Supplementary Material for: Inference for nonstationary time series of counts with application to change-point problems

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Appendix: Supplement to the proof of Theorem 3.2

Under the assumptions of Theorem 3.2, let us prove that :

1.
$$\frac{1}{n}\Big\|\frac{\partial^2}{\partial\theta\partial\theta'}\widehat{L}\big(T_{k^*+1,k^*+n},\theta\big)-\frac{\partial^2}{\partial\theta\partial\theta'}\tilde{L}\big(T_{k^*+1,k^*+n},\theta\big)\Big\|_\Theta \xrightarrow[n\to\infty]{\texttt{a.s.}} 0; \tag{A.1}$$

2.
$$\frac{1}{n}\Big\|\sum_{t=k^*+1}^{k^*+n}\frac{1}{\widehat{f}^t_\theta}\big(\frac{\partial}{\partial\theta}\widehat{f}^t_\theta\big)\big(\frac{\partial}{\partial\theta}\widehat{f}^t_\theta\big)'-\sum_{t=k^*+1}^{k^*+n}\frac{1}{\tilde{f}^t_\theta}\big(\frac{\partial}{\partial\theta}\tilde{f}^t_\theta\big)\big(\frac{\partial}{\partial\theta}\tilde{f}^t_\theta\big)'\Big\|_\Theta \xrightarrow[n\to\infty]{\texttt{a.s.}} 0; \tag{A.2}$$

3.
$$\mathbb{E}\Big(\frac{1}{\sqrt{n}}\Big\|\frac{\partial}{\partial\theta}\widehat{L}\big(T_{k^*+1,k^*+n},\theta\big)-\frac{\partial}{\partial\theta}\tilde{L}\big(T_{k^*+1,k^*+n},\theta\big)\Big\|_\Theta\Big) \xrightarrow[n\to\infty]{} 0. \tag{A.3}$$

In the sequel, C denotes a positive constant whom value may differ from an inequality to another.

Proof of A.1 We have,

$$\frac{1}{n}\Big\|\frac{\partial^2}{\partial\theta\partial\theta'}\widehat{L}\big(T_{k^*+1,k^*+n},\theta\big)-\frac{\partial^2}{\partial\theta\partial\theta'}\tilde{L}\big(T_{k^*+1,k^*+n},\theta\big)\Big\|_\Theta \leq \frac{1}{n}\sum_{t=k^*+1}^{k^*+n}\Big\|\frac{\partial^2\widehat{\ell}_t(\theta)}{\partial\theta\partial\theta'}-\frac{\partial^2\tilde{\ell}_t(\theta)}{\partial\theta\partial\theta'}\Big\|_\Theta; \tag{A.4}$$

with $\widehat{\ell}_t(\theta)=Y_t\log\widehat{\lambda}_t(\theta)-\widehat{\lambda}_t(\theta)$, $\widehat{\lambda}_t(\theta):=\widehat{f}^t_\theta:=f_\theta(Y_{t-1},\dots,Y_1,0,\dots)$, $\tilde{\ell}_t(\theta)=\tilde{Y}_t\log\tilde{\lambda}_t(\theta)-\tilde{\lambda}_t(\theta)$, $\tilde{\lambda}_t(\theta):=\tilde{f}^t_\theta:=f_\theta(\tilde{Y}_{t-1},\dots)$.

Let $i,j\in\{1,\cdots,d\}$, it suffices to prove that

$$\frac{1}{n}\sum_{t=k^*+1}^{k^*+n}\Big\|\frac{\partial^2\widehat{\ell}_t(\theta)}{\partial\theta_i\partial\theta_j}-\frac{\partial^2\tilde{\ell}_t(\theta)}{\partial\theta_i\partial\theta_j}\Big\|_\Theta\xrightarrow[n\to\infty]{\text{a.s}}0. \tag{A.5}$$

Let $0<r<1$. According to Kounias and Weng (1969), it is enough to show that

$$\sum_{\ell\geq1}\big(\frac{1}{\ell}\big)^r\mathbb{E}\Big[\Big\|\frac{\partial^2\widehat{\ell}_{k^*+\ell}(\theta)}{\partial\theta_i\partial\theta_j}-\frac{\partial^2\tilde{\ell}_{k^*+\ell}(\theta)}{\partial\theta_i\partial\theta_j}\Big\|_\Theta^r\Big]<\infty. \tag{A.6}$$

We have for all $\theta\in\Theta$,

$$\frac{\partial\widehat{\ell}_{k^*+\ell}(\theta)}{\partial\theta_i}=Y_{k^*+\ell}\frac{1}{\widehat{f}_\theta^{k^*+\ell}}\frac{\partial\widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i}-\frac{\partial\widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i}, \tag{A.7}$$

$$\begin{aligned}\frac{\partial^2\widehat{\ell}_{k^*+\ell}(\theta)}{\partial\theta_i\partial\theta_j}&=\frac{\partial}{\partial\theta_j}\Big(Y_{k^*+\ell}\frac{1}{\widehat{f}_\theta^{k^*+\ell}}\frac{\partial\widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i}-\frac{\partial\widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i}\Big)\\&=Y_{k^*+\ell}\left[\frac{\partial}{\partial\theta_j}\left(\frac{1}{\widehat{f}_\theta^{k^*+\ell}}\right)\times\frac{\partial\widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i}+\frac{1}{\widehat{f}_\theta^{k^*+\ell}}\times\frac{\partial^2\widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i\partial\theta_j}\right]-\frac{\partial^2\widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i\partial\theta_j}\\&=Y_{k^*+\ell}\left[-\frac{1}{\big(\widehat{f}_\theta^{k^*+\ell}\big)^2}\left(\frac{\partial\widehat{f}_\theta^{k^*+\ell}}{\partial\theta_j}\right)\times\frac{\partial\widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i}+\frac{1}{\widehat{f}_\theta^{k^*+\ell}}\times\frac{\partial^2\widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i\partial\theta_j}\right]-\frac{\partial^2\widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i\partial\theta_j}\\&=-\frac{Y_{k^*+\ell}}{\big(\widehat{f}_\theta^{k^*+\ell}\big)^2}\left(\frac{\partial\widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i}\right)\left(\frac{\partial\widehat{f}_\theta^{k^*+\ell}}{\partial\theta_j}\right)+\frac{\Big(Y_{k^*+\ell}-\widehat{f}_\theta^{k^*+\ell}\Big)}{\widehat{f}_\theta^{k^*+\ell}}\frac{\partial^2\widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i\partial\theta_j}.\end{aligned} \tag{A.8}$$

