in \mathbb{R}} \log E\left[\exp\left(\lambda h_i(X_1, t)\right)\right] < 0, \ t \in F_i \cap K.$$

Let

$$I_i(t) := -\inf_{\lambda \in \mathbb{R}} \log E\left[\exp\left(\lambda h_i(X_1, t)\right)\right], \ t \in F_i \cap K.$$

Then, $I_i(t) > 0$, $t \in F_i \cap K$ and this function is lower-semicontinuous. Hence,

$$\inf_{t\in F_i\cap K}I_i(t)>0.$$

By this, (17), and (18),

$$\limsup_{n \to \infty} \frac{1}{n} \log P^* \left(\frac{1}{n} \sum_{j=1}^n h_i(X_j, t) \in C_i \right) \le -\inf_{t \in F_i \cap K} I_i(t) < 0, \ i = 1, 2.$$

Thus we have the assertion.

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Proof of Theorem 4 We adopt Theorem 3 in Shen (2001). The condition being LDconsistent in the statement of Theorem 3 in Shen (2001) is equivalent with Proposition 14. Let $\psi(x, \theta) := \log p(x;(\mu, \sigma))$. Then, we can show conditions (C1) and (C2) in Theorem 3 in Shen (2001) by some calculations for the first and second orders of the partial derivatives of ψ with respect to μ and σ . Now we can apply Theorem 3 in Shen (2001), and then, the assertion follows from this and Proposition 2 in Shen (2001).

6.3 Proofs of assertions in Section 4

Proof of Theorem 5 We see that for every $C_1 \in (0, 1)$ and $\epsilon > 0$,

$$P(|Z_n - \theta| > \epsilon) \le P(|Z_n - \hat{\theta}_n| > c_1\epsilon) + P(|\hat{\theta}_n - \theta| > (1 - c_1)\epsilon).$$

By (4),

$$\lim_{c_1 \to +0} b((1 - c_1)\epsilon, \theta) = b(\epsilon, \theta).$$

By this and Theorem 4, we see that for every $\eta > 0$, there exists $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0)$ and $c_1 \in (0, \epsilon_0)$,

$$P\left(\left|\hat{\theta}_n - \theta\right| > (1 - c_1)\epsilon\right) = O(\exp\left(-n(b(\epsilon, \theta) - \eta)\right)),$$

where the large order depends on c_1 , η and ϵ .

By this and (5), it suffices to show that for sufficiently small $\epsilon > 0$

$$P(\left|Z_n - \hat{\theta}_n\right| > \epsilon) = O(\exp(-n\epsilon^{3/2})).$$
⁽¹⁹⁾

By using the fact that $\hat{\theta}_n$ is MLE and (15), we see that

$$Z_n - \hat{\theta}_n = Y_n - \hat{\theta}_n + \frac{2\mathrm{Im}(Y_n)i}{n} \sum_{j=1}^n \left(\frac{X_j - Y_n}{X_j - \overline{Y_n}} - \frac{X_j - \hat{\theta}_n}{X_j - \overline{\theta}_n} \right)$$
$$= \frac{Y_n - \hat{\theta}_n}{n} \sum_{j=1}^n \frac{X_j - Y_n}{X_j - \overline{Y_n}} + \frac{\overline{Y_n - \hat{\theta}_n}}{n} \sum_{j=1}^n \frac{X_j - \hat{\theta}_n}{X_j - \overline{\theta}_n}$$
$$- \frac{\overline{Y_n - \hat{\theta}_n}}{n} \sum_{j=1}^n \frac{X_j - \hat{\theta}_n}{X_j - \overline{\theta}_n} \frac{X_j - \gamma_n}{X_j - \overline{\theta}_n}.$$

