Supplementary Material

Whittle estimation for stationary state space models

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7 Proofs for the adjusted Whittle estimator in Section 4

7.1 Proofs of Section 4.1

The proof of Theorem 3 is similar to the proof of Theorem 1. Therefore, we simply adapt the parts which are not the same, namely Proposition 2 and Proposition 3. We start by stating that $W_n^{(A)}$ converges almost surely uniformly to

$$W^{(A)}(\vartheta) := \int_{-\pi}^{\pi} |\Pi(e^{i\omega},\vartheta)|^2 f_Y^{(\Delta)}(\omega)d\omega$$

which can be shown in the same way as the uniform convergence of $W_n^{(1)}$ in Proposition 2.

Proposition 7 *Let Assumptions* $(A1)$–$(A4)$ *hold. Then, as* $n\to\infty$,

$$\sup_{\vartheta\in\Theta} |W_n^{(A)}(\vartheta) - W^{(A)}(\vartheta)| \xrightarrow{n\to\infty} 0 \quad \mathbb{P}\text{-}a.s.$$

Proposition 8 *Let Assumptions* $(A1)$–$(A4)$ *and* $(\widetilde{A6})$ *hold. Then,* $W^{(A)}$ *has a unique minimum in* ϑ_0.

Proof Let $\vartheta \neq \vartheta_0$. Due to the definition of the linear innovation and assumption $(\widetilde{\mathrm{A6}})$, we have

$$V^{(\Delta)} = \mathbb{E}[\varepsilon_k^{(\Delta)2}] = \mathbb{E}\left[\Pi(\mathsf{B})Y_k^{(\Delta)}\right]^2 < \mathbb{E}\left[\Pi(\mathsf{B},\vartheta)Y_k^{(\Delta)}\right]^2 = \int_{-\pi}^{\pi} |\Pi(e^{i\omega},\vartheta)|^2 f_Y^{(\Delta)}(\omega)d\omega = W^{(A)}(\vartheta),$$

where for the second last equality we used Brockwell and Davis (1991), Theorem 11.8.3 as well. Furthermore, $V^{(\Delta)} = \mathbb{E}[(\Pi(\mathsf{B})Y_k^{(\Delta)})^2] = W^{(A)}(\vartheta_0)$ holds. □

7.2 Proofs of Section 4.2

The proof of the asymptotic normality of the adjusted Whittle estimator is similar to the proof of the asymptotic normality of the original Whittle estimator. We start to prove an adapted version of Proposition 5.

Proposition 9 *Let Assumptions* $(A2)$–$(A4)$ *and* $(B2)$ *hold. Suppose* $\eta : [-\pi,\pi] \to \mathbb{C}$ *is a symmetric function with Fourier coefficients* $(\mathfrak{f}_u)_{u\in\mathbb{Z}}$ *satisfying* $\sum_{u=-\infty}^{\infty} |\mathfrak{f}_u||u|^{1/2} < \infty$ *and*

$$\int_{-\pi}^{\pi} \left|\Pi^{-1}(e^{i\omega})\right|^2 \eta(\omega)d\omega = 0.$$

Then, as $n\to\infty$,

$$\frac{\pi}{\sqrt{n}} \sum_{j=-n+1}^{n} \eta(\omega_j) I_n(\omega_j) \xrightarrow{\mathscr{D}} \mathscr{N}(0,\Sigma_\eta),$$

where

$$\begin{aligned}\Sigma_\eta &= 4\pi \int_{-\pi}^{\pi} \eta(\omega)^2 f_Y^{(\Delta)}(\omega)^2 d\omega + \frac{1}{4\pi^2}\int_{-\pi}^{\pi} \eta(\omega)\,\mathrm{vec}\left(\Phi(e^{-i\omega})^\top \Phi(e^{i\omega})\right)^\top d\omega \\ &\quad \cdot \left(\mathbb{E}\left[N_1^{(\Delta)} N_1^{(\Delta)\top} \otimes N_1^{(\Delta)} N_1^{(\Delta)\top}\right] - 3\Sigma_N^{(\Delta)} \otimes \Sigma_N^{(\Delta)}\right) \int_{-\pi}^{\pi} \eta(\omega)\,\mathrm{vec}\left(\Phi(e^{i\omega})^\top \Phi(e^{-i\omega})\right) d\omega.\end{aligned}$$