Also,

$$\frac{\partial^2\tilde{\ell}_{k^*+\ell}(\theta)}{\partial\theta_i\partial\theta_j}=-\frac{\tilde{Y}_{k^*+\ell}}{\big(\tilde{f}_\theta^{k^*+\ell}\big)^2}\left(\frac{\partial\tilde{f}_\theta^{k^*+\ell}}{\partial\theta_i}\right)\left(\frac{\partial\tilde{f}_\theta^{k^*+\ell}}{\partial\theta_j}\right)+\frac{\Big(\tilde{Y}_{k^*+\ell}-\tilde{f}_\theta^{k^*+\ell}\Big)}{\tilde{f}_\theta^{k^*+\ell}}\frac{\partial^2\tilde{f}_\theta^{k^*+\ell}}{\partial\theta_i\partial\theta_j}. \tag{A.9}$$

Therefore, by using the inequalities $|a_1b_1c_1-a_2b_2c_2|\leq|a_1-a_2||b_1||c_1|+|b_1-b_2||a_2||c_1|+|c_1-c_2||a_2||b_2|$ and $|a_1b_1c_1d_1-a_2b_2c_2d_2|\leq|a_1-a_2||b_1||c_1||d_1|+|b_1-b_2||a_2||c_1||d_1|+|c_1-c_2||a_2||b_2||d_1|+|d_1-d_2||a_2||b_2||c_2|$, we get from (A.8) and (A.9),

$$\begin{aligned}\Big\|\frac{\partial^2\widehat{\ell}_{k^*+\ell}(\theta)}{\partial\theta_i\partial\theta_j}-\frac{\partial^2\tilde{\ell}_{k^*+\ell}(\theta)}{\partial\theta_i\partial\theta_j}\Big\|_\Theta&\leq|Y_{k^*+\ell}-\tilde{Y}_{k^*+\ell}|\left\|\frac{1}{\big(\widehat{f}_\theta^{k^*+\ell}\big)^2}\right\|_\Theta\left\|\frac{\partial\widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i}\right\|_\Theta\left\|\frac{\partial\widehat{f}_\theta^{k^*+\ell}}{\partial\theta_j}\right\|_\Theta\\&+\left\|\frac{1}{\big(\widehat{f}_\theta^{k^*+\ell}\big)^2}-\frac{1}{\big(\tilde{f}_\theta^{k^*+\ell}\big)^2}\right\|_\Theta|\tilde{Y}_{k^*+\ell}|\left\|\frac{\partial\widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i}\right\|_\Theta\left\|\frac{\partial\widehat{f}_\theta^{k^*+\ell}}{\partial\theta_j}\right\|_\Theta+\left\|\frac{\partial\widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i}-\frac{\partial\tilde{f}_\theta^{k^*+\ell}}{\partial\theta_i}\right\|_\Theta|\tilde{Y}_{k^*+\ell}|\left\|\frac{1}{\big(\tilde{f}_\theta^{k^*+\ell}\big)^2}\right\|_\Theta\left\|\frac{\partial\widehat{f}_\theta^{k^*+\ell}}{\partial\theta_j}\right\|_\Theta\\&+\left\|\frac{\partial\widehat{f}_\theta^{k^*+\ell}}{\partial\theta_j}-\frac{\partial\tilde{f}_\theta^{k^*+\ell}}{\partial\theta_j}\right\|_\Theta|\tilde{Y}_{k^*+\ell}|\left\|\frac{1}{\big(\tilde{f}_\theta^{k^*+\ell}\big)^2}\right\|_\Theta\left\|\frac{\partial\tilde{f}_\theta^{k^*+\ell}}{\partial\theta_i}\right\|_\Theta\\&+\Big(|Y_{k^*+\ell}-\tilde{Y}_{k^*+\ell}|+\Big\|\widehat{f}_\theta^{k^*+\ell}-\tilde{f}_\theta^{k^*+\ell}\Big\|_\Theta\Big)\left\|\frac{1}{\widehat{f}_\theta^{k^*+\ell}}\right\|_\Theta\left\|\frac{\partial^2\widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i\partial\theta_j}\right\|_\Theta+\left\|\frac{1}{\widehat{f}_\theta^{k^*+\ell}}-\frac{1}{\tilde{f}_\theta^{k^*+\ell}}\right\|_\Theta\Big\|\tilde{Y}_{k^*+\ell}-\tilde{f}_\theta^{k^*+\ell}\Big\|_\Theta\left\|\frac{\partial^2\widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i\partial\theta_j}\right\|_\Theta\\&+\left\|\frac{\partial^2\widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i\partial\theta_j}-\frac{\partial^2\tilde{f}_\theta^{k^*+\ell}}{\partial\theta_i\partial\theta_j}\right\|_\Theta\Big\|\tilde{Y}_{k^*+\ell}-\tilde{f}_\theta^{k^*+\ell}\Big\|_\Theta\left\|\frac{1}{\tilde{f}_\theta^{k^*+\ell}}\right\|_\Theta\end{aligned}$$

Under the assumption D(Θ), $f_\theta^{k^*+\ell}, \tilde{f}_\theta^{k^*+\ell} \geq \underline{c}$. Hence,