Hence,

$$\begin{split} P\Big(\Big|Z_n - \hat{\theta}_n\Big| > \epsilon\Big) &\leq P\bigg(|Y_n - \hat{\theta}_n| \cdot \left|\frac{1}{n} \sum_{j=1}^n \frac{X_j - Y_n}{X_j - \overline{Y_n}}\right| > \frac{\epsilon}{3}\bigg) \\ &+ P\bigg(|Y_n - \hat{\theta}_n| \cdot \left|\frac{1}{n} \sum_{j=1}^n \frac{X_j - \hat{\theta}_n}{X_j - \overline{\theta}_n}\right| > \frac{\epsilon}{3}\bigg) \\ &+ P\bigg(|Y_n - \hat{\theta}_n| \cdot \left|\frac{1}{n} \sum_{j=1}^n \frac{X_j - \hat{\theta}_n}{X_j - \overline{\theta}_n} \frac{X_j - Y_n}{X_j - \overline{Y_n}}\right| > \frac{\epsilon}{3}\bigg). \end{split}$$

By Theorems 1 and 4 , we see that there exists a constant $c_2 > 0$ such that for sufficiently small $\eta > 0$,

$$P(|Y_n - \theta| > \eta) = O(\exp(-c_2 n\eta^2))$$
(20)

and

$$P(\left|\hat{\theta}_n - \theta\right| > \eta) = O(\exp(-c_2 n\eta^2)).$$
(21)

Hence, there exists a constant $c_3 > 0$ such that for sufficiently small $\eta > 0$,

$$P\left(\left|Y_n - \hat{\theta}_n\right| > \eta\right) = O(\exp(-c_3 n \eta^2)).$$
(22)

By the Cramér–Chernoff method, we also obtain that there exists a constant $c_4 > 0$ such that for sufficiently small $\eta > 0$,

$$P\left(\left|\frac{1}{n}\sum_{j=1}^{n}\frac{X_{j}-\theta}{X_{j}-\overline{\theta}}\right|>\eta\right)=O(\exp(-c_{4}n\eta^{2})).$$
(23)

and

$$P\left(\left|\frac{1}{n}\sum_{j=1}^{n}\left(\frac{X_{j}-\theta}{X_{j}-\overline{\theta}}\right)^{2}\right|>\eta\right)=O(\exp(-c_{4}n\eta^{2})).$$
(24)

In the above we have used that

$$E\left[\frac{X_j-\theta}{X_j-\overline{\theta}}\right] = E\left[\left(\frac{X_j-\theta}{X_j-\overline{\theta}}\right)^2\right] = 0,$$

both of which follow from the Cauchy integral formula.

By (16), we see that

$$\frac{1}{n}\sum_{j=1}^{n}\frac{X_{j}-Y_{n}}{X_{j}-\overline{Y_{n}}} \le \left|\frac{1}{n}\sum_{j=1}^{n}\frac{X_{j}-\theta}{X_{j}-\overline{\theta}}\right| + 2\frac{|Y_{n}-\theta|}{\operatorname{Im}(\theta)}$$

and

$$\left|\frac{1}{n}\sum_{j=1}^{n}\frac{X_{j}-\hat{\theta}_{n}}{X_{j}-\overline{\hat{\theta}_{n}}}\right| \leq \left|\frac{1}{n}\sum_{j=1}^{n}\frac{X_{j}-\theta}{X_{j}-\overline{\theta}}\right| + 2\frac{\left|\hat{\theta}_{n}-\theta\right|}{\operatorname{Im}(\theta)}$$

By these estimates, (22), (23), and (20), we see that there exists a constant $c_5 > 0$ such that for sufficiently small $\epsilon > 0$,

.

$$\begin{split} & P\left(\left|Y_n - \hat{\theta}_n\right| \cdot \left|\frac{1}{n} \sum_{j=1}^n \frac{X_j - Y_n}{X_j - \overline{Y_n}}\right| > \frac{\epsilon}{3}\right) \\ & \leq 2P\left(\left|Y_n - \hat{\theta}_n\right| > \sqrt{\frac{\epsilon}{6}}\right) + P\left(\left|\frac{1}{n} \sum_{j=1}^n \frac{X_j - \theta}{X_j - \overline{\theta}}\right| > \sqrt{\frac{\epsilon}{6}}\right) \\ & + P\left(2\frac{|Y_n - \theta|}{\operatorname{Im}(\theta)} > \sqrt{\frac{\epsilon}{6}}\right) \\ & = O\left(\exp(-c_5 n\epsilon)\right). \end{split}$$

