Supplementary material for “Robust model selection with covariables missing at random”

Zhongqi Liang\textsuperscript{1}, Qihua Wang\textsuperscript{2} and Yuting Wei\textsuperscript{3}

School of Statistics and Mathematics, Zhejiang Gongshang University\textsuperscript{1}, Hangzhou 310018, Zhejiang, China
School of Statistics and Mathematics, Zhejiang Gongshang University\textsuperscript{2}, Hangzhou 310018, Zhejiang, China
Academy of Mathematics and Systems Science, Chinese Academy of Sciences\textsuperscript{3}, Beijing 100190, China
Department of Statistics and Finance, University of Science and Technology of China\textsuperscript{1}, Hefei 230026, China

Lemmas

The proofs of Lemma 1–Lemma 2 are completely similar to Lemma S3–Lemma S4 in Wang et al. (2020). For convenience of review, we give the details.

**Lemma 1.** Provided that Conditions (C2)-(C8) hold, and further assume that $\hat{\alpha}_n - \alpha^* = O_p(n^{-1/2})$, we have

\[
Q_{n1} = n^{-1} \sum_{i=1}^{n} \{ \delta_i \log g_M(Y_i|X_i; Z_i; \theta_M) \}\{ \tilde{q}_{\alpha_n, b_n}(\phi_\pi(Y_i, Z_i; \hat{\alpha}_n)) - \tilde{q}_{\alpha^*, b_n}(\phi_\pi(Y_i, Z_i; \alpha^*)) \}
\]

\[= n^{-1} \sum_{i=1}^{n} \{ \delta_i \log g_M(Y_i|X_i; Z_i; \theta_M) \}\{ \partial q_{\alpha^*}(\phi_\pi(Y_i, Z_i; \alpha^*)) / \partial \alpha \}(\hat{\alpha}_n - \alpha^*) + o_p(n^{-1/2}). \quad (S.1)\]

**Proof.** For simplicity, we denote $T_i = (Y_i, Z_i)$ and denote

\[
B_n(y, z; \alpha, \alpha^*) = \tilde{q}_{\alpha_n, n}(\phi_\pi(y, z; \alpha)) \tilde{r}_{\alpha_n, n}(\phi_\pi(y, z; \alpha)) - \tilde{q}_{\alpha^*, n}(\phi_\pi(y, z; \alpha^*)) \tilde{r}_{\alpha^*, n}(\phi_\pi(y, z; \alpha^*))
\]

\[
\Gamma_n(y, z; \alpha, \alpha^*) = \tilde{r}_{\alpha_n, n}(\phi_\pi(y, z; \alpha)) - \tilde{r}_{\alpha^*, n}(\phi_\pi(y, z; \alpha^*))
\]

\[
\Gamma_{b_n}(y, z; \alpha, \alpha^*) = \tilde{r}_{\alpha_n, b_n}(\phi_\pi(y, z; \alpha)) - \tilde{r}_{\alpha^*, b_n}(\phi_\pi(y, z; \alpha^*))
\]

and

\[
A_n(y, z; \alpha) = q_{\alpha_n}(\phi_\pi(y, z; \alpha)) \tilde{r}_{\alpha_n}(\phi_\pi(y, z; \alpha)) - q_{\alpha}(\phi_\pi(y, z; \alpha)) r_{\alpha}(\phi_\pi(y, z; \alpha))
\]

\[
\Delta_n(y, z; \alpha) = \tilde{r}_{\alpha_n}(\phi_\pi(y, z; \alpha)) - r_{\alpha}(\phi_\pi(y, z; \alpha))
\]

\[
\Delta_{b_n}(y, z; \alpha) = \tilde{r}_{\alpha_n, b_n}(\phi_\pi(y, z; \alpha)) - r_{\alpha_n, b_n}(\phi_\pi(y, z; \alpha))
\]

Then, according to the similar arguments to lemma 2 of Li et al. (2011) with (C.3), (C.5), (C.6) and (C.7) we have

\[
\sup_{y, z, \alpha} |A_n(y, z; \alpha)| = O_p(h_n^k + \frac{\log n}{\sqrt{nh_n^2}}) = o_p(b_n), \quad (S.2)
\]