$$
\begin{aligned}
&\Big\| \frac{\partial^2 \widehat{\ell}_{k^*+\ell}(\theta)}{\partial\theta_i\partial\theta_j} - \frac{\partial^2 \tilde{\ell}_{k^*+\ell}(\theta)}{\partial\theta_i\partial\theta_j} \Big\|_\Theta \leq C \Bigg(\Big\| \frac{\partial \widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i} \Big\|_\Theta \Big\| \frac{\partial \widehat{f}_\theta^{k^*+\ell}}{\partial\theta_j} \Big\|_\Theta + \big\| \widehat{f}_\theta^{k^*+\ell} + \tilde{f}_\theta^{k^*+\ell} \big\|_\Theta |\tilde{Y}_{k^*+\ell}| \Big\| \frac{\partial \widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i} \Big\|_\Theta \Big\| \frac{\partial \widehat{f}_\theta^{k^*+\ell}}{\partial\theta_j} \Big\|_\Theta \\
&+ |\tilde{Y}_{k^*+\ell}| \Big\| \frac{\partial \widehat{f}_\theta^{k^*+\ell}}{\partial\theta_j} \Big\|_\Theta + |\tilde{Y}_{k^*+\ell}| \Big\| \frac{\partial \tilde{f}_\theta^{k^*+\ell}}{\partial\theta_i} \Big\|_\Theta + \Big\| \frac{\partial^2 \widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i\partial\theta_j} \Big\|_\Theta + \big\| \tilde{Y}_{k^*+\ell} - \tilde{f}_\theta^{k^*+\ell} \big\|_\Theta \Big\| \frac{\partial^2 \widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i\partial\theta_j} \Big\|_\Theta + \big\| \tilde{Y}_{k^*+\ell} - \tilde{f}_\theta^{k^*+\ell} \big\|_\Theta \Bigg) \\
&\times \Bigg(|Y_{k^*+\ell} - \tilde{Y}_{k^*+\ell}| + \big\| \widehat{f}_\theta^{k^*+\ell} - \tilde{f}_\theta^{k^*+\ell} \big\|_\Theta + \Big\| \frac{\partial \widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i} - \frac{\partial \tilde{f}_\theta^{k^*+\ell}}{\partial\theta_i} \Big\|_\Theta + \Big\| \frac{\partial \widehat{f}_\theta^{k^*+\ell}}{\partial\theta_j} - \frac{\partial \tilde{f}_\theta^{k^*+\ell}}{\partial\theta_j} \Big\|_\Theta + \Big\| \frac{\partial^2 \widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i\partial\theta_j} - \frac{\partial^2 \tilde{f}_\theta^{k^*+\ell}}{\partial\theta_i\partial\theta_j} \Big\|_\Theta \Bigg) \\
&\leq C \Big(1 + \big\| \widehat{f}_\theta^{k^*+\ell} \big\|_\Theta + \big\| \tilde{f}_\theta^{k^*+\ell} \big\|_\Theta \Big) \big(1 + \tilde{Y}_{k^*+\ell} \big) \Bigg(1 + \Big\| \frac{\partial \widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i} \Big\|_\Theta + \Big\| \frac{\partial \tilde{f}_\theta^{k^*+\ell}}{\partial\theta_i} \Big\|_\Theta \Bigg) \Bigg(1 + \Big\| \frac{\partial \widehat{f}_\theta^{k^*+\ell}}{\partial\theta_j} \Big\|_\Theta + \Big\| \frac{\partial^2 \widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i\partial\theta_j} \Big\|_\Theta \Bigg) \\
&\times \Bigg(|Y_{k^*+\ell} - \tilde{Y}_{k^*+\ell}| + \big\| \widehat{f}_\theta^{k^*+\ell} - \tilde{f}_\theta^{k^*+\ell} \big\|_\Theta + \Big\| \frac{\partial \widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i} - \frac{\partial \tilde{f}_\theta^{k^*+\ell}}{\partial\theta_i} \Big\|_\Theta + \Big\| \frac{\partial \widehat{f}_\theta^{k^*+\ell}}{\partial\theta_j} - \frac{\partial \tilde{f}_\theta^{k^*+\ell}}{\partial\theta_j} \Big\|_\Theta + \Big\| \frac{\partial^2 \widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i\partial\theta_j} - \frac{\partial^2 \tilde{f}_\theta^{k^*+\ell}}{\partial\theta_i\partial\theta_j} \Big\|_\Theta \Bigg)
\end{aligned}
\tag{A.10}
$$

Therefore, from the Hölder's inequality, we get

$$
\begin{aligned}
&\mathbb{E}\Big\| \frac{\partial^2 \widehat{\ell}_{k^*+\ell}(\theta)}{\partial\theta_i\partial\theta_j} - \frac{\partial^2 \tilde{\ell}_{k^*+\ell}(\theta)}{\partial\theta_i\partial\theta_j} \Big\|_\Theta^r \leq C \Bigg(\mathbb{E}\Bigg[\Big(1 + \big\| \widehat{f}_\theta^{k^*+\ell} \big\|_\Theta + \big\| \tilde{f}_\theta^{k^*+\ell} \big\|_\Theta \Big) \big(1 + \tilde{Y}_{k^*+\ell} \big) \Bigg(1 + \Big\| \frac{\partial \widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i} \Big\|_\Theta + \Big\| \frac{\partial \tilde{f}_\theta^{k^*+\ell}}{\partial\theta_i} \Big\|_\Theta \Bigg) \\
&\Bigg(1 + \Big\| \frac{\partial \widehat{f}_\theta^{k^*+\ell}}{\partial\theta_j} \Big\|_\Theta + \Big\| \frac{\partial^2 \widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i\partial\theta_j} \Big\|_\Theta \Bigg) \Bigg]^{\frac{r}{1-r}} \Bigg)^{1-r} \times \Bigg(\mathbb{E}\Bigg[|Y_{k^*+\ell} - \tilde{Y}_{k^*+\ell}| + \big\| \widehat{f}_\theta^{k^*+\ell} - \tilde{f}_\theta^{k^*+\ell} \big\|_\Theta + \Big\| \frac{\partial \widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i} - \frac{\partial \tilde{f}_\theta^{k^*+\ell}}{\partial\theta_i} \Big\|_\Theta \\
&\qquad + \Big\| \frac{\partial \widehat{f}_\theta^{k^*+\ell}}{\partial\theta_j} - \frac{\partial \tilde{f}_\theta^{k^*+\ell}}{\partial\theta_j} \Big\|_\Theta + \Big\| \frac{\partial^2 \widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i\partial\theta_j} - \frac{\partial^2 \tilde{f}_\theta^{k^*+\ell}}{\partial\theta_i\partial\theta_j} \Big\|_\Theta \Bigg] \Bigg)^r
\end{aligned}
\tag{A.11}
$$