Remark 9 For an Ornstein-Uhlenbeck process (CAR(1) process), Σ_η reduces to

$$\Sigma_\eta = 4\pi \int_{-\pi}^{\pi} \eta(\omega)^2 f_Y^{(\Delta)}(\omega)^2 d\omega,$$

since $\Pi^{-1}(e^{i\omega},\vartheta) = \Phi(e^{i\omega},\vartheta)\ \forall\ (\omega,\vartheta) \in [-\pi,\pi]\times\Theta$ implies

$$\int_{-\pi}^{\pi} \eta(\omega)\,\mathrm{vec}\left(\Phi(e^{-i\omega})^\top \Phi(e^{i\omega})\right)^\top d\omega = \int_{-\pi}^{\pi} \left|\Pi^{-1}(e^{i\omega})\right|^2 \eta(\omega) d\omega = 0.$$

Proof of Proposition 9. Note that

$$\sqrt{n}\int_{-\pi}^{\pi} f_Y^{(\Delta)}(\omega)\eta(\omega)d\omega = \frac{\sqrt{n}V^{(\Delta)}}{2\pi}\int_{-\pi}^{\pi} \left|\Pi^{-1}(e^{i\omega})\right|^2 \eta(\omega)d\omega = 0.$$

Therefore, an application of Lemma 8 gives

$$\frac{\pi}{\sqrt{n}}\sum_{j=-n+1}^{n} \eta(\omega_j) I_n(\omega_j) = \frac{\pi}{\sqrt{n}}\sum_{j=-n+1}^{n} \eta(\omega_j)\left(I_n(\omega_j) - f_Y^{(\Delta)}(\omega_j)\right) + o(1)$$

and Proposition 5 leads to the statement. □

Proposition 10 *Let Assumptions* (A2)–(A4)*,* $(\widetilde{A6})$ *and* (B2)–(B3) *hold. Then, as* $n\to\infty$*,*

$$\sqrt{n}\left[\nabla_\vartheta W_n^{(A)}(\vartheta_0)\right]^\top \xrightarrow{\mathscr{D}} \mathscr{N}(0,\Sigma_{\nabla W^{(A)}}).$$

Proof Similar to the proof of Proposition 6, we make use of the Cramér Wold Theorem. For $\lambda = (\lambda_1,\ldots,\lambda_r)^\top \in \mathbb{R}^r$, we get

$$\begin{aligned}\sqrt{n}\left[\nabla_\vartheta W_n^{(A)}(\vartheta_0)\right]\lambda &= \frac{\pi}{\sqrt{n}}\sum_{j=-n+1}^{n}\sum_{t=1}^{r}\lambda_t \frac{\partial}{\partial\vartheta_t}\left|\Pi(e^{i\omega_j},\vartheta_0)\right|^2 I_n(\omega_j)\\ &= \frac{\pi}{\sqrt{n}}\sum_{j=-n+1}^{n}\sum_{t=1}^{r}\lambda_t \frac{\partial}{\partial\vartheta_t}\left(f_Y^{(\Delta)}(\omega,\vartheta_0)^{-1}\frac{V^{(\Delta)}(\vartheta_0)}{2\pi}\right) I_n(\omega_j).\end{aligned}$$

We define η_λ by

$$\eta_\lambda(\omega) = \sum_{t=1}^{r}\lambda_t\frac{\partial}{\partial\vartheta_t}\left(f_Y^{(\Delta)}(\omega,\vartheta_0)^{-1}\frac{V^{(\Delta)}(\vartheta_0)}{2\pi}\right),\quad \omega\in[-\pi,\pi],$$

