Supplementary material to "Asymptotics for function derivatives estimators based on stationary and ergodic discrete time processes"

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Abstract This supplementary material file contains some additional mathematical details needed to prove the main results in the manuscript entitled "Asymptotics for function derivatives estimators based on stationary and ergodic discrete time processes".

Additional Mathematical Details

The following additional results are required to complete the proofs of the results in the main manuscript.

Lemma 1 *Under the assumptions* (K.1)(iii) *and* (C.3)(ii), *we have*

$$\sup_{\mathbf{x}\in\mathbf{J}}\left|\mathbb{E}\left(D^{|\mathbf{s}|}r_n(\psi;\mathbf{x},h_n)\right)-D^{|\mathbf{s}|}r(\psi;\mathbf{x})\right|=O(h_n^{\ell/p}).$$

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Proof of Lemma 1.

Notice that, under conditions (C.3)(ii), we have

$$\begin{aligned}
&\mathbb{E}\left(D^{|\mathbf{s}|}r_n(\psi;\mathbf{x},h_n)\right)\\
&=\mathbb{E}\left(\frac{1}{nh_n^{1+|\mathbf{s}|/p}}\sum_{i=1}^{n}\psi(\mathbf{Y}_i)D^{|\mathbf{s}|}K\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n^{1/p}}\right)\right)\\
&=\mathbb{E}\left(\frac{1}{h_n^{1+|\mathbf{s}|/p}}\psi(\mathbf{Y}_i)D^{|\mathbf{s}|}K\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n^{1/p}}\right)\right)\\
&=\frac{1}{h_n^{1+|\mathbf{s}|/p}}\int_{\mathbb{R}^p}\int_{\mathbb{R}^q}\psi(\mathbf{v})D^{|\mathbf{s}|}K\left(\frac{\mathbf{x}-\mathbf{u}}{h_n^{1/p}}\right)f_{\mathbf{X},\mathbf{Y}}(\mathbf{u},\mathbf{v})d\mathbf{u}d\mathbf{v}\\
&=\frac{1}{h_n}\int_{\mathbb{R}^p}\int_{\mathbb{R}^q}\psi(\mathbf{v})K\left(\frac{\mathbf{x}-\mathbf{u}}{h_n^{1/p}}\right)D^{|\mathbf{s}|}f_{\mathbf{X},\mathbf{Y}}(\mathbf{u},\mathbf{v})d\mathbf{u}d\mathbf{v}\\
&=\int_{\mathbb{R}^p}\int_{\mathbb{R}^q}\psi(\mathbf{v})K\left(\mathbf{s}\right)D^{|\mathbf{s}|}f_{\mathbf{X},\mathbf{Y}}(\mathbf{x}-h_n^{1/p}\mathbf{s},\mathbf{v})d\mathbf{s}d\mathbf{v}\\
&=\int_{\mathbb{R}^p}\int_{\mathbb{R}^q}\psi(\mathbf{v})K\left(\mathbf{s}\right)D^{|\mathbf{s}|}f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{v})d\mathbf{s}d\mathbf{v}\\
&\quad+(-1)^{\ell}\frac{h_n^{\ell/p}}{\ell!}\int_{\mathbb{R}^p}\int_{\mathbb{R}^q}\sum_{k_1+\cdots+k_p=\ell}s_1^{k_1}\ldots s_p^{k_p}\frac{\partial^{\ell}D^{|\mathbf{s}|}f_{\mathbf{X},\mathbf{Y}}(\mathbf{x}-h\theta\mathbf{s},\mathbf{v})}{\partial x_1^{k_1}\ldots\partial x_p^{k_p}}\psi(\mathbf{v})K(\mathbf{s})d\mathbf{s}d\mathbf{v},
\end{aligned}$$

where $\theta=(\theta_1,\ldots,\theta_p)$ and $0<\theta_i<1$, $i=,1,\ldots,p$. This, in turn, implies that

$$\begin{aligned}
&\left|\mathbb{E}\left(D^{|\mathbf{s}|}r_n(\psi;\mathbf{x},h_n)\right)-D^{|\mathbf{s}|}r(\psi;\mathbf{x})\right|\\
&=\frac{h_n^{\ell/p}}{\ell!}\left|\int_{\mathbb{R}^p}\int_{\mathbb{R}^q}\sum_{k_1+\cdots+k_p=\ell}s_1^{k_1}\ldots s_p^{k_p}\frac{\partial^{\ell}D^{|\mathbf{s}|}f_{\mathbf{X},\mathbf{Y}}(\mathbf{x}-h\theta\mathbf{s},\mathbf{v})}{\partial x_1^{k_1}\ldots\partial x_p^{k_p}}\psi(\mathbf{v})K(\mathbf{s})d\mathbf{s}d\mathbf{v}\right|.
\end{aligned}$$