\textsuperscript{1}E-mail: 2945155436@qq.com(Zhongqi Liang)

\textsuperscript{2}Corresponding Author: Tel.: +86 15101512088; E-mail: qhwang@amss.ac.cn (Qihua Wang)

\textsuperscript{3}E-mail: ytwei@mail.ustc.edu.cn (Yuting Wei)
To prove (S.1), we show that

\[
\sup_{y, z, \alpha} |\Delta_n(y, z; \alpha)| = O_p(h_n^k + \frac{\log n}{\sqrt{n h_n^*}}) = o_p(b_n). \quad (S.3)
\]

And due to

\[
\sup_{y, z, \alpha} |\Delta_n(y, z; \alpha)| \leq \sup_{y, z, \alpha} |\Delta_n(y, z; \alpha)|,
\]

then if \( n \) is large enough, we have

\[
|\hat{r}_{\alpha, b_n}(\phi_\pi(y, z; \alpha))| \geq |r_{\alpha, b_n}(\phi_\pi(y, z; \alpha))| - o_p(b_n) \geq cb_n. \quad (S.4)
\]

Note that,

\[
n^{-1} \sum_{i=1}^{n} C_i \{ \hat{q}_{\alpha, b_n}(\phi_\pi(Y_i, Z_i; \hat{\alpha}_n)) - \hat{q}_{\alpha, b_n}(\phi_\pi(Y_i, Z_i; \alpha^*)) \}
\]

\[
= n^{-1} \sum_{i=1}^{n} C_i \frac{B_n(T_i; \hat{\alpha}_n, \alpha^*)}{r_{\alpha, b_n}(\phi_\pi(T_i; \alpha^*))} - n^{-1} \sum_{i=1}^{n} C_i \frac{\Delta_n(T_i; \alpha^*) B_n(T_i; \hat{\alpha}_n, \alpha^*)}{r_{\alpha, b_n}(\phi_\pi(T_i; \alpha^*)) r_{\alpha, b_n}(\phi_\pi(T_i; \alpha^*))}
\]

\[
- n^{-1} \sum_{i=1}^{n} C_i \frac{q_{\alpha^*}(\phi_\pi(T_i; \alpha^*)) r_{\alpha^*}(\phi_\pi(T_i; \alpha^*)) \Gamma_n(T_i; \hat{\alpha}_n, \alpha^*)}{r_{\alpha, b_n}(\phi_\pi(T_i; \alpha^*)) r_{\alpha^*, b_n}(\phi_\pi(T_i; \alpha^*))}
\]

\[
+ n^{-1} \sum_{i=1}^{n} C_i \frac{q_{\alpha^*}(\phi_\pi(T_i; \alpha^*)) r_{\alpha^*}(\phi_\pi(T_i; \alpha^*)) \Delta_n(T_i; \alpha^*) \Gamma_n(T_i; \hat{\alpha}_n, \alpha^*)}{r_{\alpha^*, b_n}(\phi_\pi(T_i; \alpha^*)) r_{\alpha^*, b_n}(\phi_\pi(T_i; \alpha^*))}
\]

\[
- n^{-1} \sum_{i=1}^{n} C_i \frac{A_n(T_i; \alpha^*) \Gamma_n^2(T_i; \hat{\alpha}_n, \alpha^*)}{r_{\alpha^*, b_n}(\phi_\pi(T_i; \hat{\alpha}_n)) r_{\alpha^*, b_n}(\phi_\pi(T_i; \alpha^*))}
\]

\[
+ n^{-1} \sum_{i=1}^{n} C_i \frac{A_n(T_i; \alpha^*) \Gamma_n^2(T_i; \hat{\alpha}_n, \alpha^*)}{r_{\alpha, b_n}(\phi_\pi(T_i; \alpha^*)) r_{\alpha^*, b_n}(\phi_\pi(T_i; \alpha^*))}
\]

\[
+ n^{-1} \sum_{i=1}^{n} C_i \frac{q_{\alpha^*}(\phi_\pi(T_i; \alpha^*)) r_{\alpha^*}(\phi_\pi(T_i; \alpha^*)) \Gamma_n^2(T_i; \hat{\alpha}_n, \alpha^*)}{r_{\alpha^*, b_n}(\phi_\pi(T_i; \hat{\alpha}_n)) r_{\alpha, b_n}(\phi_\pi(T_i; \alpha^*))} := \sum_{i=1}^{9} G_{ni}.
\]