and obtain

$$\begin{aligned}&\int_{-\pi}^{\pi}\eta_\lambda(\omega)\left|\Pi^{-1}(e^{i\omega})\right|^2 d\omega\\ &= \int_{-\pi}^{\pi}\sum_{t=1}^{r}\lambda_t\left(\frac{\frac{\partial}{\partial\vartheta_t}V^{(\Delta)}(\vartheta_0)}{2\pi} f_Y^{(\Delta)}(\omega)^{-1} - \frac{\frac{\partial}{\partial\vartheta_t}f_Y^{(\Delta)}(\omega,\vartheta_0)}{f_Y^{(\Delta)}(\omega,\vartheta_0)^2}\frac{V^{(\Delta)}(\vartheta_0)}{2\pi}\right)\left|\Pi^{-1}(e^{i\omega})\right|^2 d\omega\\ &= \left[2\pi\nabla_\vartheta\log(V^{(\Delta)}(\vartheta_0)) - \int_{-\pi}^{\pi}\nabla_\vartheta\log(f_Y^{(\Delta)}(\omega,\vartheta_0))d\omega\right]\lambda.\end{aligned}$$

Under Assumption (B3), the Leibniz rule and Theorem 3''', Chapter 3, of Hannan (2009) can be applied, which results in

$$\begin{aligned}&\left[2\pi\nabla_\vartheta\log(V^{(\Delta)}(\vartheta_0)) - \int_{-\pi}^{\pi}\nabla_\vartheta\log(f_Y^{(\Delta)}(\omega,\vartheta_0))d\omega\right]\lambda\\ &= \nabla_\vartheta\left[2\pi\log(V^{(\Delta)}(\vartheta_0)) - 2\pi\log(V^{(\Delta)}(\vartheta_0)) + 2\pi\log(2\pi)\right]\lambda = 0.\end{aligned}$$

As in Proposition 6, this transformation leads to the applicability of Proposition 9. Therefore, we get

$$\sqrt{n}\left[\nabla_\vartheta W_n^{(A)}(\vartheta_0)\right]\lambda \xrightarrow{\mathscr{D}} \mathscr{N}(0,\Sigma_{\nabla W^{(A)}\lambda})$$

with

$$\begin{aligned}\Sigma_{\nabla W^{(A)}\lambda} =& 4\pi\int_{-\pi}^{\pi}\eta_\lambda(\omega)^2 f_Y^{(\Delta)}(\omega)^2 d\omega+\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\eta_\lambda(\omega)\operatorname{vec}\left(\Phi(e^{-i\omega})^\top\Phi(e^{i\omega})\right)^\top d\omega\\ &\left(\mathbb{E}\left[N_1^{(\Delta)}N_1^{(\Delta)\top}\otimes N_1^{(\Delta)}N_1^{(\Delta)\top}\right]-3\Sigma_N^{(\Delta)}\otimes\Sigma_N^{(\Delta)}\right)\int_{-\pi}^{\pi}\eta_\lambda(\omega)\operatorname{vec}\left(\Phi(e^{i\omega})^\top\Phi(e^{-i\omega})\right)d\omega.\end{aligned}$$

The representation $\eta_\lambda(\omega)=\nabla_\vartheta|\Pi(e^{i\omega},\vartheta_0)|^2\lambda$ completes the proof. □

To prove Theorem 4, we need an analog result to Proposition 4. Since the following proposition can be shown completely analogously, the proof will be restricted to the transformation of the limit matrix.

Proposition 11 *Let Assumptions (A1)–(A4),$(\widetilde{A6})$ and (B3) hold. Furthermore, let $(\vartheta_n^*)_{n\in\mathbb{N}}$ be a sequence in Θ with $\vartheta_n^* \xrightarrow{a.s.} \vartheta_0$ as $n\to\infty$. Then, as $n\to\infty$,*

$$\nabla_\vartheta^2 W_n^{(A)}(\vartheta_n^*) \xrightarrow{a.s.} \Sigma_{\nabla^2 W^{(A)}}.$$

Proof Some straightforward calculation yields

$$\Sigma_{\nabla^2 W^{(A)}} = \int_{-\pi}^{\pi}\left[\nabla_\vartheta^2|\Pi(e^{i\omega},\vartheta_0)|^2\right] f_Y^{(\Delta)}(\omega)d\omega.$$