Thus a straightforward application of the Lebesgue dominated convergence theorem gives, for n large enough,

$$\sup_{\mathbf{x}\in\mathbf{J}}\left|\mathbb{E}\left(D^{|\mathbf{s}|}r_n(\psi;\mathbf{x},h_n)\right)-D^{|\mathbf{s}|}r(\psi;\mathbf{x})\right|=O(h_n^{\ell/p}). \tag{0.1}$$

Hence the proof is complete. □

Lemma 2 *Under assumptions* (K.1)(i), (K.1)(ii), (C.1), (C.2), (C.3), (R.1)(i), (R.1)(iii). *For all* $\mathbf{x}\in\mathbf{J}$, *we have, as* $n\to\infty$,

$$\sqrt{nh_n^{1+2\left(\frac{|\mathbf{s}|}{p}\right)}}\left(D^{|\mathbf{s}|}r_n(\psi;\mathbf{x},h_n)-\widetilde{D}^{|\mathbf{s}|}r_n(\psi;\mathbf{x},h_n)\right)\xrightarrow{\mathcal{D}}\mathcal{N}(0,\sigma_\psi^2(\mathbf{x})), \tag{0.2}$$

where we recall

$$\sigma_\psi^2(\mathbf{x})=\Psi_2(\mathbf{x})f_{\mathbf{X}}(\mathbf{x})\left(\int_{\mathbb{R}^p}\left(D^{|\mathbf{s}|}K\left(\mathbf{v}\right)\right)^2d\mathbf{v}\right),$$

and

$$\Psi_2(\mathbf{x})=\mathbb{E}(\psi^2(\mathbf{Y})|\mathbf{X}=\mathbf{x}).$$

Proof of Lemma 2.

Let us introduce the notation

$$\eta_{ni}(\mathbf{x},\psi) = \left(\frac{1}{nh_n}\right)^{1/2} \psi(\mathbf{Y}_i) D^{|\mathbf{s}|} K\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n^{1/p}}\right), \quad \xi_{ni}(\mathbf{x},\psi) = \sum_{i=1}^{n} \left(\eta_{ni} - \mathbb{E}\left[\eta_{ni} \mid \mathcal{G}_{i-1}\right]\right).$$

We first observe that

$$\sqrt{nh_n^{1+2\left(\frac{|\mathbf{s}|}{p}\right)}}\left(D^{|\mathbf{s}|} r_n(\psi;\mathbf{x},h_n) - \widetilde{D}^{|\mathbf{s}|} r_n(\psi;\mathbf{x},h_n)\right) = \sum_{i=1}^{n} \xi_{ni}, \tag{0.3}$$

where, for any fixed $\mathbf{x}$ in $\mathbf{J}$, the summands statement (0.3) form a triangular array of stationary ergodic martingale differences with respect to the σ-field $\mathcal{G}_{i-1}$. This allows us to apply the central limit theorem for discrete-time arrays of real-valued martingales (see, Hall and Heyde (1980), page 23) to establish the asymptotic normality. This can be done if we establish the following statements:

(a) $\sum_{i=1}^{n} \mathbb{E}\left[\xi_{ni}^2(\mathbf{x},\psi) \mid \mathcal{G}_{i-1}\right] \xrightarrow{\mathbb{P}} \sigma_\psi^2(\mathbf{x})$,
(b) $n\mathbb{E}[\xi_{ni}^2(\mathbf{x},\psi)\mathbb{1}_{\{|\xi_{ni}(\mathbf{x},\psi)|>\epsilon\}}] = o(1)$ holds, for any $\epsilon > 0$. (Lindeberg condition).

Proof of part (a). We first observe that

$$\left|\sum_{i=1}^{n} \mathbb{E}\left[\eta_{ni}^2(\mathbf{x},\psi) \mid \mathcal{G}_{i-1}\right] - \sum_{i=1}^{n} \mathbb{E}\left[\xi_{ni}^2(\mathbf{x},\psi) \mid \mathcal{G}_{i-1}\right]\right| = \left|\sum_{i=1}^{n} \left(\mathbb{E}\left[\eta_{ni}(\mathbf{x},\psi) \mid \mathcal{G}_{i-1}\right]\right)^2\right|.$$

