Supplementary material for"Robust model selection with covariables missing at random"

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Lemmas

The proofs of Lemma 1–Lemma 2 are completely similar to Lemma S3–Lemma S4 in Wang et al. (2020). For convenience of review, we give the details.

Lemma 1. *Provided that Conditions (C2)-(C8) hold, and further assume that* $\hat{\alpha}_n - \alpha^* = O_p(n^{-1/2})$*, we have*

$$
\begin{aligned}
Q_{n1} &= n^{-1}\sum_{i=1}^{n}\{\delta_i \log g_M(Y_i|X_i,Z_i;\theta_M)\}\{\hat{q}_{\hat{\alpha}_n,b_n}(\phi_\pi(Y_i,Z_i;\hat{\alpha}_n)) - \hat{q}_{\alpha^*,b_n}(\phi_\pi(Y_i,Z_i;\alpha^*))\} \\
&= n^{-1}\sum_{i=1}^{n}\{\delta_i \log g_M(Y_i|X_i,Z_i;\theta_M)\}\{\partial q_{\alpha^*}(\phi_\pi(Y_i,Z_i;\alpha^*))/\partial\alpha\}(\hat{\alpha}_n-\alpha^*) + o_p(n^{-1/2}). \quad (\text{S.1})
\end{aligned}
$$

Proof. For simplicity, we denote $T_i = (Y_i, Z_i)$ and denote

$$
\begin{aligned}
B_n(y,z;\alpha,\alpha^*) &= \hat{q}_{\alpha,n}(\phi_\pi(y,z;\alpha))\hat{r}_{\alpha,n}(\phi_\pi(y,z;\alpha)) - \hat{q}_{\alpha^*,n}(\phi_\pi(y,z;\alpha^*))\hat{r}_{\alpha^*,n}(\phi_\pi(y,z;\alpha^*)), \\
\Gamma_n(y,z;\alpha,\alpha^*) &= \hat{r}_{\alpha,n}(\phi_\pi(y,z;\alpha)) - \hat{r}_{\alpha^*,n}(\phi_\pi(y,z;\alpha^*)), \\
\Gamma_{b_n}(y,z;\alpha,\alpha^*) &= \hat{r}_{\alpha,b_n}(\phi_\pi(y,z;\alpha)) - \hat{r}_{\alpha^*,b_n}(\phi_\pi(y,z;\alpha^*)),
\end{aligned}
$$

and

$$
\begin{aligned}
A_n(y,z;\alpha) &= \hat{q}_{\alpha,n}(\phi_\pi(y,z;\alpha))\hat{r}_{\alpha,n}(\phi_\pi(y,z;\alpha)) - q_\alpha(\phi_\pi(y,z;\alpha))r_\alpha(\phi_\pi(y,z;\alpha)), \\
\Delta_n(y,z;\alpha) &= \hat{r}_{\alpha,n}(\phi_\pi(y,z;\alpha)) - r_\alpha(\phi_\pi(y,z;\alpha)), \\
\Delta_{b_n}(y,z;\alpha) &= \hat{r}_{\alpha,b_n}(\phi_\pi(y,z;\alpha)) - r_{\alpha,b_n}(\phi_\pi(y,z;\alpha)).
\end{aligned}
$$

Then, according to the similar arguments to lemma 2 of Li et al. (2011) with (C.3), (C.5), (C.6) and (C.7) we have

$$
\sup_{y,z,\alpha}|A_n(y,z;\alpha)| = O_p(h_n^k + \frac{\log n}{\sqrt{nh_n^J}}) = o_p(b_n), \quad (\text{S.2})
$$

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$$\sup_{y,z,\alpha} |\Delta_n(y,z;\alpha)| = O_p(h_n^k + \frac{\log n}{\sqrt{nh_n^J}}) = o_p(b_n). \tag{S.3}$$