Applications of (10), the Leibniz rule and Theorem 3''' in Chapter 3 of Hannan (2009) give the representation

$$\begin{aligned}\Sigma_{\nabla^2 W^{(A)}} &= \int_{-\pi}^{\pi}\nabla_\vartheta^2\left[\frac{V^{(\Delta)}(\vartheta_0)}{2\pi}f_Y^{(\Delta)}(\omega,\vartheta_0)^{-1}\right]f_Y^{(\Delta)}(\omega)d\omega\\ &= \nabla_\vartheta^2 V^{(\Delta)}(\vartheta_0)-2\nabla_\vartheta V^{(\Delta)}(\vartheta_0)^\top\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\nabla_\vartheta\log(f_Y^{(\Delta)}(\omega,\vartheta_0))d\omega\right)\\ &\quad -\frac{V^{(\Delta)}}{2\pi}\int_{-\pi}^{\pi}\nabla_\vartheta^2\log(f_Y^{(\Delta)}(\omega,\vartheta_0))d\omega+\frac{V^{(\Delta)}}{2\pi}\int_{-\pi}^{\pi}\nabla_\vartheta\log(f_Y^{(\Delta)}(\omega,\vartheta_0))^\top\nabla_\vartheta\log(f_Y^{(\Delta)}(\omega,\vartheta_0))d\omega\\ &= \frac{V^{(\Delta)}}{2\pi}\int_{-\pi}^{\pi}\nabla_\vartheta\log(f_Y^{(\Delta)}(\omega,\vartheta_0))^\top\nabla_\vartheta\log(f_Y^{(\Delta)}(\omega,\vartheta_0))d\omega\\ &\quad -V^{(\Delta)}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\nabla_\vartheta\log(f_Y^{(\Delta)}(\omega,\vartheta_0))d\omega\right)^\top\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\nabla_\vartheta\log(f_Y^{(\Delta)}(\omega,\vartheta_0))d\omega\right)\\ &= \frac{V^{(\Delta)}}{2\pi}\int_{-\pi}^{\pi}\nabla_\vartheta\log(|\Pi(e^{i\omega},\vartheta_0)|^{-2})^\top\nabla_\vartheta\log(|\Pi(e^{i\omega},\vartheta_0)|^{-2})d\omega.\end{aligned}$$

□

The proof of Theorem 4 now matches the proof of Theorem 2, where Proposition 6 is replaced by Proposition 10 and Proposition 4 is replaced by Proposition 11.

8 Auxiliary Results

The following property motivates why the Whittle estimator is based on the frequencies $\{-\frac{\pi(n-1)}{n},\ldots,\pi\}$.

Lemma 4 *Let $h\in\mathbb{Z}$. Then,*

$$\frac{1}{2n}\sum_{j=-n+1}^{n}e^{-ih\omega_j}=\mathbb{1}_{\{\exists z\in\mathbb{Z}:\ h=2zn\}}.$$

8.1 The behavior of the sample autocovariance

We state and prove results concerning the asymptotic behavior of the estimators of the various arising covariance matrices.

Lemma 5 *Define the empirical sample autocovariance function*

$$\overline{\Gamma}_n^{(\Delta)}(h) = \frac{1}{n}\sum_{k=1}^{n-h} Y_{k+h}^{(\Delta)} Y_k^{(\Delta)\top} \quad \text{and} \quad \overline{\Gamma}_n^{(\Delta)}(-h) = \overline{\Gamma}_n^{(\Delta)}(h)^\top, \quad 0 \leq h \leq n.$$

Suppose $(A2)$ *and* $(A3)$ *hold. Then, for* $h \in \mathbb{Z}$ *and* $n \to \infty$,

$$\overline{\Gamma}_n^{(\Delta)}(h) \xrightarrow{a.s.} \Gamma^{\Delta}(h)$$

and $\sum_{h=-\infty}^{\infty} \|\Gamma^{(\Delta)}(h)\| < \infty$.