Using assumption (R.1)(i), by a first order Taylor series expansion and the change of variable $h_n^{1/p}\mathbf{v} = \mathbf{x} - \mathbf{u}$, we have

$$
\begin{aligned}
&\mathbb{E}\left[\eta_{ni}(\mathbf{x},\psi) \mid \mathcal{G}_{i-1}\right] \\
&= \left(\frac{1}{nh_n}\right)^{1/2} \mathbb{E}\left[\psi(\mathbf{Y}_i) D^{|\mathbf{s}|} K\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n^{1/p}}\right) \mid \mathcal{G}_{i-1}\right] \\
&= \left(\frac{1}{nh_n}\right)^{1/2} \mathbb{E}\left[\mathbb{E}\left[\psi(\mathbf{Y}_i)\middle|\mathcal{S}_{i-1}\right] \middle| D^{|\mathbf{s}|} K\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n^{1/p}}\right) \mathcal{G}_{i-1}\right] \\
&= \left(\frac{1}{nh_n}\right)^{1/2} \int_{\mathbb{R}^p} m(\mathbf{u},\psi) D^{|\mathbf{s}|} K\left(\frac{\mathbf{x}-\mathbf{u}}{h_n^{1/p}}\right) f_{\mathbf{X}_i}^{\mathcal{G}_{i-1}}(\mathbf{u}) d\mathbf{u} \\
&\leq \left(\frac{1}{nh_n}\right)^{1/2} \left(\sup_{\mathbf{x}\in\mathbf{J}} \left| \sup_{\|\mathbf{u}-\mathbf{x}\|\leq\lambda h_n} |m(\mathbf{u},\psi) - m(\mathbf{x},\psi)| + m(\mathbf{x},\psi) \right|\right) \\
&\quad \times \int_{\mathbb{R}^p} D^{|\mathbf{s}|} K\left(\frac{\mathbf{x}-\mathbf{u}}{h_n^{1/p}}\right) f_{\mathbf{X}_i}^{\mathcal{G}_{i-1}}(\mathbf{u}) d\mathbf{u} \\
&\leq \left(\frac{h_n^{\frac{1}{2}+|\mathbf{s}|/p}}{\sqrt{n}}\right) \mathbf{C_m} \int_{\mathbb{R}^p} K(\mathbf{v}) D^{|\mathbf{s}|}(\mathbf{x} - h_n^{1/p}\mathbf{v}) d\mathbf{v} \\
&= \left(\frac{h_n^{\frac{1}{2}+|\mathbf{s}|/p}}{\sqrt{n}}\right) \mathbf{C_m} \Bigg[D^{|\mathbf{s}|} f_{\mathbf{X}_i}^{\mathcal{G}_{i-1}}(\mathbf{x}) \\
&\quad + (-1)^r \frac{h_n^{r/p}}{r!} \int_{\mathbb{R}^p} \sum_{k_1+\cdots+k_p=r} r_1^{k_1} \ldots r_p^{k_p} \frac{\partial^r D^{|\mathbf{s}|} f_{\mathbf{X}_i}^{\mathcal{G}_{i-1}}(\mathbf{x} - h\theta\mathbf{r})}{\partial x_1^{k_1} \ldots \partial x_p^{k_p}} K(\mathbf{r}) d\mathbf{r} \Bigg] \\
&= O\left(\frac{h_n^{\frac{1}{2}+|\mathbf{s}|/p}}{\sqrt{n}}\right) \left[D_{\mathbf{X}_i}^{|\mathbf{s}|}(\mathbf{x}) + O\left(h_n^{r/p}\right)\right],
\end{aligned}
$$

where

$$
\mathbf{C_m} = \sup_{\mathbf{x}\in\mathbf{J}} \left| \sup_{\|\mathbf{u}-\mathbf{x}\|\leq\lambda h_n} |m(\mathbf{u},\psi) - m(\mathbf{x},\psi)| + m(\mathbf{x},\psi) \right|.
$$

Making use of the condition (C.2), we have

$$
\begin{aligned}
\sum_{i=1}^{n} \left(\mathbb{E}\left[\eta_{ni}(\mathbf{x},\psi) \mid \mathcal{G}_{i-1}\right]\right)^2 &= O\left(h_n^{1+2(|\mathbf{s}|/p)}\right) \left(\frac{1}{n}\sum_{i=1}^{n} \left(D^{|\mathbf{s}|} f_{\mathbf{X}_i}^{\mathcal{G}_{i-1}}(\mathbf{x})\right)^2 + O\left(\frac{h_n^{r/p}}{n}\right)\right) \\
&= O\left(h_n^{1+2(|\mathbf{s}|/p)}\right).
\end{aligned}
$$