And due to

$$\sup_{y,z,\alpha} |\Delta_{b_n}(y,z;\alpha)| \leq \sup_{y,z,\alpha} |\Delta_n(y,z;\alpha)|,$$

then if n is large enough, we have

$$|\hat{r}_{\alpha,b_n}(\phi_\pi(y,z;\alpha))| \geq |r_{\alpha,b_n}(\phi_\pi(y,z;\alpha))| - o_p(b_n) \geq cb_n. \tag{S.4}$$

Note that,

$$\begin{aligned}
&n^{-1}\sum_{i=1}^n C_i\{\hat{q}_{\hat{\alpha}_n,b_n}\left(\phi_\pi\left(Y_i,Z_i;\hat{\alpha}_n\right)\right) - \hat{q}_{\alpha^*,b_n}(\phi_\pi(Y_i,Z_i;\alpha^*))\}\\
=&n^{-1}\sum_{i=1}^n C_i\frac{B_n(T_i;\hat{\alpha}_n,\alpha^*)}{r_{\alpha^*,b_n}(\phi_\pi(T_i;\alpha^*))} - n^{-1}\sum_{i=1}^n C_i\frac{\Delta_{b_n}(T_i;\alpha^*)B_n(T_i;\hat{\alpha}_n,\alpha^*)}{\hat{r}_{\alpha^*,b_n}(\phi_\pi(T_i;\alpha^*))r_{\alpha^*,b_n}(\phi_\pi(T_i;\alpha^*))}\\
-&n^{-1}\sum_{i=1}^n C_i\frac{q_{\alpha^*}(\phi_\pi(T_i;\alpha^*))r_{\alpha^*}(\phi_\pi(T_i;\alpha^*))\Gamma_{b_n}(T_i;\hat{\alpha}_n,\alpha^*)}{r^2_{\alpha^*,b_n}(\phi_\pi(T_i;\alpha^*))}\\
+&n^{-1}\sum_{i=1}^n C_i\frac{q_{\alpha^*}(\phi_\pi(T_i;\alpha^*))r_{\alpha^*}(\phi_\pi(T_i;\alpha^*))\Delta_{b_n}(T_i;\alpha^*)\Gamma_{b_n}(T_i;\hat{\alpha}_n,\alpha^*)}{r^2_{\alpha^*,b_n}(\phi_\pi(T_i;\alpha^*))\hat{r}_{\alpha^*,b_n}(\phi_\pi(T_i;\alpha^*))}\\
+&n^{-1}\sum_{i=1}^n C_i\frac{q_{\alpha^*}(\phi_\pi(T_i;\alpha^*))r_{\alpha^*}(\phi_\pi(T_i;\alpha^*))\Delta_{b_n}(T_i;\alpha^*)\Gamma_{b_n}(T_i;\hat{\alpha}_n,\alpha^*)}{r_{\alpha^*,b_n}(\phi_\pi(T_i;\alpha^*))\hat{r}^2_{\alpha^*,b_n}(\phi_\pi(T_i;\alpha^*))}\\
-&n^{-1}\sum_{i=1}^n C_i\frac{A_n(T_i;\alpha^*)\Gamma_{b_n}(T_i;\hat{\alpha}_n,\alpha^*)}{\hat{r}^2_{\alpha^*,b_n}(\phi_\pi(T_i;\alpha^*))} - n^{-1}\sum_{i=1}^n C_i\frac{B_n(T_i;\hat{\alpha}_n,\alpha^*)\Gamma_{b_n}(T_i;\hat{\alpha}_n,\alpha^*)}{\hat{r}_{\alpha^*,b_n}(\phi_\pi(T_i;\alpha^*))\hat{r}_{\hat{\alpha}_n,b_n}(\phi_\pi(T_i;\hat{\alpha}_n))}\\
+&n^{-1}\sum_{i=1}^n C_i\frac{A_n(T_i;\alpha^*)\Gamma^2_{b_n}(T_i;\hat{\alpha}_n,\alpha^*)}{\hat{r}_{\hat{\alpha}_n,b_n}(\phi_\pi(T_i;\hat{\alpha}_n))\hat{r}^2_{\alpha^*,b_n}(\phi_\pi(T_i;\alpha^*))}\\
+&n^{-1}\sum_{i=1}^n C_i\frac{q_{\alpha^*}(\phi_\pi(T_i;\alpha^*))r_{\alpha^*}(\phi_\pi(T_i;\alpha^*))\Gamma^2_{b_n}(T_i;\hat{\alpha}_n,\alpha^*)}{\hat{r}_{\hat{\alpha}_n,b_n}(\phi_\pi(T_i;\hat{\alpha}_n))\hat{r}_{\alpha^*,b_n}(\phi_\pi(T_i;\alpha^*))} := \sum_{i=1}^9 G_{ni}.
\end{aligned}$$