Proof Due to Proposition 3.34 of Marquardt and Stelzer (2007) the process Y is ergodic. Therefore, Theorem 4.3 of Krengel (2011) implies that the sampled process $Y^{(\Delta)}$ is ergodic as well. Moreover, $\Gamma_Y(h) = C^\top e^{Ah}\Sigma_N^{(\Delta)}C$ due to Marquardt and Stelzer (2007). Since the eigenvalues of A have strictly negative real parts

$$\sum_{h\in\mathbb{Z}} \|\Gamma^{\Delta}(h)\| = \sum_{h\in\mathbb{Z}} \|\Gamma_Y(\Delta h)\| < \infty.$$

Birkhoff‘s Ergodic Theorem now leads to

$$\overline{\Gamma}_n^{(\Delta)}(h) \xrightarrow{a.s.} \mathbb{E}\left[Y_h^{(\Delta)} Y_0^{(\Delta)\top}\right] = \Gamma^{\Delta}(h).$$

□

Remark 10 Similarly, one can show that in the situation of Lemma 5 the sample autocovariance function of $N^{(\Delta)}$ as introduced in Proposition 1 behaves in the same way, i.e.

$$\overline{\Gamma}_{n,N}(h) \xrightarrow{a.s.} \Gamma_N(h) \quad \forall\, h \in \mathbb{Z}.$$

Obviously, $\Gamma_N(h) = 0$ for $h \neq 0$ and $\Gamma_N(0) = \Sigma_N^{(\Delta)}$.

Under the stronger assumption of an i.i.d. white noise, the sample autocovariance function has an asymptotic normal distribution.

Lemma 6 *Let* $(Z_k)_{k\in\mathbb{N}}$ *be an* N*-dimensional i.i.d. white noise with* $\mathbb{E}\|Z_1\|^4 < \infty$ *and covariance matrix* Σ_Z. *Define*

$$\overline{\Gamma}_{n,Z}(h) = \frac{1}{n}\sum_{j=1}^{n-h} Z_{j+h}Z_j^\top, \quad n \geq h \geq 0,$$

Then, for fixed $\ell \in \mathbb{N}$,

$$\sqrt{n}\left(\begin{bmatrix} \operatorname{vec}\left(\overline{\Gamma}_{n,Z}(0)\right) \\ \operatorname{vec}\left(\overline{\Gamma}_{n,Z}(1)\right) \\ \vdots \\ \operatorname{vec}\left(\overline{\Gamma}_{n,Z}(\ell)\right) \end{bmatrix} - \begin{bmatrix} \operatorname{vec}(\Sigma_Z) \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) \xrightarrow{\mathscr{D}} \mathscr{N}(0, \Sigma_{\Gamma_Z}(\ell)),$$

where

$$\Sigma_{\Gamma_Z}(\ell) = \left(\begin{array}{c|c} \mathbb{E}[Z_1Z_1^\top \otimes Z_1Z_1^\top] - \Sigma_Z \otimes \Sigma_Z & 0_{N^2\times \ell N^2} \\ \hline 0_{\ell N^2 \times N^2} & I_\ell \otimes \Sigma_Z \otimes \Sigma_Z \end{array}\right).$$

Proof The proof is similar to the proof of Proposition 4.4 in Lütkepohl (2005) and is therefore omitted. □

8.2 Convergence rate of the integral approximation

To prove the uniform convergence of the Whittle function, it is necessary to guarantee that the deterministic part of the Whittle function converges uniformly.

Lemma 7 *Let Θ be a compact parameter space and let $g : [-\pi,\pi]\times\Theta\to\mathbb{C}$ be differentiable in the first component. Assume further that $\frac{\partial}{\partial\omega}g(\omega,\vartheta)$ is continuous on $[-\pi,\pi]\times\Theta$. Then,*

$$\sup_{\vartheta\in\Theta}\left|\frac{1}{2n}\sum_{j=-n+1}^{n}g(\omega_j,\vartheta)-\frac{1}{2\pi}\int_{-\pi}^{\pi}g(\omega,\vartheta)d\omega\right|\overset{n\to\infty}{\longrightarrow}0.$$

Proof Follows by an application of the mean value theorem. □

Lemma 8 *Let $g : [-\pi,\pi]\to\mathbb{C}$ be continuously differentiable. Then,*

$$\frac{1}{\sqrt{n}}\sum_{j=-n+1}^{n}g(\omega_j)-\frac{\sqrt{n}}{\pi}\int_{-\pi}^{\pi}g(\omega)d\omega\overset{n\to\infty}{\longrightarrow}0$$

holds.