Therefore, statement (**a**) follows if we show

$$
\sum_{i=1}^{n} \mathbb{E}\left[\eta_{ni}^2(\mathbf{x},\psi) \mid \mathcal{G}_{i-1}\right] \xrightarrow{\mathbb{P}} \sigma_\psi^2(\mathbf{x}).
$$

Observe that by assumptions (K.1)(i) and (R.1)(iii), we have

$$\begin{aligned}
&\sum_{i=1}^{n}\mathbb{E}\left[\eta_{ni}^{2}(\mathbf{x},\psi)\mid\mathcal{G}_{i-1}\right]\\
&=\left(\frac{1}{nh_n}\right)\sum_{i=1}^{n}\mathbb{E}\left[\psi^{2}(\mathbf{Y}_i)\left(D^{|\mathbf{s}|}K\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n^{1/p}}\right)\right)^{2}\mid\mathcal{G}_{i-1}\right]\\
&=\left(\frac{1}{nh_n}\right)\sum_{i=1}^{n}\mathbb{E}\left[\mathbb{E}\left[\psi^{2}(\mathbf{Y}_i)\mid\mathcal{S}_{i-1}\right]\left(D^{|\mathbf{s}|}K\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n^{1/p}}\right)\right)^{2}\mid\mathcal{G}_{i-1}\right]\\
&=\left(\frac{1}{nh_n}\right)\sum_{i=1}^{n}\mathbb{E}\left[\Psi_2(\mathbf{X}_i)\left(D^{|\mathbf{s}|}K\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n^{1/p}}\right)\right)^{2}\mid\mathcal{G}_{i-1}\right]\\
&=I_{n1}+I_{n2},
\end{aligned}$$

where

$$I_{n1}=\left(\frac{1}{nh_n}\right)\Psi_2(\mathbf{x})\sum_{i=1}^{n}\mathbb{E}\left[\left(D^{|\mathbf{s}|}K\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n^{1/p}}\right)\right)^{2}\mid\mathcal{G}_{i-1}\right]$$

and

$$I_{n2}=\left(\frac{1}{nh_n}\right)\sum_{i=1}^{n}\mathbb{E}\left[\left(\Psi_2(\mathbf{X}_i)-\Psi_2(\mathbf{x})\right)\left(D^{|\mathbf{s}|}K\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n^{1/p}}\right)\right)^{2}\mid\mathcal{G}_{i-1}\right].$$

Using the condition (C.1)(ii) and (R.1)(iii) and a Taylor series expansion up to order one, we can write

$$\begin{aligned}
I_{n2}\leq&\sup_{\|\mathbf{u}-\mathbf{x}\|\leq\lambda h_n}|\Psi_2(\mathbf{u})-\Psi_2(\mathbf{x})|\left(\frac{1}{nh_n}\right)\sum_{i=1}^{n}\mathbb{E}\left[\left(D^{|\mathbf{s}|}K\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n^{1/p}}\right)\right)^{2}\mid\mathcal{G}_{i-1}\right]\\
=&\,o(1)\times\left(\frac{1}{nh_n}\right)\sum_{i=1}^{n}\mathbb{E}\left[\left(D^{|\mathbf{s}|}K\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n^{1/p}}\right)\right)^{2}\mid\mathcal{G}_{i-1}\right]\\
=&\,o(1)\times\left(\frac{1}{nh_n}\right)\sum_{i=1}^{n}\int_{\mathbb{R}^p}\left(D^{|\mathbf{s}|}K\left(\frac{\mathbf{x}-\mathbf{u}}{h_n^{1/p}}\right)\right)^{2}f_{\mathbf{X}_i}^{\mathcal{G}_{i-1}}(\mathbf{u})d\mathbf{u}\\
=&\,o(1)\times\left(\frac{1}{n}\right)\sum_{i=1}^{n}\int_{\mathbb{R}^p}\left(D^{|\mathbf{s}|}K\left(\mathbf{v}\right)\right)^{2}f_{\mathbf{X}_i}^{\mathcal{G}_{i-1}}(\mathbf{x}-\mathbf{v})d\mathbf{v}\\
=&\,o(1)\times\left[\left(\frac{1}{n}\right)\left(\sum_{i=1}^{n}f_{\mathbf{X}_i}^{\mathcal{G}_{i-1}}(\mathbf{x})\right)\int_{\mathbb{R}^p}\left(D^{|\mathbf{s}|}K\left(\mathbf{v}\right)\right)^{2}d\mathbf{v}\right.\\
&\left.+(-1)^{\Bbbk}\frac{h_n^{\Bbbk/p}}{\Bbbk!}\int_{\mathbb{R}^p}\sum_{k_1+\cdots+k_p=\Bbbk}r_1^{k_1}\ldots r_p^{k_p}\frac{\partial^{\Bbbk}f_{\mathbf{X}_i}^{\mathcal{G}_{i-1}}(\mathbf{x}-h\theta\mathbf{r})}{\partial x_1^{k_1}\ldots\partial x_p^{k_p}}\left(D^{|\mathbf{s}|}K\left(\mathbf{r}\right)\right)^{2}d\mathbf{r}\right].
\end{aligned}$$