To prove (S.1), we show that

\[
G_{n1} + G_{n3}
\]

\[
n^{-1} \sum_{i=1}^{n} \{ \delta_i \log g_M(Y_i, X_i, Z_i; \theta_M) \} \{ \partial q_{\alpha^*}(\phi_\pi(T_i; \alpha^*)) / \partial \alpha \} (\hat{\alpha}_n - \alpha^*) + o_p(n^{-1/2}), \quad (S.5)
\]
and
\[ G_{ni} = o_p(n^{-1/2}), \quad i = 2, 4, 5, 6, 7, 8, 9. \] (S.6)

First, we prove (S.5). Denote
\[ W_n(T_i, T_j; \alpha) = \frac{C_i}{r_{\alpha^*, \hat{\alpha}_n}(\phi_{\alpha}(T_i; \alpha) - \phi_{\alpha}(T_j; \alpha))} \left\{ K\left( \frac{\phi_{\alpha}(T_i; \alpha) - \phi_{\alpha}(T_j; \alpha)}{h_n} \right) - K\left( \frac{\phi_{\alpha}(T_i; \alpha^*) - \phi_{\alpha}(T_j; \alpha^*)}{h_n} \right) \right\}. \]

Then
\[
G_{n1} = h_n^{-j} n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} W_n(T_i, T_j; \hat{\alpha}_n)
= h_n^{-j} \left\{ n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} W_n(T_i, T_j; \hat{\alpha}_n) - n^{-1} \sum_{i=1}^{n} \int W_n(T_i, t; \hat{\alpha}_n) dF_T(t) \right\}
- n^{-1} \sum_{j=1}^{n} \int W_n(t, T_j; \hat{\alpha}_n) dF_T(t) + \int \int W_n(t_1, t_2; \hat{\alpha}_n) dF_T(t_1) dF_T(t_2) \right\} + \left\{ h_n^{-j} \left[ n^{-1} \sum_{j=1}^{n} W_n(t, T_j; \hat{\alpha}_n) dF_T(t) - \int \int W_n(t_1, t_2; \hat{\alpha}_n) dF_T(t_1) dF_T(t_2) \right] \right\}
+ \left\{ h_n^{-j} n^{-1} \sum_{i=1}^{n} W_n(T_i, t; \hat{\alpha}_n) dF_T(t) \right\} := G_{n11} + G_{n12} + G_{n13},
\]

where $F_T(\cdot)$ denotes the distribution function of $T$. According to similar statements as the MAIN COROLLARY in Sherman (1994), we have
\[
E\left[ n \sup_{\|\alpha - \alpha^*\| = O(n^{-1/2})} \left| n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} W_n(T_i, T_j; \hat{\alpha}_n) - n^{-1} \sum_{i=1}^{n} \int W_n(T_i, t; \hat{\alpha}_n) dF_T(t) \right| \right]
- n^{-1} \sum_{j=1}^{n} \int W_n(t, T_j; \hat{\alpha}_n) dF_T(t) + \int \int W_n(t_1, t_2; \hat{\alpha}_n) dF_T(t_1) dF_T(t_2) \right| \right] \leq c(E[ \sup_{\|\hat{\alpha}_n - \alpha^*\| = O(n^{-1/2})} W_n(T_1, T_2; \hat{\alpha}_n)^2])^{1/2};
\]

