# Supplementary material
# to
# Wigner and Wishart Ensembles for sparse Vinberg models

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## 1. Description of Supplementary material

In this Supplementary material we give all technical details of proofs and more illustrations.

In order to facilitate using the Supplementary material, we include in it the main text of the article and keep the same numbering.

## 2. Preliminaries

We begin this paper with recalling the definition of the empirical eigenvalue distribution of a symmetric matrix. Let $\mathrm{Sym}(n,\mathbb{R})$ be the space of symmetric matrices of size $n$ and $\mathrm{Sym}(n,\mathbb{R})^+$ the open convex cone of positive definite symmetric matrices in $\mathrm{Sym}(n,\mathbb{R})$. Let $\lambda_1(X) \geq \cdots \geq \lambda_n(X)$ be the ordered eigenvalues of $X \in \mathrm{Sym}(n,\mathbb{R})$ with counting multiplicities. Denote by $\delta_a$ the Dirac measure at $a$. Then, the empirical eigenvalue distribution $\mu_X$ of $X$ is defined by $\mu_X = \frac{1}{n}\sum_{i=1}^n \delta_{\lambda_i(X)}$.

If $\{X_n\}_{n=1}^\infty$ ($X_n \in \mathrm{Sym}(n,\mathbb{R})$) is a sequence of Gaussian, Wigner or Wishart matrices, then it is well known that there exists a limit $\mu$ of $\mu_{X_n}$ as $n \to \infty$, and the sequence of random measures $\mu_{X_n}$ converges almost surely weakly to the semi-circle law or the Marchenko-Pastur law, respectively (see for example Bai and Silverstein (2010); Bordenave (2019)). The limits $\mu$ of $\mu_{X_n}$, in the almost sure weak sense, are said to be the "limiting eigenvalue distributions $\mu$ of $X_n$." For simplicity, we will say "i.i.d. matrices" instead of "matrices with independent and identically distributed non-null terms".

### 2.1. Generalized dual Vinberg cones and Vinberg matrices.

Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be non-decreasing sequences of positive integers such that $a_n + b_n = n$ and the ratio $a_n/n$ converges to $c \in [0,1]$. Then, we introduce the matrix space $\mathcal{U}_n$ as a subspace of $\mathrm{Sym}(n,\mathbb{R})$ defined by

$$\mathcal{U}_n := \left\{ U = \begin{pmatrix} x & y \\ y^\top & d \end{pmatrix};\ \begin{array}{l} x \in \mathrm{Sym}(a_n,\mathbb{R}),\ y \in \mathrm{Mat}(a_n \times b_n,\mathbb{R}), \\ d \text{ is a diagonal matrix of size } b_n \end{array} \right\},$$

where $\mathrm{Mat}(a_n \times b_n;\mathbb{R})$ denotes the space of $a_n \times b_n$ matrices. Set

$$P_n := \mathcal{U}_n \cap \mathrm{Sym}(n,\mathbb{R})^+.$$

Then, $P_n$ is an open convex cone in $\mathcal{U}_n$. Moreover, the cone $P_n$ admits a transitive group action, *i.e.* $P_n$ is a *homogeneous cone*, since the following triangular group

$$H_n := \left\{ h = \begin{pmatrix} h_1 & y \\ 0 & d \end{pmatrix} \in GL(n,\mathbb{R});\ \begin{array}{l} h_1 \in GL(a_n,\mathbb{R}) \text{ is upper triangular}, \\ y \in \mathrm{Mat}(a_n \times b_n;\mathbb{R}), \\ d\text{: diagonal of size } b_n \end{array} \right\}$$

acts on $P_n$ transitively by the quadratic action $\rho(h)U := hUh^\top$ for $h \in H_n$ and $U \in P_n$. This is easily verified by using the Cholesky decomposition (cf. Ishi (2016, p. 3)). For definition and basic properties of homogeneous cones, see Vinberg (1963); Ishi (2014).

If $n = 3$ and $(a_n, b_n) = (1,2)$, then $P_3$ is the dual Vinberg cone (see Example 1) so that, in this paper, we call $P_n$ a *generalized dual Vinberg cone* and elements $U \in \mathcal{U}_n$ *Vinberg matrices*. On the other hand, if we set $a_n = n-1$ and $b_n = 1$, then $\mathcal{U}_n$ is the space $\mathrm{Sym}(n,\mathbb{R})$ of symmetric matrices of size $n$, and hence our discussion covers the classical results. In what follows, we introduce two kinds of random matrices related to the homogeneous cones $P_n$, that is, Gaussian and Wigner matrices and Wishart quadratic (covariance) matrices.

### 2.2. Gaussian and Wigner matrices in $\mathcal{U}_n$.

Analogously to the classical Wigner matrices, we say that $U_n = (u_{ij}) \in \mathcal{U}_n$ is a Wigner random matrix if

$$\begin{cases} \bullet \text{ the diagonal terms } (u_{ii}) \text{ are independent of the off-diagonal terms } (u_{ij})_{i<j}, \\ \bullet \text{ the diagonal } u_{ii}\text{'s are centered i.i.d. variables with variance } v' \text{ and fourth moment } M_4', \\ \bullet \text{ the non-nul off-diagonal } u_{ij}\text{'s, } i<j, \text{ are centered i.i.d. variables with variance } v \text{ and fourth} \\ \quad \text{moment } M_4, \end{cases} \tag{1}$$

where $v, v', M_4, M_4'$ are fixed positive real numbers. If the non-nul terms $u_{ij}$ are Gaussian, with $v = 1$ and $v' = 2$, the matrices $U_n$ form a Gaussian Orthogonal Ensemble of Vinberg matrices. In Section 3, we consider empirical eigenvalue distributions of rescaled Wigner matrices $U_n/\sqrt{n} \in \mathcal{U}_n$.

2.3. **Quadratic construction of Wishart (covariance) matrices in $\mathcal{U}_n$.** Recall that sample covariance matrices, essential in multivariate statistical analysis, are defined as a quadratic map $\frac{1}{n}VV^\top$ of the observed centered sample vector $V$. Consequently, Wishart matrices are constructed quadratically both in Random Matrix Theory and in statistics. In this section we define, by a quadratic construction, Wishart (covariance) matrices in $\mathcal{U}_n$.

We first recall the notion of a direct sum of quadratic maps. Let $Q_i\colon \mathbb{R}^{m_i} \to \mathbb{R}^m$ $(i = 1, \dots, k)$ be quadratic maps. Then, the direct sum $Q_1 \oplus \cdots \oplus Q_k$ is an $\mathbb{R}^m$-valued quadratic map on $\mathbb{R}^{m_1} \oplus \cdots \oplus \mathbb{R}^{m_k}$ given by

$$Q(x) := Q_1(x_1) + \cdots + Q_k(x_k) \quad \text{where} \quad x = x_1 + \cdots + x_k \quad \big(x_i \in \mathbb{R}^{m_i}\big).$$

If $Q_1 = \cdots = Q_k$, then the direct sum $Q$ is denoted by $Q_1^{\oplus k}$. As showed in Graczyk and Ishi (2014), any homogeneous cone $\Omega$ admits a canonical family of the so-called *basic quadratic maps* $q_j$ $(j = 1, \dots, r)$ defined for each $j$ on a suitable finite dimensional vector space $E_j$ and with values in the closure $\overline{\Omega}$ of $\Omega$. The number $r$ is called the rank of $\Omega$ and $r = n$ for the cones $\mathcal{U}_n$. Using the basic quadratic maps $q_j$, one constructs quadratic maps $Q_{\underline{k}}$ for $\underline{k} \in \mathbb{Z}^r_{\geq 0}$ by

$$Q_{\underline{k}} := q_1^{\oplus k_1} \oplus \cdots \oplus q_r^{\oplus k_r},$$

defined on $E_{\underline{k}} := E_1^{\oplus k_1} \oplus \cdots \oplus E_r^{\oplus k_r}$. The maps $Q_{\underline{k}}$ are $\Omega$-positive, *i.e.* if $\xi \in E_{\underline{k}} \setminus \{\mathbf{0}\}$, then $Q_{\underline{k}}(\xi) \in \overline{\Omega} \setminus \{\mathbf{0}\}$.

In our case $\Omega = P_n$, the basic quadratic maps are given as follows (cf. Graczyk and Ishi (2014)). For $j = 1, \dots, n$, define $E_j \subset \mathbb{R}^n$ by

$$E_j = \left\{ \begin{pmatrix} \boldsymbol{\xi} \\ \mathbf{0} \end{pmatrix} \in \mathbb{R}^n;\ \boldsymbol{\xi} \in \mathbb{R}^j \right\} \quad (j \leq a_n),$$
$$E_j = \left\{ \begin{pmatrix} \boldsymbol{\xi} \\ \mathbf{0} \end{pmatrix} + \xi' \boldsymbol{e}_j \in \mathbb{R}^n;\ \boldsymbol{\xi} \in \mathbb{R}^{a_n},\ \xi' \in \mathbb{R} \right\} \quad (j > a_n),$$

where $\boldsymbol{e}_i$ $(i = 1, \dots, n)$ is the vector in $\mathbb{R}^n$ having 1 on the $i$-th position and zeros elsewhere. We note that each $E_j$ corresponds to the $j$-th column of the Lie algebra $\mathfrak{h}_n$ of $H_n$, that is, we have $\mathfrak{h}_n = \big\{H = (\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_n);\ \boldsymbol{\xi}_j \in E_j\big\}$. Then, the basic quadratic maps $q_j\colon E_j \to \mathcal{U}_n$ of the cone $P_n$ are defined by

$$q_j(\boldsymbol{\xi}_j) := \boldsymbol{\xi}_j \boldsymbol{\xi}_j^\top \in \mathcal{U}_n \quad (\boldsymbol{\xi}_j \in E_j).$$

Let $\underline{k} \in \mathbb{Z}^n_{\geq 0}$. Then, $E_{\underline{k}}$ can be viewed as a subspace of $\mathrm{Mat}(n \times (k_1 + \cdots + k_n); \mathbb{R})$ of the form

$$\left\{ \eta = \Big( \overbrace{\boldsymbol{\xi}_1^{(1)}, \dots, \boldsymbol{\xi}_1^{(k_1)}}^{k_1}, \dots, \overbrace{\boldsymbol{\xi}_{n-1}^{(1)}, \dots, \boldsymbol{\xi}_{n-1}^{(k_{n-1})}}^{k_{n-1}}, \overbrace{\boldsymbol{\xi}_n^{(1)}, \dots, \boldsymbol{\xi}_n^{(k_n)}}^{k_n} \Big);\ \begin{array}{l} \boldsymbol{\xi}_j^{(i)} \in E_j, \\ 1 \leq j \leq n, \\ 1 \leq i \leq k_j \end{array} \right\},$$

and then $Q_{\underline{k}}(\eta) = \eta\,\eta^\top$ for $\eta \in E_{\underline{k}}$. In order to simplify formulas when we apply the so-called variance profile method in §4, we do not multiply $\frac{1}{2}$ in definition of $Q_{\underline{k}}(\eta)$.

When $\eta \in E_{\underline{k}}$ is an i.i.d. random matrix whose non-null terms have the normal law $N(0, v)$, the law of $Q_{\underline{k}}(\eta)$ is a *Wishart law* $\gamma_{Q_{\underline{k}}, 1/(2v)\mathrm{Id}_n}$ on the cone $P_n$. For the definition of all Wishart laws on the cone $P_n$, see Graczyk and Ishi (2014). More generally, in this paper, we consider eigenvalue distributions of rescaled matrix $Q_{\underline{k}}(\eta)/n$ under the assumption that $\eta \in E_{\underline{k}}$ is a centered rectangular i.i.d. matrix whose non-null terms have variance $v$ and finite fourth moments $M_4$.

We consider two-dimensional multiparameters $\underline{k} = \underline{k}(n) \in \mathbb{Z}^n_{\geq 0}$ of the form

$$\underline{k} = m_1(1, \dots, 1) + m_2(\overbrace{0, \dots, 0}^{a_n}, \overbrace{1, \dots, 1}^{b_n}) \quad (m_1, m_2 \in \mathbb{Z}_{\geq 0}). \tag{2}$$

**Example 1.** Let $n = 3$, $a_3 = 1$ and $b_3 = 2$. In this case, $P_3$ is the dual Vinberg cone (cf. Vinberg (1963, p. 397), Ishi (2001, §5.2)):

$$P_3 = \left\{ x = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & 0 \\ x_{13} & 0 & x_{33} \end{pmatrix};\ x \text{ is positive definite} \right\}.$$

Consider $m_1 = m_2 = 1$, so $\underline{k} = (1, 2, 2)$. Then $E_{\underline{k}} = E_{(1,2,2)}$ can be written as

$$E_{(1,2,2)} = \left\{ \eta = \begin{pmatrix} x & y_{11} & y_{12} & z_{11} & z_{12} \\ 0 & y_{21} & y_{22} & 0 & 0 \\ 0 & 0 & 0 & z_{21} & z_{22} \end{pmatrix};\ x, y_{ij}, z_{ij} \in \mathbb{R} \right\},$$

and $Q_{(1,2,2)}(\eta) = \eta\,\eta^\top$ is given as

$$Q_{(1,2,2)}(\eta) = \begin{pmatrix} x^2 + y_{11}^2 + y_{12}^2 + z_{11}^2 + z_{12}^2 & y_{11}y_{21} + y_{12}y_{22} & z_{11}z_{21} + z_{12}z_{22} \\ y_{11}y_{21} + y_{12}y_{22} & y_{21}^2 + y_{22}^2 & 0 \\ z_{11}z_{21} + z_{12}z_{22} & 0 & z_{21}^2 + z_{22}^2 \end{pmatrix}.$$

If $x, y_{ij}, z_{ij}$ are $N(0, v)$ i.i.d. Gaussian variables, the random matrix $Q_{(1,2,2)}(\eta)$ has a Wishart law on $P_3$.

The form (2) of the Wishart multiparameter $\underline{k}$ englobes and generalizes the following cases. In both cases, with rescaling $1/n$, the limiting eigenvalue distribution is known.

(i) The classical Wishart Ensemble $MM^\top$ on $\mathrm{Sym}(n,\mathbb{R})^+$, where $M = M_{n\times N}$ is an i.i.d. matrix with finite fourth moment $M_4$, with parameter $C := \lim_n \frac{N}{n} > 0$ (see Anderson et.al. (2010); Faraut (2014)) for $(a_n, b_n) = (n-1, 1)$, $m_1 = 0$ and $m_2 \sim Cn$. The limiting eigenvalue distribution is the Marchenko-Pastur law $\mu_C$ with parameter $C$, i.e. denoting $a = \left(\sqrt{C}-1\right)^2, b = \left(\sqrt{C}+1\right)^2$ and $[x]_+ := \max(x,0)$ $(x \in \mathbb{R})$,

$$\mu_C = [1-C]_+\delta_0 + \frac{\sqrt{(t-a)(b-t)}}{2\pi t}\,\chi_{[a,b]}(t)dt.$$

(ii) The Wishart Ensemble related to the Triangular Gaussian Ensemble (Dykema and Haagerup (2004); Cheliotis (2018)) for $(a_n, b_n) = (n-1, 1)$, $m_1 = 1$ and $m_2 = 0$. When $v = 1$, the limiting eigenvalue distribution, which we call the *Dykema-Haagerup measure* $\chi_1$, is absolutely continuous with respect to Lebesgue measure and has support equal to the interval $[0, e]$. Its density function $\phi$ is defined on the interval $(0, e]$ by the implicit formula (Dykema and Haagerup (2004, Theorem 8.9))

$$\phi\left(\frac{\sin x}{x}\exp(x\cot x)\right) = \frac{1}{\pi}\sin x\exp(-x\cot x) \qquad (0 \le x < \pi), \tag{3}$$

with $\phi(0+) = \infty$ and $\phi(e) = 0$. For $v \neq 1$, the limiting measure $\chi_v$ has density $\phi(y/v)/v$ on the segment $(0, ve]$.

2.4. **Resolvent method for Wigner ensembles with a variance profile $\sigma$.** Let $\mathbb{C}^+$ denote the upper half plane in $\mathbb{C}$. In this paper, the Stieltjes transform $S(z) = S_\mu(z)$ of a finite measure or a non-negative $L^1$-function $\mu$ on $\mathbb{R}$ is defined to be

$$S(z) = \int_{\mathbb{R}} \frac{\mu(dt)}{t-z} \quad (z \in \mathbb{C}^+).$$

In the sequel, we will need the following properties of the Stieltjes transform, which are not difficult to prove.

**Proposition 2.** *1. Suppose that $S(z)$ is the Stieltjes transform of a finite measure $\nu$ on $\mathbb{R}$. If for all $x \in \mathbb{R}$ it holds*

$$\lim_{y\to 0+} \operatorname{Im} S(x+iy) = 0$$

*then $S(z) \equiv 0$ and $\nu$ is a null measure ($\nu(B) = 0$ for any Borel set $B$).*
*2. Suppose $f \ge 0$ and $f \in L^1(\mathbb{R})$. Let $S(z)$ be the Stieltjes transform of $f$. If $f$ is continuous at $x$ then*

$$\lim_{y\to 0+} \frac{1}{\pi}\operatorname{Im} S(x+iy) = f(x). \tag{4}$$

*If $f$ is continuous on an interval $[a,b]$, $a < b$, the convergence (4) is uniform for $x \in [a,b]$.*

Recall that if $\mu$ is a probabilistic measure on $\mathbb{R}$, with Stieltjes transform $S(z)$ and the absolutely continuous part of $\mu$ has density $f$, then (4) holds for almost all $x$ (Lemma 3.2 (iii) of Bordenave (2019)).

We present now the following, slightly strengthened result from the Lecture Notes of Bordenave (2019, §3.2), that will be a main tool of proofs in this paper.

Let $\sigma\colon [0,1]\times[0,1] \to [0,\infty)$ be a bounded Borel measurable symmetric function. For each integer $n$, we partition the interval $[0,1]$ into $n$ equal intervals $J_i$, $i = 1,\dots,n$. Put $Q_{ij} := J_i \times J_j$, which is a partition of $[0,1]\times[0,1]$. We assume that $Y_{ij}$ $(i \le j)$ are independent centered real variables, defined on a common probability space, with variance

$$\mathbb{E}Y_{ij}^2 = \frac{1}{n}\left(\int_{Q_{ij}} \frac{\sigma(x,y)}{|Q_{ij}|}\,dx\,dy + \delta_{ij}(n)\right), \tag{5}$$

for a sequence $\delta_{ij}(n)$. We note that the law of $Y_{ij}$ depends on $n$. We set $Y_{ji} := Y_{ij}$ and we consider the symmetric matrix $Y_n := (Y_{ij})_{1\le i,j\le n}$. We note that, if $\sigma$ is continuous, then, up to a perturbation $\delta_{ij}(n)$, the variance of $\sqrt{n}Y_{ij}$ is approximatively $\sigma(i/n, j/n)$, and hence we call $\sigma$ a variance profile in this paper.

**Theorem 3.** *Let $\delta_0(n) := \frac{1}{n^2}\sum_{i,j\le n} |\delta_{ij}(n)|$. Assume (5) and suppose that*

$$\lim_n \delta_0(n) = 0 \quad \text{and} \quad \max_{i,j\le n} \frac{\mathbb{E}(Y_{ij}^4)}{n(\mathbb{E}Y_{ij}^2)^2} = o(1) \quad (Y_{ij} \neq 0). \tag{6}$$

*Let $\mu_{Y_n}$ be the empirical eigenvalue distribution of $Y_n$. Then, there exists a probability measure $\mu_\sigma$ depending on $\sigma$ such that $\mu_{Y_n}$ converges weakly to $\mu_\sigma$ almost surely. The Stieltjes transform $S_\sigma$ of $\mu_\sigma$ is given as follows.*
*(a) For each $z$ with $\operatorname{Im} z > \sqrt{\sup\sigma}$, there exists a unique $\mathbb{C}^+$-valued $L^1$-solution $\eta_z : [0,1] \mapsto \mathbb{C}^+$, of the equation*

$$\eta_z(x) = -\left(z + \int_0^1 \sigma(x,y)\,\eta_z(y)\,dy\right)^{-1} \quad (\textit{for almost all } x \in [0,1]), \tag{7}$$

*and the function $z \mapsto \eta_z(x)$ extends to an analytic $\mathbb{C}^+$-valued function on $\mathbb{C}^+$, for almost all $x \in [0,1]$. Then,*

$$S_\sigma(z) = \int_0^1 \eta_z(x)\,dx.$$

*(b) The function $x \to \eta_z(x)$ is also a solution of (7) for $0 < \operatorname{Im} z \le \sqrt{\sup\sigma}$.*

*Proof.* The proof is the same as the proof of Bordenave (2019, Theorem 3.1), where a stronger assumption $|\delta_{ij}(n)| \leq \delta(n)$ is required for some sequence $\delta(n)$ going to 0. It is replaced by the first condition of (6). Detailed analysis of the proof of the approximate fixed point equation in Bordenave (2019, page 42) shows that the second condition of (6) is the weakest assumption on the fourth moments $\mathbb{E}Y_{ij}^4$ ensuring the concentration of the conditional variance related to the Schur complement of the Stieltjes transform of the approximating matrix of $Y_n$. The property (b) is observed in Bordenave (2019, page 39) by analiticity. □ □

Since now we assume that $\sigma\colon [0,1]^2 \to [0,+\infty)$ is bounded (not obligatorily by 1 as in Bordenave (2019)). Consequently, the condition on $z$ should be $\operatorname{Im} z > \sqrt{\sup \sigma}$, not $\operatorname{Im} z > 1$. In fact, if $M^2 = \sup \sigma > 0$, we have for $\widetilde{\sigma}(x,y) := \sigma(x,y)/M^2$

$$\begin{aligned}
\widetilde{\eta}_z(x) &= -\left(z + \int_0^1 \widetilde{\sigma}(x,y)\widetilde{\eta}_z(y)\,dy\right)^{-1} \\
&= -M\left(Mz + \int_0^1 \sigma(x,y)\frac{\widetilde{\eta}_z(y)}{M}\,dy\right)^{-1} \\
\iff \frac{\widetilde{\eta}_z(x)}{M} &= -\left(Mz + \int_0^1 \sigma(x,y)\frac{\widetilde{\eta}_z(y)}{M}\,dy\right)^{-1}.
\end{aligned}$$

Note that the last equation has a unique solution when $\operatorname{Im} z > 1$ by Bordenave (2019). On the other hand, with respect to $\sigma$, we have

$$\eta_w(x) = -\left(w + \int_0^1 \sigma(x,y)\eta_w(y)\,dy\right)^{-1},$$

and thus if we set $w = Mz$, then we see that $\eta_w(x) = \frac{1}{M}\widetilde{\eta}_{w/M}(x)$ is a unique solution when $\operatorname{Im} w/M > 1$, i,e., $\operatorname{Im} w > M = \sqrt{\sup \sigma}$.

Theorem 3 shows that, to each variance profile function $\sigma$, one associates uniquely a Stieltjes transform $S_\sigma(z)$ of a probability measure. For the correspondence between $\sigma$ and $S_\sigma$, the conditions (6) are not needed. We define $S_\sigma(z)$ as the *Stieltjes transform associated to $\sigma$.*

**Remark 4.** A prototype of the variance profile method for Wigner ensembles was given by Anderson and Zeitouni (2006, Theorem 3.2). Theorem 3.1 of Bordenave (2019) and Theorem 3 provide a simple general approach. Special cases of variance profile convergence results for Wigner matrices were studied before, as discussed below in (i) and (ii).

(i) If we set $\sigma(x,y) = 1$ for all $x, y$, then $\sqrt{n}Y$ is a Wigner ensemble with $v = v' = 1$. Let $S_{\mathrm{sc}}(z)$ be the Stieltjes transform of the semi-circle law on $[-2,2]$. Then, the functions $x \to \eta_z(x)$ do not depend on $x$ (but do on $z$) and the functional equation (7) gives the equation $S_{\mathrm{sc}}(z) = -(z + S_{\mathrm{sc}}(z))^{-1}$, which is well known from the detailed study of resolvent matrices (see Tao (2012, §2.4.3)).

(ii) The paper Anderson and Zeitouni (2006) deals primarily with a variance profile $\sigma$ such that $\int \sigma(x,y)\,dy = 1$ for any $x$, corresponding to a band matrix model. For band matrix ensembles, see also Erdös et al. (2012a,b); Nica et al. (2002); Shlyakhtenko (1996).

2.4.1. *Proofs of Proposition 2 and Theorem 3.*

**Proof of Proposition 2**

1. The zero limit means that the Stieltjes transform $s(z)$ has no discontinuity on $\mathbb{R}$, so $s(z)$ is holomorphic on $\mathbb{C}$ and has decay $1/z$ when $|z| \to \infty$, so is bounded. By Liouville theorem, this implies that $s(z) = \mathit{const} = 0$ and, by unicity of the Stieltjes transform, $\nu = 0$.