Proof The lemma is a consequence of the definition of the Riemann integral and the continuously differentiability of g. □

9 Extended simulation study

In addition to the simulation study of Section 5, we investigate bivariate MCAR(1) processes and CAR(3) processes for both the Brownian motion and the NIG driven setting. The parametrization of the MCAR(1) model is given in Table 1 of Schlemm and Stelzer (2012a) and it is

$$A(\vartheta)=\begin{pmatrix}\vartheta_1 & \vartheta_2\\ \vartheta_3 & \vartheta_4\end{pmatrix}=B(\vartheta),\quad C(\vartheta)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad \Sigma_L(\vartheta)=\begin{pmatrix}\vartheta_5 & \vartheta_6\\ \vartheta_6 & \vartheta_7\end{pmatrix}$$

in which we choose the parameter

$$\vartheta_0^{(3)}=(1,-2,3,-4,0.7513,-0.3536,0.3536).$$

The results of this simulation study are summarized in Table 5 and Table 6, respectively. Likewise, as for the MCARMA(2,1) model in Table 1 and Table 2 of Section 5, the Whittle estimator and the QMLE converge very fast. Furthermore, we use the parameter

$$\vartheta_0^{(4)}=(-0.01,0,7,-1,0.7513,-0.3536,0.3536)$$

in this model class. One eigenvalue of $A(\vartheta_0^{(4)})$ is close to zero. An eigenvalue equal to zero results in a non-stationary MCARMA process. Table 7 shows the results for this setting for $n_2=2000$, and both the Brownian and the NIG driven model. The Whittle estimator and the QMLE estimate the parameters very well. But it is striking that the bias of several parameters of the QMLE even vanish.

For the univariate CAR(3) processes with parametrization

$$A(\vartheta)=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ \vartheta_1 & \vartheta_2 & \vartheta_3\end{pmatrix},\qquad B(\vartheta)=\begin{pmatrix}0\\ 0\\ \vartheta_1\end{pmatrix},\quad C(\vartheta)=(1\;0\;0).$$

and

$$\vartheta_0^{(5)}=(-6,-11,-6),$$

we choose once again the Brownian motion and the NIG Lévy process as driving processes. The results are documented in Table 8 and Table 9. They correspond to the results of Table 3 and Table 4, respectively for CARMA(2,1) processes.

$n_1 = 500$						
	Whittle			QMLE		
ϑ_0	mean	bias	std.	mean	bias	std.
1	1.0018	0.0018	0.0301	1.0045	0.0045	0.0362
-2	-2.0063	0.0063	0.0321	-2.0068	0.0068	0.0357
3	2.9966	0.0034	0.0399	3.0055	0.0055	0.0604
-4	-3.9980	0.0020	0.0399	-4.0019	0.0019	0.0565
0.7513	0.7543	0.0030	0.0516	0.7522	0.0009	0.0923
-0.3536	-0.3573	0.0037	0.0463	-0.3531	0.0005	0.0674
0.3536	0.3685	0.0149	0.0510	0.3704	0.0168	0.0714
$n_2 = 2000$						
	Whittle			QMLE		
ϑ_0	mean	bias	std.	mean	bias	std.
1	1.0035	0.0035	0.0150	1.0039	0.0039	0.0181
-2	-2.0067	0.0067	0.0165	-2.0066	0.0066	0.0192
3	2.9991	0.0009	0.0192	3.0021	0.0021	0.0286
-4	-3.9987	0.0013	0.0223	-4.0003	0.0003	0.0302
0.7513	0.7532	0.0019	0.0257	0.7514	0.0001	0.0401
-0.3536	-0.3603	0.0067	0.0248	-0.3574	0.0038	0.0352
0.3536	0.3675	0.0139	0.0280	0.3706	0.0170	0.0376
$n_3 = 5000$						
	Whittle			QMLE		
ϑ_0	mean	bias	std.	mean	bias	std.
1	1.0042	0.0042	0.0101	1.0050	0.0050	0.0117
-2	-2.0062	0.0062	0.0106	-2.0074	0.0074	0.0111
3	-2.9996	0.0004	0.0114	3.0021	0.0021	0.0169
-4	-3.9965	0.0035	0.0158	-4.0013	0.0013	0.0196
0.7513	0.7537	0.0024	0.0173	0.7549	0.0036	0.0258
-0.3536	-0.3596	0.0060	0.0166	-0.3559	0.0023	0.0201
0.3536	0.3663	0.0027	0.0169	0.3693	0.0157	0.0200