According to the assumption $\mathbf{A}_0(\Theta)$, we have for all $\ell \geq 1$,

$$
\begin{aligned}
\big\| \widehat{f}_\theta^{k^*+\ell} \big\|_\Theta &\leq \big\| \widehat{f}_\theta^{k^*+\ell} - \widehat{f}_\theta(0) \big\|_\Theta + \big\| \widehat{f}_\theta(0) \big\|_\Theta \leq \| f_\theta(Y_{k^*+\ell-1}, \cdots, Y_1, 0, \cdots) - f_\theta(0) \|_\Theta + \| f_\theta(0) \|_\Theta \\
&\leq \sum_{j=1}^{k^*+\ell-1} \alpha_j^{(0)} Y_{k^*+\ell-j} + \| f_\theta(0) \|_\Theta \leq \sum_{j\geq 1} \alpha_j^{(0)} Y_{k^*+\ell-j} + \| f_\theta(0) \|_\Theta .
\end{aligned}
\tag{A.12}
$$

Hence, from Proposition 2.1, we have for all $s > 0$,

$$
\Big\| \big\| \widehat{f}_\theta^{k^*+\ell} \big\|_\Theta \Big\|_s \leq \sum_{j\geq 1} \alpha_j^{(0)} \| Y_{k^*+\ell-j} \|_s + \| f_\theta(0) \|_\Theta \leq C \sum_{j\geq 1} \alpha_j^{(0)} + \| f_\theta(0) \|_\Theta \leq C. \tag{A.13}
$$

Recall that the process $\tilde{Y} = (\tilde{Y}_t)_{t\in\mathbb{Z}}$ is stationary with finite moment of any order. The same arguments used to get (A.13) with Proposition 2.1, assumptions $\mathbf{A}_1(\Theta)$ and $\mathbf{A}_2(\Theta)$ yield for any $s > 0$, we can find a constant $C > 0$ such that $\mathbb{E}\tilde{Y}_{k^*+\ell}^s \leq C$, $\mathbb{E}\big\| \widehat{f}_\theta^{k^*+\ell} \big\|_\Theta^s \leq C$, $\mathbb{E}\big\| \tilde{f}_\theta^{k^*+\ell} \big\|_\Theta^s \leq C$, $\mathbb{E}\big\| \partial \widehat{f}_\theta^{k^*+\ell} / \partial\theta_i \big\|_\Theta^s \leq C$, $\mathbb{E}\big\| \partial \tilde{f}_\theta^{k^*+\ell} / \partial\theta_i \big\|_\Theta^s \leq C$, $\mathbb{E}\big\| \partial \widehat{f}_\theta^{k^*+\ell} / \partial\theta_j \big\|_\Theta^s \leq C$, $\mathbb{E}\big\| \partial^2 \widehat{f}_\theta^{k^*+\ell} / \partial\theta_i\partial\theta_j \big\|_\Theta^s \leq C$ for all $\ell \geq 1$. Therefore, (A.11) implies

$$
\begin{aligned}
\mathbb{E}\Big\| \frac{\partial^2 \widehat{\ell}_{k^*+\ell}(\theta)}{\partial\theta_i\partial\theta_j} - \frac{\partial^2 \tilde{\ell}_{k^*+\ell}(\theta)}{\partial\theta_i\partial\theta_j} \Big\|_\Theta^r &\leq C \Bigg(\mathbb{E}\Bigg[|Y_{k^*+\ell} - \tilde{Y}_{k^*+\ell}| + \big\| \widehat{f}_\theta^{k^*+\ell} - \tilde{f}_\theta^{k^*+\ell} \big\|_\Theta + \Big\| \frac{\partial \widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i} - \frac{\partial \tilde{f}_\theta^{k^*+\ell}}{\partial\theta_i} \Big\|_\Theta \\
&\qquad + \Big\| \frac{\partial \widehat{f}_\theta^{k^*+\ell}}{\partial\theta_j} - \frac{\partial \tilde{f}_\theta^{k^*+\ell}}{\partial\theta_j} \Big\|_\Theta + \Big\| \frac{\partial^2 \widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i\partial\theta_j} - \frac{\partial^2 \tilde{f}_\theta^{k^*+\ell}}{\partial\theta_i\partial\theta_j} \Big\|_\Theta \Bigg] \Bigg)^r .
\end{aligned}
\tag{A.14}
$$

Lemma 7.1 yields, for all $\ell \geq 1$,

$$\mathbb{E}|Y_{k^*+\ell} - \tilde{Y}_{k^*+\ell}| \leq C\Big(\inf_{1 \leq p \leq \ell} \big\{ (\alpha^{(0)})^{\ell/p} + \sum_{k \geq p} \alpha_k^{(0)} \big\} \Big) \quad \text{with } \alpha^{(0)} = \sum_{k \geq 1} \alpha_k^{(0)}. \tag{A.15}$$

From the proof of Theorem 3.1, we get

$$\mathbb{E}\|\widehat{f}_\theta^{k^*+\ell} - \tilde{f}_\theta^{k^*+\ell}\|_\Theta \leq C\Big(\inf_{1 \leq p \leq \ell/2} \big\{ (\alpha^{(0)})^{\ell/(2p)} + \sum_{i \geq p} \alpha_i^{(0)} \big\} + \sum_{j \geq \ell/2} \alpha_j^{(0)} \Big). \tag{A.16}$$

Also, according to the assumption $\mathbf{A}_1(\Theta)$, Proposition 2.1 and Lemma 7.1, we have

$$\begin{aligned}
\mathbb{E}\left\| \frac{\partial \widehat{f}_\theta^{k^*+\ell}}{\partial \theta_i} - \frac{\partial \tilde{f}_\theta^{k^*+\ell}}{\partial \theta_i} \right\|_\Theta &\leq \sum_{j \geq 1} \alpha_j^{(1)} \mathbb{E}|Y_{k^*+\ell-j} - \tilde{Y}_{k^*+\ell-j}| \leq \sum_{j=1}^{\ell/2-1} \alpha_j^{(1)} \mathbb{E}|Y_{k^*+\ell-j} - \tilde{Y}_{k^*+\ell-j}| + C \sum_{j \geq \ell/2} \alpha_j^{(1)} \\
&\leq C \sum_{j=1}^{\ell/2-1} \alpha_j^{(1)} \Big(\inf_{1 \leq p \leq \ell-j} \big\{ (\alpha^{(0)})^{(\ell-j)/p} + \sum_{i \geq p} \alpha_i^{(0)} \big\} \Big) + C \sum_{j \geq \ell/2} \alpha_j^{(1)} \\
&\leq C\Big(\inf_{1 \leq p \leq \ell/2} \big\{ (\alpha^{(0)})^{\ell/(2p)} + \sum_{i \geq p} \alpha_i^{(0)} \big\} + \sum_{j \geq \ell/2} \alpha_j^{(1)} \Big).
\end{aligned} \tag{A.17}$$