and
\[
E\left[ n^{1/2} \sup_{\|\alpha - \alpha^*\| = O(n^{-1/2})} \left| n^{-1} \sum_{j=1}^{n} \int W_n(t, T_j; \hat{\alpha}_n) dF_T(t) \right| \right] - \int \int W_n(t_1, t_2; \hat{\alpha}_n) dF_T(t_1) dF_T(t_2) \right| \right] \leq c(E[ \sup_{\|\hat{\alpha}_n - \alpha^*\| = O(n^{-1/2})} W_n(T_1, T_2; \hat{\alpha}_n)^2])^{1/2}.
\]
Further, since $E[\log y_M^2(Y|X,Z;\theta_M)] < \infty$, we have
\[
E[ \sup_{\|\alpha - \alpha^*\| = O(n^{-1/2})} W_n(T_1, T_2; \hat{\alpha}_n)^2 ]
\leq c_n b_n^{-2} \int \sup_{\|\alpha - \alpha^*\| = O(n^{-1/2})} \left[ K \left( \frac{\phi_\pi(t_1; \alpha) - \phi_\pi(t_2; \alpha)}{h_n} \right) - K \left( \frac{\phi_\pi(t_1; \alpha^*) - \phi_\pi(t_1; \alpha^*)}{h_n} \right) \right]^2 dt_1 dt_2
\leq c_n b_n^{-2} \int \sup_{\|\alpha - \alpha^*\| = O(n^{-1/2})} \left( \frac{\phi_\pi(t_1; \alpha) - \phi_\pi(t_2; \alpha) - \phi_\pi(t_1; \alpha^*) + \phi_\pi(t_2; \alpha^*)}{h_n} \right)^2 dt_1 dt_2
\leq c_n b_n^{-2} h_n^{-2} \int \sup_{\|\alpha - \alpha^*\| = O(n^{-1/2})} \left( \|l(t_1) + l(t_2)\|^2 \|\alpha - \alpha^*\|^2 \right) \cdot f_T(t_1) f_T(t_2) dt_1 dt_2
\leq \frac{1}{nb_n^2 h_n^2},
\]
where the last third inequality is due to (C.6), and the last second inequality is due to (C.5)/(i). Thus, according to (C.7), we have $G_{n11} = o_p(n^{-1/2})$ and $G_{n12} = o_p(n^{-1/2})$. To $G_{n13}$, recalling that $f_\alpha(t) = q_\alpha(t) r_\alpha(t)$, we have
\[
\sup_{t, \alpha} \left| h_n^{-1} \int K \left( \frac{\phi_\pi(T_1; \alpha) - t}{h_n} \right) f_\alpha(t) dt - q_\alpha(\phi_\pi(t; \alpha)) r_\alpha(\phi_\pi(t; \alpha)) \right| = O(h_n^k).
\]
Then we can obtain
\[
\left| G_{n13} - n^{-1} \sum_{i=1}^n C_i \left\{ q_\alpha(\phi_\pi(T_i; \hat{\alpha}_n)) r_\alpha(\phi_\pi(T_i; \hat{\alpha}_n)) - q_\alpha^*(\phi_\pi(T_i; \alpha^*)) r_\alpha^*(\phi_\pi(T_i; \alpha^*)) \right\} \right| = O_p(h_n^k/b_n).
\]
Thus we have
\[
G_{n1} = n^{-1} \sum_{i=1}^n C_i \frac{q_\alpha(\phi_\pi(T_i; \hat{\alpha}_n)) r_\alpha(\phi_\pi(T_i; \hat{\alpha}_n)) - q_\alpha^*(\phi_\pi(T_i; \alpha^*)) r_\alpha^*(\phi_\pi(T_i; \alpha^*))}{r_\alpha^*, b_n(\phi_\pi(T_i; \alpha^*))} + o_p(n^{-1/2}).
\]
(S.7)