2. is given in the following lemma.

**Lemma S1.** *Let $f$ be an $L^1$-function on $\mathbb{R}$: $\int |f(x)|dx = F < +\infty$ and let $S$ be its Stieltjes transform.*
*(a) If $f$ is continuous at $x = x_0$, then we have*

$$\lim_{y \to +0} \frac{1}{\pi} \operatorname{Im} S(x_0 + yi) = f(x_0). \tag{S.1}$$

*(b) If $f$ is continuous on an interval $[a,b]$, $a < b$, then the convergence in* (S.1) *is uniform for $x \in [a,b]$.*

*Proof.* Since $S(\bar{z}) = \overline{S(z)}$, we have

$$\begin{aligned}
\operatorname{Im} S(x+yi) &= \frac{1}{2i}\left(\int_{\mathbb{R}} \frac{f(t)}{t-x-yi}dt - \int_{\mathbb{R}} \frac{f(t)}{t-x+yi}dt\right) \\
&= y\int_{\mathbb{R}} \frac{f(t)}{(t-x)^2+y^2}dt = \int_{\mathbb{R}} \frac{f(x+yu)}{1+u^2}du.
\end{aligned}$$

In the third equality, we change variable $t - x = yu$.

(a) Let $y > 0$. We consider

$$\frac{1}{\pi}\operatorname{Im} S(x_0+yi) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(x_0+yu)}{u^2+1}du.$$

Let us take an enough small $\varepsilon > 0$. Then, there exists $\delta > 0$ such that if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \varepsilon$. We divide the integral into two parts: $I_1 = \{u;\ |(x_0 + yu) - x_0| = |yu| < \delta\}$ and its complement $I_2 = \{u;\ |(x_0 + yu) - x_0| = |yu| \geq \delta\}$:

$$\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(x_0 + yu)}{u^2+1}du = \frac{1}{\pi}\int_{I_1} \frac{f(x_0 + yu)}{u^2+1}du + \frac{1}{\pi}\int_{I_2} \frac{f(x_0 + yu)}{u^2+1}du =: J_1 + J_2.$$

Let us consider $J_1$. Since $|yu| < \delta$ for $u \in I_1$, we have $f(x_0) - \varepsilon < f(x_0 + yu) < f(x_0) + \varepsilon$ so that

$$\frac{f(x_0) - \varepsilon}{\pi}\int_{|u|<\frac{\delta}{y}} \frac{du}{1+u^2} \leq J_1 \leq \frac{f(x_0) + \varepsilon}{\pi}\int_{|u|<\frac{\delta}{y}} \frac{du}{1+u^2}.$$

Set

$$A = A_{y,\delta} = \frac{1}{\pi}\int_{|u|<\frac{\delta}{y}} \frac{du}{1+u^2} = \frac{2}{\pi}\mathrm{Arctan}\,\frac{\delta}{y} \leq 1.$$

Then, the above inequality means

$$|J_1 - f(x_0)A| \leq \varepsilon A \leq \varepsilon$$

Next we consider $J_2$. By changing variable $v = yu$, we have

$$\begin{aligned}|J_2| &= \left| y \cdot \int_{|v|\geq\delta} \frac{f(x_0 + v)}{v^2 + y^2}dv \right| \leq y \cdot \int_{|v|\geq\delta} \frac{|f(x_0 + v)|}{v^2 + y^2}dv \leq y \cdot \int_{|v|\geq\delta} \frac{|f(x_0 + v)|}{\delta^2 + y^2}dv \\ &\leq \frac{y}{\delta^2 + y^2}\int_{\mathbb{R}} |f(x_0 + v)|dv = \frac{Fy}{\delta^2 + y^2} \leq \frac{F}{\delta^2}\cdot y.\end{aligned}$$

Since we can choose $y_0 > 0$ such that if $0 < y < y_0$ then

$$|f(x_0)| \cdot |A - 1| \leq \varepsilon, \quad \frac{F}{\delta^2}\cdot y \leq \varepsilon$$

(Note that $A_{\delta,y} \to 1$ as $y \to +0$ when $\delta$ is fixed), we see that

$$|J_1 + J_2 - f(x_0)| \leq |J_1 - f(x_0)| + |J_2| \leq |J_1 - f(x_0)A| + |f(x_0)| \cdot |A - 1| + |J_2| \leq \varepsilon + \varepsilon + \varepsilon = 3\varepsilon.$$

Since $\varepsilon$ is arbitrary, we conclude that $\displaystyle\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(x_0 + yu)}{u^2+1}du \to f(x_0)$ as $y \to +0$.

(b) The proof is the same, using the uniform continuity of $f$ on $[a, b]$. We choose the same $\delta$ for all $x \in [a, b]$ and $y_0$ such that $\|f\mathbf{1}_{[a,b]}\|_\infty |A - 1| < \epsilon$ for $0 < y < y_0$. □

Note that the proof of 2. is shorter when $f$ is bounded continuous. Since $f(x)$ is continuous, $\lim_{y\to0+} \frac{f(x+yu)}{1+u^2} = \frac{f(x)}{1+u^2}$ and all these functions are bounded by $\frac{\|f\|_\infty}{1+u^2}$ integrable, we can change the limit and the integral by the dominated convergence theorem so that

$$\lim_{y\to+0} \mathrm{Im}\, S(x + yi) = \lim_{y\to+0}\int_{\mathbb{R}} \frac{f(x + yu)}{1+u^2}du = \int_{\mathbb{R}} \lim_{y\to+0}\frac{f(x + yu)}{1+u^2}du = \int_{\mathbb{R}} \frac{f(x)}{1+u^2}du = \pi f(x). \quad \square$$

**Proof of Theorem 3**

To give a proof of Theorem 3, we first prepare some basic lemmas on matrices. For Hermitian symmetric matrix $A$, we set

$$\|A\|_F^2 = \mathrm{tr}(A^2), \quad \|A\| = \sup_{|x|=1}\frac{|Ax|}{x}.$$

Note that $\|A\|_F$ is called the Frobenius norm of $A$. For $X, Y \in \mathbb{C}^n$, we set $\langle X \,|\, Y \rangle = X^\top Y$, which is a complex bilinear form.

**Lemma S2.** *Let $A$ be a Hermitian symmetric matrix of size $n$ and $R = (A - zI_n)^{-1}$ its resolvent. Then, for any $z \in \mathbb{C}^+$, one has*
*(i) $\|R(z)\|_F^2 \leq \dfrac{n}{(\mathrm{Im}\, z)^2}$ and $\|R(z)\|^2 \leq \frac{1}{(\mathrm{Im}\, z)^2}$,*
*(ii) $R_{ij}(z) \in \mathbb{C}^+$ for any $i, j$,*
*(iii) $\langle X \,|\, R(z)X \rangle \in \mathbb{C}^+$ for any $X \in \mathbb{R}^n$.*

*Proof.* Since $A$ is symmetric, there exists an orthogonal matrix $O = (v_1, \ldots, v_r) \in O(n)$ such that

$$A = O\Lambda O^\top, \quad \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n), \quad \lambda_j \in \mathbb{R}.$$

Then, we have

$$R(z) = (O\Lambda O^\top - zI_n)^{-1} = O(\Lambda - zI_n)^{-1}O^\top = \sum_{j=1}^n \frac{1}{\lambda_j - z}\, v_j{}^t v_j,$$

and thus

$$\|R(z)\|_F^2 = \sum_{j=1}^n \frac{1}{|\lambda_j - z|^2} \leq \sum_{j=1}^n \frac{1}{(\mathrm{Im}\, z)^2} = \frac{n}{(\mathrm{Im}\, z)^2}.$$

Moreover, since $v_j v_j^\top$ are real matrices and

$$\frac{1}{\lambda - z} = \frac{\lambda - \bar{z}}{|\lambda - z|^2} \in \mathbb{C}^+,$$

each $R_{ij}(z)$ has positive imaginary parts. We have

$$\langle X \,|\, R(z)X \rangle = X^\top O(\Lambda - zI_n)^{-1} O^\top X = Y^\top(\Lambda - zI_n)Y = \sum_{j=1}^{n} \frac{y_j^2}{\lambda - z} \in \mathbb{C}^+,$$

where we set $Y = (y_j) = O^\top X$. □

**Lemma S3.** *Let $n \geq 2$. Let $A$ be a symmetric matrix of size $n$ and $R$ its resolvent.*
(i) *(Resolvent complement formula) For $i = 1, \dots, n$, one has*

$$R_{ii} = -\left(z - A_{ii} + \left\langle X^{(i)} \,|\, R^{(i)} X^{(i)} \right\rangle\right)^{-1},$$

*where $X^{(i)} = (A_{ji})_{j \neq i}$ and $R^{(i)}$ is the resolvent of the matrix $A^{(i)}$ obtained from $A$ removing the $i$-th row and column.*
(ii) Moreover,

$$|R_{ii}|^2 \leq \frac{1}{(\operatorname{Im} z)^2}.$$

*Proof.* Note that there exists a permutation matrix $P$ such that

$$A = P \begin{pmatrix} A^{(i)} & X^{(i)} \\ X^{(i)\top} & A_{ii} \end{pmatrix} P^\top,$$

and thus it is enough to consider the case $i = n$. Set $A' = A^{(n)}$, $X' = X^{(n)}$. We have

$$\begin{pmatrix} A' & X' \\ (X')^\top & A_{nn} \end{pmatrix} = \begin{pmatrix} I_{n-1} & 0 \\ ((A')^{-1}X')^\top & 1 \end{pmatrix} \begin{pmatrix} A' & 0 \\ 0 & A_{nn} - (X')^\top (A')^{-1} X' \end{pmatrix} \begin{pmatrix} I_{n-1} & (A')^{-1}X' \\ 0 & 1 \end{pmatrix},$$

whence

$$\begin{pmatrix} A' - zI_{n-1} & X' \\ (X')^\top & A_{nn} - z \end{pmatrix}^{-1} = \begin{pmatrix} I_{n-1} & -(A' - zI_{n-1})^{-1}X' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (A' - zI_{n-1})^{-1} & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} I_{n-1} & 0 \\ ((A' - zI_{n-1})^{-1}X')^\top & 1 \end{pmatrix},$$

where

$$\alpha = (A_{nn} - z - (X')^\top (A' - zI_{n-1})^{-1} X')^{-1} = -\left(z - A_{nn} + \left\langle X' \,|\, (A' - zI_{n-1})^{-1} X' \right\rangle\right)^{-1}.$$

By Lemma S2(iii), we have

$$w = a + bi := -A_{nn} + \left\langle X' \,|\, (A' - zI_{n-1})^{-1} X' \right\rangle \in \mathbb{C}^+.$$

Then, the $(n, n)$ entry of $R = \begin{pmatrix} A' - zI_{n-1} & X' \\ (X')^\top & A_{nn} - z \end{pmatrix}^{-1}$ is given by $\alpha = -\frac{1}{z+w}$. Therefore, by setting $z = x + yi$,

$$|R_{ii}|^2 = \frac{1}{|z + w|^2} = \frac{1}{(x + a)^2 + (y + b)^2} \leq \frac{1}{(y + b)^2} \leq \frac{1}{y^2}$$

since $b > 0$. Thus we obtain the lemma. □

Theorem 3 is a slightly strengthened version of Theorem 3.1 in Bordenave (2019). Our assumptions (6) are different from the assumptions of Theorem 3.1 in Bordenave (2019). The proof is similar to the proof of Theorem 3.1 in Bordenave (2019). Below we point out the places where our assumptions intervene and justify their sufficiency. In this proof, we use the notation $\sigma^2$ of Bordenave (2019) for variance profile (to simplify, in our paper we use $\sigma$ for variance profile).

Since now we assume that $\sigma\colon [0, 1]^2 \to [0, +\infty)$ is bounded (not obligatorily by 1 as in Bordenave (2019)). Consequently, the condition on $z$ should be $\operatorname{Im} z > \sqrt{\sup \sigma}$, not $\operatorname{Im} z > 1$.

Bordenave (2019, P.41, line 11): an upper estimate of

$$\mathbb{E} \int \lambda^2 d\mu_Y \leq \|\sigma^2\|_1 + \delta_0(n) = O(1).$$

Bordenave (2019, P.42, line 5): Estimation of

$$\frac{1}{n^2} \sum_{i,j} |\rho^2(\frac{i}{n}, \frac{j}{n}) - nVar(Y_{ij})| \tag{S.2}$$

Here $\rho$ is a function depending on $L$, i.e. $\rho = \rho_L$ and is constant on squares $P_{kl}$ of size $1/L^2$.
(1) The first idea is to replace each $\rho^2(\frac{i}{n}, \frac{j}{n})$ by $\frac{1}{|Q_{ij}|} \int_{Q_{ij}} \rho^2(x, y) dxdy$.

Suppose $n > L$. Note that if $Q_{ij} \subset P_{kl}$ then

$$\rho^2(\frac{i}{n}, \frac{j}{n}) = \frac{1}{|Q_{ij}|} \int_{Q_{ij}} \rho^2(x,y) dxdy.$$

The difference between the last terms may be not zero only if $Q_{ij}$ intersects $P_{kl}$, but is not included in $P_{kl}$. This happens on squares $Q_{ij}$ of size $1/n$ along $2(L-1)$ segments $x = \frac{i}{L}$ and $y = \frac{i}{L}$, $i = 1, \dots, L-1$ in the unit square.

Denote the union of such error-generating rectangles $Q_{ij}$ by $E$. There are less than $2nL$ error-generating rectangles in $E$. In order to control the error we perform the following estimations.

Recall that $\rho_{kl} = L^2 \int_{P_{kl}} \sigma dxdy$ and that $0 \le \sigma$ is bounded. We will suppose without loss of generality that $\sigma \le 1$. Thus $\max_{k,l} \rho_{kl}^2 \le 1$.

Suppose $n \ge L^2$. We have

$$\frac{1}{n^2} \sum_{Q_{ij} \subset E} \rho^2(\frac{i}{n}, \frac{j}{n}) \le \frac{1}{n^2} \cdot 2nL \le \frac{2L}{n} \le \frac{2}{L};$$

$$\begin{aligned} \frac{1}{n^2} \sum_{Q_{ij} \subset E} \frac{1}{|Q_{ij}|} \int_{Q_{ij}} \rho^2(x,y) dxdy &= \sum_{Q_{ij} \subset E} \int_{Q_{ij}} \rho^2(x,y) dxdy = \int_E \rho^2(x,y) dxdy \\ &\le \lambda(E) \le \frac{2L}{n} \le \frac{2}{L}. \end{aligned}$$

Finally, when $n \ge L^2$,

$$\sum_{ij} \left| \frac{1}{|Q_{ij}|} \int_{Q_{ij}} \rho^2(x,y) dxdy - \rho^2(\frac{i}{n}, \frac{j}{n}) \right| \le \frac{4}{L} = O(\frac{1}{L}).$$

(2) One replaces

$$n\mathrm{Var}(Y_{ij}) = \int_{Q_{ij}} \frac{\sigma(x,y)^2}{|Q_{ij}|}\, dx\, dy + \delta_{ij}(n)$$

(3) one uses triangular inequality to get

$$\begin{aligned} &\frac{1}{n^2} \sum_{i,j} |\rho^2(\frac{i}{n}, \frac{j}{n}) - nVar(Y_{ij})| \\ \le\ & \frac{1}{n^2} \sum_{i,j} \left| \frac{1}{|Q_{ij}|} \int_{Q_{ij}} (\rho^2(x,y) - \sigma(x,y)^2) dxdy \right| + \delta_0(n) + O(\frac{1}{L}) \\ \le\ & \sum_{i,j} \int_{Q_{ij}} |\rho^2(x,y) - \sigma(x,y)^2| dxdy + \delta_0(n) + O(\frac{1}{L}) \\ =\ & \int_{[0,1]^2} |\rho^2(x,y) - \sigma(x,y)^2| dxdy + \delta_0(n) + O(\frac{1}{L}) \end{aligned}$$

The hypothesis $\delta_0(n) \to 0$ allows to conclude like in Bordenave (2019, p.42, l.5).

Bordenave (2019, Page 42, lines -3 / -1): For two vectors $X, Y$, we set

$$\langle X \,|\, Y \rangle = \sum_j X_j Y_j.$$

Take $z \in \mathbb{C}^+$. Set

$$Z = (Z_{ij}), \quad Z_{ij} = \begin{cases} \frac{Y_{ij}}{\sqrt{n\mathrm{Var}(Y_{ij})}} \rho(\frac{i}{n}, \frac{j}{n}) & (\mathrm{Var}(Y_{ij}) \neq 0) \\ 0 & (\mathrm{Var}(Y_{ij}) = 0) \end{cases}$$

and

$$R = \left(R_{ij}\right)_{1 \le i,j \le n} = (Z - zI_n)^{-1}.$$

Note that

$$\mathbb{E}|Z_{ij}|^2 = \mathbb{E}\left[\frac{Y_{ij}}{\sqrt{n\mathrm{Var}(Y_{ij})}} \rho(\tfrac{i}{n}, \tfrac{j}{n})\right]^2 = \rho(\tfrac{i}{n}, \tfrac{j}{n})^2 \frac{\mathbb{E}|Y_{ij}|^2}{n\mathrm{Var}(Y_{ij})} = \frac{\rho(\frac{i}{n}, \frac{j}{n})^2}{n}.$$

Fix an integer $i$ such that $1 \le i \le n$. Let $X^{(i)} = \left(Z_{ji}\right)_{j \neq i} \in \mathbb{R}^{n-1}$ and $Z^{(i)}$ be the matrix obtained from $Z$ where the $i$-th row and $i$-th column have been removed. Setting $R^{(i)} = (R^{(i)}_{jk})_{j,k} = (Z^{(i)} - zI_{n-1})^{-1}$, we have by Lemma S3

$$R_{ii} = -\left(z - Z_{ii} + \left\langle X^{(i)} \,|\, R^{(i)} X^{(i)} \right\rangle\right)^{-1}.$$

For three complex numbers $z, w, w' \in \mathbb{C}^+$ with positive imaginary parts, we have

$$\left| \frac{1}{z+w} - \frac{1}{z+w'} \right| = \frac{|w'-w|}{|z+w| \cdot |z+w'|} \le \frac{|w-w'|}{(\mathrm{Im}\, z)^2}.$$

By Lemma S2, we obtain $-Z_{ii} + \left\langle X^{(i)} \,|\, R^{(i)} X^{(i)} \right\rangle \in \mathbb{C}^+$ and $R^{(i)}_{jj} \in \mathbb{C}^+$, and hence

$$\begin{aligned}
LHS &:= \left| R_{ii} + \left( z + \frac{1}{n} \sum_{j \neq i} \rho(\tfrac{i}{n}, \tfrac{j}{n})^2 R^{(i)}_{jj} \right)^{-1} \right| \\
&= \left| -\left( z - Z_{ii} + \left\langle X \,|\, R^{(i)} X \right\rangle \right)^{-1} + \left( z + \frac{1}{n} \sum_{j \neq i} \rho(\tfrac{i}{n}, \tfrac{j}{n})^2 R^{(i)}_{jj} \right)^{-1} \right| \\
&\leq \frac{1}{(\operatorname{Im} z)^2} \left| Z_{ii} - \left\langle X^{(i)} \,|\, R^{(i)} X^{(i)} \right\rangle + \frac{1}{n} \sum_{j \neq i} \rho(\tfrac{i}{n}, \tfrac{j}{n})^2 R^{(i)}_{jj} \right| \\
&\leq \frac{1}{(\operatorname{Im} z)^2} \left( |Z_{ii}| + \left| \left\langle X^{(i)} \,|\, R^{(i)} X^{(i)} \right\rangle - \frac{1}{n} \sum_{j \neq i} \rho(\tfrac{i}{n}, \tfrac{j}{n})^2 R^{(i)}_{jj} \right| \right) \\
&\overset{(1)}{=} \frac{1}{(\operatorname{Im} z)^2} \left( |Z_{ii}| + \left| \left\langle X^{(i)} \,|\, R^{(i)} X^{(i)} \right\rangle - \sum_{j \neq i} (\mathbb{E}|Z_{ij}|^2) R^{(i)}_{jj} \right| \right) \\
&\overset{(2)}{=} \frac{1}{(\operatorname{Im} z)^2} \left( |Z_{ii}| + \left| \left\langle X^{(i)} \,|\, R^{(i)} X^{(i)} \right\rangle - \mathbb{E}_i \left\langle X^{(i)} \,|\, R^{(i)} X^{(i)} \right\rangle \right| \right).
\end{aligned}$$

Here, $\mathbb{E}_i = \mathbb{E}(\,\cdot\,| R^{(i)})$ is the conditional expectation with respect to $R^{(i)}$ . We use $\mathbb{E}|Z_{ij}|^2 = \frac{1}{n}\rho(\frac{i}{n}, \frac{j}{n})^2$ in the equality (1), and in the equality (2) we use (S.3) below.

The objective, stated by Bordenave (2019) in the last two lines of p.42, is to show that, for fixed $z$ and $i$,

$$\mathbb{E}(LHS)^2 \to 0 \quad \text{when } n \to \infty.$$

By the last inequality, it is sufficient to show that

$$\mathbb{E}Z_{ii}^2 \to 0 \quad \text{and } \mathbb{E} \left| \left\langle X^{(i)} \,|\, R^{(i)} X^{(i)} \right\rangle - \mathbb{E}_i \left\langle X^{(i)} \,|\, R^{(i)} X^{(i)} \right\rangle \right|^2 \to 0 \qquad \text{when } n \to \infty.$$

The convergence $\mathbb{E}Z_{ii}^2 \to 0$ follows from $\mathbb{E}Z_{ii}^2 \leq \frac{1}{n}$.

Let $\mathrm{Var}_i$ be the variance with respect to $R^{(i)}$. We note that

$$\begin{aligned}
&\mathbb{E} \left| \left\langle X^{(i)} \,|\, R^{(i)} X^{(i)} \right\rangle - \mathbb{E}_i \left\langle X^{(i)} \,|\, R^{(i)} X^{(i)} \right\rangle \right|^2 \\
&\quad = \mathbb{E}(\mathbb{E}_i \left| \left\langle X^{(i)} \,|\, R^{(i)} X^{(i)} \right\rangle - \mathbb{E}_i \left\langle X^{(i)} \,|\, R^{(i)} X^{(i)} \right\rangle \right|^2) \\
&\quad = \mathbb{E}(\mathrm{Var}_i \left\langle X^{(i)} \,|\, R^{(i)} X^{(i)} \right\rangle).
\end{aligned}$$

We will apply (the proof of) the concentration inequality in Bordenave (2019, Lemma 3.6) in order to estimate $\mathrm{Var}_i \left\langle X^{(i)} \,|\, R^{(i)} X^{(i)} \right\rangle$ and next the $\mathbb{E}$ of it.