Table 5 Estimation results for a Brownian motion driven bivariate MCAR(1) process with parameter $\vartheta_0^{(3)}$.

$n_1 = 500$						
	Whittle			QMLE		
ϑ_0	mean	bias	std.	mean	bias	std.
1	0.9905	0.0095	0.0407	0.9806	0.0194	0.0460
-2	-1.9871	0.0129	0.0531	-2.0038	0.0038	0.0579
3	2.9920	0.0080	0.0579	2.9240	0.0760	0.0842
-4	-3.9409	0.0591	0.1027	-3.9918	0.0082	0.0894
0.7513	0.7281	0.0232	0.1869	0.7125	0.0388	0.0568
-0.3536	-0.3366	0.0170	0.0302	-0.3251	0.0285	0.0497
0.3536	0.3381	0.0155	0.0335	0.3182	0.0354	0.0486

$n_2 = 2000$						
	Whittle			QMLE		
ϑ_0	mean	bias	std.	mean	bias	std.
1	0.9916	0.0084	0.0261	0.9839	0.0161	0.0316
-2	-1.9892	0.0110	0.0321	-2.0072	0.0072	0.0320
3	2.9797	0.0203	0.0416	2.9377	0.0623	0.0576
-4	-3.9700	0.0300	0.0767	-4.0051	0.0051	0.0561
0.7513	0.7489	0.0024	0.1392	0.7210	0.0303	0.0351
-0.3536	-0.3603	0.0067	0.0241	-0.3224	0.0312	0.0312
0.3536	0.3417	0.0119	0.0224	0.3352	0.0184	0.0300

$n_3 = 5000$						
	Whittle			QMLE		
ϑ_0	mean	bias	std.	mean	bias	std.
1	0.9952	0.0048	0.0186	0.9810	0.0190	0.0240
-2	-1.9890	0.0110	0.0253	-2.0086	0.0086	0.0289
3	2.9789	0.0211	0.0365	2.9341	0.0659	0.0478
-4	-3.9849	0.0151	0.0611	-4.0064	0.0064	0.0516
0.7513	0.7500	0.0013	0.0749	0.6912	0.0601	0.0428
-0.3536	-0.3600	0.0064	0.0148	-0.3412	0.0124	0.0237
0.3536	0.3499	0.0037	0.0201	0.3208	0.0328	0.0238

Table 6 Estimation results for a NIG driven bivariate MCAR(1) process with parameter $\vartheta_0^{(3)}$.

Brownian motion driven, $n_2 = 2000$						
	Whittle			QMLE		
ϑ_0	mean	bias	std.	mean	bias	std.
-0.01	-0.0099	0.0001	0.0005	-0.0103	0.0003	0
0	0	0	0	0	0	0.1891
7	6.9245	0.0755	0.0853	7	0	0.0012
-1	-1.0442	0.0442	0.1915	-1	0	0.0019
0.7513	0.8574	0.1061	0.2193	0.7513	0	0.0031
-0.3536	-0.3492	0.0044	0.0587	-0.3535	0.0001	0.0013
0.3536	0.7958	0.4422	0.4160	0.3536	0	0.0005
NIG driven, $n_2 = 2000$						
	Whittle			QMLE		
ϑ_0	mean	bias	std.	mean	bias	std.
-0.01	-0.0125	0.0025	0.0534	-0.099	0.0001	0.0001
0	-0.0084	0.0084	0.0507	0	0	0.1805
7	7.0137	0.0137	0.1081	7	0	0.0180
-1	-0.8731	0.1269	0.1354	-1	0	0.0049
0.7513	1.4557	0.7045	0.0959	0.7513	0	0.0027
-0.3536	0.1189	0.4724	0.1675	-0.3536	0	0.0017
0.3536	0.7397	0.3862	0.0524	0.3535	0.0001	0.0008

Table 7 Estimation results for a bivariate MCAR(1) process with parameter $\vartheta_0^{(4)}$ close to the non-stationary case.