Similarly, we can obtain that
\[
G_{n3} = -n^{-1} \sum_{i=1}^n C_i \frac{q_\alpha^*(\phi_\pi(T_i; \alpha^*)) \left\{ r_\alpha(\phi_\pi(T_i; \hat{\alpha}_n)) - r_\alpha^*(\phi_\pi(T_i; \alpha^*)) \right\}}{r_\alpha^*, b_n(\phi_\pi(T_i; \alpha^*))} + o_p(n^{-1/2}).
\]
(S.8)
Thus, combining with (S.4), (S.7) and (S.8), we have

\[ G_{n1} + G_{n3} = n^{-1} \sum_{i=1}^{n} C_i \{ q_{\alpha_n}(\phi_\pi(T_i; \hat{\alpha}_n)) - q_{\alpha^*}(\phi_\pi(T_i; \alpha^*)) \} r_{\alpha_n}(\phi_\pi(T_i; \hat{\alpha}_n)) + o_p(n^{-1/2}) \]

\[ = n^{-1} \sum_{i=1}^{n} C_i \{ q_{\alpha_n}(\phi_\pi(T_i; \hat{\alpha}_n)) - q_{\alpha^*}(\phi_\pi(T_i; \alpha^*)) \} r_{\alpha_n}(\phi_\pi(T_i; \hat{\alpha}_n)) + o_p(n^{-1/2}) \]

\[ = n^{-1} \sum_{i=1}^{n} C_i q_{\alpha_n}(\phi_\pi(T_i; \hat{\alpha}_n)) - q_{\alpha^*}(\phi_\pi(T_i; \alpha^*)) + o_p(n^{-1/2}) \]

\[ = n^{-1} \sum_{i=1}^{n} C_i \{ q_{\alpha_n}(\phi_\pi(T_i; \alpha^*)) / \partial \alpha \} (\hat{\alpha}_n - \alpha^*) + o_p(n^{-1/2}). \]

In what follows, we prove (S.6). According to (S.2) and (S.3), we have

\[ \sup_{t, \alpha} |A_n(t; \alpha)| = O_p(h_n + \log n \sqrt{n h_n}), \quad \sup_{t, \alpha} |\Delta_{bn}(t; \alpha)| = O_p(h_n + \log n \sqrt{n h_n}). \]  

(S.9)

And according to (C.5)(ii), we have

\[ n^{-1} \sum_{i=1}^{n} |C_i| \cdot q_{\alpha_n}(\phi_\pi(T_i; \hat{\alpha}_n)) r_{\alpha_n}(\phi_\pi(T_i; \hat{\alpha}_n)) - q_{\alpha^*}(\phi_\pi(T_i; \alpha^*)) r_{\alpha^*}(\phi_\pi(T_i; \alpha^*)) \]

\[ = O_p(\|\hat{\alpha}_n - \alpha^*\|). \]  

(S.10)

Thus, to \( G_{n2} \), by (S.9) and (S.10), we have

\[ |G_{n2}| \leq n^{-1} \sum_{i=1}^{n} \frac{|C_i|}{b_n^2} \sup_t |\Delta_{bn}(t; \alpha^*)| \sup_{t, \alpha} |A_n(t; \alpha) - A_n(t; \alpha^*)| \]

\[ + n^{-1} \sum_{i=1}^{n} \frac{|C_i|}{b_n^2} \sup_t |\Delta_{bn}(t, \alpha^*)| \]

\[ \times |q_{\alpha_n}(\phi_\pi(T_i; \hat{\alpha}_n)) r_{\alpha_n}(\phi_\pi(T_i; \hat{\alpha}_n)) - q_{\alpha^*}(\phi_\pi(T_i; \alpha^*)) r_{\alpha^*}(\phi_\pi(T_i; \alpha^*))| = o_p(n^{-1/2}). \]

And the \( G_{ni} = o_p(n^{-1/2}) \), for \( i = 4, 5, 6 \) can be proved by similar arguments. And to \( G_{n7} \),
by (S.2), (S.3) and (C.7), we have

\[
|G_{nT}| \leq n^{-1} \sum_{i=1}^{n} \left| \frac{C_i}{b_n^2} \sup_{t, \alpha} |A_{n}(t; \alpha) - A_{n}(t; \alpha^*)| \right|
\]

\[
+ |q_{\hat{a}_n}(\phi_{\pi}(T_i; \hat{\alpha}_n)) r_{\hat{a}_n}(\phi_{\pi}(T_i; \hat{\alpha}_n)) - q_{a}(\phi_{\pi}(T_i; \alpha^*)) r_{a}(\phi_{\pi}(T_i; \alpha^*))| \]

\[
\times \left[ \sup_{t, \alpha} |\Delta_{n}(t; \alpha) - \Delta_{n}(t; \alpha^*)| + |r_{\hat{a}_n}(\phi_{\pi}(T_i; \hat{\alpha}_n)) - r_{a}(\phi_{\pi}(T_i; \alpha^*))| \right]
\]

\[
= o_p(n^{-1/2}).
\]