Let us consider $\mathrm{Var}_i \left\langle X^{(i)} \,|\, R^{(i)} X^{(i)} \right\rangle$. We have

$$\left\langle X^{(i)} \,|\, R^{(i)} X^{(i)} \right\rangle = \sum_{j,k} R^{(i)}_{jk} X_j X_k.$$

Here, the sum taken over all $j, k$ different from $i$, and we use this notation in the sequel. By definition, the vector $X^{(i)}$ is independent of $R^{(i)}$ because there is no variables of $X^{(i)}$ in $R^{(i)}$. Then,

$$\mathbb{E}_i \left\langle X^{(i)} \,|\, R^{(i)} X^{(i)} \right\rangle = \mathbb{E}_i \sum_{j,k} R^{(i)}_{jk} X_j X_k = \sum_{j} R^{(i)}_{jj} \mathbb{E}_i X_j^2 = \sum_{j} (\mathbb{E}Z_{ij}^2) R^{(i)}_{jj}. \tag{S.3}$$

Similarly as in the proof of Bordenave (2019, Lemma 3.6), we have

$$\begin{aligned}
&\mathrm{Var}_i \left\langle X^{(i)} \,|\, R^{(i)} X^{(i)} \right\rangle \\
&= \mathbb{E}_i \left( \sum_{j_1, j_2, k_1, k_2} R^{(i)}_{j_1 k_1} \overline{R^{(i)}_{j_2 k_2}} X_{j_1} X_{k_1} X_{j_2} X_{k_2} \right) - \left| \mathbb{E}_i \sum_{j,k} R^{(i)}_{jk} X_j X_k \right|^2 \\
&= \sum_{j_1, j_2, k_1, k_2} R^{(i)}_{j_1 k_1} \overline{R^{(i)}_{j_2 k_2}} \mathbb{E}\Big( X_{j_1} X_{k_1} X_{j_2} X_{k_2} \Big) - \sum_{j,k} R^{(i)}_{jj} \overline{R^{(i)}_{kk}} (\mathbb{E}|X_j|^2)(\mathbb{E}|X_k|^2).
\end{aligned}$$

The first sum is non zero only if

$$\begin{aligned}
&\text{(i) } j_1 = j_2 = k_1 = k_2, \quad \text{(ii) } (j_1, k_1) = (j_2, k_2), \\
&\text{(iii) } (j_1, k_1) = (k_2, j_2), \quad \text{(iv) } (j_1, j_2) = (k_1, k_2)
\end{aligned}$$

so that, noting that by independence of $R^{(i)}$ and $X^{(i)}$ we have $\mathbb{E}_i(X_j^4) = \mathbb{E}(X_j^4)$, $\mathrm{Var}_i X_j^2 = \mathrm{Var} X_j^2$ etc.

$$\begin{aligned}
\mathrm{Var}_i \left\langle X^{(i)} \,|\, R^{(i)} X^{(i)} \right\rangle &= \overset{\text{(i)}}{\sum_j} \left| R^{(i)}_{jj} \right|^2 \mathbb{E}(X_j^4) + \overset{\text{(ii)}}{\sum_{j_1 \neq k_1}} \left| R^{(i)}_{j_1 k_1} \right|^2 \mathbb{E}(X_{j_1}^2 X_{k_1}^2) \\
&\quad + \overset{\text{(iii)}}{\sum_{j_1 \neq k_1}} R^{(i)}_{j_1 k_1} \overline{R^{(i)}_{k_1 j_1}} \mathbb{E}(X_{j_1}^2 X_{k_1}^2) + \overset{\text{(iv)}}{\sum_{j_1 \neq j_2}} R^{(i)}_{j_1 j_1} \overline{R^{(i)}_{j_2 j_2}} \mathbb{E}(X_{j_1}^2 X_{j_2}^2) \\
&\quad - \sum_j \left| R^{(i)}_{jj} \right|^2 \left( \mathbb{E} X_j^2 \right)^2 - \sum_{j \neq k} R^{(i)}_{jj} \overline{R^{(i)}_{kk}} (\mathbb{E} X_j^2)(\mathbb{E} X_k^2) \\
&= \sum_j \left| R^{(i)}_{jj} \right|^2 \left( \mathbb{E}(X_j^4) - \left( \mathbb{E} X_j^2 \right)^2 \right) + 2 \sum_{j \neq k} \left| R^{(i)}_{jk} \right|^2 (\mathbb{E} X_j^2)(\mathbb{E} X_k^2) \\
&= \sum_j \left| R^{(i)}_{jj} \right|^2 \mathrm{Var}(X_j^2) + 2 \sum_{j \neq k} \left| R^{(i)}_{jk} \right|^2 (\mathbb{E} X_j^2)(\mathbb{E} X_k^2).
\end{aligned}$$

(In the first line, the numbers (i)–(iv) on the summation mean the correspondence to the case of $j_1, j_2, k_1, k_2$.) Recall that $X_j = Z_{ji}$. Note that $\max_{j,k} \rho_{jk} \le 1$. Then,

$$\mathbb{E} X_j^2 = \mathbb{E} |Z_{ji}|^2 = \frac{1}{n} \rho(\tfrac{i}{n}, \tfrac{j}{n})^2 \le \frac{1}{n},$$

which implies, using the estimate of the Frobenius matrix norm and by Lemma S2 (i), $\|R^{(i)}\|_F^2 \le (n-1) \|R^{(i)}\|^2 \le \frac{n-1}{(\operatorname{Im} z)^2}$

$$2 \sum_{j \neq k} \left| R^{(i)}_{jk} \right|^2 (\mathbb{E} X_j^2)(\mathbb{E} X_k^2) \le \frac{2}{n^2} \sum_{j \neq k} \left| R^{(i)}_{jk} \right|_F^2 = \frac{2}{n^2} \left\| R^{(i)} \right\|^2 \le \frac{2}{(\operatorname{Im} z)^2} \cdot \frac{1}{n},$$

Here, for real symmetric matrices $H$ we set $\|H\|^2 = \mathrm{tr} H^2 = \sum_{jk} |H_{jk}|^2$.

Using $\displaystyle\sum_k |R_{kk}|^2 \le \|R\|_F^2 \le \frac{n}{(\operatorname{Im} z)^2}$ we get

$$\sum_j \left| R^{(i)}_{jj} \right|^2 \mathrm{Var}(X_j^2) \le \frac{n}{(\operatorname{Im} z)^2} \max_j \mathrm{Var}(X_j^2).$$

In the last estimates the dependence on $R^{(i)}$ vanishes, so they provide desired upper bounds for $\mathbb{E}(\mathrm{Var}_i \left\langle X^{(i)} \,|\, R^{(i)} X^{(i)} \right\rangle)$. We have

$$\mathrm{Var}(X_j^2) = \mathbb{E}(X_j^4) - \left( \mathbb{E} X_j^2 \right)^2 \le \mathbb{E}(X_j^4) = \frac{\rho(\frac{i}{n}, \frac{j}{n})^4}{n^2 \mathrm{Var}(Y_{ij})^2} \mathbb{E}(Y_{ij}^4) \le \frac{1}{n^2} \cdot \frac{\mathbb{E}(Y_{ij}^4)}{(\mathbb{E} Y_{ij}^2)^2}$$

$$\sum_j \left| R^{(i)}_{jj} \right|^2 \mathrm{Var}(X_j^2) \le \frac{n}{(\operatorname{Im} z)^2} \max_j \mathrm{Var}(X_j^2) \le \frac{1}{n (\operatorname{Im} z)^2} \max_j \frac{\mathbb{E}(Y_{ij}^4)}{(\mathbb{E} Y_{ij}^2)^2}.$$

We see that the weakest sufficient condition on the 4th moments is:

$$\max_{i,j} \frac{\mathbb{E}(Y_{ij}^4)}{n (\mathbb{E} Y_{ij}^2)^2} = o(1), \quad \text{equivalently: } \max_{i,j} \frac{\mathbb{E}(Y_{ij}^4)}{(\mathbb{E} Y_{ij}^2)^2} = o(n).$$

## 2.5. **Properties of the Stieltjes transform.**

**Lemma S4.** *1. Assume that $f(x)$ has a pole at $x = x_0$, and is continuous elsewhere. Then $\lim_{y\to 0+} \operatorname{Im} s(x_0+iy) = \infty$.*
*2. Let $\mu$ be a finite positive measure on $\mathbb{R}$ with Stieltjes transform $s(z)$. Suppose that $\mu$ has no atoms different from 0. If $\lim_{y\to 0+} \operatorname{Im} s(x+iy) = 0$ for all $x \neq 0$ uniformly on compact intervals of $\mathbb{R}^*$, then $\mu = c\delta_0$ for a $c > 0$ or $\mu = 0$.*
*3. Let $\mu$ be a finite positive measure on $\mathbb{R}$ with Stieltjes transform $s(z)$. Suppose that $F$ is a finite subset of $\mathbb{R}$ and that $\mu$ has no atoms different from elements of $F$. If $\lim_{y\to 0+} \operatorname{Im} s(x+iy) = 0$ for all $x \notin F$, uniformly on compact intervals of $\mathbb{R} \setminus F$, then $\mu = \sum_{a\in F} c_a\delta_a$ for some $c_a \geq 0, a \in F$ (this includes the case $\mu = 0$).*

*Proof.* Proof of 1. Assume that $f(x)$ has a pole at $x = x_0$, and is continuous elsewhere. We consider

$$\int_{\mathbb{R}} \frac{f(x_0+yu)}{1+u^2}du$$

($f(x)$ has a pole at $x = x_0$: for any $L > 0$ there exists $\varepsilon > 0$ such that if $0 < |y - x_0| < \delta$ then $f(y) > L$.) Take large $L > 0$ and the corresponding $\varepsilon > 0$. Set $y = \varepsilon > 0$. Then, since the integrand is non-negative,

$$\int_{\mathbb{R}} \frac{f(x_0+\varepsilon u)}{1+u^2}du \geq \int_{-1}^{1} \frac{f(x_0+\varepsilon u)}{1+u^2}du \geq \frac{1}{2}\int_{-1}^{1} f(x_0+\varepsilon u)du = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} f(x_0+v)dv.$$

In the second inequality, we use the fact $\frac{1}{1+u^2} \geq \frac{1}{2}$ on $[-1, 1]$. In the last equality, we change variable $v = \varepsilon u$. Then, since $|(x_0 + v) - x_0| < \varepsilon$ for $-\varepsilon < v < \varepsilon$, we have $f(x_0 + v) > L$ in the same interval so that

$$\int_{\mathbb{R}} \frac{f(x_0+\varepsilon u)}{1+u^2}du \geq \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} f(x_0+v)dv \geq \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} Ldv = L.$$

Since we can take $L$ arbitrary large enough, we conclude that the integral diverges.
Proof of 2. and 3. Let $[a, b]$ be a segment included in $\mathbb{R} \setminus F$. By the assumption, $\mu(\{a\}) = \mu(\{b\}) = 0$. By Theorem 2.4.3 in Anderson et.al. (2010) and by dominated convergence, we have

$$\mu([a,b]) = \frac{1}{\pi}\lim_{y\to 0+}\int_a^b s(x+iy)dx = \frac{1}{\pi}\int_a^b \lim_{y\to 0+} s(x+iy)dx = 0,$$

so that $\mu(\mathbb{R} \setminus F) = 0$. If $\mu \neq 0$ then $\mu$ is purely atomic with atoms in $F$. □

**Lemma S5.** *If $S(z)$ is odd, then $\operatorname{Im} S(-x+yi) = \operatorname{Im} S(x+yi)$ and $S_{im}(x) := \lim_{y\to+0} \operatorname{Im} S(x+yi)$ is even.*

*Proof.* We know that $S(\bar{z}) = \overline{S(z)}$ so that

$$S_{im}(-x) = \lim_{y\to+0} \operatorname{Im} S(-x+yi) = -\lim_{y\to+0} \operatorname{Im} S(x-yi) = -\lim_{y\to+0} \operatorname{Im} S(\overline{x+yi}) = -\Big(-S_{im}(x)\Big) = S_{im}(x).$$

In the second equality, we use the assumption that $S(z)$ is odd. □

**Lemma S6.**
*Let $\mu$ be a probability measure and $S$ its Stieltjes transform. Then, for any $x \in \mathbb{R}$, one has $\mu(\{x\}) = \lim_{y\to+0} y\operatorname{Im} S(x+yi)$.*

## 3. Wigner Ensembles of Vinberg Matrices

In this section, we give explicitly the limiting eigenvalue distributions $\mu$ for the Wigner matrices $U_n \in \mathcal{U}_n$ defined by (1). Let $\chi_I$ denote the indicator function of a subset $I \subset \mathbb{R}$. For a real number $a$, its cubic root is denoted by $\sqrt[3]{a} \in \mathbb{R}$ and set $[\,a\,]_+ = \max(a, 0)$. We introduce two real numbers $\alpha_c$, $\beta_c$ depending on $c \in [0,1)$ by

$$\alpha_c = \frac{8+4c-13c^2-\sqrt{c(8-7c)^3}}{8(1-c)}, \quad \beta_c = \frac{8+4c-13c^2+\sqrt{c(8-7c)^3}}{8(1-c)}. \tag{8}$$

It is clear that $\alpha_0 = \beta_0 = 1$, $\alpha_c < \beta_c$ and $\beta_c > 0$ for all $c \in (0,1)$. We note that $\alpha_{1/2} = 0$, $\alpha_c < 0$ when $c > 1/2$, $\lim_{c\to 1-} \alpha_c = -\infty$, $\lim_{c\to 1-}(1-c)\alpha_c = -1/4$ and $\lim_{c\to 1-} \beta_c = 4$, so that we set $\beta_1 = 4$. It can be shown that $c \mapsto \alpha_c$ is strictly decreasing and $c \mapsto \beta_c$ is strictly increasing on $[0,1]$ (see Figure 1).

**Theorem 5.** *Let $U_n$ be a Wigner matrix on $\mathcal{U}_n$ defined by (1). Assume that $\lim_{n\to+\infty} a_n/n = c \in (0,1)$. Then, the limiting eigenvalue distribution $\mu$ of the rescaled matrices $U_n/\sqrt{n}$ exists and is given for $c \in (0,1)$ as*

$$\mu = f_c(t)\,dt + [1-2c]_+\delta_0$$

*with*

$$f_c(t) := \frac{\sqrt[3]{R_+(t/\sqrt{v};\,c)} - \sqrt[3]{R_-(t/\sqrt{v};\,c)}}{2\sqrt{3}\pi\, t}\,\chi_{[\alpha_c,\beta_c]}\left(\frac{t^2}{v}\right), \tag{9}$$

*where, for $x^2 \in [\alpha_c, \beta_c]$,*

$$\begin{aligned} R_\pm(x;\,c) := x^6 - 3(c+1)x^4 + \tfrac{3}{2}(5c^2-2c+2)x^2 + (2c-1)^3 \\ \pm 3c\sqrt{3-3c}\cdot x\sqrt{(x^2-\alpha_c)(\beta_c-x^2)}. \end{aligned}$$

*The support of $\mu$ is given as*

$$\operatorname{supp}\mu = \begin{cases} \left[\,-\sqrt{v\beta_c},\, -\sqrt{v\alpha_c}\,\right] \cup \{0\} \cup \left[\,\sqrt{v\alpha_c},\, \sqrt{v\beta_c}\,\right] & (\textit{if}\ \ c \in (0, \tfrac{1}{2})) \\ \left[\,-\sqrt{v\beta_c},\, \sqrt{v\beta_c}\,\right] & (\textit{if}\ \ c \in [\tfrac{1}{2}, 1)). \end{cases} \tag{10}$$

*If $c = 0$, then $\mu = \delta_0$. If $c = 1$, then $\mu$ is the semicircle law on $[-2\sqrt{v}, 2\sqrt{v}]$.*

**Remark 6.** The formula (9) is valid for the extreme cases $c = 0$ or $c = 1$. If $c = 0$ then there is no density and $\mu = \delta_0$. If $c = 1$, then it can be checked that $\sqrt[3]{R_+(x;\,1)} - \sqrt[3]{R_-(x;\,1)} = \sqrt{3}x\sqrt{4-x^2}$ so that, for $v = 1$ we get the semicircle law $\mu(dt) = (1/2\pi)\sqrt{4-t^2}\chi_{[-2,2]}(t)dt$ of Wigner (1955).

**Remark 7.** Note that the limiting measure $\mu$ does not depend on the diagonal variance $v'$. This phenomenon already holds for the classical Wigner ensemble. In terms of the variance profile method, it may be explained by the fact that the variance profile (11) does not depend on $v'$ because the difference $|v - v'|$ on the diagonal is absorbed by the perturbation terms $\delta_{ii}$.

**Remark 8.** An intuitive explanation of the fact that if $c < \frac{1}{2}$ then $\mu$ has an atom at 0 and $\mu((0, \sqrt{v\alpha_c})) = 0$ is that small eigenvalues are strongly attracted by the zero eigenvalue and asymptotically vanish. Note that if $c = 0$, the model is asymptotically diagonal. For the diagonal Wigner matrices, the empirical eigenvalue distribution converges to 0 by the Strong Law of Large Numbers.

3.1. **Properties of functions $c \mapsto \alpha_c, \beta_c$.** The limit $\lim_{c\to 1+} \beta_c$ is computed easily by the De l'Hospital rule. In order to prove that $\beta_c > 0$, we write $\beta_c = R(c) - S(c)$ with $R(c) = \sqrt{c(8-7c)^3}$ an $S(c) = 13c^2 - 4c - 8$ and we show that $R(c) > S(c)$ on $[0,1)$. The function $R(c) \geq 0$, whereas $S(c)$ changes the sign from negative to positive at $c_S = (2+6\sqrt{3})/13$, and grows on $[c_S, 1]$ from 0 to 1. On the interval $[c_S, 1]$ the function $R(c)$ is decreasing, so $R(c) \geq R(1) = 1$ and $R(c) - S(c) > 0$.
In order to show that $c \mapsto \alpha_c$ is strictly decreasing and $c \mapsto \beta_c$ is strictly increasing on $[0,1]$, we compute the derivatives of these functions. Set

$$S(c) = 8 + 4c - 13c^2, \quad T(c) = \sqrt{c(8-7c)^3}, \quad f_\varepsilon(c) := \frac{8+4c-13c^2+\varepsilon\sqrt{c(8-7c)^3}}{8(1-c)} \quad (\varepsilon = \pm).$$

Of course we have $\alpha_c = f_-(c)$ and $\beta_c = f_+(c)$. Then we have

$$S'(c) = 4 - 26c, \quad T'(c) = \frac{(8-7c)^3 + c\cdot 3(8-7c)^2\cdot(-7)}{2\sqrt{c(8-7c)^3}} = \frac{4-14c}{\sqrt{c}}\sqrt{8-7c},$$

so that

$$
\begin{aligned}
f'_\varepsilon(c) &= \frac{(S'+\varepsilon T')(1-c)-(S+\varepsilon T)\cdot(-1)}{8(1-c)^2} \\
&= \frac{\left(4-26c+\varepsilon\frac{4-14c}{\sqrt{c}}\sqrt{8-7c}\right)(1-c)+8+4c-13c^2+\varepsilon\sqrt{c}(8-7c)\sqrt{8-7c}}{8(1-c)^2} \\
&= \frac{(4-26c)(1-c)+8+4c-13c^2+\varepsilon\sqrt{\frac{8-7c}{c}}\Big((4-14c)(1-c)+c(8-7c)\Big)}{8(1-c)^2} \\
&= \frac{13c^2-26c+12+\varepsilon\sqrt{\frac{8-7c}{c}}\,(7c^2-10c+4)}{8(1-c)^2}.
\end{aligned}
$$

Put

$$A = 13c^2-26c+12, \quad B = 7c^2-10c+4.$$

Notice that $B>0$ because $B = 7(c-\frac{5}{7})^2+\frac{3}{7}$. What we want to show is that

$$8(1-c)^2\cdot f'_+(c) = A+\sqrt{\frac{8-7c}{c}}\,B \geq 0, \quad 8(1-c)^2\cdot f'_-(c) = A-\sqrt{\frac{8-7c}{c}}\,B \leq 0.$$

Let us consider

$$\left(\frac{A}{B}\right)^2-\frac{8-7c}{c} = \frac{cA^2-(8-7c)B^2}{cB^2}.$$

By using a calculator, we can factorize the numerator $cA^2-(8-7c)B^2$ so that we obtain the following inequality

$$\left(\frac{A}{B}\right)^2-\frac{8-7c}{c} = \frac{cA^2-(8-7c)B^2}{cB^2} = -128\frac{(1-c)^3(2c-1)^2}{cB^2} < 0.$$

Since $\frac{8-7c}{c}>0$ for $c\in(0,1)$, this shows the following inequality

$$-\sqrt{\frac{8-7c}{c}} \leq \frac{A}{B} \leq \sqrt{\frac{8-7c}{c}}$$

and since $B>0$ we obtain

$$-B\sqrt{\frac{8-7c}{c}} \leq A \leq B\sqrt{\frac{8-7c}{c}},$$

whence we obtain $f'_+(c)\geq 0$ and $f'_-(c)\leq 0$ for $c\in[0,1)$.

In the Figures 2–6 we present graphical comparison between simulations for $n=4000$ and the limiting densities, when $c=1/5, 2/5, 1/2, 3/5, 4/5$.

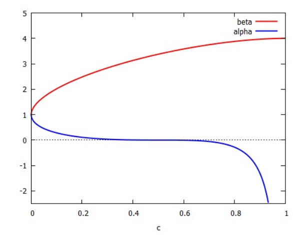



FIGURE 1. Graphs of $\alpha_c$ and $\beta_c$

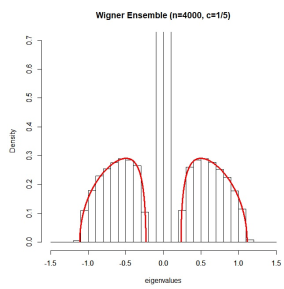



FIGURE 2. Simulation for $c=1/5$

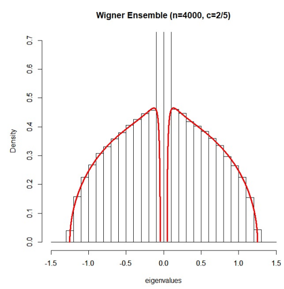



FIGURE 3. Simulation for $c=2/5$

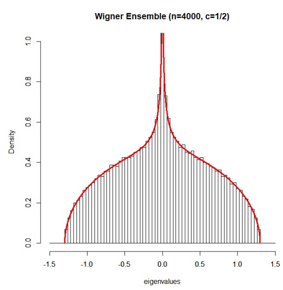



FIGURE 4. Simulation for $c=1/2$

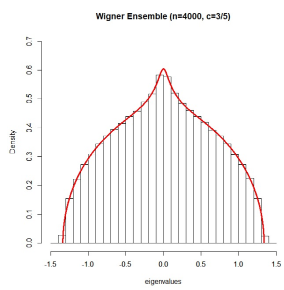



FIGURE 5. Simulation for $c=3/5$

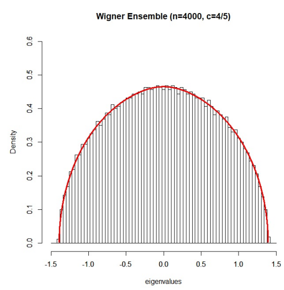



FIGURE 6. Simulation for $c=4/5$

3.2. **Proof of Theorem 5.** We first derive the Stieltjes transform of the limiting eigenvalue distribution by applying Theorem2.3 to $Y_n = U_n/\sqrt{n}$. Let $U_n = (U_{ij})_{1\le i,j\le n}$, so that $Y_{ij} = (1/\sqrt{n})U_{ij}$. The variance profile is given as

$$\sigma(x,y) = \begin{cases} v & \text{if } (x,y)\in\mathcal{C}, \\ 0 & \text{otherwise,} \end{cases} \qquad \mathcal{C} := \left\{(x,y)\in[0,1]^2;\ \min(x,y)\le c\right\}. \tag{11}$$

Note that

$$I_{ij} := \int_{Q_{ij}} \frac{\sigma(x,y)}{|Q_{ij}|}\,dx\,dy = v\frac{|C\cap Q_{ij}|}{|Q_{ij}|}.$$

The perturbation term equals $\delta_{ij}(n) = n\mathbb{E}Y_{ij}^2 - I_{ij} = \mathbb{E}U_{ij}^2 - I_{ij}$ and we have $\delta_{ij}(n) = 0$ unless $i = j$ or $i,j$ are such that $\emptyset \ne C\cap Q_{ij} \ne Q_{ij}$ . There are at most $3n$ perturbation terms $\delta_{ij}(n)\ne 0$, and they are all bounded by $M := \max\{|v-v'|, v', v\}$. It follows that the first condition $\lim_n \delta_0(n) = 0$ of the consition (2.6) is satisfied:

$$\delta_0(n) = \frac{1}{n^2}\sum_{i,j}\delta_{ij}(n) \le \frac{3Mn}{n^2}.$$

The second condition in (2.6) is evident since, by (11),

$$\max_{i,j}\frac{\mathbb{E}(Y_{ij}^4)}{n(\mathbb{E}Y_{ij}^2)^2} \le \frac{\max\{\kappa,\kappa'\}}{n\min\{v,v'\}} = o(1).$$

Assume that $\operatorname{Im} z > 0$. The functional equation (2.7) becomes

$$\eta_z(x) = -\left(z + v\int_0^1 \eta_z(y)\,dy\right)^{-1} \quad (x\le c), \qquad \eta_z(x) = -\left(z + v\int_0^c \eta_z(y)\,dy\right)^{-1} \quad (x > c).$$

Note that the right-hand sides are independent of $x$. We integrate both sides of these equations to obtain

$$\int_0^c \eta_z(x)\,dx = -c\left(z + v\int_0^1 \eta_z(y)\,dy\right)^{-1}, \qquad \int_c^1 \eta_z(x)\,dx = -(1-c)\left(z + v\int_0^c \eta_z(y)\,dy\right)^{-1},$$

so that by setting $A = \int_0^1 \eta_z(x)\,dx$ and $B = \int_0^c \eta_z(x)\,dx$, we obtain the following simultaneous equations

$$B = \frac{-c}{z+vA} \quad \text{(a)}, \qquad A - B = \frac{c-1}{z+vB} \quad \text{(b)} \tag{12}$$

Note that $A$ is the desired Stieltjes transform $S(z)$.

If $c = 0$, then we have $A = -1/z$ so that the limiting measure is $\mu = \delta_0$. If $c = 1$ then the equation (12) reduces to the equation $A = -(z+vA)^{-1}$, which corresponds to the Stieltjes transform of the semi-circular law (cf. Tao (2012, p.178)). Thus we assume $0 < c < 1$ in what follows.