To prove (S.1), we show that

$$\begin{aligned}
&G_{n1} + G_{n3}\\
=&n^{-1}\sum_{i=1}^n\{\delta_i \log g_M(Y_i|X_i,Z_i;\theta_M)\}\{\partial q_{\alpha^*}(\phi_\pi(T_i;\alpha^*))/\partial\alpha\}(\hat{\alpha}_n - \alpha^*) + o_p(n^{-1/2}),
\end{aligned} \tag{S.5}$$

and

$$G_{ni} = o_p(n^{-1/2}), \quad i = 2,4,5,6,7,8,9. \tag{S.6}$$

First, we prove (S.5). Denote

$$\begin{aligned}
&W_n(T_i, T_j; \alpha) \\
&= \frac{C_i}{r_{\alpha^*, b_n}(\phi_\pi(T_i; \alpha^*))} \left\{ K\left(\frac{\phi_\pi(T_i; \alpha) - \phi_\pi(T_j; \alpha)}{h_n} \right) - K\left(\frac{\phi_\pi(T_i; \alpha^*) - \phi_\pi(T_j; \alpha^*)}{h_n} \right) \right\}.
\end{aligned}$$

Then

$$\begin{aligned}
G_{n1} &= h_n^{-J} n^{-2} \sum_{i=1}^n \sum_{j=1}^n W_n(T_i, T_j; \hat{\alpha}_n) \\
&= h_n^{-J} \Big\{ n^{-2} \sum_{i=1}^n \sum_{j=1}^n W_n(T_i, T_j; \hat{\alpha}_n) - n^{-1} \sum_{i=1}^n \int W_n(T_i, t; \hat{\alpha}_n) dF_T(t) \\
&\quad - n^{-1} \sum_{j=1}^n \int W_n(t, T_j; \hat{\alpha}_n) dF_T(t) + \iint W_n(t_1, t_2; \hat{\alpha}_n) dF_T(t_1) dF_T(t_2) \Big\} \\
&\quad + \left\{ h_n^{-J} \left[n^{-1} \sum_{j=1}^n \int W_n(t, T_j; \hat{\alpha}_n) dF_T(t) - \iint W_n(t_1, t_2; \hat{\alpha}_n) dF_T(t_1) dF_T(t_2) \right] \right\} \\
&\quad + \left\{ h_n^{-J} n^{-1} \sum_{i=1}^n \int W_n(T_i, t; \hat{\alpha}_n) dF_T(t) \right\} := G_{n11} + G_{n12} + G_{n13},
\end{aligned}$$

where $F_T(\cdot)$ denotes the distribution function of T. According to similar statements as the MAIN COROLLARY in Sherman (1994), we have