In the same manner, by noting (22), (23), and (21), we see that there exists a constant $c_6 > 0$ such that for sufficiently small $\epsilon > 0$,

$$P\left(\left|Y_n-\hat{\theta}_n\right|\cdot\left|\frac{1}{n}\sum_{j=1}^n\frac{X_j-\hat{\theta}_n}{X_j-\overline{\hat{\theta}_n}}\right|>\frac{\epsilon}{3}\right)=O\left(\exp(-c_6n\epsilon)\right).$$

We remark that

$$\begin{aligned} \frac{X_j - \hat{\theta}_n}{X_j - \overline{\hat{\theta}_n}} \frac{X_j - Y_n}{X_j - \overline{Y_n}} - \left(\frac{X_j - \theta}{X_j - \overline{\theta}}\right)^2 \\ &= \left(\frac{X_j - \hat{\theta}_n}{X_j - \overline{\hat{\theta}_n}} - \frac{X_j - \theta}{X_j - \overline{\theta}}\right) \frac{X_j - Y_n}{X_j - \overline{Y_n}} + \left(\frac{X_j - Y_n}{X_j - \overline{Y_n}} - \frac{X_j - \theta}{X_j - \overline{\theta}}\right) \frac{X_j - \theta}{X_j - \overline{\theta}}.\end{aligned}$$

By this, (16) and the fact that $|X_j - \theta| = |X_j - \overline{\theta}|$, we see that

$$\begin{aligned} \left| \frac{1}{n} \sum_{j=1}^{n} \frac{X_{j} - \hat{\theta}_{n}}{X_{j} - \overline{\hat{\theta}_{n}}} \frac{X_{j} - Y_{n}}{X_{j} - \overline{Y_{n}}} \right| \\ \leq \left| \frac{1}{n} \sum_{j=1}^{n} \left(\frac{X_{j} - \theta}{X_{j} - \overline{\theta}} \right)^{2} \right| + 2 \frac{|Y_{n} - \theta|}{\operatorname{Im}(\theta)} + 2 \frac{|\hat{\theta}_{n} - \theta|}{\operatorname{Im}(\theta)} + 4 \frac{|Y_{n} - \theta| \cdot |\hat{\theta}_{n} - \theta|}{\operatorname{Im}(\theta)^{2}} \end{aligned}$$

By these estimates, we see that for every $\epsilon > 0$,

$$\begin{split} & P\left(|Y_n - \hat{\theta}_n| \cdot \left|\frac{1}{n} \sum_{j=1}^n \frac{X_j - \hat{\theta}_n}{X_j - \overline{\theta}_n} \frac{X_j - Y_n}{X_j - \overline{Y}_n}\right| > \frac{\epsilon}{3}\right) \\ & \leq 4P\left(\left|Y_n - \hat{\theta}_n\right| > \sqrt{\frac{\epsilon}{12}}\right) + P\left(\left|\frac{1}{n} \sum_{j=1}^n \left(\frac{X_j - \theta}{X_j - \overline{\theta}}\right)^2\right| > \sqrt{\frac{\epsilon}{12}}\right) \\ & + P\left(2\frac{|Y_n - \theta|}{\operatorname{Im}(\theta)} > \sqrt{\frac{\epsilon}{12}}\right) + P\left(2\frac{|\hat{\theta}_n - \theta|}{\operatorname{Im}(\theta)} > \sqrt{\frac{\epsilon}{12}}\right) \\ & + P\left(4\frac{|Y_n - \theta| \cdot |\hat{\theta}_n - \theta|}{\operatorname{Im}(\theta)^2} > \sqrt{\frac{\epsilon}{12}}\right). \end{split}$$

We remark that for sufficiently small $\epsilon > 0$,

$$\begin{split} & P\left(4\frac{|Y_n-\theta|\cdot|\hat{\theta}_n-\theta|}{\operatorname{Im}(\theta)^2} > \sqrt{\frac{\epsilon}{12}}\right) \\ & \leq P\left(2\frac{|Y_n-\theta|}{\operatorname{Im}(\theta)} > 1\right) + P\left(4\frac{|Y_n-\theta|\cdot|\hat{\theta}_n-\theta|}{\operatorname{Im}(\theta)^2} > \sqrt{\frac{\epsilon}{12}}, \ 2\frac{|Y_n-\theta|}{\operatorname{Im}(\theta)} \le 1\right) \\ & \leq P\left(2\frac{|Y_n-\theta|}{\operatorname{Im}(\theta)} > \sqrt{\frac{\epsilon}{12}}\right) + P\left(2\frac{|\hat{\theta}_n-\theta|}{\operatorname{Im}(\theta)} > \sqrt{\frac{\epsilon}{12}}\right). \end{split}$$