Let us eliminate $B$ from these equations. Substituting (a) into (b), we obtain

$$\begin{aligned} & A - \frac{-c}{z+vA} = \frac{c-1}{z + v\frac{-c}{z+vA}} \\ \Leftrightarrow\quad & \frac{(z+vA)A + c}{z+vA} = \frac{(c-1)(z+vA)}{z(z+vA) - cv} \\ \Leftrightarrow\quad & \big((z+vA)A + c\big)\big(z(z+vA) - cv\big) = (c-1)(z+vA)^2, \end{aligned}$$

and hence

$$v^2 zA^3 + \big(2vz^2 + (1-2c)v^2\big)A^2 + \big(z^2 + 2v(1-c)\big)zA + z^2 - c^2 v = 0. \tag{13}$$

If we set

$$z_v := \frac{z}{\sqrt{v}}, \quad A_v := \sqrt{v}A,$$

then we have

$$v\Big(z_v A_v^3 + \big(2z_v^2 + (1-2c)\big)A_v^2 + \big(z_v^2 + 2(1-c)\big)z_v A_v + (z_v^2 - c^2)\Big) = 0,$$

or

$$\left(\frac{A_v}{z_v}\right)^3 + \left(2 + \frac{1-2c}{z_v^2}\right)\left(\frac{A_v}{z_v}\right)^2 + \left(1 + \frac{2(1-c)}{z_v^2}\right)\frac{A_v}{z_v} + \frac{z_v^2 - c^2}{z_v^4} = 0.$$

We now use the Cardano method. The last equation (13) is reduced to

$$Y^3 + p\,(z_v)\,Y + q\,(z_v) = 0, \tag{14}$$

where we set $z_v := z/\sqrt{v}$,

$$Y = Y(z) := \frac{vA}{z} + \frac{2}{3} - \frac{(2c-1)v}{3z^2}. \tag{15}$$

Here, the coefficients $p, q$ are given by the following analytical rational functions on $\mathbb{C}^* := \mathbb{C}\setminus\{0\}$

$$\begin{aligned} p(z_v) &= \left(1 + \frac{2(1-c)}{z_v^2}\right) - \frac{1}{3}\left(2 + \frac{1-2c}{z_v^2}\right)^2 = \frac{1}{3}\left(3 + \frac{6-6c}{z_v^2} - 4 - \frac{4-8c}{z_v^2} - \frac{(1-2c)^2}{z_v^4}\right) \\ &= -\frac{1}{3}\left(1 - \frac{2(c+1)}{z_v^2} + \frac{(2c-1)^2}{z_v^4}\right) \end{aligned}$$

and

$$
\begin{aligned}
q(z_v) &= \frac{z_v^2-c^2}{z_v^4} - \frac{1}{3}\left(1+\frac{2(1-c)}{z_v^2}\right)\left(2+\frac{1-2c}{z_v^2}\right) + \frac{2}{27}\left(2+\frac{1-2c}{z_v^2}\right)^3 \\
&= \left(\frac{1}{z_v^2}-\frac{c^2}{z_v^4}\right) - \frac{1}{3}\left(2+\frac{5-6c}{z_v^2}+\frac{2-6c+4c^2}{z_v^4}\right) + \frac{2}{27}\left(8+\frac{12-24c}{z_v^2}+\frac{6(1-4c+4c^2)}{z_v^4}+\frac{(1-2c)^3}{z_v^6}\right) \\
&= -\frac{2}{27} + \frac{6c+6}{27z_v^2} + \frac{-15c^2+6c-6}{27z_v^4} + \frac{2}{27}\frac{(1-2c)^3}{z_v^6} \\
&= -\frac{2}{27}\left(1-\frac{3c+3}{z_v^2}+\frac{3(5c^2-2c+2)}{2z_v^4}-\frac{(1-2c)^3}{z_v^6}\right).
\end{aligned}
$$

Define, for $z \neq 0$,

$$
F_c(z) := \frac{z^6 - 3(c+1)z^4 + \frac{3}{2}(5c^2-2c+2)z^2 + (2c-1)^3}{z^6}.
$$

Then, we have

$$
Y = \frac{vA}{z} + \frac{2}{3} - \frac{(2c-1)v}{3z^2}, \quad p(z) = -\frac{1}{3}\left(1-\frac{2(c+1)}{z^2}+\frac{(2c-1)^2}{z^4}\right), \quad q(z) = -\frac{2F_c(z)}{27}, \quad z_v = \frac{z}{\sqrt{v}}
$$

with

$$
Y^3 + p(z_v)\, Y + q(z_v) = 0.
$$

Cardano's method tells us that the solutions of the equation have the form $Y(z) = U_+(z_v) + U_-(z_v)$ where $U_\pm(z)$ satisfy

$$
U_\pm(z)^3 = -\frac{q(z)}{2} \pm \sqrt{\left(\frac{q(z)}{2}\right)^2 + \left(\frac{p(z)}{3}\right)^3}, \quad U_+(z)\cdot U_-(z) = -\frac{1}{3}p(z), \tag{S.4}
$$

and accordingly, $A$ is described as

$$
A = \frac{zY(z)}{v} - \frac{2z}{3v} + \frac{2c-1}{3z}. \tag{S.5}
$$

Let us calculate $\left(\frac{q(z)}{2}\right)^2 + \left(\frac{p(z)}{3}\right)^3$. By a simple but little bit cumbersome computation, we have

$$
\begin{aligned}
\left(\frac{q(z)}{2}\right)^2 = \frac{1}{27^2}\Bigg(&1 - \frac{6(c+1)}{z^2} + \frac{15c^2-6c+6+9(c+1)^2}{z^4} - \frac{2(1-2c)^3+9(c+1)(5c^2-2c+2)}{z^6} \\
&+ \frac{9(5c^2-2c+2)^2+24(c+1)(1-2c)^3}{4z^8} - \frac{3(5c^2-2c+2)(1-2c)^3}{z^{10}} + \frac{(1-2c)^6}{z^{12}}\Bigg)
\end{aligned}
$$

and

$$
\begin{aligned}
\left(\frac{p(z)}{3}\right)^3 = -\frac{1}{9^3}\Bigg(&1 - \frac{6(c+1)}{z^2} + \frac{3(2c-1)^2+12(c+1)^2}{z^4} - \frac{12(c+1)(2c-1)^2+8(c+1)^3}{z^6} \\
&+ \frac{3(2c-1)^4+12(c+1)^2(2c-1)^2}{z^8} - \frac{6(c+1)(2c-1)^4}{z^{10}} + \frac{(2c-1)^6}{z^{12}}\Bigg).
\end{aligned}
$$

Put $\frac{1}{27^2}$ in factor. The coefficients of $1/z^k$ ($k = 0, 2, 12$) are zero. Since the coefficients of $1/z^k$ ($k = 4, 6, 8, 10$) are

$$
\begin{aligned}
&\frac{1}{z^4}:\ (15c^2-6c+6+9(c+1)^2) - (3(2c-1)^2+12(c+1)^2) && = 0, \\
&\frac{1}{z^6}:\ -(2(1-2c)^3+9(c+1)(5c^2-2c+2)) + 12(c+1)(2c-1)^2 + 8(c+1)^3 && = 27c^2(c-1), \\
&\frac{1}{z^8}:\ \big(9(5c^2-2c+2)^2+24(c+1)(1-2c)^3\big)/4 - \big(3(2c-1)^4+12(c+1)^2(2c-1)^2\big) && = -27c^2(13c^2-4c-8)/4 \\
&\frac{1}{z^{10}}:\ -3(5c^2-2c+2)(1-2c)^3 + 6(c+1)(2c-1)^4 && = 27c^2(2c-1)^3,
\end{aligned}
$$

so that

$$
\begin{aligned}
\left(\frac{q(z)}{2}\right)^2 + \left(\frac{p(z)}{3}\right)^3 &= \frac{c^2}{27z^6}\left(c-1-\frac{(13c^2-4c-8)}{4z^2}+\frac{(2c-1)^3}{z^4}\right) \\
&= -\frac{c^2(1-c)}{27z^{10}}\left(z^4 + \frac{13c^2-4c-8}{4(1-c)}z^2 - \frac{(2c-1)^3}{1-c}\right).
\end{aligned}
$$

The last formula implies that

$$
\alpha_c\beta_c = -\frac{(2c-1)^3}{1-c}. \tag{S.6}
$$

Here, since

$$\begin{aligned}\left(\frac{13c^2-4c-8}{4(1-c)}\right)^2-4\left(-\frac{(2c-1)^3}{1-c}\right) &= \frac{(13c^2-4c-8)^2+4^3(1-c)(2c-1)^3}{(4(1-c))^2}\\ &= \frac{(169c^4-104c^3-192c^2+64c+64)+64(-8c^4+20c^3-18c^2+7c-1)}{(4(1-c))^2}\\ &= \frac{-343c^4+1176c^3-1344c^2+512c}{(4(1-c))^2} = \frac{c(-7^3c^3+3\cdot 7^2\cdot 8c^2-3\cdot 7\cdot 8^2c+8^3)}{(4(1-c))^2}\\ &= \frac{c(8-7c)^3}{(4(1-c))^2},\end{aligned}$$

we have

$$\left(\frac{q(z)}{2}\right)^2+\left(\frac{p(z)}{3}\right)^3=-\frac{c^2(1-c)}{27z^{10}}(z^2-\alpha_-)(z^2-\alpha_+)=:-\frac{D_c(z)^2}{27},$$

where

$$\alpha_\pm=\frac{1}{2}\left(-\frac{13c^2-4c-8}{4(1-c)}\pm\sqrt{\frac{c(8-7c)^3}{(4(1-c))^2}}\right)=\frac{8+4c-13c^2\pm\sqrt{c(8-7c)^3}}{8(1-c)}\qquad(=\alpha_c \text{ or } \beta_c).$$

Hence we have

$$U_\pm(z)^3=\frac{1}{27}\left(F_c(z)\pm i\,D_c(z)\right).$$

Since $A$ is the Stieltjes transform $S(z)$ of a probability measure, by (S.5) we have, with $u_\pm(z)=3U_\pm(z)$,

$$S(z)=\frac{z(u_+(z)+u_-(z))}{3v}-\frac{2z}{3v}+\frac{2c-1}{3z};\quad u_\pm(z):=\left(F_c(z_v)\pm i\,D_c(z_v)\right)^{\frac{1}{3}}, \tag{16}$$

where convenient branches of the cube root are chosen for $u_\pm(z)$ to be such that $S(z)$ is holomorphic on $\mathbb{C}^+$ and

$$u_+(z)\cdot u_-(z)=-3p(z),\quad \text{and}\quad \operatorname{Im}S(z)>0\quad (z\in\mathbb{C}^+). \tag{17}$$

are satisfied on $\mathbb{C}^+$. Let $\mathcal{E}=\{z\in\mathbb{C};\ z=0 \text{ or } \operatorname{Disc}(z)=0\}$ be the set of exceptional points.

**Lemma S7.** *One has $\mathcal{E}=\{0,\pm\sqrt{\alpha_c},\pm\sqrt{\beta_c}\}$. More precisely,*

$$\mathcal{E}=\begin{cases}\{0,\pm\sqrt{\alpha_c},\pm\sqrt{\beta_c}\} & (0<c<\frac{1}{2}),\\ \{0,\pm\sqrt{\beta_c}\} & (c=\frac{1}{2}),\\ \{0,\pm i\sqrt{|\alpha_c|},\pm\sqrt{\beta_c}\} & (\frac{1}{2}<c<1).\end{cases}$$

Set $J:=\{x\in\mathbb{R};\ x\notin\mathcal{E}\}$ and

$$D:=\begin{cases}\mathbb{C}^+\cup\{x+iy;\ x\notin\mathcal{E},\ -1<y\le 0\} & (0<c\le\frac{1}{2}),\\ \mathbb{C}^+\cup\{x+iy;\ x\notin\mathcal{E},\ -1<y\le 0\}\setminus(i\sqrt{|\alpha_c|}+i\mathbb{R}_{\ge 0}) & (\frac{1}{2}<c<1).\end{cases}$$

Then, $D$ is a connected and simply connected domain containing no exceptional points of (14), and $J\subset D$.

**Lemma S8.** *[Palka (1991, Theorem X.3.7)] Let $z_0\in D$ and $X_0\in\mathbb{C}$ a solution of (14) at $z_0$. Then there exists a function $s(z)$ holomorphic on $D$ such that $s(z)$ is a solution of (14) on $D$ and $s(z_0)=X_0$. Such function $s$ is unique.*

*Proof.* This is because $D$ is a connected and simply connected domain containing no exceptional points $\mathcal{E}$ of (14), and hence we can use Palka (1991, Theorem X.3.7). □

**Proposition 9.** *For each $x\in\mathbb{R}^*$, there exists the limit $S(x)=\lim_{y\to+0}S(x+yi)$. The function $S$ is continuous on $\mathbb{R}^*$ and $S(x)$ is a solution of (13) on $\mathbb{R}^*$.*

*Proof.* It is sufficient to prove it for a solution $U(z)$ of the reduced equation (14) on $\mathbb{C}^+$, such that $U(z)$ is holomorphic on $\mathbb{C}^+$. We apply (Palka, 1991, Theorem X.3.7) to a convenient connected and simply connected domain $D$ avoiding the set $\mathcal{E}$. By the discussion of (Ahlfors, 1979, p.304), $U$ has at most an ordinary algebraic singularity at a non-zero exceptional point, so $U(z)$ is continuous on $\mathbb{R}^*$. □

Note that the branches of the cube root in $u_\pm(z)$ may be different on different subregions of $\mathbb{C}^+$. This is because the functions $u_\pm(z)^3$ in the cubic roots may pass through the slit $\mathbb{R}^-$ so that the cubic root functions need to change branches in order that $S(z)$ is analytic. We also note that the definition of square root is not essential. In fact, in the above solution, two square roots $\pm D_c(z)$ of $D_c(z)^2$ appear symmetrically so that changing definition of square roots induces at most switching a role of $u_+(z)$ and $u_-(z)$.

Without loss of generality, we suppose $v = 1$. We first assume that $x = 0$. The detailed local analysis of (16) and (17) that is presented below, shows that

$$
\begin{array}{llll}
(Z1) & \text{if } 0 < c < \frac{1}{2}, & \lim\limits_{y\to+0} y\mathrm{Im}\, S(yi) = 1-2c, & \text{so } \mu \text{ has an atom at 0 with the mass } 1-2c<1,\\
(Z2) & \text{if } c = \frac{1}{2}, & \lim\limits_{y\to+0} \mathrm{Im}\, S(yi) = +\infty, & \lim\limits_{y\to+0} y\mathrm{Im}\, S(yi) = 0 \text{ so } \mu \text{ does not have an atom at 0},\\
(Z3) & \text{if } \frac{1}{2} < c < 1, & \lim\limits_{y\to+0} \mathrm{Im}\, S(yi) = c(2c-1)^{-1/2} = \pi f_c(0), & \text{so } \mu \text{ does not have an atom at 0.}
\end{array}
$$

Next we consider the case $x \neq 0$. Combining the fact that $S(z)$ is an odd function as a function on $\mathbb{C}\setminus\mathbb{R}$ by (16) and the property $S(\overline{z}) = \overline{S(z)}$ of the Stieltjes transform, we obtain $\mathrm{Im}\, S(-x+iy) = \mathrm{Im}\, S(x+iy)$ so that $\mathrm{Im}\, S(-x) = \mathrm{Im}\, S(x)$ (cf. Lemma S5). Thus we can assume that $x > 0$.

Suppose $\mathrm{Disc}(x) \geq 0$. Since the coefficients $p, q$ of (14) are real on $\mathbb{R}^*$, the equation (14) has only real solutions (cf. Ronald (2004)). Therefore, $S(x)$ is real so that the density of $\mu$ vanishes at such points.

Next we assume that $\mathrm{Disc}(x) < 0$. By Proposition 9, $S(x)$ is a solution of the cubic equation (13) and $U(x) = (u_+(x) + u_-(x))/3$ is a solution of the reduced equation (14). In particular, the formulas (16) and (17) hold for $S(x)$, with convenient choices of branches of cubic roots and square roots. Consequently, we have

$$\{F_c(x) + iD_c(x), F_c(x) - iD_c(x)\} = \{R'_+(x),\, R'_-(x)\}$$

as a set, where $R'_\pm(x) := R_\pm(x;\, c)/x^6 \in \mathbb{R}$. Let $\omega = e^{2i\pi/3}$ denote the cube root of 1 with positive imaginary part. Then, (16) yields that the sum $u_+(x) + u_-(x)$ has the following form

$$u_+(x) + u_-(x) = \omega^{k_+}\sqrt[3]{R'_+(x)} + \omega^{k_-}\sqrt[3]{R'_-(x)} \quad \text{with} \quad k_+, k_- \in \{0, 1, 2\}.$$

By the first condition in (17), as $p(x) \in \mathbb{R}$, we need to have $k_+ + k_- \equiv 0 \bmod 3$, that is, $(k_+, k_-) = (0,0)$, $(1,2)$ and $(2,1)$. Using the fact that $R'_+(x) > R'_-(x)$ when $x > 0$ and $\mathrm{Disc}(x) < 0$, we see that the imaginary part of $u_+(x) + u_-(x)$ and of $\lim_{y\to 0+} S(x+iy)$ is, respectively, nul, positive and negative in these three cases. Since $\mathrm{Im}\, S(z) > 0$, the last case is impossible. Set $h(x) := \mathrm{Im}\big(\omega\sqrt[3]{R'_+(x)} + \omega^2\sqrt[3]{R'_-(x)}\big)$. Notice that $h$ is a strictly positive continuous function on the set $\{x \in \mathbb{R};\ \mathrm{Disc}(x) < 0\}$ and that $\frac{1}{\pi}h(t) = f_c(t)$, the density part of $\mu$ in the formula (9). Since the function $\mathrm{Im}\, S$ is continuous on $\mathbb{R}^*$ by Proposition 9, we have $\mathrm{Im}\, S \equiv h$ or $\mathrm{Im}\, S \equiv 0$ on the set $\{x \in \mathbb{R}^*;\ \mathrm{Disc}(x) < 0\}$.

We now show that the latter case $\mathrm{Im}\, S \equiv 0$ is impossible. Note that $\mu$ has no atoms different from zero because $S(z)$ is continuous on $\overline{\mathbb{C}^+}\setminus\{0\}$. By Anderson et.al. (2010, Theorem 2.4.3) and by the dominated convergence, we have for closed intervals $[a,b] \subset \mathbb{R}^*$

$$\mu([a,b]) = \frac{1}{\pi}\lim_{y\to 0+}\int_a^b S(x+iy)\,dx = \frac{1}{\pi}\int_a^b \lim_{y\to 0+} S(x+iy)\,dx = 0, \tag{18}$$

so that $\mu(0,\infty) = 0$ and, symmetrically, $\mu(-\infty, 0) = 0$. Since $\mu$ is a probability measure, we get $\mu = \delta_0$. This contradicts properties (Z1-3) proven in the case $x = 0$. Thus, we have $\mathrm{Im}\, S \equiv h$ on the set $\{x \in \mathbb{R}^*;\ \mathrm{Disc}(x) \leq 0\}$ and, for $x \in \mathbb{R}^*$, $\lim\limits_{y\to 0+}\frac{1}{\pi}\mathrm{Im}\, S(x+iy) = \frac{1}{\pi}h(x) = f_c(x)$. Note that $f_c$ has a compact support $\{\mathrm{Disc}(x) \leq 0\}$. For $c \neq \frac{1}{2}$, the function $f_c$ is continuous on $\mathbb{R}$. For $c = \frac{1}{2}$, a detailed analysis shows that $\lim_{x\to 0} f_c(0) = \infty$, with $f_c(x) \sim |x|^{-1/2}$ at $x = 0$ and $f_c$ is continuous on $\mathbb{R}^*$. By property (Z3), if $c > \frac{1}{2}$ then $\lim_{y\to 0+}\mathrm{Im}\, S(iy) = \pi f_c(0)$. When $c \neq 1/2$, Proposition 2.1 implies that $\mu = f_c(t)\,dt + [1-2c]_+\delta_0$. Actually, if $s(z)$ is the Stieltjes transform of $\mu - f_c(t)\,dt - [1-2c]_+\delta_0$, then, using Proposition 2.2, we get $\lim_{y\to 0+}\mathrm{Im}\, s(x+iy) = 0$ for all $x \in \mathbb{R}$. When $c = 1/2$, by Proposition 2.2, we get $\lim_{y\to 0+}\mathrm{Im}\, s(x+iy) = 0$ for all $x \in \mathbb{R}^*$, uniformly on compact intervals $[a,b] \subset \mathbb{R}^*$. Like in (18), we conclude by Theorem 2.4.3 in Anderson et.al. (2010) that $\mu = f_c(t)\,dt$. The support formula (10) follows by $\mathrm{supp}\, f_c = \{\mathrm{Disc}(x) \leq 0\}$. □

**Detailed analysis of the case $x = 0$.**

**(Z1) the case $0 < c < \frac{1}{2}$.** In this case, $\alpha_c, \beta_c \geq 0$. Note that by (S.6), $\alpha_c\beta_c = \dfrac{(1-2c)^3}{1-c}$. Then, we have

$$
\begin{aligned}
D_c(z) &= \frac{3c\sqrt{3-3c}}{z^5}\sqrt{z^2-\alpha_c}\,\sqrt{z^2-\beta_c} = \frac{3c\sqrt{3-3c}}{z^5}\cdot\sqrt{-\alpha_c}\sqrt{-\beta_c}\sqrt{1-\frac{z^2}{\alpha_c}}\sqrt{1-\frac{z^2}{\beta_c}}\\
&= -\frac{3c\sqrt{3-3c}}{z^5}\cdot\frac{(1-2c)^{\frac{3}{2}}}{\sqrt{1-c}}\sqrt{1-\frac{z^2}{\alpha_c}}\sqrt{1-\frac{z^2}{\beta_c}} = -\frac{3\sqrt{3}\,c(1-2c)^{\frac{3}{2}}}{z^5}\sqrt{1-\frac{z^2}{\alpha_c}}\sqrt{1-\frac{z^2}{\beta_c}},
\end{aligned}
$$

and hence around $z = 0$

$$z^6 D_c(z) = -3\sqrt{3}\,c\,(1-2c)^{\frac{3}{2}}(z + o(z)).$$

On the other hand,

$$
\begin{aligned}
z^6 F_c(z) &= (2c-1)^3 + \frac{3}{2}(5c^2-2c+2)z^2 - 3(c+1)z^4 + z^6\\
&= (2c-1)^3\left(1 + \frac{3(5c^2-2c+2)}{2(2c-1)^3}z^2 - \frac{3(c+1)}{(2c-1)^3}z^4 + \frac{z^6}{(2c-1)^3}\right)
\end{aligned}
$$

and hence, around $z=0$

$$z^6 F_c(z) = (2c-1)^3(1+o(z)). \tag{S.7}$$

Combining those, we obtain

$$\begin{aligned}(F_c(z)+\varepsilon i D_c(z))^{\frac{1}{3}} &= \left(\frac{(2c-1)^3 - \varepsilon i\cdot 3\sqrt{3}\,c\,(1-2c)^{\frac{3}{2}}z + o(z)}{z^6}\right)^{\frac{1}{3}} \\ &= \frac{2c-1}{z^2}\left(1+\varepsilon i\cdot\frac{3\sqrt{3}c}{(1-2c)^{\frac{3}{2}}}z+o(z)\right)^{\frac{1}{3}} \\ &= \frac{2c-1}{z^2}\omega^{k(\varepsilon)}\left(1+\varepsilon i\cdot\frac{\sqrt{3}c}{(1-2c)^{\frac{3}{2}}}z+o(z)\right)\end{aligned}$$

around $z=0$. Here, $\varepsilon=\pm1$ and $k(\varepsilon)\in\{0,1,2\}$. Let us consider the first condition in (17). Recall that

$$-3p(z) = \frac{z^4-2(c+1)z^2+(2c-1)^2}{z^4} = \frac{(2c-1)^2}{z^4}(1+o(z)).$$

Therefore, since

$$(F_c(z)+iD_c(z))^{\frac{1}{3}}\cdot(F_c(z)-iD_c(z))^{\frac{1}{3}} = \frac{(2c-1)^2}{z^4}\omega^{k(+)+k(-)}\left(1+o(z)\right),$$

$k(+)+k(-)\equiv 0 \bmod 3$. Next, let us consider the latter condition in (17). By (16), we have (recall that $v=1$)

$$\begin{aligned}S(z) &= \frac{z}{3}\left((F_c(z)+iD_c(z))^{\frac{1}{3}}+(F_c(z)-iD_c(z))^{\frac{1}{3}}\right)-\frac{2z}{3}+\frac{2c-1}{3z} \\ &= \frac{2c-1}{3z}\left(\omega^{k(+)}\left(1+i\cdot\frac{\sqrt{3}c}{(1-2c)^{\frac{3}{2}}}z\right)+\omega^{k(-)}\left(1-i\cdot\frac{\sqrt{3}c}{(1-2c)^{\frac{3}{2}}}z\right)+o(z)\right)-\frac{2z}{3}+\frac{2c-1}{3z} \\ &= \frac{2c-1}{3z}(\omega^{k(+)}+\omega^{k(-)}+1)+\frac{2c-1}{3}\cdot i\frac{\sqrt{3}c}{(1-2c)^{\frac{3}{2}}}(\omega^{k(+)}-\omega^{k(-)})-\frac{2z}{3}+o(1).\end{aligned}$$

Here, since $k(+)+k(-)\equiv 0 \bmod 3$, we have $\operatorname{Im} i(\omega^{k(+)}-\omega^{k(-)})=0$ for any choice. Now we assume that $x=0$, we can set $z=yi$ and then

$$\operatorname{Im} S(yi) = i\left(\frac{1-2c}{3y}(\omega^{k(+)}+\omega^{K(-)}+1)-\frac{2}{3}y\right).$$

If $(k(+),k(-))=(1,2)$ or $(2,1)$, then $\omega^{k(+)}+\omega^{k(-)}+1=0$ so that $\operatorname{Im} S(z)=-\frac{2}{3}y<0$, which is not suitable. Therefore $(k(+),k(-))=(0,0)$ and

$$\lim_{y\to+0} y\operatorname{Im} S(yi) = (1-2c)\lim_{y\to+0} y\cdot\frac{1}{y} = 1-2c,$$

and hence $\mu$ has an atomic component $(1-2c)\delta_0$ by Lemma S6.