$$\begin{aligned}
&E\Big[n \sup_{\|\alpha - \alpha^*\| = O(n^{-1/2})} \Big| n^{-2} \sum_{i=1}^n \sum_{j=1}^n W_n(T_i, T_j; \hat{\alpha}_n) - n^{-1} \sum_{i=1}^n \int W_n(T_i, t; \hat{\alpha}_n) dF_T(t) \\
&\quad - n^{-1} \sum_{j=1}^n \int W_n(t, T_j; \hat{\alpha}_n) dF_T(t) + \iint W_n(t_1, t_2; \hat{\alpha}_n) dF_T(t_1) dF_T(t_2) \Big| \Big] \\
&\quad \le c(E[\sup_{\|\hat{\alpha}_n - \alpha^*\| = O(n^{-1/2})} W_n(T_1, T_2; \hat{\alpha}_n)^2])^{1/2},
\end{aligned}$$

and

$$\begin{aligned}
&E\Big[n^{1/2} \sup_{\|\alpha - \alpha^*\| = O(n^{-1/2})} \Big| n^{-1} \sum_{j=1}^n \int W_n(t, T_j; \hat{\alpha}_n) dF_T(t) \\
&- \iint W_n(t_1, t_2; \hat{\alpha}_n) dF_T(t_1) dF_T(t_2) \Big| \Big] \le c(E[\sup_{\|\hat{\alpha}_n - \alpha^*\| = O(n^{-1/2})} W_n(T_1, T_2; \hat{\alpha}_n)^2])^{1/2}.
\end{aligned}$$

Further, since $E[\log g_M^2(Y|X,Z;\theta_M)] < \infty$, we have

$$
\begin{aligned}
& E[\sup_{\|\alpha-\alpha^*\|=O(n^{-1/2})} W_n(T_1,T_2;\hat{\alpha}_n)^2] \\
\leq & cb_n^{-2} \int \sup_{\|\alpha-\alpha^*\|=O(n^{-1/2})} \left[K\left(\frac{\phi_\pi(t_1;\alpha)-\phi_\pi(t_2;\alpha)}{h_n}\right) - K\left(\frac{\phi_\pi(t_1;\alpha^*)-\phi_\pi(t_1;\alpha^*)}{h_n}\right)\right]^2 \\
& \times f_T(t_1) f_T(t_2) dt_1 dt_2 \\
\leq & cb_n^{-2} \int \sup_{\|\alpha-\alpha^*\|=O(n^{-1/2})} \left| \frac{\phi_\pi(t_1;\alpha)-\phi_\pi(t_2;\alpha)-\phi_\pi(t_1;\alpha^*)+\phi_\pi(t_2;\alpha^*)}{h_n} \right|^2 \\
& \times f_T(t_1) f_T(t_2) dt_1 dt_2 \\
\leq & cb_n^{-2} h_n^{-2} \int \sup_{\|\alpha-\alpha^*\|=O(n^{-1/2})} [\{l(t_1)+l(t_2)\}^2 \|\alpha-\alpha^*\|^2] \cdot f_T(t_1) f_T(t_2) dt_1 dt_2 \\
\leq & \frac{1}{nb_n^2 h_n^2},
\end{aligned}
$$

where the last third inequality is due to (C.6), and the last second inequality is due to (C.5)*(i)*. Thus, according to (C.7), we have $G_{n11} = o_p(n^{-1/2})$ and $G_{n12} = o_p(n^{-1/2})$. To G_{n13}, recalling that $f_\alpha(t) = q_\alpha(t) r_\alpha(t)$, we have

$$
\sup_{t,\alpha} \left| h_n^{-J} \int K\left(\frac{\phi_\pi(T;\alpha)-t}{h_n}\right) f_\alpha(t) dt - q_\alpha(\phi_\pi(t;\alpha)) r_\alpha(\phi_\pi(t;\alpha)) \right| = O(h_n^k).
$$