The same arguments yield

$$\mathbb{E}\left\| \frac{\partial \widehat{f}_\theta^{k^*+\ell}}{\partial \theta_j} - \frac{\partial \tilde{f}_\theta^{k^*+\ell}}{\partial \theta_j} \right\|_\Theta \leq C\Big(\inf_{1 \leq p \leq \ell/2} \big\{ (\alpha^{(0)})^{\ell/(2p)} + \sum_{i \geq p} \alpha_i^{(0)} \big\} + \sum_{j \geq \ell/2} \alpha_j^{(1)} \Big), \tag{A.18}$$

$$\mathbb{E}\left\| \frac{\partial^2 \widehat{f}_\theta^{k^*+\ell}}{\partial \theta_i \partial \theta_j} - \frac{\partial^2 \tilde{f}_\theta^{k^*+\ell}}{\partial \theta_i \partial \theta_j} \right\|_\Theta \leq C\Big(\inf_{1 \leq p \leq \ell/2} \big\{ (\alpha^{(0)})^{\ell/(2p)} + \sum_{i \geq p} \alpha_i^{(0)} \big\} + \sum_{j \geq \ell/2} \alpha_j^{(2)} \Big). \tag{A.19}$$

Thus, (A.14), (A.15), (A.16), (A.17), (A.18), (A.19) imply

$$\begin{aligned}
\mathbb{E}\Big\| \frac{\partial^2 \widehat{\ell}_{k^*+\ell}(\theta)}{\partial \theta_i \partial \theta_j} - \frac{\partial^2 \tilde{\ell}_{k^*+\ell}(\theta)}{\partial \theta_i \partial \theta_j} \Big\|_\Theta^r &\leq C \Bigg(\inf_{1 \leq p \leq \ell} \big\{ (\alpha^{(0)})^{\ell/p} + \sum_{k \geq p} \alpha_k^{(0)} \big\} + \inf_{1 \leq p \leq \ell/2} \big\{ (\alpha^{(0)})^{\ell/(2p)} + \sum_{i \geq p} \alpha_i^{(0)} \big\} + \sum_{j \geq \ell/2} \alpha_j^{(0)} \\
&\quad + \inf_{1 \leq p \leq \ell/2} \big\{ (\alpha^{(0)})^{\ell/(2p)} + \sum_{i \geq p} \alpha_i^{(0)} \big\} + \sum_{j \geq \ell/2} \alpha_j^{(1)} + \inf_{1 \leq p \leq \ell/2} \big\{ (\alpha^{(0)})^{\ell/(2p)} + \sum_{i \geq p} \alpha_i^{(0)} \big\} + \sum_{j \geq \ell/2} \alpha_j^{(2)} \Bigg)^r \\
&\leq C \Bigg(\inf_{1 \leq p \leq \ell/2} \big\{ (\alpha^{(0)})^{\ell/(2p)} + \sum_{j \geq p} \alpha_j^{(0)} \big\} + \sum_{j \geq \ell/2} \alpha_j^{(0)} + \sum_{j \geq \ell/2} \alpha_j^{(1)} + \sum_{j \geq \ell/2} \alpha_j^{(2)} \Bigg)^r \\
&\leq C \Bigg((\alpha^{(0)})^{\ell/(2p_\ell)} + \sum_{j \geq p_\ell} \alpha_j^{(0)} + \sum_{j \geq p_\ell} \alpha_j^{(1)} + \sum_{j \geq p_\ell} \alpha_j^{(2)} \Bigg)^r \\
&\leq C \Bigg((\alpha^{(0)})^{r\ell/(2p_\ell)} + \Big(\sum_{j \geq p_\ell} \alpha_j^{(0)} \Big)^r + \Big(\sum_{j \geq p_\ell} \alpha_j^{(1)} \Big)^r + \Big(\sum_{j \geq p_\ell} \alpha_j^{(2)} \Big)^r \Bigg),
\end{aligned} \tag{A.20}$$

with $p_\ell = \ell/\log \ell$. Therefore,

$$\begin{aligned}
&\sum_{\ell\geq 1}\left(\frac{1}{\ell}\right)^r \mathbb{E}\left[\left\|\frac{\partial^2 \widehat{\ell}_{k^*+\ell}(\theta)}{\partial\theta_i\partial\theta_j} - \frac{\partial^2 \tilde{\ell}_{k^*+\ell}(\theta)}{\partial\theta_i\partial\theta_j}\right\|_\Theta^r\right] \\
&\leq C\sum_{\ell\geq 1}\left(\frac{1}{\ell}\right)^r \left[(\alpha^{(0)})^{r\ell/(2p_\ell)} + \left(\sum_{j\geq p_\ell}\alpha_j^{(0)}\right)^r + \left(\sum_{j\geq p_\ell}\alpha_j^{(1)}\right)^r + \left(\sum_{j\geq p_\ell}\alpha_j^{(2)}\right)^r\right] \\
&\leq C\sum_{\ell\geq 1}\left(\frac{1}{\ell}\right)^r (\alpha^{(0)})^{\frac{r}{2}\log \ell} + C\sum_{\ell\geq 1}\left(\frac{1}{\ell}\right)^r \left[\left(\sum_{j\geq p_\ell}\alpha_j^{(0)}\right)^r + \left(\sum_{j\geq p_\ell}\alpha_j^{(1)}\right)^r + \left(\sum_{j\geq p_\ell}\alpha_j^{(2)}\right)^r\right] \\
&\leq C\sum_{\ell\geq 1}\frac{1}{\ell^{r-\frac{r}{2}\log\alpha^{(0)}}} + C\sum_{\ell\geq 1}\left(\frac{1}{\ell}\right)^r\left(\frac{1}{(p_\ell)^{\gamma-1}}\right)^r \qquad \text{(A.21)} \\
&\leq C\sum_{\ell\geq 1}\frac{1}{\ell^{\frac{r}{2}(2-\log\alpha^{(0)})}} + C\sum_{\ell\geq 1}\frac{(\log \ell)^{r(\gamma-1)}}{\ell^{r\gamma}}, \qquad \text{(A.22)}
\end{aligned}$$

where (A.21) holds from the assumption that for $i = 0, 1, 2$, $\alpha_j^{(i)} = O(j^{-\gamma})$ with $\gamma > 3/2$. For any $r \in (\max(\frac{2}{3}, \frac{2}{2-\log\alpha^{(0)}}), 1)$, the sums on the right-hand side of (A.22) are finite. Thus, (A.6) is established; which completes the proof of (A.1).