**(Z2) the case $\frac{1}{2}<c<1$.** In this case, we have $\alpha_c<0$ and $\beta_c>0$. Note that $-\alpha_c\beta_c=\dfrac{(2c-1)^3}{1-c}$. Then we have

$$\begin{aligned}D_c(z) &= \frac{3c\sqrt{3-3c}}{z^5}\cdot\sqrt{-\alpha_c}\sqrt{-\beta_c}\sqrt{1-\frac{z^2}{\alpha_c}}\sqrt{1-\frac{z^2}{\beta_c}} = i\cdot\frac{3c\sqrt{3-3c}}{z^5}\cdot\frac{(2c-1)^{\frac{3}{2}}}{\sqrt{1-c}}\sqrt{1-\frac{z^2}{\alpha_c}}\sqrt{1-\frac{z^2}{\beta_c}} \\ &= i\cdot\frac{3\sqrt{3}\,c(2c-1)^{\frac{3}{2}}}{z^5}\sqrt{1-\frac{z^2}{\alpha_c}}\sqrt{1-\frac{z^2}{\beta_c}},\end{aligned}$$

and hence around $z=0$

$$z^6 D_c(z) = i\cdot 3\sqrt{3}\,c\,(2c-1)^{\frac{3}{2}}(z+o(z)).$$

By (S.7), we obtain

$$\begin{aligned}(F_c(z)+\varepsilon i D_c(z))^{\frac{1}{3}} &= \left(\frac{(2c-1)^3+\varepsilon i\cdot i\cdot 3\sqrt{3}\,c\,(2c-1)^{\frac{3}{2}}z+o(z)}{z^6}\right)^{\frac{1}{3}} \\ &= \frac{2c-1}{z^2}\left(1-\varepsilon\cdot\frac{3\sqrt{3}c}{(2c-1)^{\frac{3}{2}}}z+o(z)\right)^{\frac{1}{3}} \\ &= \frac{2c-1}{z^2}\omega^{k(\varepsilon)}\left(1-\varepsilon\cdot\frac{\sqrt{3}c}{(2c-1)^{\frac{3}{2}}}z+o(z)\right)\end{aligned}$$

around $z=0$. Here, $\varepsilon=\pm1$ and $k(\varepsilon)\in\{0,1,2\}$. Let us consider the first condition in (17). Since

$$(F_c(z)+iD_c(z))^{\frac{1}{3}}\cdot(F_c(z)-iD_c(z))^{\frac{1}{3}} = \frac{(2c-1)^2}{z^4}\omega^{k(+)+k(-)}\left(1+o(z)\right),$$

$k(+) + k(-) \equiv 0 \bmod 3$. Next, let us consider the latter condition in (17). By (16), we have

$$\begin{aligned} S(z) &= \frac{z}{3}\left((F_c(z)+iD_c(z))^{\frac{1}{3}} + (F_c(z)-iD_c(z))^{\frac{1}{3}}\right) - \frac{2z}{3} + \frac{2c-1}{3z} \\ &= \frac{2c-1}{3z}\left(\omega^{k(+)}\left(1-\frac{\sqrt{3}c}{(2c-1)^{\frac{3}{2}}}z\right)+\omega^{k(-)}\left(1+\frac{\sqrt{3}c}{(2c-1)^{\frac{3}{2}}}z\right)+o(z)\right) - \frac{2z}{3} + \frac{2c-1}{3z} \\ &= \frac{2c-1}{3z}(\omega^{k(+)}+\omega^{k(-)}+1) + \frac{2c-1}{3}\cdot\frac{\sqrt{3}c}{(2c-1)^{\frac{3}{2}}}(\omega^{k(-)}-\omega^{k(+)}) - \frac{2z}{3} + o(1). \end{aligned}$$

Let $z = yi$ with $y > 0$. Then, since

$$\frac{2c-1}{3z} = -\frac{2c-1}{3y}i$$

and $-(2c-1) < 0$, we need to have $\omega^{k(+)} + \omega^{K(-)} + 1 = 0$, that is, $(k(+), k(-)) = (1, 2)$ or $(2, 1)$. In this case, the second term above can be described as

$$\frac{2c-1}{3}\cdot\frac{\sqrt{3}c}{(2c-1)^{\frac{3}{2}}}(\omega^{k(-)}-\omega^{k(+)}) = \frac{c}{\sqrt{3}\sqrt{2c-1}}\cdot\varepsilon'\sqrt{3}\,i \quad (\varepsilon' = \pm 1),$$

and hence we obtain $(k(+), k(-)) = (2, 1)$. Thus,

$$\lim_{y\to+0}\operatorname{Im} S(yi) = \frac{c}{\sqrt{3}\sqrt{2c-1}}\cdot\sqrt{3} - \lim_{y\to+0}\frac{2y}{3} = \frac{c}{\sqrt{2c-1}}.$$

We note that the density $f_c$ of $\mu$ in (9) satisfies

$$\lim_{x\to 0} f_c(x) = \frac{c}{\pi\sqrt{2c-1}}.$$

**(Z3) the case $c = \frac{1}{2}$.** In this case, we have $\alpha_{1/2} = 0$, $\beta := \beta_{1/2} = \frac{27}{8} = (\frac{3}{2})^3$. Moreover, since

$$F(z) := F_{1/2}(z) = \frac{\beta - \frac{9}{2}z^2 + z^4}{z^4} = \frac{1}{z^4}\left(\beta - \frac{9}{2}z^2 + z^4\right)$$

and

$$D(z) := D_{1/2}(z) = \frac{\sqrt{\beta}}{z^4}\sqrt{z^2-\beta} = i\cdot\frac{\beta}{z^4}\sqrt{1-\frac{z^2}{\beta}} = i\cdot\frac{\beta}{z^4}\left(1-\frac{z^2}{2\beta}-\frac{z^4}{8\beta^2}+o(z^4)\right)$$

around $z = 0$, we obtain

$$\begin{aligned} F(z)+iD(z) &= \frac{1}{z^4}\left(\beta-\frac{9}{2}z^2+z^4-\beta\left(1-\frac{z^2}{2\beta}-\frac{z^4}{8\beta^2}+o(z^4)\right)\right) = \frac{1}{z^4}\left(-4z^2+\left(1+\frac{1}{8\beta}\right)z^4+o(z^4)\right) \\ &= \frac{-4}{z^2}\left(1-\frac{8\beta+1}{32\beta}z^2+o(z^2)\right) = -\frac{4}{z^2}\left(1-\frac{7}{27}z^2+o(z^2)\right) \end{aligned}$$

and

$$\begin{aligned} F(z)-iD(z) &= \frac{1}{z^4}\left(\beta-\frac{9}{2}z^2+z^4+\beta\left(1-\frac{z^2}{2\beta}-\frac{z^4}{8\beta^2}+o(z^4)\right)\right) = \frac{1}{z^4}\left(2\beta-5z^2+\left(1-\frac{1}{8\beta}\right)z^4+o(z^4)\right) \\ &= \frac{2\beta}{z^4}\left(1-\frac{5}{2\beta}z^2+\frac{8\beta-1}{16\beta^2}z^4+o(z^4)\right) = \frac{27}{4z^4}\left(1-\frac{20}{27}z^2+\frac{13\cdot 8}{27^2}z^4+o(z^4)\right). \end{aligned}$$

Thus,

$$(F(z)+iD(z))^{\frac{1}{3}} = -\omega^{k_+}\frac{\sqrt[3]{4}}{z^{\frac{2}{3}}}\left(1-\frac{7}{81}z^2+o(z^2)\right), \quad (F(z)-iD(z))^{\frac{1}{3}} = \omega^{k_-}\frac{3}{\sqrt[3]{4}z^{\frac{4}{3}}}\left(1-\frac{20}{81}z^2+o(z^2)\right),$$

where $k_+, k_- \in \{0, 1, 2\}$. Let us consider the first condition in (17). Since

$$(F(z)+iD(z))^{\frac{1}{3}}\cdot(F(z)-iD(z))^{\frac{1}{3}} = -\omega^{k_++k_-}\frac{3}{z^2}\left(1-\frac{z^2}{3}+o(z^2)\right) = \omega^{k_++k_-}\left(-\frac{3}{z^2}+1+o(1)\right)$$

and

$$-3p(z) = 1 - \frac{3}{z^2},$$

we have $k_+ + k_- \equiv 0 \bmod 3$. Next, let us consider the latter condition in (17). By (16), we have

$$S(z) = \frac{z}{3}\left(-\omega^{k_+}\frac{\sqrt[3]{4}}{z^{\frac{2}{3}}}\left(1-\frac{7}{81}z^2+o(z^2)\right)+\omega^{k_-}\frac{3}{\sqrt[3]{4}z^{\frac{4}{3}}}\left(1-\frac{20}{81}z^2+o(z^2)\right)\right) - \frac{2z}{3} = \frac{\omega^{k_-}}{\sqrt[3]{4}}z^{-\frac{1}{3}} + O(z^{\frac{1}{3}})$$

Now $z = yi$ with $y > 0$, $z^{-\frac{1}{3}} = (1/\sqrt[3]{y})e^{-\pi i/6}$ so that $k_-$ must be equal to 1. In fact, in this case, $\omega^{k_-}z^{-\frac{1}{3}} = i/\sqrt[3]{y}$ and thus

$$\operatorname{Im} S(z) = \frac{1}{\sqrt[3]{4y}} + O(y^{\frac{1}{3}}) > 0 \quad (\text{if } y \text{ enough small})$$

and

$$\mu(\{0\}) = \lim_{y\to+0}\operatorname{Im} yS(x+yi) = \lim_{y\to+0}\sqrt[3]{\frac{y^2}{4}} + O(y^{\frac{4}{3}}) = 0.$$

By Lemma S6, this formula also yields that

$$\lim_{y\to+0} \operatorname{Im} S(yi) = +\infty \quad \text{and } \mu \text{ does not have an atom at } x = 0. \tag{S.8}$$

3.3. **Supplement for Remark 6 (the case of $c = 1$).** If we take $c \to 1-0$, then we have

$$\lim_{c\to1-0} 3c\sqrt{3-3c}\,x\sqrt{(x^2-\alpha_c)(\beta_c-x^2)} = \lim_{c\to1-0} 3\sqrt{3}cx\sqrt{((1-c)x^2-(1-c)\alpha_c)(\beta_c-x^2)} = 3\sqrt{3}x\sqrt{\frac{1}{4}(4-x^2)},$$

and hence

$$R_\pm(x;\,1) = x^6 - 6x^4 + \frac{15}{2}x^2 + 1 \pm \frac{3\sqrt{3}x}{2}\sqrt{4-x^2}.$$

Since $R_\pm(x;\,1)$ can be factored as

$$R_\pm(x;\,1) = \left(-\frac{1}{2}x^2 + 1 \pm \frac{\sqrt{3}x}{2}\sqrt{4-x^2}\right)^3,$$

we obtain

$$\sqrt[3]{R_+(x;\,1)} - \sqrt[3]{R_-(x;\,1)} = \left(-\frac{1}{2}x^2 + 1 + \frac{\sqrt{3}x}{2}\sqrt{4-x^2}\right) - \left(-\frac{1}{2}x^2 + 1 - \frac{\sqrt{3}x}{2}\sqrt{4-x^2}\right) = \sqrt{3}x\,\sqrt{4-x^2},$$

and hence

$$\mu(dt) = \frac{\sqrt{3}(t/\sqrt{v})\,\sqrt{4-t^2/v}}{2\sqrt{3}\pi t}\chi(t) = \frac{1}{2\pi v}\sqrt{4v-t^2}\;\chi(t).$$

**Remark 10.** The Wigner case may be considered in a framework of operator-valued free probability theory by methods of the rectangular free probability (cf. Mingo and Speicher (2017, Chapter 9), Benaych-Georges (2009)).

## 4. Wishart Ensembles of Vinberg Matrices

In this section, we shall consider the quadratic Wishart (covariance) matrices introduced in §2.3. We first prepare some special functions which we need later. They generalize the Lambert $W$ function appearing (see Cheliotis (2018)) in the case $P_n = \mathrm{Sym}(n,\mathbb{R})^+$ and $\underline{m} = (1,\dots,1)$.

### 4.1. **Lambert–Tsallis $W$ function and Lambert–Tsallis function $W_{\kappa,\gamma}$.**

For a non-zero real number $\kappa$, we set

$$\exp_\kappa(z) := \Big(1+\frac{z}{\kappa}\Big)^\kappa \quad (1+\frac{z}{\kappa}\in\mathbb{C}\setminus\mathbb{R}_{\le 0}),\quad \log^{\langle\kappa\rangle}(z) := \frac{z^\kappa - 1}{\kappa} \quad (z\in\mathbb{C}\setminus\mathbb{R}_{\le 0}),$$

where we take the main branch of the power function when $\kappa$ is not integer. If $\kappa = \frac{1}{1-q}$, then it is exactly the so-called Tsallis *$q$-exponential function* and *$q$-logarithm*, respectively (cf. Amari and Ohara (2011); Zhang et al. (2018)). We have the following relationship between these two functions:

$$\log^{\langle 1/\kappa\rangle}\circ\exp_\kappa(z) = z \quad \Big(-\pi < \kappa\mathrm{Arg}\Big(1+\frac{z}{\kappa}\Big) < \pi\Big). \tag{19}$$

Since $\lim_{\kappa\to\infty}\exp_\kappa(z) = e^z$, we regard $\exp_\infty(z) = e^z$ and $\log^{\langle 0\rangle}(z) = \log(z)$.

For two real numbers $\kappa,\gamma$ such that $\gamma \le \frac{1}{\kappa} \le 1$ and $\gamma < 1$, we introduce a holomorphic function $f_{\kappa,\gamma}(z)$, which we call *generalized Tsallis function*, by

$$f_{\kappa,\gamma}(z) := \frac{z}{1+\gamma z}\exp_\kappa(z) \quad (1+\frac{z}{\kappa}\in\mathbb{C}\setminus\mathbb{R}_{\le 0}).$$

We note that $\kappa\in(-\infty,0)\cup[1,+\infty)$. Analogously to Tsallis $q$-exponential, we also consider $f_{\infty,\gamma}(z) = \frac{ze^z}{1+\gamma z}$ $(z\in\mathbb{C})$. In particular, $f_{\infty,0}(z) = ze^z$.

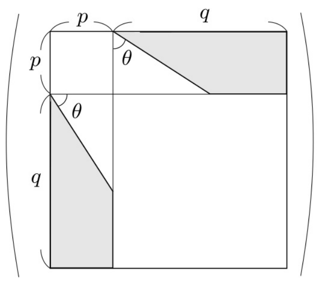
In our work it is crucial to consider an inverse function to $f_{\kappa,\gamma}$. A multivariate inverse function of $f_{\infty,0}(z) = ze^z$ is called the Lambert $W$ function and studied in Corless et al. (1996). Hence, we call an inverse function to $f_{\kappa,\gamma}$ the *Lambert–Tsallis $W$ function.*

The function $f_{\kappa,\gamma}(z)$ has the inverse function $w_{\kappa,\gamma}$ in a neighborhood of $z=0$, because we have $f'_{\kappa,\gamma}(0) = 1 \neq 0$ by

$$f'_{\kappa,\gamma}(z) = \frac{\gamma z^2 + \big(1+1/\kappa\big)z + 1}{(1+\gamma z)^2}\Big(1+\frac{z}{\kappa}\Big)^{\kappa-1}.$$

The condition on $\kappa$ and $\gamma$ comes from the variance profile $\sigma$ of the form

$\sigma =$




with $p+q=1,\ p,q>0$, $0\le\tan\theta=\alpha\le\frac{q}{p}$

Then, we are going to deal with the function $f_{\kappa,\gamma}(z)$ for the parameters

$$\kappa = \frac{1}{1-\alpha},\quad \gamma = \frac{p-q}{p} = \frac{2p-1}{p}.$$

By definition of $\kappa$ and $\gamma$ and by the range of $\tan\theta$, we have

$$1 \ge \frac{1}{\kappa} = 1-\tan\theta \ge 1 - \frac{1-p}{p} = \frac{2p-1}{p} = \gamma \quad\text{and}\quad -\infty < \gamma < 1.$$

Thus the condition we consider is

$$\gamma < 1 \text{ and } 1 \ge \frac{1}{\kappa} \ge \gamma, \quad\text{or equivalently}\quad \gamma<1,\ \frac{1}{\kappa}-\gamma \ge 0 \text{ and } \frac{1}{\kappa}\le 1$$

(see Figure 7). If $\alpha\in[0,1)$, or equivalently $0\le\alpha<1$, then $\kappa\in[1,\infty)$ and $\kappa\gamma\le 1$. If $\alpha>1$, or equivalently $\alpha>1$, then $\kappa\in(-\infty,0)$, and by setting $\kappa' = -\kappa > 0$ and $\gamma' = \gamma - 1/\kappa$, they satisfy

$$\kappa'\gamma' = -\kappa(\gamma - 1/\kappa) = 1-\kappa\gamma \le 0$$

so that this case is reduced to the case $\kappa>0$ (see §6.5). In the case of $\alpha = 1$, we consider $f_{\infty,\gamma} = \frac{x}{1+\gamma x}e^x$. In this case we have $\gamma\le 0$.

Let us present some properties of $f_{\kappa,\gamma}$. When $\gamma\kappa\neq 1$, the function $f_{\kappa,\gamma}$ has a pole at $x = -\frac{1}{\gamma}$. By the condition on $\kappa$ and $\gamma$, the function $\gamma z^2 + (1+1/\kappa)z + 1$ has two real roots, say $\alpha_1\le\alpha_2$ when $\gamma\neq 0$. If $\gamma = 0$, there is only one real root, that we denote $\alpha_2 = -\frac{\kappa}{\kappa+1}$.

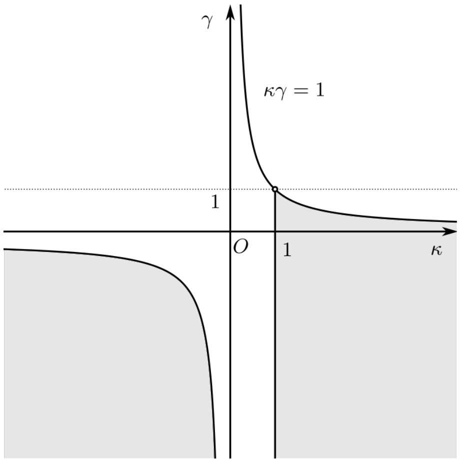



FIGURE 7. Region of $\kappa$ and $\gamma$

$f'_{\kappa,\gamma}(z) = 0$ implies $z = \alpha_i$ $(i = 1, 2)$, or $z = -\kappa$ if $\kappa > 1$. For the case $\kappa < 0$, it is convenient to change the variable by a homographic action $z' = \frac{z}{1+\frac{z}{\kappa}}$. Then

$$f_{\kappa,\gamma}(z) = f_{\kappa',\gamma'}(z') \quad \text{where} \quad \kappa' = -\kappa > 0, \quad \gamma' = \gamma - \frac{1}{\kappa}.$$

Since a homographic action by element in $SL(2, \mathbb{R})$ leaves $\mathbb{C}^+$ invariant, the analysis of the case $\kappa < 0$ reduces to the case $\kappa' > 0$ and $\gamma' \leq 0$. Then, the set $\mathcal{S} := \mathbb{R} \setminus f_{\kappa,\gamma}(\mathbb{R})$ has the following possibilities.

**Theorem S9.** The set $\mathcal{S} := \mathbb{R} \setminus f_{\kappa,\gamma}(\mathbb{R})$ is expressed by following formulas.

(S1) $\mathcal{S} = (f_{\kappa,\gamma}(\alpha_2), f_{\kappa,\gamma}(\alpha_1))$, where $f_{\kappa,\gamma}(\alpha_2) < f_{\kappa,\gamma}(\alpha_1) < 0$. It occurs when $\kappa \in [1, +\infty]$ and $\gamma < 0$, and when $\kappa < 0$ and $\gamma' = \gamma - \frac{1}{\kappa} < 0$.
(S2) $\mathcal{S} = (-\infty, f_{\kappa,\gamma}(\alpha_2))$, where $f_{\kappa,\gamma}(\alpha_2) < 0$. It occurs when $\kappa > 1$ and $\gamma \geq 0$ and when $(\kappa, \gamma) = (1, 0)$.
(S3) $\mathcal{S} = (-\infty, f_{\kappa,\gamma}(\alpha_1))$, where $f_{\kappa,\gamma}(\alpha_1) < 0$. It occurs when $\kappa < 0$ and $\gamma' = \gamma - \frac{1}{\kappa} = 0$.
(S4) $\mathcal{S} = (f_{\kappa,\gamma}(\alpha_1), f_{\kappa,\gamma}(\alpha_2))$, where $f_{\kappa,\gamma}(\alpha_1) < f_{\kappa,\gamma}(\alpha_2) < 0$. It occurs when $\kappa = 1$ and $\gamma > 0$.

We study in detail the cases (S1,S2,S3). The case (S4) appears in the well known Wishart Ensemble case.

**Theorem 11.** *Let $\mathcal{S}$ be an interval or half-line given by (S1)-(S4) above, and $\overline{\mathcal{S}} \subset (-\infty, 0)$ its closure. Then, there exists a complex domain $\Omega \subset \mathbb{C}$, symmetric with respect to the real axis and containing 0, such that $f_{\kappa,\gamma}$ maps $\Omega$ bijectively to $\mathbb{C} \setminus \overline{\mathcal{S}}$. Consequently, the function $w_{\kappa,\gamma}$ can be continued in a unique way to a holomorphic function $W_{\kappa,\gamma}$ defined on $\mathbb{C} \setminus \overline{\mathcal{S}}$. The codomain of $W_{\kappa,\gamma}$ is $\Omega$, that is, $W_{\kappa,\gamma}(\mathbb{C} \setminus \overline{\mathcal{S}}) = \Omega$.*

**Definition 12.** The unique holomorphic extension $W_{\kappa,\gamma}$ of $w_{\kappa,\gamma}$ to $\mathbb{C} \setminus \overline{\mathcal{S}}$ is called *the main branch* of Lambert-Tsallis $W$ function. In this paper, we only study and use $W_{\kappa,\gamma}$ among other branches so that we call $W_{\kappa,\gamma}$ the *Lambert–Tsallis function* for short. Note that in our terminology the Lambert-Tsallis $W$ function is multivalued and the Lambert-Tsallis function $W_{\kappa,\gamma}$ is single-valued.

We summarize the basic properties of the Lambert-Tsallis function that we need later.