Then we can obtain

$$
\begin{aligned}
& \Bigg| G_{n13} - n^{-1} \sum_{i=1}^n \frac{C_i}{r_{\alpha^*,b_n}(\phi_\pi(T_i;\alpha^*))} \{ q_{\hat{\alpha}_n}(\phi_\pi(T_i;\hat{\alpha}_n)) r_{\hat{\alpha}_n}(\phi_\pi(T_i;\hat{\alpha}_n)) \\
& - q_\alpha^*(\phi_\pi(T_i;\alpha^*)) r_\alpha^*(\phi_\pi(T_i;\alpha^*)) \} \Bigg| = O_p(h_n^k / b_n).
\end{aligned}
$$

Thus we have

$$
G_{n1} = n^{-1} \sum_{i=1}^n C_i \frac{q_{\hat{\alpha}_n}(\phi_\pi(T_i;\hat{\alpha}_n)) r_{\hat{\alpha}_n}(\phi_\pi(T_i;\hat{\alpha}_n)) - q_\alpha^*(\phi_\pi(T_i;\alpha^*)) r_\alpha^*(\phi_\pi(T_i;\alpha^*))}{r_{\alpha^*,b_n}(\phi_\pi(T_i;\alpha^*))} + o_p(n^{-1/2}). \tag{S.7}
$$

Similarly, we can obtain that

$$
G_{n3} = -n^{-1} \sum_{i=1}^n C_i \frac{q_\alpha^*(\phi_\pi(T_i;\alpha^*)) \{ r_{\hat{\alpha}_n}(\phi_\pi(T_i;\hat{\alpha}_n)) - r_\alpha^*(\phi_\pi(T_i;\alpha^*)) \}}{r_{\alpha^*,b_n}(\phi_\pi(T_i;\alpha^*))} + o_p(n^{-1/2}). \tag{S.8}
$$

Thus, combining with (S.4), (S.7) and (S.8), we have

$$
\begin{aligned}
&G_{n1}+G_{n3}\\
=&n^{-1}\sum_{i=1}^{n}C_i\frac{\{q_{\hat\alpha_n}(\phi_\pi(T_i;\hat\alpha_n))-q_{\alpha^*}(\phi_\pi(T_i;\alpha^*))\}r_{\hat\alpha_n}(\phi_\pi(T_i;\hat\alpha_n))}{r_{\alpha^*,b_n}(\phi_\pi(T_i;\alpha^*))}+o_p(n^{-1/2})\\
=&n^{-1}\sum_{i=1}^{n}C_i\frac{\{q_{\hat\alpha_n}(\phi_\pi(T_i;\hat\alpha_n))-q_{\alpha^*}(\phi_\pi(T_i;\alpha^*))\}r_{\alpha^*}(\phi_\pi(T_i;\hat\alpha_n))}{r_{\alpha^*,b_n}(\phi_\pi(T_i;\alpha^*))}+o_p(n^{-1/2})\\
=&n^{-1}\sum_{i=1}^{n}C_i\{q_{\hat\alpha_n}(\phi_\pi(T_i;\hat\alpha_n))-q_{\alpha^*}(\phi_\pi(T_i;\alpha^*))\}+o_p(n^{-1/2})\\
=&n^{-1}\sum_{i=1}^{n}C_i\{\partial q_{\alpha^*}(\phi_\pi(T_i;\alpha^*))/\partial\alpha\}(\hat\alpha_n-\alpha^*)+o_p(n^{-1/2}).
\end{aligned}
$$

In what follows, we prove (S.6). According to (S.2) and (S.3), we have

$$
\sup_{t,\alpha}|A_n(t;\alpha)|=O_p(h_n^k+\frac{\log n}{\sqrt{nh_n^J}}),\quad \sup_{t,\alpha}|\Delta_{b_n}(t;\alpha)|=O_p(h_n^k+\frac{\log n}{\sqrt{nh_n^J}}). \tag{S.9}
$$