And the \(G_{ni} = o_p(n^{-1/2})\), for \(i = 8, 9\) can be proved by similar statements. Thus, the proof of Lemma 1 is completed.

**Lemma 2.** Provided that Conditions (C2)-(C8) hold, we have

\[
Q_{n2} = n^{-1} \sum_{i=1}^{n} \left\{ \delta_i \log g_{M}(Y_i|X_i, Z_i; \theta_{M}) \right\} \{ \hat{q}_{a} - b_n(\phi_{\pi}(Y_i, Z_i; \alpha^*)) - q_{a}(\phi_{\pi}(Y_i, Z_i; \alpha^*)) \}
\]

\[
= n^{-1} \sum_{i=1}^{n} \left\{ 1 - \delta_i q_{a}(\phi_{\pi}(Y_i, Z_i; \alpha^*)) \right\} q_{a}(\phi_{\pi}(Y_i, Z_i; \alpha^*)) \delta_i \log g_{M}(Y_i|X_i, Z_i; \theta_{M}) + o_p(n^{-1/2}).
\]

(S.11)

**Proof.** Using the notations in Lemma 1 and denote

\[
\eta_{n}(t; \alpha^*) = \left( nh_{n} \right)^{-1} \sum_{j=1}^{n} \left( 1 - \delta_j q_{a}(\phi_{\pi}(T_j; \alpha^*)) \right) K \left( \frac{\phi_{\pi}(t; \alpha^*) - \phi_{\pi}(T_j; \alpha^*)}{h_{n}} \right),
\]

\[
\xi_{n}(t; \alpha^*) = \left( nh_{n} \right)^{-1} \sum_{j=1}^{n} \left( \delta_j q_{a}(\phi_{\pi}(T_j; \alpha^*)) - \delta_j q_{a}(\phi_{\pi}(t; \alpha^*)) \right) K \left( \frac{\phi_{\pi}(t; \alpha^*) - \phi_{\pi}(T_j; \alpha^*)}{h_{n}} \right).
\]
We first prove that
\[ n^{-1} \sum_{i=1}^{n} \left\{ \delta_i \log g_M(Y_i | X_i, Z_i; \theta_M) \right\} \{ \hat{q}_{n, b_n}(\phi_\pi(Y_i, Z_i; \alpha^*)) - q_{n, b_n}(\phi_\pi(Y_i, Z_i; \alpha^*)) \} \]

\[ = n^{-1} \sum_{i=1}^{n} \frac{C_i}{r_{n, b_n}(\phi_\pi(T_i; \alpha^*))} + n^{-1} \sum_{i=1}^{n} \frac{\xi_n(T_i; \alpha^*)}{r_{n, b_n}(\phi_\pi(T_i; \alpha^*))} \]

\[ + n^{-1} \sum_{i=1}^{n} C_i q_{n, b_n}(\phi_\pi(T_i; \alpha^*)) \Delta_n(T_i; \alpha^*) - n^{-1} \sum_{i=1}^{n} C_i q_{n, b_n}(\phi_\pi(T_i; \alpha^*)) \Delta_{n}(T_i; \alpha^*) \]

\[ - n^{-1} \sum_{i=1}^{n} C_i (\hat{g}_{n, b_n}(\phi_\pi(T_i; \alpha^*)) - r_{n, b_n}(\phi_\pi(T_i; \alpha^*)) \Delta_{n}(T_i; \alpha^*) \]

\[ + n^{-1} \sum_{i=1}^{n} C_i (\hat{g}_{n, b_n}(\phi_\pi(T_i; \alpha^*)) - r_{n, b_n}(\phi_\pi(T_i; \alpha^*)) \Delta_{n}(T_i; \alpha^*) \]

\[ := \sum_{i=1}^{6} Q_{mi}. \]