In view of the assumptions (K.1)(i) and (C.2), we deduce, almost surely, as $n\to\infty$,

$$I_{n2}=o(1)\times\left(f(\mathbf{x})\left(\int_{\mathbb{R}^p}\left(D^{|\mathbf{s}|}K\left(\mathbf{v}\right)\right)^{2}d\mathbf{v}\right)+O\left(h_n^{r/p}\right)\right)\to 0. \tag{0.4}$$

Considering now the term I_{n1}, making use of assumption (C.1), we have

$$\begin{aligned}
I_{n1} &= \left(\frac{1}{nh_n}\right)\Psi_2(\mathbf{x})\sum_{i=1}^{n}\mathbb{E}\left[\left(D^{|\mathbf{s}|}K\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n^{1/p}}\right)\right)^2 \mid \mathcal{G}_{i-1}\right]\\
&= \Psi_2(\mathbf{x})\left(\frac{1}{nh_n}\right)\sum_{i=1}^{n}\int_{\mathbb{R}^p}\left(D^{|\mathbf{s}|}K\left(\frac{\mathbf{x}-\mathbf{u}}{h_n^{1/p}}\right)\right)^2 f_{\mathbf{X}_i}^{\mathcal{G}_{i-1}}(\mathbf{u})d\mathbf{u}\\
&= \Psi_2(\mathbf{x})\left(\frac{1}{n}\right)\sum_{i=1}^{n}\int_{\mathbb{R}^p}\left(D^{|\mathbf{s}|}K\left(\mathbf{v}\right)\right)^2 f_{\mathbf{X}_i}^{\mathcal{G}_{i-1}}(\mathbf{x}-\mathbf{v})d\mathbf{v}\\
&= \Psi_2(\mathbf{x})\left[\left(\frac{1}{n}\sum_{i=1}^{n}f_{\mathbf{X}_i}^{\mathcal{G}_{i-1}}(\mathbf{x})\right)\int_{\mathbb{R}^p}\left(D^{|\mathbf{s}|}K\left(\mathbf{v}\right)\right)^2 d\mathbf{v}\right.\\
&\quad\left.+(-1)^r\frac{h_n^{r/p}}{r!}\int_{\mathbb{R}^p}\sum_{k_1+\cdots+k_p=r}r_1^{k_1}\dots r_p^{k_p}\frac{\partial^r f_{\mathbf{X}}(\mathbf{x}-h\theta\mathbf{r})}{\partial x_1^{k_1}\dots\partial x_p^{k_p}}\left(D^{|\mathbf{s}|}K\left(\mathbf{r}\right)\right)^2 d\mathbf{r}\right].
\end{aligned}$$

Making use of assumptions (K.1)(i), (K.1)(iv) and (C.2), we readily obtain that

$$I_{n1} = \Psi_2(\mathbf{x})\left(f_{\mathbf{X}}(\mathbf{x})\left(\int_{\mathbb{R}^p}\left(D^{|\mathbf{s}|}K\left(\mathbf{v}\right)\right)^2 d\mathbf{v}\right) + O\left(h_n^{r/p}\right)\right).$$

We have then

$$\lim_{n\to\infty} I_{n1} = \Psi_2(\mathbf{x})f(\mathbf{x})\left(\int_{\mathbb{R}^p}\left(D^{|\mathbf{s}|}K\left(\mathbf{v}\right)\right)^2 d\mathbf{v}\right). \tag{0.5}$$

Combining (0.4) and (0.5), we obtain

$$\lim_{n\to\infty}\sum_{i=1}^{n}\mathbb{E}\left[\eta_{ni}^2(\mathbf{x},\psi)\mid\mathcal{G}_{i-1}\right] = \Psi_2(\mathbf{x})f_{\mathbf{X}}(\mathbf{x})\left(\int_{\mathbb{R}^p}\left(D^{|\mathbf{s}|}K\left(\mathbf{v}\right)\right)^2 d\mathbf{v}\right). \tag{0.6}$$