And according to (C.5)*(ii)*, we have

$$
\begin{aligned}
&n^{-1}\sum_{i=1}^{n}|C_i|\cdot|q_{\hat\alpha_n}(\phi_\pi(T_i;\hat\alpha_n))r_{\hat\alpha_n}(\phi_\pi(T_i;\hat\alpha_n))-q_\alpha^*(\phi_\pi(T_i;\alpha^*))r_\alpha^*(\phi_\pi(T_i;\alpha^*))|\\
=&O_p(\|\hat\alpha_n-\alpha^*\|).
\end{aligned} \tag{S.10}
$$

Thus, to G_{n2}, by (S.9) and (S.10), we have

$$
\begin{aligned}
|G_{n2}|\le{}& n^{-1}\sum_{i=1}^{n}\frac{|C_i|}{b_n^2}\sup_t|\Delta_{b_n}(t,\alpha^*)|\sup_{t,\alpha}|A_n(t;\alpha)-A_n(t;\alpha^*)|\\
&+n^{-1}\sum_{i=1}^{n}\frac{|C_i|}{b_n^2}\sup_t|\Delta_{b_n}(t,\alpha^*)|\\
&\quad\times|q_{\hat\alpha_n}(\phi_\pi(T_i;\hat\alpha_n))r_{\hat\alpha_n}(\phi_\pi(T_i;\hat\alpha_n))-q_\alpha^*(\phi_\pi(T_i;\alpha^*))r_\alpha^*(\phi_\pi(T_i;\alpha^*))|=o_p(n^{-1/2}).
\end{aligned}
$$

And the $G_{ni}=o_p(n^{-1/2})$, for $i=4,5,6$ can be proved by similar arguments. And to G_{n7}, by (S.2), (S.3) and (C.7), we have

$$
\begin{aligned}
|G_{n7}| \leq & \; n^{-1}\sum_{i=1}^{n}\frac{|C_i|}{b_n^2}[\sup_{t,\alpha}|A_n(t;\alpha)-A_n(t;\alpha^*)| \\
& + |q_{\hat{\alpha}_n}(\phi_\pi(T_i;\hat{\alpha}_n))r_{\hat{\alpha}_n}(\phi_\pi(T_i;\hat{\alpha}_n)) - q_{\alpha^*}(\phi_\pi(T_i;\alpha^*))r_{\alpha^*}(\phi_\pi(T_i;\alpha^*))|] \\
& \times [\sup_{t,\alpha}|\Delta_n(t;\alpha)-\Delta_n(t;\alpha^*)| + |r_{\hat{\alpha}_n}(\phi_\pi(T_i;\hat{\alpha}_n)) - r_{\alpha^*}(\phi_\pi(T_i;\alpha^*))|] \\
& = o_p(n^{-1/2}).
\end{aligned}
$$

And the $G_{ni} = o_p(n^{-1/2})$, for $i = 8, 9$ can be proved by similar statements. Thus, the proof of Lemma 1 is completed. □

Lemma 2. *Provided that Conditions (C2)-(C8) hold, we have*

$$
\begin{aligned}
Q_{n2} = & \; n^{-1}\sum_{i=1}^{n}\{\delta_i \log g_M(Y_i|X_i,Z_i;\theta_M)\}\{\hat{q}_{\alpha^*,b_n}(\phi_\pi(Y_i,Z_i;\alpha^*)) - q_{\alpha^*,b_n}(\phi_\pi(Y_i,Z_i;\alpha^*))\} \\
= & n^{-1}\sum_{i=1}^{n}\{1-\delta_i q_{\alpha^*}(\phi_\pi(Y_i,Z_i;\alpha^*))\}q_{\alpha^*}(\phi_\pi(Y_i,Z_i;\alpha^*))\delta_i \log g_M(Y_i|X_i,Z_i;\theta_M) + o_p(n^{-1/2}).
\end{aligned}
\tag{S.11}
$$