**Proposition 13.** *(i) Let $D = \Omega \cap \mathbb{C}^+$. The function $f_{\kappa,\gamma}$ is continuous and injective on the closure $\overline{D}$. Consequently, $W_{\kappa,\gamma}$ extends continuously from $\mathbb{C}^+$ to $\mathbb{C}^+ \cup \mathbb{R}$, and one has $f_{\kappa,\gamma}(\partial\Omega \cap \mathbb{C}^+) = \mathcal{S}$.*
*(ii) The Lambert-Tsallis function $W_{\kappa,\gamma}$ has the following properties.*

(a) *Suppose that $\kappa \geq 1$ and $\gamma < 0$, or $\kappa < 0$ and $\gamma' \leq 0$. In these cases, the set $D = \Omega \cap \mathbb{C}^+$ is bounded. If $\kappa \geq 1$ then we have $D \subset \left\{z \in \mathbb{C}^+;\ \mathrm{Arg}\left(1 + \frac{z}{\kappa}\right) \in (0, \frac{\pi}{\kappa+1})\right\}$ and $z \in D$ satisfies $\mathrm{Re}\, z > -\kappa$. If $\kappa = \infty$, then one has $\mathrm{Im}\, W_{\kappa,\gamma}(z) \in (0, \pi)$ for $z \in \mathbb{C}^+$. If $\kappa < 0$ then we have $D \subset \left\{z \in \mathbb{C}^+;\ \mathrm{Arg}\Big(\left(1 + \frac{z}{\kappa}\right)^{-1}\Big) \in (0, \frac{\pi}{|\kappa|+1})\right\}$. Moreover, $\lim_{|z|\to+\infty} W_{\kappa,\gamma}(z) = -\frac{1}{\gamma}$ (recall that $-\frac{1}{\gamma}$ is a pole of $f_{\kappa,\gamma}$).*
(b) *Suppose $\kappa \in [1, +\infty]$ and $\gamma = 0$. The set $D = \Omega \cap \mathbb{C}^+$ is unbounded and $f_{\kappa,0}(\infty) = \infty$. If $\kappa \in [1, +\infty)$ then $D \subset \left\{z \in \mathbb{C}^+;\ \mathrm{Arg}\left(1 + \frac{z}{\kappa} \in (0, \frac{\pi}{\kappa+1})\right)\right\}$. If $\kappa = \infty$, then $W_{\infty,0}(z)$ is the classical Lambert function, and one has $\mathrm{Im}\, W_{\infty,0}(z) \in (0, \pi)$ for $z \in \mathbb{C}^+$.*
(c) *Suppose $\gamma > 0$. In this case we have $\kappa \in [1, \frac{1}{\gamma}]$. The set $D = \Omega \cap \mathbb{C}^+$ is unbounded and $f_{\kappa,\gamma}(\infty) = \infty$. Moreover, one has $D = \left\{z \in \mathbb{C}^+;\ \mathrm{Arg}\left(1 + \frac{z}{\kappa}\right) \in (0, \frac{\pi}{\kappa})\right\}$.*

The proofs of Theorem S9, Theorem 11 and Proposition 13 will be given in Appendix (see page 34).

**Remark 14.** It is worth underlying that we consider the main branch of the complex power function in the Tsallis $q$-exponential $\exp_\kappa(z)$ appearing inside the generalized Tsallis function $f_{\kappa,\gamma}$. Consequently, the main branch $W_{\kappa,\gamma}$ is the unique one such that $W(0) = 0$. A complete study of all branches of the Lambert-Tsallis $W$ function will be interesting to do. The study of the Lambert-Tsallis function $W_{\kappa,\gamma}$ in the full range of parameters $\kappa, \gamma$ is also an interesting open problem. We exclude the case $\kappa\gamma > 1$ with $\kappa > 0$ because we do not need it later. We note that, when $\kappa\gamma > 1$ and $\kappa > 1$ with a condition $(1+\kappa)^2 - 4\gamma\kappa^2 > 0$, then $f_{\kappa,\gamma}$ maps a subregion of $\mathbb{C}^+$ onto $\mathbb{C}^+$.

Applying the Lagrange inversion theorem, we see that the Taylor series of the function $W_{\kappa,\gamma}$ near $z = 0$ is

$$W_{\kappa,\gamma}(z) = z + (\gamma - 1)z^2 + \left(\gamma^2 - 3\gamma + \frac{3\kappa + 1}{\kappa}\right) z^3 + o(z^3). \tag{S.9}$$

4.2. **Quadratic Wishart matrices.** We will now study eigenvalues of Wishart (covariance) matrices in $P_n \subset \mathcal{U}_n$, defined in Section 2.3. We apply the approach of Bordenave (2019, Cor.3.5), based on the variance profile method (Theorem 3).

In this subsection, we first consider the case of $a_n = n - 1$ and $b_n = 1$, that is, $P_n$ is the symmetric cone $\mathrm{Sym}(n, \mathbb{R})^+$ of positive definite symmetric matrices of size $n$. Let $\xi_n = (\xi_{ij})$ be a rectangular matrix of size $n \times N$, where the entries $\xi_{ij}$ are centered i.i.d. variables with variance $v$ and fourth moment $M_4$. In order to study eigenvalue distributions of $X_n = \xi_n \xi_n^\top$, we equivalently consider Wigner matrices of the form

$$Y_n := \begin{pmatrix} 0 & \xi_n \\ \xi_n^\top & 0 \end{pmatrix} \in \mathrm{Sym}(n + N, \mathbb{R}). \tag{20}$$

If $X_n$ has eigenvalues $\lambda_j \geq 0$ $(j = 1, \ldots, n)$, then those of $Y_n$ are exactly $\pm\sqrt{\lambda_j}$ $(j = 1, \ldots, n)$ and zeros with multiplicity $|N - n|$.

This is because, by the singular value decomposition, there exist orthogonal matrices $U, V \in O(n)$ and non-negative $\mu_1, \ldots, \mu_n \geq 0$ such that

$$\xi_n = U\big(D_n\, 0\big)V, \quad D_n = \mathrm{diag}(\mu_1, \ldots, \mu_n).$$

Here we assume that $N \geq n$ for simplicity. Since

$$X_n = \xi_n\, \xi_n^\top = U\big(D_n\, 0\big)VV^\top \begin{pmatrix} D_n \\ 0 \end{pmatrix} U^\top = UD_n^2U^\top,$$

we see that $\lambda_j$ is one of $\mu_k^2$ for some $k$, and we can assume that $\lambda_j = \mu_j^2$ because we can arrange the ordering of eigenvalues by the action of $O(n)$. Since

$$Y_n = \begin{pmatrix} 0 & \xi_n \\ \xi_n^\top & 0 \end{pmatrix} = \begin{pmatrix} 0 & U\big(D_n\, 0\big)V \\ V^\top \begin{pmatrix} D_n \\ 0 \end{pmatrix} U^\top & 0 \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & V^\top \end{pmatrix} \begin{pmatrix} 0 & D_n & 0 \\ D_n & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} U^\top & 0 \\ 0 & V \end{pmatrix}$$

(in the right hand side, the matrix in the center is a block matrix with partition $n$, $n$ and $N-n$), the characteristic polynomial $g(t)$ of $Y_n$ is given as

$$g(t) = t^{N-n} \prod_{i=1}^{n} (t^2 - \mu_i^2), \quad \text{so that eigenvalues of } Y_n \text{ are} \quad \pm\,\mu_i = \pm\sqrt{\lambda_i} \text{ and } 0.$$

Let $T_n$ denote the Stieltjes transform of the empirical eigenvalue distribution of rescaled $X_n/n$ and $S_n$ the Stieltjes transform of rescaled $Y_n/\sqrt{n+N}$. Then, it is easy to see that these Stieltjes transforms satisfy

$$T_n\left(\frac{z^2}{p_n}\right) = \frac{1}{2z}\left(\frac{1 - 2p_n}{z} + S_n(z)\right), \tag{21}$$

where $p_n := \frac{n}{n+N}$ and $q_n = \frac{N}{n+N}$.

In fact, we have for $n \leq N$

$$\begin{aligned}
S_n(z) &= \frac{1}{n+N}\left(\frac{N-n}{0-z} + \sum_{j=1}^{n} \frac{1}{\frac{\sqrt{\lambda_j}}{\sqrt{n+N}} - z} + \frac{1}{-\frac{\sqrt{\lambda_j}}{\sqrt{n+N}} - z}\right) \\
&= -\frac{1}{n+N} \cdot \frac{n+N-2n}{z} + \frac{1}{n+N} \sum_{j=1}^{n} \frac{-2z}{z^2 - \frac{\lambda_j}{n+N}} \\
&= -\frac{1-2p_n}{z} + \frac{n}{n+N} \cdot \frac{1}{n} \sum_{j=1}^{n} \frac{-2z}{z^2 - \frac{n}{n+N} \cdot \frac{\lambda_j}{n}} \\
&= -\frac{1-2p_n}{z} + \frac{2p_n z}{n} \sum_{j=1}^{n} \frac{1}{p_n \cdot \frac{\lambda_j}{n} - z^2} \\
&= -\frac{1-2p_n}{z} + 2z \cdot \frac{1}{n} \sum_{j=1}^{n} \frac{1}{\frac{1}{n}\lambda_j - \frac{z^2}{p_n}} \\
&= -\frac{1-2p_n}{z} + 2zT_n\left(\frac{z^2}{p_n}\right),
\end{aligned}$$

and for $n \leq N$

$$\begin{aligned}
S_n(z) &= \frac{1}{n+N}\left(\sum_{j=1}^{N}\frac{1}{\sqrt{\lambda_j}/\sqrt{n+N}-z}+\frac{1}{-\sqrt{\lambda_j}/\sqrt{n+N}-z}+\frac{n-N}{0-z}\right)\\
&= \frac{1}{n+N}\sum_{j=1}^{N}\frac{-2z}{z^2-\lambda_j/(n+N)}-\frac{n-N}{n+N}\cdot\frac{1}{z}=\frac{1}{n}\sum_{j=1}^{N}\frac{2z}{\frac{\lambda_j}{n}-\frac{n+N}{n}z^2}-\frac{p_n-q_n}{z}\\
&= 2z\left\{\frac{1}{n}\left(\sum_{j=1}^{N}\frac{1}{\frac{\lambda_j}{n}-\frac{z^2}{p_n}}+\frac{n-N}{0-\frac{z^2}{p_n}}\right)-\frac{1}{n}\cdot\frac{n-N}{0-\frac{z^2}{p_n}}\right\}-\frac{p_n-q_n}{z}\\
&= 2z\left(T_n\Big(\frac{z^2}{p_n}\Big)+\frac{p_n-q_n}{p_n}\cdot\frac{p_n}{z^2}\right)-\frac{p_n-q_n}{z}=2zT_n\left(\frac{z^2}{p_n}\right)+\frac{2(p_n-q_n)}{z}-\frac{p_n-q_n}{z}\\
&= 2zT_n\left(\frac{z^2}{p_n}\right)+\frac{p_n-q_n}{z}.
\end{aligned}$$

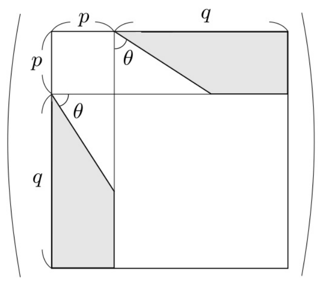
In order to study eigenvalue distributions of covariance matrices from Section 2.3, with parameters $\underline{k}$ as in (2), we introduce a trapezoidal variance profile $\sigma$ as follows. Let $p, \alpha$ be real numbers such that $0 < p < 1$ and $0 \leq \alpha \leq (1-p)/p$. Then, $\sigma$ is defined by

$$\sigma(x,y) := v \quad \text{if } (x,y) \in \mathcal{C}, \quad \sigma(x,y) := 0 \quad \text{(otherwise)}, \tag{22}$$

where $\mathcal{C}$ is given as

$$\mathcal{C} = \left\{(x,y) \in [0,1]^2;\ \begin{array}{ll}\text{(i)} & x < p \text{ and } y \geq p + \alpha x,\\ \text{(ii)} & x \geq p \text{ and } 0 \leq y \leq \min\{(x-p)/\alpha, p\}\end{array}\right\}.$$

The perturbation term $\delta_{ij}(n)$ in (5) equals $\delta_{ij}(n) = \mathbb{E}U_{ij}^2 - v\frac{|\mathcal{C}\cap Q_{ij}|}{|Q_{ij}|}$. Graphically, $\mathcal{C}$ is of the form

$\mathcal{C} =$ 

with $\begin{array}{l} p+q=1,\ p,q>0 \\ 0 \leq \tan\theta = \alpha \leq \frac{q}{p}\end{array}$ (23)

If $\lim_n p_n = p$, by Theorem 3, this variance profile determines the limiting distribution of empirical eigenvalue distributions of the Wigner matrices $Y_n$ in (20). Recall that, to a variance profile $\sigma$, Theorem 3 associates the Stieltjes transform $S_\sigma(z)$. It will be determined in Theorem 15. Analogously, to a variance profile $\sigma$ of $\xi_n$, we associate the "covariance Stieltjes transform" $T_\sigma(z)$ of the corresponding covariance matrices $Q_{\underline{k}}(\xi_n) = \xi_n\xi_n^\top$. The covariance Stieltjes transform $T_\sigma(z)$ is related to $S_\sigma(z)$ by the formula (21). It will be determined in Proposition 17.

**Theorem 15.** *Let $\sigma$ be a variance profile given in* (22)*, and set $\kappa := 1/(1-\alpha)$ and $\gamma := (2p-1)/p = 1-(q/p)$. Then, the Stieltjes transform $S_\sigma(z)$ associated to $\sigma$ is given as*

$$S_\sigma(z) = -\frac{2p}{zW_{\kappa,\gamma}\left(-\frac{vp}{z^2}\right)}+\frac{1-2p}{z}-\frac{2z}{v} \quad (z \in \mathbb{C}^+), \tag{S.10}$$

*where $W_{\kappa,\gamma}$ is the Lambert-Tsallis function defined in Section 4.1.*

*Proof.* We use Theorem 3. Take $z \in \mathbb{C}^+$ such that $\operatorname{Im} z$ is large enough. By definition of $\sigma$ and $\eta_z$, we have

$$\eta_z(x) = \begin{cases} -\left(z + v\displaystyle\int_{p+\alpha x}^{1}\eta_z(y)\,dy\right)^{-1} & (0 \leq x \leq p),\\ -\left(z + v\displaystyle\int_{0}^{\alpha^{-1}(x-p)}\eta_z(y)\,dy\right)^{-1} & (p < x \leq p+\alpha p),\\ -\left(z + v\displaystyle\int_{0}^{p}\eta_z(y)\,dy\right)^{-1} & (p+\alpha p < x \leq 1).\end{cases} \tag{S.11}$$

For $z$ fixed, we set

$$a(t) := \eta_z(t), \quad t \in [0,p], \qquad b(t) := \eta_z(p+\alpha t), \quad t \in (0,p].$$

By differentiating both sides in the above equations, we obtain a differential equation

$$\left\{\begin{array}{l} a'(t) = -v\alpha a(t)^2 b(t),\\ b'(t) = va(t)b(t)^2,\end{array}\right. \tag{24}$$

with initial data

$$a(p) = -\left(z + v\int_{p+\alpha p}^{1} \eta_z(y)\,dy\right)^{-1}, \quad b(0+) = -\frac{1}{z}.$$

Note that the third line of definition of $\eta_z$ ensures that $\eta_z$ is constant on the interval $[p+\alpha p, 1]$ so that we have

$$a(p) = -\Big(z + v(1-p-\alpha p)b(p)\Big)^{-1}. \tag{25}$$

In what follows, we shall show that, if $\alpha \neq 1$ then

$$a(t) = -zw(z)X(t)^{\alpha\kappa}, \quad b(t) = -\frac{1}{z}\cdot X(t)^{-\kappa},$$

where $w(z) := -\frac{1}{vp}W_{\kappa,\gamma}\left(-\frac{vp}{z^2}\right)$ and $X(t) := 1 - \frac{vw(z)}{\kappa}t$ satisfy (24). Here, we choose the main branches for complex power functions. If $\alpha = 1$ then

$$a(t) = -zw(z)e^{-vw(z)t}, \quad b(t) = -\frac{1}{z}\cdot e^{vw(z)t}.$$

We omit the proof for $\alpha = 1$ because it can be done by a similar argument. Recall that we can take $z \in \mathbb{C}^+$ such that $-vp/z^2$ is in a neighbourhood of 0. By (S.9), we obtain

$$a(t) = -\frac{1}{z} + \frac{(\gamma-1)vp + \alpha vt}{z^3} + o(1/z^3), \quad b(t) = -\frac{1}{z} - \frac{vt}{z^3} + o(1/z^3). \tag{S.12}$$

In fact, by (S.9), we have

$$w(z) = -\frac{1}{vp}W_{\kappa,\gamma}\Big(-\frac{vp}{z^2}\Big) = -\frac{1}{vp}\left(-\frac{vp}{z^2} + (\gamma-1)\left(-\frac{vp}{z^2}\right)^2 + o(1/z^4)\right) = \frac{1}{z^2} - \frac{vp(\gamma-1)}{z^4} + o(1/z^4),$$

and thus

$$-zw(z) = -\frac{1}{z} - \frac{vp(\gamma-1)}{z^3} + o(1/z^3).$$

On the other hand, by the Taylor expansion of the complex power function we have

$$X(t)^{\alpha\kappa} = \left(1 - \frac{vwt}{\kappa}\right)^{\alpha\kappa} = \left(1 - \frac{vt}{\kappa}\Big(\frac{1}{z^2} + o(1/z^2)\Big)\right)^{\alpha\kappa} = 1 - \frac{\alpha vt}{z^2} + o(1/z^2)$$

so that

$$a(t) = -zwX(t)^{\alpha\kappa} = \left(-\frac{1}{z} - \frac{vp(\gamma-1)}{z^3} + o(1/z^3)\right)\left(1 - \frac{\alpha vt}{z^2} + o(1/z^2)\right) = -\frac{1}{z} + \frac{(\gamma-1)vp + \alpha vt}{z^3} + o(1/z^3).$$

Similarly, we obtain

$$b(t) = -\frac{1}{z}\left(1 - \frac{vw(z)t}{\kappa}\right)^{-\kappa} = -\frac{1}{z}\left(1 - \frac{vt}{\kappa}\cdot\frac{1}{z^2} + o(1/z^2)\right)^{-\kappa} = -\frac{1}{z}\left(1 + \frac{vt}{z^2} + o(1/z^2)\right) = -\frac{1}{z} - \frac{vt}{z^3} + o(1/z^3).$$

Since $\eta_z(x)$ is independent of $x$ when $x \in [p+\alpha p, 1]$, we see that $\eta_z(x) = b(p)$ for $x \in (p+\alpha p, 1]$. We deduce from (S.12) that when $\operatorname{Im} z$ is large enough, then $\eta_z(x) \in \mathbb{C}^+$ for all $x \in [0,1]$. Actually, we have $\operatorname{Im} -1/z > 0$ if $z \in \mathbb{C}^+$. If $\operatorname{Im} z$ is large enough, then $\operatorname{Im}(o(1/z))$ is small compared with $-1/z$ so that $\operatorname{Im}(-1/z + o(1/z)) > 0$.

Since $W_{\kappa,\gamma}$ is holomorphic around $z = 0$ and $W_{\kappa,\gamma}(0) = 0$, we can choose $z \in \mathbb{C}^+$ such that

$$\sup_t |\mu \operatorname{Arg} X(t)| < \pi \quad \text{for all} \quad \mu = 2\alpha\kappa, -2\kappa, \alpha\kappa - 1, -\kappa - 1, 2\alpha\kappa - \kappa, \alpha\kappa - 2\kappa.$$

This means that we are able to calculate $X(t)^{\mu}X(t)^{\nu'} = X(t)^{\mu+\mu'}$ for $\mu$,$\mu'$ being any of numbers in the above list. By differentiating $a(t)$ and $b(t)$, we obtain

$$a'(t) = -zw(z)\cdot\Big(-\frac{v\alpha\kappa w(z)}{\kappa}X(t)^{\alpha\kappa-1}\Big) = v\alpha zw(z)^2X(t)^{\alpha\kappa-1},$$
$$b'(t) = -\frac{1}{z}\cdot\Big(-\frac{-v\kappa w(z)}{\kappa}X(t)^{-\kappa-1}\Big) = -\frac{vw(z)}{z}X(t)^{-\kappa-1}.$$

On the other hand, since we take the main branch of complex power functions, we have by $\alpha\kappa = \kappa - 1$

$$-v\alpha a(t)^2b(t) = -v\alpha zw(z)^2X(t)^{\alpha\kappa-1} \quad \text{and} \quad va(t)b(t)^2 = -\frac{vw(z)}{z}X(t)^{-\kappa-1}.$$

Therefore, we confirm that $a'(t) = -v\alpha a(t)^2b(t)$ and $b'(t) = va(t)b(t)^2$. Next we consider the initial conditions. It is obvious that $b(0) = -\frac{1}{z}$. Since $f_{\kappa,\gamma}(-vpw(z)) = -\frac{vp}{z^2}$, we have, setting $w = w(z)$ and $X = X(p)$ for simplicity,

$$
\begin{aligned}
\frac{wX^\kappa}{1+v(1-2p)w} = \frac{1}{z^2} \quad &\Longleftrightarrow \quad wz^2X^\kappa = 1+v(1-2p)w \\
&\Longleftrightarrow \quad wz^2X^\kappa = 1-\frac{vwp}{\kappa}-(p+\alpha p-1)vw \qquad \left(\because \kappa=\frac{1}{1-\alpha}\right) \\
&\Longleftrightarrow \quad X = z^2wX^\kappa+(p+\alpha p-1)vw \qquad \left(\because X = 1-\frac{vwp}{\kappa}\right) \\
&\Longleftrightarrow \quad 1 = zwX^{\kappa-1}\left(z+(p+\alpha p-1)\frac{v}{z}\cdot X^{-\kappa}\right) \\
&\Longleftrightarrow \quad -zwX^{\kappa-1} = -\left(z+\frac{v(p+\alpha p-1)}{z}\cdot X^{-\kappa}\right)^{-1}.
\end{aligned}
$$

Since $a(p) = -zwX^{\alpha\kappa} = -zwX^{\kappa-1}$ by $\alpha\kappa = \kappa-1$, we see that

$$
a(p) = -\left(z+v\cdot\frac{p+\alpha p-1}{zX^\kappa}\right)^{-1}.
$$

On the other hand, since $\eta_z(x)$ is independent of $x$ when $x\in[p+\alpha p,1]$, we have

$$
\int_{p+\alpha p}^1 \eta_z(y)\,dy = (1-p-\alpha p)\eta_z(p+\alpha p) = (1-p-\alpha p)b(p) = \frac{p+\alpha p-1}{zX^\kappa}.
$$

Thus we conclude that $a(t)$ satisfies the initial condition, and hence $a(t)$ and $b(t)$ give indeed a solution of (24) and of (S.11). The property $\eta_z(x)\in\mathbb{C}^+$ and the unicity part of Theorem 3 imply that $a(t)$ and $b(t)$ give the $\mathbb{C}^+$-valued solution $\eta_z(x)$ of (S.11) such that the desired Stieltjes transform equals $S_\sigma(z) = \int_0^1 \eta_z(x)dx$. Then, we have

$$
S_\sigma(z) = \int_0^1 \eta_z(x)\,dx = \left(\int_0^p + \int_p^{p+\alpha p} + \int_{p+\alpha p}^1\right)\eta_z(x)\,dx = \int_0^p a(t)\,dt + \alpha\int_0^p b(t)\,dt + \int_{p+\alpha p}^1 \eta_z(x)\,dx.
$$

By formulas $f_{\kappa,\gamma}(-vpw(z)) = -\frac{vp}{z^2}$ and $a(p) = -zwX^{\kappa-1}$, we obtain

$$
\int_0^p a(t)\,dt = \frac{z}{v}\left(X^\kappa-1\right) = \frac{z}{v}\left(\frac{1}{wz^2}+\frac{(1-2p)v}{z^2}-1\right), \quad \int_0^p b(t)\,dt = \frac{1}{v\alpha zw}(1-X^{1-\kappa}) = \frac{1}{v\alpha zw}\left(1+\frac{wz}{a(p)}\right),
$$

and by the initial data of $a(t)$

$$
\int_{p+\alpha p}^1 \eta_z(x)\,dx = -\frac{1}{v}\left(\frac{1}{a(p)}+z\right).
$$

Thus, we have

$$
S_\sigma(z) = \frac{z}{v}(X^\kappa-1)+\frac{1}{vzw}\left(1-X^{1-\kappa}\right)+\int_{p+\alpha p}^1 \eta_z(x)\,dx = -\frac{2p}{zW_{\kappa,\gamma}\left(-\frac{vp}{z^2}\right)}+\frac{1-2p}{z}-\frac{2z}{v}. \tag{S.13}
$$

Since the image of $\mathbb{C}^+$ with respect to the map $z\mapsto -vp/z^2$ is $\mathbb{C}\setminus\mathbb{R}_{\le 0}$, we see that $-\frac{vp}{z^2}$ $(z\in\mathbb{C}^+)$ is included in $\mathbb{C}\setminus\overline{\mathcal{S}}$, the domain of $W_{\kappa,\gamma}$, because $\overline{\mathcal{S}}\subset(-\infty,0)$ by Theorem 11. Therefore, the formula (S.13) is valid for all $z\in\mathbb{C}^+$, and hence $S_\sigma(z)$ can be analytically continued to a holomorphic function on $\mathbb{C}^+$. We conclude that $S_\sigma(z)$ is given as (S.10). □

**Remark 16.** We call the parameter $\kappa$ of Lambert-Tsallis functions the *angle parameter* since it depends only on the angle of the trapeze in (23). If $\kappa=1$, then we have $\alpha=0$ so that the trapeze reduces to a rectangle. If $\alpha=q/p$, i.e. $\kappa=p/(p-q)=1/\gamma$, then the trapeze reduces to a triangle. On the other hand, the parameter $\gamma=\frac{2p-1}{p}=1-C$ depends directly on the shape parameter $C=q/p$. We call $\gamma$ the *shape parameter* of the Lambert-Tsallis function. Note that the geometric condition $0\le\alpha\le\frac{p}{q}$ is equivalent to the condition $\frac{1}{\kappa}\ge\gamma$. The formula $\gamma=1-\frac{q}{p}$ shows that $\gamma\in(-\infty,1)$. We have

$$
\kappa\in[1,\tfrac{1}{\gamma}] \text{ if } 0\le\gamma<1, \quad\text{and}\quad \kappa\in[1,\infty]\cup(-\infty,\tfrac{1}{\gamma}] \text{ if } \gamma<0.
$$

Now we give the covariance Stieltjes transform $T_\sigma(z)$ for the profile $\sigma$. We note that, corresponding to a probability measure $\mu$, there exists the so-called $R$-transform $R(z)$ which plays an important role in the field of free probability (cf. Mingo and Speicher (2017, Chapter 3)). It satisfies a relation $R(z)=S^{-1}(-z)-1/z$ where $S(z)$ is the Stieltjes transform of $\mu$. We also give the corresponding $R$-transform $R_\sigma$ related to $\sigma$ in a view of future studies.