Proof of part (b). The Lindeberg condition results from Corollary 9.5.2 in Chow (1998), page 349, which implies that

$$n\mathbb{E}[\xi_{ni}^2(\mathbf{x},\psi)\mathbb{1}_{\{|\xi_{ni}(\mathbf{x},\psi)|>\epsilon\}}] \leq 4n\mathbb{E}[\eta_{ni}^2(\mathbf{x},\psi)\mathbb{1}_{\{|\eta_{ni}(\mathbf{x},\psi)|>\epsilon/2\}}].$$

Let $a >$ and $b >$ such that $\frac{1}{a}+\frac{1}{b}=1$. Making use of Hölder and Markov inequalities and assumption (R.1)(iii) one can write, for all $\epsilon>0$

$$\begin{aligned}
&\mathbb{E}[\eta_{ni}^2(\mathbf{x},\psi)\mathbb{1}_{\{|\eta_{ni}(\mathbf{x},\psi)|>\epsilon/2\}}]\\
&\leq \frac{\mathbb{E}[\eta_{ni}^{2a}(\mathbf{x},\psi)]}{(\epsilon/2)^{2a/b}} = \frac{1}{(\epsilon/2)^{2a/b}}\left(\frac{1}{nh_n}\right)^{2a}\mathbb{E}\left[\psi_{2a}(\mathbf{Y}_i)\left(D^{|\mathbf{s}|}K\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n^{1/p}}\right)\right)^{2a}\right]\\
&= \frac{1}{(\epsilon/2)^{2a/b}}\left(\frac{1}{nh_n}\right)^{2a}\mathbb{E}\left[\mathbb{E}\left[\psi_{2a}(\mathbf{Y}_i)\mid\mathcal{S}_{i-1}\right]\left(D^{|\mathbf{s}|}K\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n^{1/p}}\right)\right)^{2a}\right]\\
&= \frac{1}{(\epsilon/2)^{2a/b}}\left(\frac{1}{nh_n}\right)^{2a}\mathbb{E}\left[\Psi_{2a}\left(\mathbf{X}_i\right)\left(D^{|\mathbf{s}|}K\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n^{1/p}}\right)\right)^{2a}\right]\\
&\leq \frac{1}{(\epsilon/2)^{2a/b}}\left(\frac{1}{nh_n}\right)^{2a}\left(\sup_{\|\mathbf{u}-\mathbf{x}\|\leq\lambda h_n}\left|\Psi_{2a}\left(\mathbf{u}\right)\Psi_{2a}\left(\mathbf{x}\right)\right| + \Psi_{2a}\left(\mathbf{x}\right)\right)\\
&\quad\times\mathbb{E}\left[\left(D^{|\mathbf{s}|}K\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n^{1/p}}\right)\right)^{2a}\right].
\end{aligned}$$

Making use of (K.1)(i) and (C.3)(i) and an first order Taylor's expansion, we infer that

$$
\begin{aligned}
&\mathbb{E}\left[\left(D^{|\mathbf{s}|}K\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n^{1/p}}\right)\right)^{2a}\right]\\
&= \int_{\mathbb{R}^p}\left(D^{|\mathbf{s}|}K\left(\frac{\mathbf{x}-\mathbf{u}}{h_n^{1/p}}\right)\right)^{2a} f_{\mathbf{X}}(\mathbf{u})d\mathbf{u}\\
&= h_n\int_{\mathbb{R}^p}\left(D^{|\mathbf{s}|}K\left(\mathbf{v}\right)\right)^{2a} f_{\mathbf{X}}(\mathbf{x}-\mathbf{v})d\mathbf{v}\\
&= h_n\int_{\mathbb{R}^p}\left(D^{|\mathbf{s}|}K\left(\mathbf{v}\right)\right)^{2a} d\mathbf{v}\\
&\quad\times\left[f_{\mathbf{X}}(\mathbf{x})+(-1)^r\frac{h_n^{r/p}}{r!}\int_{\mathbb{R}^p}\sum_{k_1+\cdots+k_p=r} r_1^{k_1}\ldots r_p^{k_p}\frac{\partial^r f_{\mathbf{X}}(\mathbf{x}-h\theta\mathbf{r})}{\partial x_1^{k_1}\ldots\partial x_p^{k_p}}\left(D^{|\mathbf{s}|}K\left(\mathbf{r}\right)\right)^2 d\mathbf{r}\right]\\
&= O(h_n)\left(\int_{\mathbb{R}^p}\left(D^{|\mathbf{s}|}K\left(\mathbf{v}\right)\right)^{2a}d\mathbf{v}\right)\left(f_{\mathbf{X}}(\mathbf{x})+O\left(h_n^{r/p}\right)\right)=O(h_n).
\end{aligned}
$$