■

Proof of A.2 Let $i, j \in \{1, \cdots, d\}$, it suffices to prove that

$$\frac{1}{n}\sum_{t=k^*+1}^{k^*+n}\left\|\frac{1}{\widehat{f}_\theta^t}\frac{\partial \widehat{f}_\theta^t}{\partial\theta_i}\frac{\partial \widehat{f}_\theta^t}{\partial\theta_j} - \frac{1}{\tilde{f}_\theta^t}\frac{\partial \tilde{f}_\theta^t}{\partial\theta_i}\frac{\partial \tilde{f}_\theta^t}{\partial\theta_j}\right\|_\Theta \xrightarrow[n\to\infty]{\texttt{a.s.}} 0;$$

According to Kounias and Weng (1969), it is enough to show that for some $r \in (0, 1)$,

$$\sum_{\ell\geq 1}\left(\frac{1}{\ell}\right)^r \mathbb{E}\left[\left\|\frac{1}{\widehat{f}_\theta^{k^*+\ell}}\frac{\partial \widehat{f}_\theta^{k^*+\ell}}{\partial\theta_i}\frac{\partial \widehat{f}_\theta^{k^*+\ell}}{\partial\theta_j} - \frac{1}{\tilde{f}_\theta^{k^*+\ell}}\frac{\partial \tilde{f}_\theta^{k^*+\ell}}{\partial\theta_i}\frac{\partial \tilde{f}_\theta^{k^*+\ell}}{\partial\theta_j}\right\|_\Theta^r\right] < \infty. \qquad \text{(A.23)}$$

(A.23) is obtained by going along similar lines as in the proof of (A.1).

■

Proof of A.3 For any $i \in \{1, \cdots, d\}$, according to (A.7), the assumption D(Θ), and by using the inequality $|a_1b_1c_1 - a_2b_2c_2| \leq |a_1 - a_2||b_1||c_1| + |b_1 - b_2||a_2||c_1| + |c_1 - c_2||a_2||b_2|$, we have

$$
\begin{aligned}
&\left\| \frac{\partial}{\partial \theta_i} \widehat{L}\left(T_{k^*+1,k^*+n}, \theta\right) - \frac{\partial}{\partial \theta_i} \tilde{L}\left(T_{k^*+1,k^*+n}, \theta\right) \right\|_\Theta \leq \sum_{t=k^*+1}^{k^*+n} \left\| \frac{\partial \widehat{\ell}_t(\theta)}{\partial \theta_i} - \frac{\partial \tilde{\ell}_t(\theta)}{\partial \theta_i} \right\|_\Theta \\
&\leq \sum_{t=k^*+1}^{k^*+n} \left\| \left(Y_t \frac{1}{\widehat{f}_\theta^t} \frac{\partial \widehat{f}_\theta^t}{\partial \theta_i} - \frac{\partial \widehat{f}_\theta^t}{\partial \theta_i} \right) - \left(\tilde{Y}_t \frac{1}{\tilde{f}_\theta^t} \frac{\partial \tilde{f}_\theta^t}{\partial \theta_i} - \frac{\partial \tilde{f}_\theta^t}{\partial \theta_i} \right) \right\|_\Theta \\
&\leq \sum_{t=k^*+1}^{k^*+n} \left(\left\| Y_t \frac{1}{\widehat{f}_\theta^t} \frac{\partial \widehat{f}_\theta^t}{\partial \theta_i} - \tilde{Y}_t \frac{1}{\tilde{f}_\theta^t} \frac{\partial \tilde{f}_\theta^t}{\partial \theta_i} \right\|_\Theta + \left\| \frac{\partial \widehat{f}_\theta^t}{\partial \theta_i} - \frac{\partial \tilde{f}_\theta^t}{\partial \theta_i} \right\|_\Theta \right) \\
&\leq \sum_{t=k^*+1}^{k^*+n} \left(|Y_t - \tilde{Y}_t| \left\| \frac{1}{\widehat{f}_\theta^t} \right\|_\Theta \left\| \frac{\partial \widehat{f}_\theta^t}{\partial \theta_i} \right\|_\Theta + \left\| \frac{1}{\widehat{f}_\theta^t} - \frac{1}{\tilde{f}_\theta^t} \right\|_\Theta |\tilde{Y}_t| \left\| \frac{\partial \widehat{f}_\theta^t}{\partial \theta_i} \right\|_\Theta + \left\| \frac{\partial \widehat{f}_\theta^t}{\partial \theta_i} - \frac{\partial \tilde{f}_\theta^t}{\partial \theta_i} \right\|_\Theta |\tilde{Y}_t| \left\| \frac{1}{\tilde{f}_\theta^t} \right\|_\Theta + \left\| \frac{\partial \widehat{f}_\theta^t}{\partial \theta_i} - \frac{\partial \tilde{f}_\theta^t}{\partial \theta_i} \right\|_\Theta \right) \\
&\leq C \sum_{t=k^*+1}^{k^*+n} \left[\left(1 + \tilde{Y}_t + \left\| \frac{\partial \widehat{f}_\theta^t}{\partial \theta_i} \right\|_\Theta + \tilde{Y}_t \left\| \frac{\partial \widehat{f}_\theta^t}{\partial \theta_i} \right\|_\Theta \right) \left(|Y_t - \tilde{Y}_t| + \left\| \widehat{f}_\theta^t - \tilde{f}_\theta^t \right\|_\Theta + \left\| \frac{\partial \widehat{f}_\theta^t}{\partial \theta_i} - \frac{\partial \tilde{f}_\theta^t}{\partial \theta_i} \right\|_\Theta \right) \right]. \qquad \text{(A.24)}
\end{aligned}
$$