By these estimates and (24), (20), (21) and (22), we see that there exists a constant $c_7 > 0$ such that for sufficiently small $\epsilon > 0$,

$$P\left(|Y_n - \hat{\theta}_n| \cdot \left| \frac{1}{n} \sum_{j=1}^n \frac{X_j - \hat{\theta}_n}{X_j - \overline{\theta}_n} \frac{X_j - Y_n}{X_j - \overline{Y}_n} \right| > \frac{\epsilon}{3} \right) = O\left(\exp(-c_7 n\epsilon)\right).$$

Thus we have (19) and this completes the proof.

Proof of Theorem 6 We see that

$$Z_n - \theta = Y_n - \theta + \frac{2\mathrm{Im}(Y_n)i}{n} \sum_{j=1}^n \left(\frac{X_j - Y_n}{X_j - \overline{Y_n}} - \frac{X_j - \theta}{X_j - \overline{\theta}}\right) + \frac{2\mathrm{Im}(Y_n)i}{n} \sum_{j=1}^n \frac{X_j - \theta}{X_j - \overline{\theta}}$$

By (16),

$$\left|Y_n - \theta + \frac{2\mathrm{Im}(Y_n)i}{n}\sum_{j=1}^n \left(\frac{X_j - Y_n}{X_j - \overline{Y_n}} - \frac{X_j - \theta}{X_j - \overline{\theta}}\right)\right| \le 9\left|Y_n - \theta\right|$$

We see that

$$\left|\frac{2\mathrm{Im}(Y_n)i}{n}\sum_{j=1}^n\frac{X_j-\theta}{X_j-\overline{\theta}}\right| \le 2\left|Y_n-\theta\right| + 2\mathrm{Im}(\theta)\left|\frac{1}{n}\sum_{j=1}^n\frac{X_j-\theta}{X_j-\overline{\theta}}\right|.$$

Hence,

$$\sqrt{n}|Z_n - \theta| \le 11\sqrt{n}|Y_n - \theta| + \frac{2\mathrm{Im}(\theta)}{\sqrt{n}} \left| \sum_{j=1}^n \frac{X_j - \theta}{X_j - \overline{\theta}} \right|.$$

Now we should recall that the generators of quasi-arithmetic means are restricted to the cases that $f(x) = \log(x + \gamma), \ \gamma \in \overline{\mathbb{H}}$ and $f(x) = \frac{x - \gamma}{x - \overline{\gamma}}, \ \gamma \in \mathbb{H}$. By Theorem 4.2(iii) and Theorem 4.4(iii) in Akaoka et al. (2021b) and (14), we can apply Theorem 3.6 in Billingsley (1999) and have that $\left\{ \left| \sqrt{n}(Y_n - \theta) \right|^2 \right\}_n$ is uniformly integrable. By the central limit theorem,

$$\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\frac{X_{j}-\theta}{X_{j}-\overline{\theta}} \Rightarrow N\left(0,\frac{1}{2}I_{2}\right), \ n \to \infty.$$

By the Cauchy integral formula, we also see that $E\left[\left|\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\frac{X_{j}-\theta}{X_{j}-\overline{\theta}}\right|\right] = 1$. Therefore we can apply Theorem 3.6 in Billingsley (1999) again and have that $\left\{\left|\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\frac{X_{j}-\theta}{X_{j}-\overline{\theta}}\right|^{2}\right\}_{n}$ is uniformly integrable. Therefore, $\left\{\left|\sqrt{n}(Z_{n}-\theta)\right|^{2}\right\}_{n}$ is also uniformly integrable By this and (12), we have the assertion.