**Proposition 17.** *Let $\sigma$ be a variance profile defined in* (22) *with parameters $p$ and $\alpha$. Set $\kappa:=\frac{1}{1-\alpha}$ and $\gamma:=\frac{2p-1}{p}=1-\frac{q}{p}$. Then, the covariance Stieltjes transform $T_\sigma(z)$ corresponding to the profile $\sigma$ is described as*

$$
T_\sigma(z) = T_{\kappa,\gamma}(z) := -\frac{1}{v}-\frac{1}{zW_{\kappa,\gamma}\left(-\frac{v}{z}\right)}-\frac{\gamma}{z} = \frac{\exp_\kappa\left(W_{\kappa,\gamma}(-v/z)\right)-1}{v} \qquad (z\in\mathbb{C}^+), \tag{26}
$$

*and its $R$-transform $R(z)$ is given as*

$$
R(z) = -\frac{1}{z}-\frac{v\gamma}{1-vz}-\frac{v}{(1-vz)\log^{\langle 1/\kappa\rangle}(1-vz)} \qquad (1-vz\in\mathbb{C}\setminus\mathbb{R}_{\le 0}).
$$

*Proof.* Let $z \in \mathbb{C}^+$ and set $W(z) = W_{\kappa,\gamma}(z)$. If $p_n \to p$ as $n \to +\infty$, the formula (21) converges as $n \to \infty$ to

$$T\Big(\frac{z^2}{p}\Big) = \frac{1}{2z}\left(\frac{1-2p}{z} + S(z)\right).$$

By Theorem 15, we obtain

$$\begin{aligned} T\Big(\frac{z^2}{p}\Big) &= \frac{1-2p}{2z^2} + \frac{1}{2z}\left(-\frac{2p}{zW\left(-vp/z^2\right)} + \frac{1-2p}{z} - \frac{2z}{v}\right) \\ &= \frac{1-2p}{2z^2} - \frac{p}{z^2 W(-vp/z^2)} + \frac{1-2p}{2z^2} - \frac{1}{v} \\ &= \frac{1-2p}{p}\cdot\frac{p}{z^2} - \frac{p}{z^2}\cdot\frac{1}{W(-v(p/z^2))} - \frac{1}{v}. \end{aligned}$$

Let $z' = z^2/p$. Then we have

$$T(z') = \frac{1-2p}{p}\cdot\frac{1}{z'} - \frac{1}{z'W(-v/z')} - \frac{1}{v}.$$

Since $z'$ runs through all elements in $\mathbb{C}^+$ and since $\gamma = \frac{2p-1}{p}$, we obtain the first equation. For the second equality, let us put $W = W(-v/z)$ for simplicity. By definition of the Lambert-Tsallis function, we have

$$-\frac{v}{z} = \frac{W}{1+\gamma W}\exp_\kappa(W) = \frac{\exp_\kappa(W)}{\gamma + 1/W}, \quad \text{and hence} \quad \gamma + \frac{1}{W} = -\frac{z}{v}\exp_\kappa(W).$$

This yields that

$$T(z) = -\frac{1}{v} - \frac{1}{zW} - \frac{\gamma}{z} = -\frac{1}{v} - \frac{1}{z}\left(\frac{1}{W} + \gamma\right) = -\frac{1}{v} - \frac{1}{z}\Big(-\frac{z}{v}\exp_\kappa(W)\Big) = -\frac{1}{v} + \frac{\exp_\kappa(W)}{v},$$

whence we obtain the second equality.

Recall the relation between the $R$-transform $R(z)$ and the Stieltjes transform $S(z)$, that is, $R(z) = S^{-1}(-z) - 1/z$ (cf. Mingo and Speicher (2017, Chapter 3)).

Let us assume that $\kappa \neq \infty$. Since we have $W_{\kappa,\gamma}(z) \in D$ for $z \in \mathbb{C}^+$, Proposition 13 (ii) tells us that $-\pi < \kappa \operatorname{Arg}\left(1 + \frac{W(z)}{\kappa}\right) < \pi$ for any $z \in \mathbb{C}^+$ so that we obtain by using (19)

$$\begin{aligned} T(z) = -\frac{1}{v} + \frac{1}{v}\left(1 + \frac{W(-v/z)}{\kappa}\right)^\kappa &\iff vT(z) + 1 = \exp_\kappa(W(-v/z)) \\ &\iff W(-v/z) = \log^{\langle 1/\kappa\rangle}(vT(z)+1) \\ &\iff -\frac{v}{z} = f_{\kappa,\gamma}(\log^{\langle 1/\kappa\rangle}(vT(z)+1)) \\ &\iff z = -\frac{v}{f_{\kappa,\gamma}(\log^{\langle 1/\kappa\rangle}(vT(z)+1))}. \end{aligned}$$

Thus, we see that

$$T^{-1}(z) = -\frac{v}{f_{\kappa,\gamma}(\log^{\langle 1/\kappa\rangle}(vz+1))},$$

and hence

$$\begin{aligned} R(z) &= T^{-1}(-z) - \frac{1}{z} = -v\left(\frac{\log^{\langle 1/\kappa\rangle}(1-vz)}{1+\gamma\log^{\langle 1/\kappa\rangle}(1-vz)} \times \exp_\kappa(\log^{\langle 1/\kappa\rangle}(1-vz))\right)^{-1} - \frac{1}{z} \\ &= -v\cdot\frac{1+\gamma\log^{\langle 1/\kappa\rangle}(1-vz)}{(1-vz)\log^{\langle 1/\kappa\rangle}(1-vz)} - \frac{1}{z} \\ &= -\frac{1}{z} - \frac{v\gamma}{1-vz} - \frac{v}{(1-vz)\log^{\langle 1/\kappa\rangle}(1-vz)}. \end{aligned}$$

By this expression, $R(z)$ can be defined on a domain such that $1 - vz \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. If $\kappa = \infty$, then we can argue similarly since Proposition 13 (ii) states that $\operatorname{Im} W_{\infty,\gamma}(z) \in (0,\pi)$ for $z \in \mathbb{C}^+$. $\square$

Recall that $\Omega$ denotes the codomain of $W_{\kappa,\gamma}$. By Proposition 13, for each $x \in \mathcal{S}$, there are exactly two solutions of $f_{\kappa,\gamma}(z) = x$ in $z \in \partial\Omega$, which are conjugate complex numbers, denoted by $K_+(x)$, $K_-(x)$, such that $\operatorname{Im} K_+(x) > 0$. Recall that $\alpha_1 \leq \alpha_2$ are zeros of the function $\gamma z^2 + (1 + 1/\kappa)z + 1$. Then, we have the following theorem.

**Theorem 18.** *Let $\sigma$ be a trapezoidal variance profile defined by (22). Let $\mu_\sigma$ be the probability measure corresponding to the associated covariance Stieltjes transform $T_\sigma$ given by (26). Then, the density function $d_\sigma$ of $\mu_\sigma$ is given as*

$$d_\sigma(x) = \begin{cases} \dfrac{1}{2\pi x i}\left(\dfrac{1}{K_-(-\frac{v}{x})} - \dfrac{1}{K_+(-\frac{v}{x})}\right) & (\text{if } -\frac{v}{x} \in \mathcal{S}), \\ 0 & (\text{if } -\frac{v}{x} \in \mathbb{R} \setminus \mathcal{S}). \end{cases} \tag{27}$$

*Moreover, one has the following possibilities.*

(1) *In the case $p < q$ and $\frac{q}{p} \neq \alpha$ (i.e. $\kappa \geq 1$ and $\gamma < 0$, or $\kappa < 0$ and $\gamma' < 0$), the measure $\mu_\sigma$ is absolutely continuous and its density $d_\sigma(x)$ is continuous on $\mathbb{R}$. In particular, $\mu_\sigma$ has no atoms. Its support is given as*
$$\operatorname{supp}\mu_\sigma = \left[-\frac{v}{f_{\kappa,\gamma}(\alpha_2)}, -\frac{v}{f_{\kappa,\gamma}(\alpha_1)}\right] = \left[\frac{v}{\alpha_2^2}\left(1+\frac{\alpha_2}{\kappa}\right)^{1-\kappa}, \frac{v}{\alpha_1^2}\left(1+\frac{\alpha_1}{\kappa}\right)^{1-\kappa}\right]. \tag{28}$$

(2) *In the case $p = q = \frac{1}{2}$ or $\frac{q}{p} = \alpha$ (i.e. $\kappa \geq 1$ and $\gamma = 0$, or $\kappa < 0$ and $\gamma' = 0$), the measure $\mu_\sigma$ is absolutely continuous. Its density $d_\sigma$ is continuous on $\mathbb{R}^*$ and $\lim_{x\to+0} d_\sigma(x) = +\infty$. In particular, $\mu_\sigma$ has no atoms. Let $\alpha_0 := \alpha_2$ if $\kappa \geq 1$ and $\alpha_0 := \alpha_1 = -1$ if $\kappa < 0$. The support of $\mu_\sigma$ is given as*
$$\operatorname{supp}\mu_\sigma = \left[0, -\frac{v}{f_{\kappa,\gamma}(\alpha_0)}\right] = \left[0, \frac{v}{\alpha_0^2}\left(1+\frac{\alpha_0}{\kappa}\right)^{1-\kappa}\right]. \tag{29}$$
*When $\kappa = \infty$, the measure $\mu_\sigma$ is the Dykema-Haagerup measure $\chi_v$ with support $[0, ve]$.*

(3) *In the case $p > q$ (i.e. $\kappa \geq 1$ and $0 < \gamma < 1$), we have $\mu_\sigma = d_\sigma(x)dx + (1 - \frac{q}{p})\delta_0$. The measure $\mu_\sigma$ has an atom at $x = 0$ with mass $1 - \frac{q}{p}$. Recall that $\kappa \in [1, 1/\gamma]$. When $\kappa > 1$, the support of $\mu_\sigma$ is given by (29). The function $d_\sigma$ is continuous on $\mathbb{R}^*$ and $\lim_{x\to+0} d_\sigma(x) = +\infty$. For $\kappa = 1$ and $-\infty < \gamma < 1$, the measure $\mu_\sigma$ is the Marchenko-Pastur law $\mu_C$ with parameter $C = \frac{q}{p} = 1 - \gamma \in (0, 1)$ and $\operatorname{supp} d_\sigma = \left[v(1-\sqrt{C})^2, v(1+\sqrt{C})^2\right]$.*

*Proof.* We use the formula of $T_\sigma(z)$ from Proposition 17. Let $z = x + yi$. By Proposition 13 (i) and the fact that $W_{\kappa,\gamma}(z) = 0$ only if $z = 0$, we see that $l(x) := \lim_{y\to+0} \operatorname{Im} T_\sigma(x + iy)$ exists when $x \neq 0$ and that $l(x) = 0$ when $-v/x \notin \mathcal{S}$.

Assume that $x \neq 0$ and $-v/x \in \mathcal{S}$. Let us set $a(x) + ib(x) := \lim_{y\to 0+} W_{\kappa,\gamma}(-v/z)$. Since the function $f_{\kappa,\gamma}$ is continuous and injective on the closure $\overline{D} \subset \overline{\mathbb{C}^+}$, the function $a + ib$ is continuous. By Proposition 13 (i), we have $b(x) > 0$ and $a(x) + ib(x) = K_+(-\frac{v}{x})$ . Since $\overline{\mathcal{S}} \subset (-\infty, 0)$ by Theorem 11, we have $-v/x < 0$, that is, $x > 0$. Thus, we obtain for $-v/x \in \mathcal{S}$ with $x \neq 0$
$$\begin{aligned} l(x) &= \lim_{y\to 0+} \operatorname{Im} T_\sigma(x + yi) = \operatorname{Im}\left(-\frac{1}{v} - \frac{1}{x(a(x) + ib(x))} - \frac{\gamma}{x}\right) \\ &= -\frac{1}{2xi}\left(\frac{1}{K_+(-\frac{v}{x})} - \frac{1}{K_-(-\frac{v}{x})}\right) = \frac{b(x)}{x(a(x)^2 + b(x)^2)} > 0, \end{aligned} \tag{30}$$
and thus $l(x)$ is a continuous function on $\mathbb{R}^*$. Therefore, $x \in \mathbb{R}^*$ is included in the support of $\mu_\sigma$ if and only if $-v/x \in \overline{\mathcal{S}}$. By (4), we have $d_\sigma(x) = \frac{1}{\pi} l(x)$, so that we obtain (27).

Let us consider the case (S1). In this case, since $\mathcal{S} = (f_{\kappa,\gamma}(\alpha_2), f_{\kappa,\gamma}(\alpha_1))$ and $f_{\kappa,\gamma}(\alpha_1) < 0$, we see that the condition $x \in \operatorname{supp}\mu$ is equivalent to
$$f_{\kappa,\gamma}(\alpha_2) \leq -\frac{v}{x} \leq f_{\kappa,\gamma}(\alpha_1) < 0 \iff -f_{\kappa,\gamma}(\alpha_2) \geq \frac{v}{x} \geq -f_{\kappa,\gamma}(\alpha_1) > 0 \iff -\frac{v}{f_{\kappa,\gamma}(\alpha_2)} \leq x \leq -\frac{v}{f_{\kappa,\gamma}(\alpha_1)}.$$
Recall that $\alpha_i$, $i = 1, 2$ are the real solutions of the equation $\gamma z^2 + (1 + 1/\kappa)z + 1 = 0$. For a solution $\alpha$ of this equation, we have by $1 + \alpha/\kappa = -\alpha(1 + \gamma\alpha)$
$$f_{\kappa,\gamma}(\alpha) = \frac{\alpha}{1+\gamma\alpha}\left(1+\frac{\alpha}{\kappa}\right)^\kappa = -\alpha^2\left(1+\frac{\alpha}{\kappa}\right)^{\kappa-1},$$
so that we arrive at the assertion 1. of the theorem. The argument for other two cases is similar, and hence we omit it.

Next we consider the case $x = 0$. We separate cases according to $\gamma$. First, let us assume that $\kappa \geq 1$ and $\gamma < 0$, or $\kappa < 0$ and $\gamma' < 0$. In this case, we know that $\lim_{|z|\to+\infty} W_{\kappa,\gamma}(z) = -\frac{1}{\gamma}$ (see Proposition 13 (ii-a)), and hence
$$\lim_{y\to+0} T_\sigma(yi) = \lim_{y\to+0} \frac{\exp_\kappa\big(W_{\kappa,\gamma}(-v/(yi))\big) - 1}{v} = \frac{\exp_\kappa(-1/\gamma) - 1}{v} \in \mathbb{R}.$$
Note that since $\gamma < 0$, we have $1 - \frac{1}{\kappa\gamma} \geq 0$, so that the condition $1 + \frac{z}{\kappa} \notin \mathbb{R}_-$ is satisfied for $z = -\frac{1}{\gamma}$. Thus, in this case, we have $l(0) = \lim_{y\to+0} \operatorname{Im} T(yi) = 0$ and the function $l$ is continuous at $x = 0$.

Next, let $\gamma = 0$. In this case, we have $\kappa \in [1, \infty)$ or $\kappa = \infty$. Consider first $\kappa \in [1, \infty)$. For $z \in \mathbb{C}^+$, let us set $re^{i\theta} = 1 + \frac{W_{\kappa,0}(-v/z)}{\kappa}$ $(r > 0,\ \theta \in (0, \pi))$. By Proposition 13 (ii-b), the set $D = \Omega \cap \mathbb{C}^+$ is unbounded and $f_{\kappa,0}(\infty) = \infty$. Consequently, if $z \to 0$ in $\mathbb{C}^+$, or equivalently $-v/z \to \infty$ in $\mathbb{C}^+$, then we have $W_{\kappa,0}(-v/z) \to \infty$ and $r \to +\infty$. Again by Proposition 13 (ii-b), we see that $\theta \in (0, \frac{\pi}{\kappa+1})$ so that $\sin\kappa\theta > 0$ when $z = -v/(iy) \in \mathbb{C}^+$, and thus
$$\begin{aligned} \operatorname{Im} T_\sigma(z) &= \operatorname{Im} \frac{\exp_\kappa\Big(W_{\kappa,0}(-v/z)\Big) - 1}{v} = \operatorname{Im} \frac{(re^{i\theta})^\kappa - 1}{v} \\ &= \operatorname{Im} \frac{r^\kappa\cos\kappa\theta - 1 + ir^\kappa\sin\kappa\theta}{v} = \frac{r^\kappa\sin\kappa\theta}{v} \to +\infty \quad (y \to +0). \end{aligned}$$
On the other hand, $\mu_\sigma$ does not have an atom at $x = 0$ because we have by $W_{\kappa,0}(-v/z) \to \infty$ and by $\gamma = 0$
$$yT_\sigma(iy) = -\frac{y}{v} - \frac{1}{iW_{\kappa,0}(-v/(yi))} - \frac{\gamma}{i} \to \gamma i = 0 \quad (y \to +0).$$

In the case $(\kappa,\gamma)=(\infty,0)$, $W(z)=W_{\infty,0}(z)$ is the classical Lambert function. If $z$ is in the image of $i\mathbb{R}^+$ by $W$, then $\operatorname{Re} ze^z=0$, i.e.

$$e^x(x\cos y - y\sin y)=0 \iff x=y\tan y.$$

We have $W(e^x(x\sin y+y\cos y)i)=x+iy=z$ so $\operatorname{Im} W(e^{y\tan y}\frac{y}{\cos y}i)=y$. This means that $\lim_{y\to+\infty}\operatorname{Im} W(iy)=\frac{\pi}{2}$. Since $W(\infty)=\infty$ by Proposition 13 (ii-b), we see that $W(-v/(iy))=a(y)+ib(y)$ satisfies $\lim_{y\to+0}a(y)=+\infty$ and $\lim_{y\to+0}b(y)=\frac{\pi}{2}$ so that

$$\lim_{y\to+0}\operatorname{Im} T(yi)=\lim_{y\to+0}\operatorname{Im}\frac{e^{W(-v/(yi))}-1}{v}=\lim_{y\to+0}\operatorname{Im}\frac{e^{a(y)}\cos b(y)-1+ie^{a(y)}\sin b(y)}{v}=\lim_{y\to+0}\frac{e^{a(y)}}{v}\sin b(y)=+\infty.$$

On the other hand, we see that $\mu$ does not have an atom at $x=0$ since

$$\operatorname{Im} yT(iy)=\operatorname{Im} y\left(-\frac{1}{v}-\frac{1}{iyW(-v/(iy))}\right)=\operatorname{Im}\left(-\frac{y}{v}+\frac{i}{a(y)+ib(y)}\right)=\frac{a(y)}{a^2(y)+b^2(y)}\to 0\quad (y\to 0+).$$

Let us consider the case $\kappa<0$ and $\gamma'=\gamma-\frac{1}{\kappa}=0$. In this case, we know that $\lim_{|z|\to\infty}W_{\kappa,\gamma}(z)=-\frac{1}{\gamma}=-\kappa$ by Proposition 13 (ii-a). Since $\kappa<0$, it is easy to verify that $\lim_{w\to-\kappa}|\exp_\kappa(w)|=\infty$ so that by continuity of $\exp_\kappa$ and $W_{\kappa,\gamma}$

$$\lim_{y\to+0}T(yi)=\lim_{y\to+0}\frac{\exp_\kappa\bigl(W_{\kappa,\gamma}(-v/(yi))\bigr)-1}{v}=\lim_{w\to-\kappa}\frac{\exp_\kappa(w)-1}{v}=\infty.$$

On the other hand, $\mu_\sigma$ does not have an atom at $x=0$ because we have by $W_{\kappa,\gamma}(-v/z)\to-\frac{1}{\gamma}$

$$yT(iy)=-\frac{y}{v}-\frac{1}{iW_{\kappa,\gamma}(-v/(yi))}-\frac{\gamma}{i}\to-\frac{1}{i(-1/\gamma)}-\frac{\gamma}{i}=0\quad (y\to+0).$$

Last, we assume that $0<\gamma<1$. If $\kappa>1$, we apply Proposition 13 (ii-c). When $z\to 0$, we have $-v/z\to\infty$ and $W_{\kappa,\gamma}(-\frac{v}{z})\to\infty$, so that we obtain

$$yT_\sigma(iy)=-\frac{y}{v}-\frac{1}{iW_{\kappa,\gamma}(-v/(iy))}-\frac{\gamma}{i}\to\gamma i\quad (y\to+0),$$

whence $\mu_\sigma$ has an atom at $x=0$ with mass $\gamma=1-\frac{q}{p}>0$. We omit the proof in the case $\kappa=1$, as it corresponds to the classical Wishart matrices with parameter $C=\frac{q}{p}<1$. Note that $\kappa=\infty$ does not occur because $\kappa\le\frac{1}{\gamma}$.

The absolute continuity of $\mu_\sigma$ follows from Proposition 2, by considering $\mu_0:=\mu_\sigma-d_\sigma(x)dx$, or, in the case with atom at $x=0$, of $\mu_0:=\mu_\sigma-d_\sigma(x)dx-\gamma\delta_0$ and using the fact that the Stieltjes transform $S_0(z)$ of $\mu_0$ satisfies $\lim_{y\to0+}\operatorname{Im} S_0(x+iy)=0$ for all $x\in\mathbb{R}$. The argument is similar as in the proof of Theorem 5. □

In the following Corollary, we give a real implicit equation for the density $d_\sigma$ analogous to the Dykema-Haagerup equation (3). To do so, we introduce the following notation

$$e_\kappa(z):=|\exp_\kappa(z)|\ge 0,\quad \theta_\kappa(z)=\kappa\operatorname{Arg}\left(1+\frac{z}{\kappa}\right)\quad (z\in\mathbb{C}^+).$$

If $\kappa=\infty$, we set $e_\kappa(z):=e^{\operatorname{Re} z}$ and $\theta_\kappa(z):=\operatorname{Im} z$. Then, we have $\exp_\kappa(z)=e_\kappa(z)\bigl(\cos(\theta_\kappa(z))+i\sin(\theta_\kappa(z))\bigr)$.