As pointed out in the proof of (A.1), for all $s > 0$, it holds that we can find a constant $C > 0$ such that $\mathbb{E} Y_t^s \leq C$, $\mathbb{E} \tilde{Y}_t^s \leq C$, $\mathbb{E} \left\| \widehat{f}_\theta^t \right\|_\Theta^s \leq C$, $\mathbb{E} \left\| \tilde{f}_\theta^t \right\|_\Theta^s \leq C$, $\mathbb{E} \left\| \partial \widehat{f}_\theta^t / \partial \theta_i \right\|_\Theta^s \leq C$, $\mathbb{E} \left\| \partial \tilde{f}_\theta^t / \partial \theta_i \right\|_\Theta^s \leq C$ for all $t > k^*$. Consequently, let $r \in (0, 1)$, from (A.24) and the Hölder's inequality, we get

$$
\begin{aligned}
&\mathbb{E} \left(\frac{1}{\sqrt{n}} \left\| \frac{\partial}{\partial \theta_i} \widehat{L}\left(T_{k^*+1,k^*+n}, \theta\right) - \frac{\partial}{\partial \theta_i} \tilde{L}\left(T_{k^*+1,k^*+n}, \theta\right) \right\|_\Theta \right) \\
&\leq C \frac{1}{\sqrt{n}} \sum_{t=k^*+1}^{k^*+n} \mathbb{E} \left[\left(1 + \tilde{Y}_t + \left\| \frac{\partial \widehat{f}_\theta^t}{\partial \theta_i} \right\|_\Theta + \tilde{Y}_t \left\| \frac{\partial \widehat{f}_\theta^t}{\partial \theta_i} \right\|_\Theta \right) \left(|Y_t - \tilde{Y}_t| + \left\| \widehat{f}_\theta^t - \tilde{f}_\theta^t \right\|_\Theta + \left\| \frac{\partial \widehat{f}_\theta^t}{\partial \theta_i} - \frac{\partial \tilde{f}_\theta^t}{\partial \theta_i} \right\|_\Theta \right) \right] \\
&\leq C \frac{1}{\sqrt{n}} \sum_{t=k^*+1}^{k^*+n} \mathbb{E} \left[\left(1 + \tilde{Y}_t + \left\| \frac{\partial \widehat{f}_\theta^t}{\partial \theta_i} \right\|_\Theta + \tilde{Y}_t \left\| \frac{\partial \widehat{f}_\theta^t}{\partial \theta_i} \right\|_\Theta \right) \left(|Y_t - \tilde{Y}_t| + \left\| \widehat{f}_\theta^t - \tilde{f}_\theta^t \right\|_\Theta + \left\| \frac{\partial \widehat{f}_\theta^t}{\partial \theta_i} - \frac{\partial \tilde{f}_\theta^t}{\partial \theta_i} \right\|_\Theta \right)^{1-r} \right. \\
&\qquad \left. \times \left(|Y_t - \tilde{Y}_t| + \left\| \widehat{f}_\theta^t - \tilde{f}_\theta^t \right\|_\Theta + \left\| \frac{\partial \widehat{f}_\theta^t}{\partial \theta_i} - \frac{\partial \tilde{f}_\theta^t}{\partial \theta_i} \right\|_\Theta \right)^r \right] \qquad \text{(A.25)} \\
&\leq C \frac{1}{\sqrt{n}} \sum_{t=k^*+1}^{k^*+n} \left(\mathbb{E} \left[\left(1 + \tilde{Y}_t + \left\| \frac{\partial \widehat{f}_\theta^t}{\partial \theta_i} \right\|_\Theta + \tilde{Y}_t \left\| \frac{\partial \widehat{f}_\theta^t}{\partial \theta_i} \right\|_\Theta \right)^{\frac{1}{1-r}} \left(|Y_t - \tilde{Y}_t| + \left\| \widehat{f}_\theta^t - \tilde{f}_\theta^t \right\|_\Theta + \left\| \frac{\partial \widehat{f}_\theta^t}{\partial \theta_i} - \frac{\partial \tilde{f}_\theta^t}{\partial \theta_i} \right\|_\Theta \right) \right] \right)^{1-r} \\
&\qquad \times \left[\mathbb{E} \left(|Y_t - \tilde{Y}_t| + \left\| \widehat{f}_\theta^t - \tilde{f}_\theta^t \right\|_\Theta + \left\| \frac{\partial \widehat{f}_\theta^t}{\partial \theta_i} - \frac{\partial \tilde{f}_\theta^t}{\partial \theta_i} \right\|_\Theta \right) \right]^r \qquad \text{(A.26)} \\
&\leq C \frac{1}{\sqrt{n}} \sum_{t=k^*+1}^{k^*+n} \left[\mathbb{E} \left(|Y_t - \tilde{Y}_t| + \left\| \widehat{f}_\theta^t - \tilde{f}_\theta^t \right\|_\Theta + \left\| \frac{\partial \widehat{f}_\theta^t}{\partial \theta_i} - \frac{\partial \tilde{f}_\theta^t}{\partial \theta_i} \right\|_\Theta \right) \right]^r \\
&\leq C \frac{1}{\sqrt{n}} \sum_{\ell=1}^{n} \left(\mathbb{E} |Y_{k^*+\ell} - \tilde{Y}_{k^*+\ell}| + \mathbb{E} \left\| \widehat{f}_\theta^{k^*+\ell} - \tilde{f}_\theta^{k^*+\ell} \right\|_\Theta + \mathbb{E} \left\| \frac{\partial \widehat{f}_\theta^{k^*+\ell}}{\partial \theta_i} - \frac{\partial \tilde{f}_\theta^{k^*+\ell}}{\partial \theta_i} \right\|_\Theta \right)^r \\