6.4 Proofs of assertions in Section 5

Proof of Theorem 8 We first remark that for sufficiently small $\epsilon > 0$,

$$\widetilde{b}(\epsilon,w) = \log\left(1 + \frac{\epsilon^2}{(1-|w|^2)(1-(|w|+\epsilon)^2)}\right).$$

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Hence,

$$\lim_{\epsilon \to +0} \frac{\widetilde{b}(\epsilon, w)}{\epsilon^2} = \frac{1}{(1 - |w|^2)^2}.$$
(25)

Let $\theta := \phi_{\alpha}^{-1}(w)$. By the complex mean-value theorem (see Theorem 2.2 in Evard and Jafari (1992)),

$$P(|W_n - w| > \epsilon) \le P\bigg(|Z_n - \theta| > \frac{\epsilon}{\sup_{z;|z - \theta| \le \epsilon} |\phi'_{\alpha}(z)|}\bigg).$$

Let

$$\epsilon' := \frac{\epsilon}{\sup_{z;|z-\theta|\leq \epsilon} |\phi'_{\alpha}(z)|}$$

Then, by (25) and (5),

$$\lim_{\epsilon \to +0} \frac{\widetilde{b}(\epsilon, w)}{b(\epsilon', \theta)} = \lim_{\epsilon \to +0} \frac{\widetilde{b}(\epsilon, w)}{\epsilon^2} \frac{(\epsilon')^2}{b(\epsilon', \theta)} \frac{\epsilon^2}{(\epsilon')^2} = \frac{4 \mathrm{Im}(\theta)^2 |\phi'_{\alpha}(\theta)|^2}{(1 - |w|^2)^2} = 1.$$

Now the assertion follows from this and Theorem 5.

Proof of Theorem 9 Let $\theta := \phi_{\alpha}^{-1}(w)$. We see that

$$nE\left[\left|\phi_{\alpha}(Z_{n})-\phi_{\alpha}(\theta)-\phi_{\alpha}'(\theta)(Z_{n}-\theta)\right|^{2}\right]$$
$$=E\left[n|Z_{n}-\theta|^{2}\left|\frac{\phi_{\alpha}(Z_{n})-\phi_{\alpha}(\theta)}{Z_{n}-\theta}-\phi_{\alpha}'(\theta)\right|^{2}\right].$$

We also have that $\sup_{z \in \mathbb{H}} |\phi'_{\alpha}(z)| \le 2\text{Im}(\alpha)$ and $\sup_{z \in \mathbb{H}} |\phi''_{\alpha}(z)| \le 4\text{Im}(\alpha)$. By using Theorem 2.2 in Evard and Jafari (1992) again,

$$\left|\frac{\phi_{\alpha}(Z_n) - \phi_{\alpha}(\theta)}{Z_n - \theta} - \phi_{\alpha}'(\theta)\right| \le 4\mathrm{Im}(\alpha)\min\left\{1, |Z_n - \theta|\right\}.$$

Let $\epsilon > 0$. Then,

$$E\left[n|Z_n - \theta|^2 \left| \frac{\phi_{\alpha}(Z_n) - \phi_{\alpha}(\theta)}{Z_n - \theta} - \phi_{\alpha}'(\theta) \right|^2 \right]$$

$$\leq 4\text{Im}(\alpha)E\left[n|Z_n - \theta|^2, |Z_n - \theta| > \epsilon\right]$$

$$+ 4\text{Im}(\alpha)\epsilon^2E\left[n|Z_n - \theta|^2, |Z_n - \theta| \le \epsilon\right]$$

By the final part of the proof of Theorem 6, $\lim_{n\to\infty} P(|Z_n - \theta| > \epsilon) = 0$ and $\{n|Z_n - \theta|^2\}_n$ is uniformly integrable. Hence,

$$\lim_{n \to \infty} E[n|Z_n - \theta|^2, |Z_n - \theta| > \epsilon] = 0.$$

Hence,

$$\limsup_{n \to \infty} E\left[n|Z_n - \theta|^2 \left| \frac{\phi_{\alpha}(Z_n) - \phi_{\alpha}(\theta)}{Z_n - \theta} - \phi_{\alpha}'(\theta) \right|^2 \right] \\ \leq 4 \operatorname{Im}(\alpha) \epsilon^2 \sup_n E[n|Z_n - \theta|^2].$$