Proof. Using the notations in Lemma 1 and denote

$$
\begin{aligned}
\eta_n(t;\alpha^*) &= (nh_n^J)^{-1}\sum_{j=1}^{n}(1-\delta_j q_{\alpha^*}(\phi_\pi(T_j;\alpha^*)))K\left(\frac{\phi_\pi(t;\alpha^*)-\phi_\pi(T_j;\alpha^*)}{h_n}\right), \\
\xi_n(t;\alpha^*) &= (nh_n^J)^{-1}\sum_{j=1}^{n}(\delta_j q_{\alpha^*}(\phi_\pi(T_j;\alpha^*)) - \delta_j q_{\alpha^*}(\phi_\pi(t;\alpha^*)))K\left(\frac{\phi_\pi(t;\alpha^*)-\phi_\pi(T_j;\alpha^*)}{h_n}\right).
\end{aligned}
$$

Then,

$$
\begin{aligned}
& n^{-1}\sum_{i=1}^{n}\{\delta_i \log g_M(Y_i|X_i,Z_i;\theta_M)\}\{\hat{q}_{\alpha^*,b_n}(\phi_\pi(Y_i,Z_i;\alpha^*)) - q_{\alpha^*,b_n}(\phi_\pi(Y_i,Z_i;\alpha^*))\}\\
=& n^{-1}\sum_{i=1}^{n} C_i \frac{\eta_n(T_i;\alpha^*)}{r_{\alpha^*,b_n}(\phi_\pi(T_i;\alpha^*))} + n^{-1}\sum_{i=1}^{n} C_i \frac{\xi_n(T_i;\alpha^*)}{r_{\alpha^*,b_n}(\phi_\pi(T_i;\alpha^*))}\\
+& n^{-1}\sum_{i=1}^{n} C_i \frac{q_{\alpha^*}(\phi_\pi(T_i;\alpha^*)\Delta_n(T_i;\alpha^*)}{r_{\alpha^*,b_n}(\phi_\pi(T_i;\alpha^*))} - n^{-1}\sum_{i=1}^{n} C_i \frac{q_{\alpha^*}(\phi_\pi(T_i;\alpha^*)r_{\alpha^*}(\phi_\pi(T_i;\alpha^*)\Delta_{b_n}(T_i;\alpha^*)}{r^2_{\alpha^*,b_n}(\phi_\pi(T_i;\alpha^*))}\\
-& n^{-1}\sum_{i=1}^{n} C_i \frac{(\hat{q}_{\alpha^*}(\phi_\pi(T_i;\alpha^*))\hat{r}_{\alpha^*}(\phi_\pi(T_i;\alpha^*)) - q_{\alpha^*}(\phi_\pi(T_i;\alpha^*))r_{\alpha^*}(\phi_\pi(T_i;\alpha^*)))\Delta_{b_n}(T_i;\alpha^*)}{r^2_{\alpha^*,b_n}(\phi_\pi(T_i;\alpha^*))}\\
+& n^{-1}\sum_{i=1}^{n} C_i \frac{\hat{q}_{\alpha^*}(\phi_\pi(T_i;\alpha^*))\hat{r}_{\alpha^*}(\phi_\pi(T_i;\alpha^*))\Delta^2_{b_n}(T_i;\alpha^*)}{r^2_{\alpha^*,b_n}(\phi_\pi(T_i;\alpha^*))\hat{r}_{\alpha^*,b_n}(\phi_\pi(T_i;\alpha^*))} := \sum_{i=1}^{6} Q_{mi}.
\end{aligned}
$$