We first prove \( Q_{m1} \), we show that

\[ Q_{m1} \]

\[ = n^{-1} \sum_{i=1}^{n} \frac{C_i}{r_{n, b_n}(\phi_\pi(T_i; \alpha^*))} \frac{1}{nh_n} \sum_{j=1}^{n} (1 - \delta_j g_{\alpha^*}(\phi_\pi(T_j; \alpha^*)) ) K \left( \frac{\phi_\pi(T_i; \alpha^*) - \phi_\pi(T_j; \alpha^*)}{h_n} \right) \]

\[ = n^{-1} \sum_{j=1}^{n} \frac{1}{h_n} \int \frac{E[C|\phi_\pi(T; \alpha^*) = t]}{r_{n, b_n}(\phi_\pi(t; \alpha^*))} K \left( \frac{t - \phi_\pi(T_j; \alpha^*)}{h_n} \right) f_\alpha(t) dt \]

\[ + n^{-1} \sum_{j=1}^{n} \frac{1}{h_n} \int \frac{E[C|\phi_\pi(T; \alpha^*) = t]}{r_{n, b_n}(\phi_\pi(t; \alpha^*))} K \left( \frac{t - \phi_\pi(T_j; \alpha^*)}{h_n} \right) f_\alpha(t) dt \]

\[ := Q_{m11} + Q_{m12}, \]

where \( C_i = \delta_i \log g_M(Y_i | X_i, Z_i; \theta_M) \). For \( Q_{m11} \), note that \( f_\alpha(t) = q_{\alpha^*}(t) r_{\alpha^*}(t) \), we have

\[ \sup_{t, \alpha} \left| n^{-1} \sum_{j=1}^{n} (1 - \delta_j g_{\alpha^*}(\phi_\pi(T_j; \alpha^*)) ) h_n^{-1} \int \frac{E[C|\phi_\pi(T; \alpha^*) = t]}{h_n} K \left( \frac{t - \phi_\pi(T_j; \alpha^*)}{h_n} \right) f_\alpha(t) dt \]

\[ - n^{-1} \sum_{i=1}^{n} (1 - \delta_j g_{\alpha^*}(\phi_\pi(T_i; \alpha^*)) ) C_i q_{\alpha^*}(\phi_\pi(t; \alpha^*)) r_{\alpha^*}(\phi_\pi(t; \alpha^*)) \right| = O(h_n^k), \]
where $k > J$. By (S.4), we know that $|r_{\alpha^*,bn}(\phi_\pi(t; \alpha^*))| \geq cb_n$, then, we have

$$Q_{m11} - n^{-1} \sum_{i=1}^{n} \frac{1 - \delta_i q_{\alpha^*}(\phi_\pi(T_i; \alpha^*))}{r_{\alpha^*,bn}(\phi_\pi(T_i; \alpha^*))} C_i q_{\alpha^*}(\phi_\pi(T_i; \alpha^*)) r_{\alpha^*,bn}(\phi_\pi(T_i; \alpha^*)) = O_p(h_n^k/b_n).$$

By standard arguments, we have $Q_{m12} = o_p(n^{-1/2})$. Thus, we have

$$Q_{m1} = n^{-1} \sum_{i=1}^{n} \{1 - \delta_i q_{\alpha^*}(\phi_\pi(T_i; \alpha^*))\} q_{\alpha^*}(\phi_\pi(T_i; \alpha^*)) C_i + o_p(n^{-1/2}).$$

According to the similar technique to $T_{ni}, i = 2, 3, 4, 5, 6$ in Lemma 1 of Wang and Rao (2002a), we can prove that $Q_{m3} + Q_{m4} = o_p(n^{-1/2})$ and $Q_{mi} = o_p(n^{-1/2})$ for $i = 2, 5, 6$. Thus, the proof of Lemma 2 is completed.

**References**