Since we can take $\epsilon > 0$ arbitrarily small, we see that

$$\lim_{n \to \infty} nE\left[\left|\phi_{\alpha}(Z_{n}) - \phi_{\alpha}(\theta) - \phi_{\alpha}'(\theta)(Z_{n} - \theta)\right|^{2}\right] = 0.$$

Hence,

$$\lim_{n \to \infty} nE \left[|\phi_{\alpha}(Z_n) - \phi_{\alpha}(\theta)|^2 \right]$$
$$= 4 \mathrm{Im}(\theta)^2 |\phi_{\alpha}'(\theta)|^2.$$

The assertion now follows from this and

$$4\mathrm{Im}(\phi_{\alpha}^{-1}(w))^{2}|\phi_{\alpha}'(\phi_{\alpha}^{-1}(w))|^{2} = (1 - |w|^{2})^{2}.$$

| I | | |
|---|--|--|
| | | |

Numerical computations

We perform simulation studies in the R Core Team (2021) to illustrate the properties of the one-step estimators $(Z_n)_n$. The version of R is 4.1.1. We deal with $nE[|Z_n - \theta|^2]/\text{Im}(\theta)^2 = nE[|Z_n - (\mu + \sigma i)|^2]/\sigma^2$ appearing in Theorem 6.

For the generators of quasi-arithmetic means, we consider the following four cases that $f_1(x) = \log x$, $f_2(x) = \log(x + i)$, $f_3(x) = 1/(x + i)$ and $f_4(x) = 1/(x + 2i)$. For the location and the scale, we consider the following four cases that $(\mu, \sigma) = (0, 10), (10, 1), (0, 10), (10, 10)$. For the sizes of samples, we consider the following five cases that n = 10, 50, 100, 500, 1000.

In each choice of the triplet (μ, σ, n) , we compute the average of the values $n|Z_n - \theta|^2/\text{Im}(\theta)^2$ for 10⁶ samples of size *n* generated by the reauchy() function in R. See Table 1 for results. In this section, we round off the outputs to three decimal places.

By these numerical computations in Table 1, we conjecture that

$$nE[|Z_n - \theta|^2] \ge 4 \operatorname{Im}(\theta)^2$$

| Table 1 Tables for simulated values for $nE[Z_n - (\mu + \sigma i) ^2]/\sigma^2$ | n | f_1 | f_2 | f_3 | f_4 | | |
|--|----------------------------|--------|--------|--------|--------|--|--|
| | $(\mu, \sigma) = (0, 1)$ | | | | | | |
| | 10 | 5.453 | 6.551 | 4.614 | 5.129 | | |
| | 50 | 4.144 | 4.397 | 4.083 | 4.208 | | |
| | 100 | 4.056 | 4.186 | 4.034 | 4.097 | | |
| | 500 | 4.016 | 4.043 | 4.013 | 4.025 | | |
| | 1000 | 4.008 | 4.022 | 4.007 | 4.013 | | |
| | $(\mu, \sigma) = (10, 1)$ | | | | | | |
| | 10 | 45.014 | 38.179 | 91.065 | 39.535 | | |
| | 50 | 29.718 | 22.231 | 48.523 | 20.440 | | |
| | 100 | 19.742 | 14.915 | 35.411 | 14.386 | | |
| | 500 | 7.420 | 6.393 | 13.280 | 6.572 | | |
| | 1000 | 5.714 | 5.206 | 8.867 | 5.313 | | |
| | $(\mu,\sigma) = (0,10)$ | | | | | | |
| | 10 | 5.463 | 5.463 | 25.377 | 9.934 | | |
| | 50 | 4.138 | 4.152 | 5.672 | 4.239 | | |
| | 100 | 4.063 | 4.072 | 4.571 | 4.076 | | |
| | 500 | 4.009 | 4.011 | 4.077 | 4.007 | | |
| | 1000 | 4.005 | 4.006 | 4.037 | 4.003 | | |
| | $(\mu, \sigma) = (10, 10)$ | | | | | | |
| | 10 | 7.066 | 6.795 | 40.737 | 15.413 | | |
| | 50 | 4.491 | 4.442 | 9.545 | 5.266 | | |
| | 100 | 4.238 | 4.214 | 6.367 | 4.559 | | |
| | 500 | 4.052 | 4.047 | 4.403 | 4.104 | | |
| | 1000 | 4.028 | 4.026 | 4.200 | 4.055 | | |
| | | | | | | | |

for every *n*. We remark that the Cramér–Rao bound cannot be applied, because Z_n might not be unbiased.