Therefore, we have

$$
\mathbb{E}[\eta_{ni}^2(\mathbf{x},\psi)\mathbb{1}_{\{|\eta_{ni}(\mathbf{x},\psi)|>\epsilon/2\}}]=O\left(\frac{1}{n^{2a}h_n^{2a-1}}\right), \tag{0.7}
$$

which concludes the proof of part (b). The proof of Lemma 2 is complete. □

Lemma 3 *Under the assumptions* (K.1)(i), (C.5), (R.1)(i)-(ii), *we have*

$$
\sqrt{nh_n^{1+2\left(\frac{|\mathbf{s}|}{p}\right)}}\left(\widetilde{D}^{|\mathbf{s}|}r_n(\psi;\mathbf{x},h_n)-\mathbb{E}D^{|\mathbf{s}|}r_n(\psi;\mathbf{x},h_n)\right)=O_{a.s}\left(h_n^{|\mathbf{s}|/p}n^{-1/2}\right). \tag{0.8}
$$

Lemma 4 *Under the assumptions* (K.1)(iii) *and* (C.3)(ii), *and the condition*

$$
n^{1/2}h_n^{(\mathbf{s}+\ell)/p+1/2}\to 0 \quad as \quad n\to\infty,
$$

we have then

$$
\sqrt{nh_n^{1+2(|s|/p)}}\left(\mathbb{E}\left(D^{|\mathbf{s}|}r_n(\psi;\mathbf{x},h_n)\right)-D^{|\mathbf{s}|}r(\psi;\mathbf{x},h_n)\right)=o(1). \tag{0.9}
$$

The proofs of Lemmas 3 and 4 will be omitted.

Proof of Theorem 1.

Theorem 1 is a direct consequence of Lemma 2 and Proposition 81. □

Proof of Theorem 2.

Consider the following decomposition

$$\begin{aligned}&\sup_{\mathbf{x}\in\mathbf{J}}\left|D^{|\mathbf{s}|}r_n(\psi;\mathbf{x},h_n)-D^{|\mathbf{s}|}r(\psi;\mathbf{x})\right|\\&=\sup_{\mathbf{x}\in\mathbf{J}}\left|D^{|\mathbf{s}|}r_n(\psi;\mathbf{x},h_n)-\mathbb{E}\left(D^{|\mathbf{s}|}r_n(\psi;\mathbf{x},h_n)\right)\right|\\&\quad+\sup_{\mathbf{x}\in\mathbf{J}}\left|\mathbb{E}\left(D^{|\mathbf{s}|}r_n(\psi;\mathbf{x},h_n)\right)-D^{|\mathbf{s}|}r(\psi;\mathbf{x})\right|.\end{aligned}$$

We achieve the proof by combining Theorem 1 and Lemma 1. □

Proof of Theorem 3.

Consider the following decomposition

$$\begin{aligned}&\sqrt{nh_n^{1+2(|s|/p)}}\left(D^{|\mathbf{s}|}r_n(\psi;\mathbf{x},h_n)-\mathbb{E}\left(D^{|\mathbf{s}|}r_n(\psi;\mathbf{x},h_n)\right)\right)\\&=\sqrt{nh_n^{1+2(|s|/p)}}\left(D^{|\mathbf{s}|}r_n(\psi;\mathbf{x},h_n)-\widetilde{D}^{|\mathbf{s}|}r_n(\psi;\mathbf{x},h_n)\right)\\&\quad+\sqrt{nh_n^{1+2(|s|/p)}}\left(\widetilde{D}^{|\mathbf{s}|}r_n(\psi;\mathbf{x},h_n)-\mathbb{E}D^{|\mathbf{s}|}r_n(\psi;\mathbf{x},h_n)\right).\end{aligned}$$

The proof is completed by combining the Lemmas 2 and 3. □

Proof of Theorem 4.

Let us consider the following decomposition

$$\begin{aligned}&\sqrt{nh_n^{1+2(|s|/p)}}\left(D^{|\mathbf{s}|}r_n(\psi;\mathbf{x},h_n)-D^{|\mathbf{s}|}r(\psi;\mathbf{x},h_n)\right)\\&=\sqrt{nh_n^{1+2(|s|/p)}}\left(D^{|\mathbf{s}|}r_n(\psi;\mathbf{x},h_n)-\mathbb{E}\left(D^{|\mathbf{s}|}r_n(\psi;\mathbf{x},h_n)\right)\right)\\&\quad+\sqrt{nh_n^{1+2(|s|/p)}}\left(\mathbb{E}\left(D^{|\mathbf{s}|}r_n(\psi;\mathbf{x},h_n)\right)-D^{|\mathbf{s}|}r(\psi;\mathbf{x},h_n)\right).\end{aligned}$$

By using Theorem 3 and Lemma 4, we achieve the proof.