&\leq C \frac{1}{\sqrt{n}} \sum_{\ell=1}^{n} \left(\inf_{1 \leq p \leq \ell} \big\{ (\alpha^{(0)})^{\ell/p} + \sum_{k \geq p} \alpha_k^{(0)} \big\} + \inf_{1 \leq p \leq \ell/2} \big\{ (\alpha^{(0)})^{\ell/(2p)} + \sum_{i \geq p} \alpha_i^{(0)} \big\} + \sum_{j \geq \ell/2} \alpha_j^{(0)} \right. \\
&\qquad \left. + \inf_{1 \leq p \leq \ell/2} \big\{ (\alpha^{(0)})^{\ell/(2p)} + \sum_{i \geq p} \alpha_i^{(0)} \big\} + \sum_{j \geq \ell/2} \alpha_j^{(1)} \right)^r \qquad \text{(A.27)}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\frac{1}{\sqrt{n}}\sum_{\ell=1}^{n}\left(\inf_{1\leq p\leq \ell/2}\left\{(\alpha^{(0)})^{\ell/(2p)}+\sum_{j\geq p}\alpha_j^{(0)}\right\}+\sum_{j\geq \ell/2}\alpha_j^{(0)}+\sum_{j\geq \ell/2}\alpha_j^{(1)}\right)^r\\
&\leq C\frac{1}{\sqrt{n}}\sum_{\ell=1}^{n^{1/3}}\left(\inf_{1\leq p\leq \ell/2}\left\{(\alpha^{(0)})^{\ell/(2p)}+\sum_{j\geq p}\alpha_j^{(0)}\right\}+\sum_{j\geq \ell/2}\alpha_j^{(0)}+\sum_{j\geq \ell/2}\alpha_j^{(1)}\right)^r\\
&\qquad\qquad+C\frac{1}{\sqrt{n}}\sum_{\ell=n^{1/3}}^{n}\left(\inf_{1\leq p\leq \ell/2}\left\{(\alpha^{(0)})^{\ell/(2p)}+\sum_{j\geq p}\alpha_j^{(0)}\right\}+\sum_{j\geq \ell/2}\alpha_j^{(0)}+\sum_{j\geq \ell/2}\alpha_j^{(1)}\right)^r\\
&\leq C\frac{1}{\sqrt{n}}\sum_{\ell=1}^{n^{1/3}}\left(\left\{(\alpha^{(0)})^{\ell/2}+\sum_{j\geq 1}\alpha_j^{(0)}\right\}+\sum_{j\geq \ell/2}\alpha_j^{(0)}+\sum_{j\geq \ell/2}\alpha_j^{(1)}\right)^r\\
&\qquad\qquad+C\frac{1}{\sqrt{n}}\sum_{\ell=n^{1/3}}^{n}\left((\alpha^{(0)})^{\ell/(2p_\ell)}+\sum_{j\geq p_\ell}\alpha_j^{(0)}+\sum_{j\geq \ell/2}\alpha_j^{(0)}+\sum_{j\geq \ell/2}\alpha_j^{(1)}\right)^r\\
&\leq C\frac{1}{\sqrt{n}}\sum_{\ell=1}^{n^{1/3}}\left((\alpha^{(0)})^{r/2}+\left(\sum_{j\geq 1}\alpha_j^{(0)}\right)^r+\left(\sum_{j\geq 1}\alpha_j^{(1)}\right)^r\right)\\
&\qquad\qquad+C\frac{1}{\sqrt{n}}\sum_{\ell=n^{1/3}}^{n}\left((\alpha^{(0)})^{r\ell/(2p_\ell)}+\left(\sum_{j\geq p_\ell}\alpha_j^{(0)}\right)^r+\left(\sum_{j\geq p_\ell}\alpha_j^{(1)}\right)^r\right)\\
&\leq C\frac{n^{1/3}}{\sqrt{n}}+C\frac{1}{\sqrt{n}}\sum_{\ell=n^{1/3}}^{n}\left((\alpha^{(0)})^{r\ell/(2p_\ell)}+\left(\frac{1}{(p_\ell)^{\gamma-1}}\right)^r\right)\leq C\frac{1}{n^{1/6}}+C\frac{1}{\sqrt{n}}\sum_{\ell=n^{1/3}}^{n}\left((\alpha^{(0)})^{r\ell^{1-\epsilon}}+\frac{1}{\ell^{r\epsilon(\gamma-1)}}\right) \qquad \text{(A.28)}\\
&\leq C\frac{1}{n^{1/6}}+C\sum_{\ell=n^{1/3}}^{n}\left(\frac{(\alpha^{(0)})^{r\ell^{1-\epsilon}}}{\sqrt{\ell}}+\frac{1}{\sqrt{\ell}\ \ell^{r\epsilon(\gamma-1)}}\right)\leq C\frac{1}{n^{1/6}}+C\sum_{\ell=n^{1/3}}^{n}\left(\frac{(\alpha^{(0)})^{r\ell^{1-\epsilon}}}{\sqrt{\ell}}+\frac{1}{\ell^{\frac{1}{2}+r\epsilon(\gamma-1)}}\right)\\
&\leq C\frac{1}{n^{1/6}}+C\sum_{\ell\geq n^{1/3}}\left(\frac{(\alpha^{(0)})^{r\ell^{1-\epsilon}}}{\sqrt{\ell}}+\frac{1}{\ell^{\frac{1}{2}+r\epsilon(\gamma-1)}}\right)\underset{n\to\infty}{\longrightarrow}0, \qquad \text{(A.29)}
\end{aligned}
$$

with $p_\ell=\frac{\ell^\epsilon}{2}$ for some $\epsilon\in(\frac{1}{2(\gamma-1)},1)$ and $r\in(\frac{1}{2\epsilon(\gamma-1)},1)$ and where the first inequality in (A.28) holds from the assumption $\alpha_j^{(0)}=O(j^{-\gamma})$, $\alpha_j^{(1)}=O(j^{-\gamma})$ with $\gamma>3/2$. Note that, since $\gamma>3/2$, $\frac{1}{2(\gamma-1)}<1$ and $\frac{1}{2\epsilon(\gamma-1)}<1$. The convergence in (A.29) holds since $\sum_{\ell\geq 1}\frac{(\alpha^{(0)})^{r\ell^{1-\epsilon}}}{\sqrt{\ell}}<\infty$ and $\sum_{\ell\geq 1}\frac{1}{\ell^{\frac{1}{2}+r\epsilon(\gamma-1)}}<\infty$ (since $\frac{1}{2}+r\epsilon(\gamma-1)>1$). This completes the proof of (A.3) ■

References

Kounias, E.G. and Weng, T.-S (1969). An inequality and almost sure convergence. *Annals of Mathematical Statistics*, 33, 1091–1093