We now consider the case of the MLE $\hat{\theta}_n$. Let $z \in \mathbb{H}$. For every $y \in \mathbb{R}$ and every t > 0, z is the maximal likelihood estimate of $\{x_1, \dots, x_n\}$ if and only if tz + x is the maximum likelihood estimate of $\{tx_1 + y, \dots, tx_n + y\}$. Furthermore, the joint distribution of $(tX_1 + y, \dots, tX_n + y)$ under P_{y+ti} is identical with the joint distribution of (X_1, \dots, X_n) under P_i . Hence, the distribution of $t\hat{\theta}_n + y$ under P_i is identical with the distribution of $\hat{\theta}_n$ under P_{ti+y} . Hence, the distribution of $\frac{\hat{\theta}_n - \mu}{\sigma}$ is identical with that

| n | V _n |
|------|-------------------------------------|
| 10 | 5.732 |
| 50 | 4.254 |
| 100 | 4.117 |
| 500 | 4.029 |
| 1000 | 4.015 |
| | n 10 50 100 500 1000 |

of $\hat{\theta}_n$ under $\theta = i$. Therefore, we can assume the distribution is the standard Cauchy distribution.

For numerical computations for $nE[|\hat{\theta}_n - \theta|^2]/\text{Im}(\theta)^2 = nE[|\hat{\theta}_n - (\mu + \sigma i)|^2]/\sigma^2$, as in Kravchuk and Pollett (2012), we use the nlminb() function in R. Here we assume that the real and imaginary parts of the initial point of the algorithm are given by the one-step estimator Z_n associated with f_3 . The results are summarized in Table 2.

By Table 2, we conjecture that

$$n\frac{E[|\theta_n - \theta|^2]}{\operatorname{Im}(\theta)^2} = 4 + O(n^{-1}), \ n \to \infty,$$

which is compatible with the asymptotic expansion for the MLE.

The one-step estimators $(Z_n)_n$ do not have such invariance as the MLE has, so the value of $nE[|Z_n - \theta|^2]/\text{Im}(\theta)^2$ depends on the value of θ . As in the case that $(\mu, \sigma) = (10, 1)$ in Table 1, if the ratio between the location and the scale is large, then the performances of the one-step estimators could be bad in particular for samples of small sizes. We consider adjusting the median in the definition of Y_n as in Kravchuk and Pollett (2012).

Let M_n be the median of $\{X_1, \ldots, X_n\}$. Let

$$\widetilde{Y}_n := M_n + f^{-1} \left(\frac{1}{n} \sum_{j=1}^n f\left(X_j - M_n \right) \right).$$

Let

$$\widetilde{Z}_n := \widetilde{Y}_n + \frac{2\mathrm{Im}(\widetilde{Y}_n)i}{n} \sum_{j=1}^n h(X_j, \widetilde{Y}_n),$$

which is the one-step estimator of $(\widetilde{Y}_n)_n$. Although we are not sure whether the conclusion of Theorem 4.2 holds or not for $(\widetilde{Z}_n)_n$, we deal with $nE[|\widetilde{Z}_n - \theta|^2]/\text{Im}(\theta)^2 = nE[|\widetilde{Z}_n - (\mu + \sigma i)|^2]/\sigma^2$. (Table 3)