□

Proof of Theorem 5.

Recall from the proof of Lemma 1 that, under conditions (K.1) and (C.3) (ii), we have

$$\begin{aligned}&\left|\mathbb{E}\left(D^{|\mathbf{s}|}r_n(\psi;\mathbf{x},h_n)\right)-D^{|\mathbf{s}|}r(\psi;\mathbf{x})\right|^2\\&=\frac{h_n^{2\ell/p}}{\ell!^2}\left|\int_{\mathbb{R}^p}\int_{\mathbb{R}^q}\sum_{k_1+\cdots+k_p=\ell}s_1^{k_1}\ldots s_p^{k_p}\frac{\partial^\ell D^{|\mathbf{s}|}f_{\mathbf{X},\mathbf{Y}}(\mathbf{x}-h\theta\mathbf{s},\mathbf{v})}{\partial x_1^{k_1}\ldots\partial x_p^{k_p}}\psi(\mathbf{v})K(\mathbf{s})d\mathbf{s}d\mathbf{v}\right|^2.\end{aligned}$$

This implies that

$$\int_{\mathbb{R}^p} \left|\mathbb{E}\left(D^{|\mathbf{s}|} r_n(\psi;\mathbf{x},h_n)\right) - D^{|\mathbf{s}|} r(\psi;\mathbf{x})\right|^2 d\mathbf{x}$$

$$= \frac{h_n^{2\ell/p}}{\ell!^2}\int_{\mathbb{R}^p}\left|\sum_{k_1+\cdots+k_p=\ell}\int_{\mathbb{R}^p} s_1^{k_1}\ldots s_p^{k_p}\int_{\mathbb{R}^q}\frac{\partial^\ell D^{|\mathbf{s}|} f_{\mathbf{X},\mathbf{Y}}(\mathbf{x}-h\theta\mathbf{s},\mathbf{v})}{\partial x_1^{k_1}\ldots\partial x_p^{k_p}}\psi(\mathbf{v})K(\mathbf{s})d\mathbf{s}d\mathbf{v}\right|^2 d\mathbf{x} = O\left(h_n^{2\ell/p}\right).$$

From the proof of (0.6) and (0.8), under the assumptions (K.1), (C.1), (C.2), (C.5) and (R.1)(i)-(ii), we have,

$$\mathrm{Var}\left(D^{|\mathbf{s}|} r_n(\psi;\mathbf{x},h_n)\right) = \frac{1}{nh_n^{1+2\frac{|\mathbf{s}|}{p}}}\Psi_2(\mathbf{x})f_{\mathbf{X}}(\mathbf{x})\left(\int_{\mathbb{R}^p}\left(D^{|\mathbf{s}|}K(\mathbf{v})\right)^2 d\mathbf{v}\right).$$

This readily implies that

$$\int_{\mathbb{R}^p}\mathrm{Var}\left(D^{|\mathbf{s}|} r_n(\psi;\mathbf{x},h_n)\right) = \frac{1}{nh_n^{1+2\frac{|\mathbf{s}|}{p}}}\left(\int_{\mathbb{R}^p}\Psi_2(\mathbf{x})f_{\mathbf{X}}(\mathbf{x})d\mathbf{x}\right)\left(\int_{\mathbb{R}^p}\left(D^{|\mathbf{s}|}K(\mathbf{v})\right)^2 d\mathbf{v}\right) = O\left(\frac{1}{nh_n^{1+2\frac{|\mathbf{s}|}{p}}}\right).$$

We finally obtain that

$$\begin{aligned}\mathrm{AMISE}\left(D^{|\mathbf{s}|} r_n(\psi;\mathbf{x},h_n)\right) &= \int_{\mathbb{R}^p}\mathrm{Bias}\left\{D^{|\mathbf{s}|} r_n(\psi;\mathbf{x},h_n)\right\}^2 d\mathbf{x} + \int_{\mathbb{R}^p}\mathrm{Var}\left(D^{|\mathbf{s}|} r_n(\psi;\mathbf{x},h_n)\right)d\mathbf{x}\\ &= O\left(h_n^{2\ell/p}\right) + O\left(\frac{1}{nh_n^{1+2\frac{|\mathbf{s}|}{p}}}\right).\end{aligned}$$

Hence the proof is complete. □

References

Chow, Y.S. Teicher, H. (1998) Probabilty Theory, 2nd ed., *Springer*, New York.

Hall, P. Heyde, C. Martingale Limit Theory and its Application, Academic Press, New York, 1980.