$n_1 = 500$									
	Whittle			adjusted Whittle			QMLE		
ϑ_0	mean	bias	std.	mean	bias	std.	mean	bias	std.
-6	-5.9230	0.0770	0.2074	-6.2266	0.2266	0.6347	-6.4357	0.4357	1.3266
-11	-10.8390	0.1610	0.4119	-11.2759	0.2759	0.9351	-11.6067	0.6067	1.6706
-6	-5.8267	0.1733	0.3585	-6.0575	0.0575	0.4800	-6.3039	0.3039	1.2821
$n_2 = 2000$									
	Whittle			adjusted Whittle			QMLE		
ϑ_0	mean	bias	std.	mean	bias	std.	mean	bias	std.
-6	-5.9886	0.0114	0.1117	-6.0410	0.0410	0.2391	-6.0549	0.0549	0.4510
-11	-10.9336	0.0664	0.2372	-11.0680	0.0680	0.4126	-11.0422	0.0422	0.6005
-6	-5.8855	0.1145	0.1755	-5.9460	0.0540	0.1924	-5.9542	0.0458	0.4464
$n_3 = 5000$									
	Whittle			adjusted Whittle			QMLE		
ϑ_0	mean	bias	std.	mean	bias	std.	mean	bias	std.
-6	-5.9856	0.0144	0.0884	-6.0455	0.0455	0.1444	-5.9861	0.0139	0.1120
-11	-10.9335	0.0665	0.1471	-11.0349	0.0349	0.1298	-10.9259	0.0741	0.1877
-6	-5.9123	0.0877	0.1262	-5.9303	0.0697	0.1104	-5.8937	0.1063	0.1406

Table 8 Estimation results for a Brownian motion driven CAR(3) process with $\vartheta_0^{(5)}$.

$n_1 = 500$									
	Whittle			adjusted Whittle			QMLE		
ϑ_0	mean	bias	std.	mean	bias	std.	mean	bias	std.
-6	-5.9449	0.0551	0.4322	-5.9238	0.0762	0.4799	-6.8247	0.8247	1.9413
-11	-10.9222	0.0778	0.5765	-10.9049	0.0951	0.6813	-12.1860	1.1860	2.3377
-6	-5.8492	0.1508	0.3455	-5.8000	0.2000	0.4239	-6.6137	0.6137	1.6559

$n_2 = 2000$									
	Whittle			adjusted Whittle			QMLE		
ϑ_0	mean	bias	std.	mean	bias	std.	mean	bias	std.
-6	-5.9611	0.0389	0.1287	-6.0737	0.0737	0.3438	-6.01035	0.1035	0.6401
-11	-10.9011	0.0989	0.2590	-11.0504	0.0504	0.4832	-11.1053	0.1053	0.8271
-6	-5.8879	0.1121	0.1988	-5.9692	0.0308	0.2175	-6.0036	0.0036	0.5522

$n_3 = 5000$									
	Whittle			adjusted Whittle			QMLE		
ϑ_0	mean	bias	std.	mean	bias	std.	mean	bias	std.
-6	-6.0313	0.0313	0.0825	-6.0622	0.0622	0.1883	-6.0087	0.0087	0.2748
-11	-10.8882	0.1118	0.1274	-11.0345	0.0345	0.1490	-10.9541	0.0459	0.3830
-6	-5.9110	0.0190	0.0885	-5.8438	0.1562	0.2144	-5.9164	0.0836	0.2513

Table 9 Estimation results for a NIG driven CAR(3) process with parameter $\vartheta_0^{(5)}$.

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