By these numerical computations, when we consider the median-adjusting, the logarithmic functions $f(x) = \log(x + \alpha)$, which are f_1 and f_2 , are better than the Möbius transformations $f(x) = 1/(x + \alpha)$, which are f_3 and f_4 , as the generators of the quasi-arithmetic means. Since $(M_n - \mu)/\sigma$ is the median of $\{(X_i - \mu)/\sigma\}_i$,

$$\frac{\widetilde{Y}_n - \mu}{\sigma} = \frac{M_n - \mu}{\sigma} + f_1^{-1} \left(\frac{1}{n} \sum_{j=1}^n f_1 \left(\frac{X_j - \mu}{\sigma} - \frac{M_n - \mu}{\sigma} \right) \right)$$

and

$$\frac{\widetilde{Z}_n - \mu}{\sigma} = \frac{\widetilde{Y}_n - \mu}{\sigma} + \frac{2\mathrm{Im}\Big((\widetilde{Y}_n - \mu)/\sigma\Big)i}{n} \sum_{j=1}^n h\bigg(\frac{X_j - \mu}{\sigma}, \frac{\widetilde{Y}_n - \mu}{\sigma}\bigg),$$

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| Table 3 Tables for simulated values for $nE[\tilde{Z}_n - (\mu + \sigma i) ^2]/\sigma^2$ | n | f_1 | f_2 | f_3 | f_4 | | |
|--|----------------------------|-------|-------|--------|--------|--|--|
| | $(\mu, \sigma) = (0, 1)$ | | | | | | |
| | 10 | 5.884 | 6.669 | 5.954 | 5.621 | | |
| | 50 | 4.279 | 4.429 | 4.168 | 4.247 | | |
| | 100 | 4.129 | 4.204 | 4.075 | 4.116 | | |
| | 500 | 4.032 | 4.047 | 4.021 | 4.029 | | |
| | 1000 | 4.016 | 4.024 | 4.011 | 4.015 | | |
| | $(\mu, \sigma) = (10, 1)$ | | | | | | |
| | 10 | 5.891 | 6.665 | 5.904 | 5.654 | | |
| | 50 | 4.284 | 4.436 | 4.173 | 4.253 | | |
| | 100 | 4.134 | 4.211 | 4.079 | 4.120 | | |
| | 500 | 4.028 | 4.043 | 4.017 | 4.025 | | |
| | 1000 | 4.009 | 4.017 | 4.004 | 4.008 | | |
| | $(\mu, \sigma) = (0, 10)$ | | | | | | |
| | 10 | 5.887 | 5.878 | 81.877 | 27.160 | | |
| | 50 | 4.273 | 4.268 | 5.270 | 4.316 | | |
| | 100 | 4.137 | 4.134 | 4.439 | 4.148 | | |
| | 500 | 4.025 | 4.024 | 4.091 | 4.028 | | |
| | 1000 | 4.013 | 4.013 | 4.047 | 4.014 | | |
| | $(\mu, \sigma) = (10, 10)$ | | | | | | |
| | 10 | 5.881 | 5.872 | 81.176 | 27.085 | | |
| | 50 | 4.276 | 4.271 | 5.264 | 4.315 | | |
| | 100 | 4.136 | 4.133 | 4.438 | 4.149 | | |
| | 500 | 4.030 | 4.030 | 4.096 | 4.033 | | |
| | 1000 | 4.019 | 4.018 | 4.053 | 4.020 | | |

the distribution of $\frac{Z_n - \mu}{\sigma}$ is identical with that of \widetilde{Z}_n under $\theta = i$. Therefore, for f_1 , $(\widetilde{Z}_n)_n$ has such invariance as the MLE has, and hence, it suffices to consider the standard Cauchy distribution only. Furthermore, the performances of $(\widetilde{Z}_n)_n$ in the case of the logarithmic functions f_1 and f_2 are similar to that of the case of MLE in Table 2.

However, there are no theoretical guarantees of these performances. We are not sure whether $(\widetilde{Y}_n)_n$ is consistent or not, and, \widetilde{Y}_n for f_1 works well only if the sample size n is even. If n is odd, then, $Im(\widetilde{Y}_n) = 0$. It would be overcome by changing the definition of the median slightly. One way is to adopt $(x_{(n-1)/2} + x_{(n+1)/2} + x_{(n+3)/2})/3$ for $\{x_1 < ... < x_n\}$. Here we do not discuss this issue further. See also Subsection 5.7 in Akaoka et al. (2021a) for some delicate issues for median-adjusting.

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