We first prove Q_{m1}, we show that

$$
\begin{aligned}
& Q_{m1}\\
=& n^{-1}\sum_{i=1}^{n}\frac{C_i}{r_{\alpha^*,b_n}(\phi_\pi(T_i;\alpha^*))}\frac{1}{nh_n^J}\sum_{j=1}^{n}(1-\delta_j q_{\alpha^*}(\phi_\pi(T_j;\alpha^*)))K\left(\frac{\phi_\pi(T_i;\alpha^*)-\phi_\pi(T_j;\alpha^*)}{h_n}\right)\\
=& n^{-1}\sum_{j=1}^{n}(1-\delta_j q_{\alpha^*}(\phi_\pi(T_j;\alpha^*)))\frac{1}{h_n^J}\int \frac{E\{C|\phi_\pi(T;\alpha^*)=t\}}{r_{\alpha^*,b_n}(\phi_\pi(t;\alpha^*))}K\left(\frac{t-\phi_\pi(T_j;\alpha^*)}{h_n}\right)f_\alpha(t)dt\\
+& n^{-1}\sum_{j=1}^{n}(1-\delta_j q_{\alpha^*}(\phi_\pi(T_j;\alpha^*)))\left[\frac{1}{nh_n^J}\sum_{i=1}^{n}\frac{C_i}{r_{\alpha^*,b_n}(\phi_\pi(T_i;\alpha^*))}K\left(\frac{\phi_\pi(T_i;\alpha^*)-\phi_\pi(T_j;\alpha^*)}{h_n}\right)\right.\\
-& \left.\frac{1}{h_n^J}\int \frac{E\{C|\phi_\pi(T;\alpha^*)=t\}}{r_{\alpha^*,b_n}(\phi_\pi(t;\alpha^*))}K\left(\frac{t-\phi_\pi(T_j;\alpha^*)}{h_n}\right)f_\alpha(t)dt\right] := Q_{m11}+Q_{m12},
\end{aligned}
$$

where $C_i = \delta_i \log g_M(Y_i|X_i,Z_i;\theta_M)$. For Q_{m11}, note that $f_\alpha(t) = q_\alpha(t)r_\alpha(t)$, we have

$$
\begin{aligned}
&\sup_{t,\alpha}\left| n^{-1}\sum_{j=1}^{n}(1-\delta_j q_{\alpha^*}(\phi_\pi(T_j;\alpha^*)))h_n^{-J}\int E\{C|\phi_\pi(T;\alpha^*)=t\}K\left(\frac{t-\phi_\pi(T_j;\alpha^*)}{h_n}\right)f_\alpha(t)dt\right.\\
&\left. - n^{-1}\sum_{i=1}^{n}(1-\delta_i q_{\alpha^*}(\phi_\pi(T_i;\alpha^*)))C_i q_{\alpha^*}(\phi_\pi(t;\alpha^*))r_{\alpha^*}(\phi_\pi(t;\alpha^*))\right| = O(h_n^k),
\end{aligned}
$$

where $k > J$. By (S.4), we know that $|r_{\alpha^*,b_n}(\phi_\pi(t;\alpha^*))| \geq cb_n$, then, we have

$$\left|Q_{m11} - n^{-1}\sum_{i=1}^{n}\frac{\{1-\delta_i q_{\alpha^*}(\phi_\pi(T_i;\alpha^*))\}C_i q_{\alpha^*}(\phi_\pi(T_i;\alpha^*))r_{\alpha^*}(\phi_\pi(T_i;\alpha^*))}{r_{\alpha^*,b_n}(\phi_\pi(T_i;\alpha^*))}\right|$$
$$= O_p(h_n^k/b_n).$$

By standard arguments , we have $Q_{m12} = o_p(n^{-1/2})$. Thus, we have

$$Q_{m1} = n^{-1}\sum_{i=1}^{n}\{1-\delta_i q_{\alpha^*}(\phi_\pi(T_i;\alpha^*))\}q_{\alpha^*}(\phi_\pi(T_i;\alpha^*))C_i + o_p(n^{-1/2}).$$

According to the similar technique to $T_{ni}, i = 2,3,4,5,6$ in Lemma 1 of Wang and Rao (2002a), we can prove that $Q_{m3} + Q_{m4} = o_p(n^{-1/2})$ and $Q_{mi} = o_p(n^{-1/2})$ for $i = 2,5,6$. Thus, the proof of Lemma 2 is completed. □

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