**Corollary 19.** (i) *Suppose $v=1$ for simplicity. For two real numbers $\kappa,\gamma$ such that $\gamma\le\frac{1}{\kappa}\le 1$ and $\gamma<1$, the density $d_\sigma$ of the limiting law $\mu_\sigma$ satisfies the equation*

$$d_\sigma\left(\frac{\sin(\theta_\kappa(z))}{b}\Bigl(1+\gamma a-\gamma b\cot(\theta_\kappa(z))\Bigr)\bigl(e_\kappa(z)\bigr)^{-1}\right)=\frac{1}{\pi}\cdot e_\kappa(z)\sin(\theta_\kappa(z))\quad (z=a+bi\in\partial D\cap\mathbb{C}^+).\tag{31}$$

*In particular, when $(\kappa,\gamma)=(\infty,0)$, the density $d_\sigma$ satisfies the equation (3) with $b=x$ and $a=-x\cot x$ $(x\in[0,\pi))$.*
(ii) *If $\kappa\in[1,\infty]$ and $\gamma<0$, then the correspondence $a\mapsto b=b(a)$ is unique for each $z=a+bi\in\partial D\cap\mathbb{C}^+$. Then, $a\in[\alpha_1,\alpha_2]$. The same is true for $\kappa=\infty$ and $\gamma=0$ with $a\in[-1,+\infty)$.*

*Proof.* (i) Let $z=a+bi\in\partial D\cap\mathbb{C}^+$. Then, it satisfies $f_{\kappa,\gamma}(z)\in\mathcal{S}$. Suppose $f_{\kappa,\gamma}(z)=-\frac{1}{x}$, and set

$$X=a+\gamma a^2+\gamma b^2,\quad Y=|1+\gamma z|^2=(1+\gamma a)^2+(\gamma b)^2.$$

Notice that $X^2+b^2=(a^2+b^2)Y$. The equation $f_{\kappa,\gamma}(z)=-\frac{1}{x}$ means that

$$\frac{e_\kappa(z)}{Y}\Bigl(X\cos(\theta_\kappa(z))-b\sin(\theta_\kappa(z))\Bigr)=-\frac{1}{x},\tag{32}$$

$$X\sin(\theta_\kappa(z))+b\cos(\theta_\kappa(z))=0.\tag{33}$$

The latter one (33) yields that $\cos(\theta_\kappa(z))=-\frac{\sin(\theta_\kappa(z))}{b}X$ so that

$$-\frac{1}{x}=-\frac{e_\kappa(z)}{Y}\cdot\frac{\sin(\theta_\kappa(z))}{b}(X^2+b^2)\iff\frac{1}{x}\cdot\frac{b}{a^2+b^2}=e_\kappa(z)\sin(\theta_\kappa(z)).$$

On the other hand, (33) can be written as $X = -b\cot\bigl(\theta_\kappa(z)\bigr)$, and using this expression together with (32), we obtain

$$-\frac{1}{x} = \frac{e_\kappa(z)}{Y}\Bigl(-b\cot\bigl(\theta_\kappa(z)\bigr)\cos\bigl(\theta_\kappa(z)\bigr) - b\sin\bigl(\theta_\kappa(z)\bigr)\Bigr) = -\frac{b}{\sin\bigl(\theta_\kappa(z)\bigr)}\cdot\frac{e_\kappa(z)}{Y} \iff x = \frac{\sin\bigl(\theta_\kappa(z)\bigr)}{b}\cdot Y\bigl(e_\kappa(z)\bigr)^{-1}.$$

It is easy to check that we have $Y = 1+\gamma a+\gamma X$. By (30), the density can be described as $d_\sigma(x) = \frac{1}{\pi x}\cdot\frac{b}{a^2+b^2}$ so that we obtain the formula (31).
(ii) Assume first that $\kappa = \infty$ so that $\gamma \le 0$. Set $z = a+bi$. Since $\mathcal{S}\subset\mathbb{R}$, $f_{\infty,\gamma}(z)\in\mathcal{S}$ means $\operatorname{Im} f_{\infty,\gamma}(z) = 0$, that is, $a+\gamma a^2+\gamma b^2+b\cot b = 0$. This equation can be rewritten as $g(b) = -a-\gamma a^2$, where $g(b) := \gamma b^2 + b\cot b$. It is easy to show that $g'(b) < 0$ for $b\in(0,\pi)$, so the function $g(b)$ is monotonic decreasing for $b\in(0,\pi)$. We have $\lim_{b\to 0+} g(b) = 1$ and $\lim_{b\to\pi-} g(b) = -\infty$. Thus, the equation $g(b) = -a-\gamma a^2$ has a solution if $-a-\gamma a^2 \le 1$, or equivalently, in case $\gamma < 0$, $\alpha_1 \le a \le \alpha_2$. Since $g$ is monotonic, for each $a\in[\alpha_1,\alpha_2]$ we can find the unique solution of the equation, which is denoted by $b(a)$. In the case $\gamma = 0$ the argument is the same with $a\in[-1,\infty)$.

Assume that $\kappa\in(1,\infty)$. Since $z = x+yi \in D = \Omega\cap\mathbb{C}^+$ satisfies $\operatorname{Arg}\bigl(1+\frac{z}{\kappa}\bigr)\in(0,\frac{\pi}{\kappa+1})$ (see Proposition 13(a)), and by the assumption $\kappa > 1$, we see that $\operatorname{Re}\Bigl(1+\frac{z}{\kappa}\Bigr) = 1+\frac{x}{\kappa} > 0$. Thus, $\theta_\kappa(x,y) = \kappa\operatorname{Arctan}\frac{y}{\kappa+x}$. Note that $\frac{\partial}{\partial y}\theta_\kappa(x,y) = \kappa\cdot\frac{\kappa+x}{(\kappa+x)^2+y^2}$. For given $x$ such that $1+\frac{x}{\kappa} > 0$, set $g(y) = y\cot(\theta_\kappa(x,y))$. We need to study the function $g(y)$ on $\mathbb{R}^+$. Set $\theta = \theta(x,y) := \operatorname{Arg}(1+\frac{x+yi}{\kappa})$ then $\theta(x,y) = \operatorname{Arctan}\frac{y}{\kappa+x}$ so that $\tan\theta = \frac{y}{\kappa+x}$ since $\theta\in(0,\frac{\pi}{2})$. Note that $\theta_\kappa(z) = \kappa\theta(x,y)$ if $z = x+yi$. Then, since

$$\frac{(\kappa+x)y}{(\kappa+x)^2+y^2} = \frac{\frac{y}{\kappa+x}}{1+\left(\frac{y}{\kappa+x}\right)^2} = \frac{\tan\theta}{1+\tan^2\theta} = \sin\theta\cos\theta = \frac{\sin 2\theta}{2},$$

we compute and estimate the derivative $g'(y)$ as follows

$$g'(y) = \cot(\theta_\kappa) + y\left(-\frac{\frac{d}{dy}\theta_\kappa(x,y)}{\sin^2(\theta_\kappa)}\right) = \frac{\sin(\theta_\kappa)\cos(\theta_\kappa) - y\frac{d}{dy}\theta_\kappa(x,y)}{\sin^2(\theta_\kappa)} = \frac{\sin(2\kappa\theta)-\kappa\sin(2\theta)}{2\sin^2(\kappa\theta)} \le 0.$$

In the last inequality we prove and use the fact that the function $H_\kappa(2\theta) := \sin(2\kappa\theta)-\kappa\sin(2\theta)$ is negative when $0<\theta<\frac{\pi}{\kappa+1}$ (see (44)). Thus, we proved that $g$ is monotonic decreasing on $\mathbb{R}^+$. Since, when $y$ is near to 0, then $\operatorname{Arctan}\frac{y}{\kappa+x} = \frac{y}{\kappa+x}+o(y)$, we see that

$$\lim_{y\to+0} g(y) = \lim_{y\to+0}\frac{y}{\sin(\kappa\operatorname{Arctan}\frac{y}{\kappa+x})}\cdot\cos(\kappa\operatorname{Arctan}\frac{y}{\kappa+x}) = \lim_{y\to+0}\frac{y}{\sin\frac{\kappa y}{\kappa+x}} = \lim_{y\to+0}\frac{\frac{\kappa y}{\kappa+x}}{\sin\frac{\kappa y}{\kappa+x}}\cdot\frac{\kappa+x}{\kappa} = 1+\frac{x}{\kappa}. \quad (S.14)$$

Our objective now is to study the function $h(y) = h(y;\,x) := x+\gamma x^2+\gamma y^2+g(y)$ for a fixed $x > -\kappa$. Recall that $h(y;\,x) = 0$ if and only if $z = x+iy\in\partial D\cap\mathbb{C}^+$. We will show that:
(a) there is exactly one solution of $h(y;\,x) = 0$ when $x\in(\alpha_1,\alpha_2)$.
(b) if $x\notin(\alpha_1,\alpha_2)$ then the equation $h(y;\,x) = 0$ does not have a solution such that $\theta(x,y)\in(0,\frac{\pi}{\kappa+1})$.

As $\gamma < 0$, we see that the function $h(y) := x+\gamma x^2+\gamma y^2+g(y)$ is decreasing on $y\in(0,y_0)$ for each fixed $x > -\kappa$. As $\kappa > 1$, there exists $y_0 > 0$ such that $\theta(x,y_0) = \operatorname{Arg}(1+\frac{x+iy_0}{\kappa}) = \frac{\pi}{\kappa+1}$. We shall show that $h(y_0;\,x) < 0$. Since $\theta_\kappa(x,y) = \kappa\theta(x,y)$ and since $\frac{\kappa\pi}{\kappa+1} = \pi-\frac{\pi}{\kappa+1}$, we have

$$\cot(\theta_\kappa(x,y_0)) = \frac{\cos\frac{\kappa\pi}{\kappa+1}}{\sin\frac{\kappa\pi}{\kappa+1}} = \frac{-\cos\frac{\pi}{\kappa+1}}{\sin\frac{\pi}{\kappa+1}} = -\frac{1}{\tan\theta(x,y_0)} = -\frac{\kappa+x}{y_0} \quad (\because \tan\theta(x,y_0) = \frac{y_0}{\kappa+x}),$$

and hence

$$h(y_0;\,x) = x+\gamma x^2+\gamma y_0^2+y_0\left(-\frac{\kappa+x}{y_0}\right) = x+\gamma x^2+\gamma y_0^2-\kappa-x = \gamma x^2+\gamma y_0^2-\kappa < 0 \quad (\because \gamma<0 \text{ and } \kappa>1).$$

By (S.14), we have $\lim_{y\to+0} h(y) = \gamma x^2+(1+\frac{1}{\kappa})x+1 = \gamma(x-\alpha_1)(x-\alpha_2)$.

(a) Suppose that $x\in(\alpha_1,\alpha_2)$, i.e. $\lim_{y\to+0} h(y) > 0$. Since $h$ is monotonic decreasing, by the intermediate value theorem, there exists a unique solution $h(y;\,x) = 0$ in $y\in(0,y_0)$ for each $x\in(\alpha_1,\alpha_2)$.

(b) If $\lim_{y\to+0} h(y;x) \le 0$ then there is no solution of $h(y) = 0$ such that $0<\theta(x,y)<\frac{\pi}{\kappa+1}$, and hence there is no $z = x+yi\in\partial D\cap\mathbb{C}^+$ such that $h(0+;\,x) < 0$.

If $\kappa = 1$, we have the classical Wishart case and we do not need to deal with it. □

**Remark 20.** Corollary 19 (ii) enables us to write the density $d_\sigma$ with one real parameter in a way similar to Dykema–Haagerup (Dykema and Haagerup, 2004, Theorem 8.9), see formula (3). In particular, in the case (a), we obtain the formula

$$d_\sigma\left(\frac{\sin b(a)}{b(a)}\Bigl(1+\gamma a-\gamma b(a)\cot b(a)\Bigr)e^{-a}\right) = \frac{1}{\pi}\cdot e^a\sin b(a) \quad (a\in[\alpha_1,\alpha_2]).$$

A natural conjecture that we always have a 1-1 correspondence $a\to b$ or $b\to a$ is not confirmed by numerical generation of the domain $\Omega$. For $\kappa = -1/3$ and $\gamma = -4$ the domain $\Omega$ is illustrated in the Figure 8. We do not have unicity of $a\to b$ nor $b\to a$.

4.3. **Applications to Wishart Ensembles of Vinberg matrices.** Now we apply Theorem 18 to the covariance matrix $X_n = Q_{\underline{k}}(\xi_n) \in P_n$ in two situations. The first (Corollary 21) is the case when $P_n$ is the symmetric cone $\mathrm{Sym}(n,\mathbb{R})^+$ with $\underline{k}$ of the form (34) below. The second situation (Theorem 24) is the general case when $P_n \subset \mathcal{U}_n$ is a dual Vinberg cone with $\underline{k}$ of the form (2). This case contains the first one, that we present separately because of the importance of the symmetric cone $\mathrm{Sym}(n,\mathbb{R})^+$.

Let us assume that $\underline{k} = \underline{k}(n)= (k_1,\dots,k_n)$ in (2) is of the form

$$\underline{k} = m_1(1,\dots,1,1) + m_2(n)(0,\dots,0,1), \quad \lim_n \frac{m_2(n)}{n} = m, \tag{34}$$

where $m_1 \in \mathbb{Z}_{\geq 0}$ is a fixed non-negative integer and $m \in \mathbb{R}_{\geq 0}$ is a non-negative real such that $m_1 + m > 0$. Set $N := k_1 + \cdots + k_n = m_1 n + m_2(n)$. We note that the case $m_1 = 0$ corresponds to the classical Wishart ensembles, and if $m_1 \geq 1$ then we have $N \geq n$.

**Corollary 21.** *Let $\underline{k}$ be as in (34). Suppose that $\xi_n \in E_{\underline{k}}$ is an i.i.d. matrix with finite fourth moments. Let $X_n = \xi_n\xi_n^\top$ and $\mu_n$ the empirical eigenvalue distribution of $X_n/n$. Then, there exists a limiting eigenvalue distribution $\mu = \lim_n \mu_n$. The Stieltjes transform $T(z)$ of $\mu$ is given by formula (26)*

$$T(z) = T_{\kappa,\gamma}(z) = \frac{\exp_\kappa\Big(W_{\kappa,\gamma}(-v/z)\Big) - 1}{v} \quad \text{with } \kappa = \frac{1}{1-m_1}, \ \gamma = 1 - m - m_1.$$

*The measure $\mu$ is absolutely continuous and has no atoms. If $m_1 = 0$ then the measure $\mu$ is the Marchenko-Pastur law with parameter $C = m$. The case $(m_1, m) = (1, 0)$ corresponds to the Dykema–Haagerup measure $\chi_v$. If $m = 0$ then the density $d$ is continuous on $\mathbb{R}^*$ and $\lim_{x\to+0} d(x) = +\infty$. When $m_1 \geq 2$ then the support of $\mu$ is $[0, v m_1^{m_1/(m_1-1)}]$. Otherwise, for $m_1, m > 0$, the density $d(x)$ of $\mu$ is continuous on $\mathbb{R}$, and its support equals $[A(\alpha_2), A(\alpha_1)]$ where $A(\alpha_i) := v\alpha_i^{-2}(1 + (1-m_1)\alpha_i)^{m_1/(m_1-1)}$ and $\alpha_1 < \alpha_2$ are roots of the function $(1 - m_1 - m)x^2 + (2 - m_1)x + 1$.*

*Proof.* We use Theorem 3. It is enough to show that the matrix $Y_n$ in (20) has the variance profile $\sigma$ in (22) and that the conditions (6) are satisfied. Since we have for $n$ large enough

$$\big|\delta_0(n)\big| \leq \frac{1}{n^2} \cdot 2v(m_1 + m + 1)n = \frac{2v(m_1+m+1)}{n} \to 0 \quad (n \to \infty)$$

and if $\mathbb{E}|Y_{ij}|^2 \neq 0$ then

$$\frac{\mathbb{E}(Y_{ij}^4)}{(n{+}N)(\mathbb{E}Y_{ij}^2)^2} = \frac{M_4}{v(n{+}N)} \to 0 \quad (n \to \infty),$$

we can easily check the conditions (6). Thus, we can apply Theorem 18. Consider $m_1 \geq 2$. Then $\kappa < 0$. When $m = 0$, then we have $\gamma' = \gamma - \frac{1}{\kappa} = 0$ so that we apply Theorem 18.2. We have $\alpha = -1$, $1 - \frac{1}{\kappa} = m_1$ and $1 - \kappa = \frac{m_1}{m_1 - 1}$. By (29), the support is given by $\operatorname{supp}\mu = \Big[0, \frac{v}{\alpha^2}\Big(1 + \frac{\alpha}{\kappa}\Big)^{1-\kappa}\Big] = \Big[0, v m_1^{m_1/(m_1-1)}\Big]$. When $m > 0$, we have $\gamma' < 0$ so that we apply Theorem 18.1. The support of $\mu$ is given by the formula (28), where $\alpha_1 \leq \alpha_2$ are roots of the function $\gamma x^2 + (1 + 1/\kappa)x + 1$. □

**Remark 22.** If $m = 0$, our results contain those of Claeys and Romano (2014, Section 4.5.1) and Cheliotis (2018, Th. 4 and (12)). The result on the limiting densities of biorthogonal ensembles in Cheliotis (2018) can be reproduced from Corollary 21. In fact, our random matrices $Q_{\underline{k}}(\xi_n)$ essentially correspond to those considered in Cheliotis (2018) through adjusting parameters $m_1 = \theta - 1$ and $m_2(n) = b - 1$ (not depending on $n$), where $\theta$ and $b$ are parameters used in that paper.

Let $x$ be an i.i.d. Gaussian random row vector in $\mathbb{R}^n$ ($x_j \sim N(0,1)$). Then, there exists an orthogonal matrix $P$ such that $xP = (0,\dots,0,|x|)$, and $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ is a random variable of chi-square distribution $\chi^2_{n/2}$ of parameter $n/2$.

Let us consider $E_{\underline{k}}$ (recall that $N = m_1 n + m_2(n)$) with each entry obeying $N(0,1)$. Each element $\xi \in E_{\underline{k}}$ can be written as $\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}$, where $\xi_k \in \mathbb{R}^N$ is a row vector of the form $\xi_k = (0,\dots,0,\eta_k)$ where $\eta_k \in \mathbb{R}^{N-(k-1)m_1} = \mathbb{R}^{(n-k+1)m_1 + m_2(n)}$. Note that the number of zeros in the $k$-th row is $(k-1)m_1$. Let us write

$$\xi = \begin{pmatrix} \xi^{[n-1]} & A_n \\ 0 & \eta_n \end{pmatrix} \quad \xi^{[n-1]} \in \mathrm{Mat}((n-1) \times ((n-1)m_1); \mathbb{R}), \quad A_n \in \mathrm{Mat}((n-1) \times (m_1 + m_2(n)); \mathbb{R}).$$

For $\eta_n$, there exists an orthogonal matrix $P_n'$ such that $\eta_n P_n' = (0,\dots,0,|\eta_n|)$, and one has

$$|\eta_n| \sim \chi^2_{(N-(n-1)m_1)/2} = \chi^2_{(m_1+m_2(n))/2}.$$

We have

$$\xi P = \begin{pmatrix} \xi^{[(n-1)]} & A_n P_n' \\ 0 & 0\cdots0|\eta_n| \end{pmatrix} \quad \text{where} \quad P = \begin{pmatrix} I_{(n-1)m_1} & 0 \\ 0 & P_n' \end{pmatrix}.$$

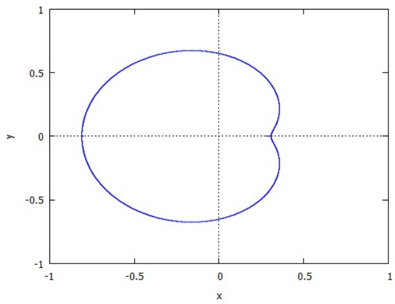



Figure 8.
Domain $\Omega$ for $(\kappa, \gamma) = (-\frac{1}{3}, -4)$.

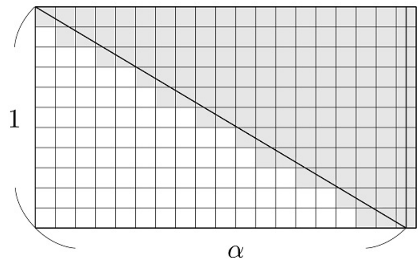



Figure 9.
Realization of non-integer $\alpha$

Since $P_n'$ is orthogonal, each element in $A_n' = A_n P_n'$ obeys $N(0,1)$. We can then apply the same argument to the matrix

$$\xi' = \begin{pmatrix} \xi^{[n-1]} & A_n'' \end{pmatrix}$$

where $A_n''$ is an $(n-1) \times (m_1 + m_2(n) - 1)$ matrix obtained from $A_n'$ removing the last column, and repeating this argument, we see that that for each $\xi \in E_{\underline{k}}$ there exists an orthogonal matrix $P$ such that $\xi P$ has the form

$$\xi P = (O_{n\times(N-n)}, T), \quad T = \begin{pmatrix} \lambda_1 & t_{12} & \cdots & t_{1n} \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & t_{n-1,n} \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}; \quad \begin{cases} \lambda_j \sim \chi^2_{((n-k+1)(m_1-1)+m_2(n)+1)/2} & (j = 1, \ldots, n), \\ t_{ij} \sim N(0,1) & (1 \le i < j \le n). \end{cases}$$

Here, $O_{n\times(N-n)}$ is the zero matrix of size $n \times (N-n)$. Thus, in the notation $\theta, b$ in Cheliotis (2018) we have $\theta = m_1 - 1$ and $b = m_2(n) + 1$. Note that we take $T$ upper triangular whereas Cheliotis (2018) lower triangular.

**Remark 23.** Until now, we assumed that $m_1 \in \mathbb{Z}_{\ge 0}$ and hence the parameter $\alpha$ of the variance profile $\sigma$ needs to be also an integer. However, we can take a sequence $\{\underline{k}(n)\}_{n=1}^{\infty}$ so that the corresponding $\alpha$ is an arbitrary given positive real number. In fact, when $\alpha > 0$ is given, we consider a right triangle with lengths 1 and $\alpha$. For an arbitrary $n$, we cover the triangle by $1/n \times 1/n$ squares as in the figure. To each $j = 1, \ldots, n$, we associate an integer $k_j(n)$ such that $\frac{k_j(n)}{n} \le \frac{j}{n}\alpha < \frac{k_j(n)+1}{n}$, or equivalently $k_j(n) \le j\alpha < k_j(n) + 1$, and we set $k(n) = (k_1(n), \ldots, k_n(n))$. Note that this condition is independent of $n$ so that $k_j(m) = k_j(n)$ when $m \ge n \ge j$, and hence $\{E_{\underline{k}(n)}\}_n$ is a sequence of vector spaces such that $E_{\underline{k}(n)} \subset E_{\underline{k}(n+1)}$.

In the Figure 9, we set $\alpha = 1.8$, $n = 11$ and $k(n) = (1, 2, 2, 1, 2, 2, 1, 2, 2, 1, 2)$.

Let us return to the quadratic Wishart case for general $P_n$ with parameter $\underline{k}$ as in (2) such that $m_1, m_2 \in \mathbb{Z}_{\ge 0}$ are fixed. Note that $m_2(n)$ in the previous discussion is now $m_2(n) = m_2 b_n$. Set $N_n := m_1 n + m_2 b_n$. We have

$$E_{\underline{k}} = \left\{ \xi = \begin{pmatrix} \eta \\ \zeta \end{pmatrix} \in \mathrm{Mat}(n \times N_n, \mathbb{R});\ \begin{matrix} \eta = (\eta_{ij}) \in \mathrm{Mat}(a_n \times N_n, \mathbb{R}),\ \zeta = (\zeta_{ij}) \in \mathrm{Mat}(b_n \times N_n, \mathbb{R}) \\ \eta_{ij} = 0 \text{ if } j \le (m_1 - 1)i, \\ \zeta_{ij} = 0 \text{ if } j - m_1 a_n - (m_1 + m_2)(i-1) \notin \{1, 2, \ldots, m_1 + m_2\} \end{matrix} \right\}.$$

**Theorem 24.** *Let $\{P_n\}_n$ be a sequence of generalized dual Vinberg cones such that $\lim_{n\to\infty} a_n/n = c \in (0,1]$. Let $\underline{k}$ be a vector as in (2) such that $m_1, m_2$ are fixed. Set $\kappa := 1/(1-m_1)$ and $\gamma := 1 - \big(m_1 + m_2(1-c)\big)/c$. Then, the Stieltjes transform $T(z)$ of the limiting eigenvalue distribution of $Q_{\underline{k}}(\xi_n)/n$ with i.i.d. matrices $\xi_n \in E_{\underline{k}}$ is given as*

$$T(z) = -\frac{1}{v} - \frac{c}{zW_{\kappa,\gamma}(-\frac{cv}{z})} - \frac{c\gamma + 1 - c}{z} = \frac{\exp_\kappa\big(W_{\kappa,\gamma}(-vc/z)\big) - 1}{v} - \frac{1-c}{z}. \quad (z \in \mathbb{C}^+)$$

The properties of absolute continuity and support of the limiting measure can be derived analogously to those obtained in Theorem 18 for $c = 1$.

*Proof.* We construct a variance profile $\sigma$ from $E_{\underline{k}}$ likely to (22). We embed the rectangular matrix $\xi_n \in E_{\underline{k}}$ in a square matrix $Y(\xi_n) = \begin{pmatrix} 0 & \xi_n \\ \xi_n^\top & 0 \end{pmatrix}$, and set $V_n = \{Y(\xi_n);\ \xi_n \in E_{\underline{k}}\}$. Set $p' = \lim\limits_{n\to\infty} \dfrac{n}{n + N_n} = \dfrac{1}{1 + m_1 + m_2(1-c)}$. Let $\sigma$ be a function $[0,1] \times [0,1] \to \mathbb{R}_{\ge 0}$ defined by $\sigma(x,y) = v$ if (i) $x < cp'$ and $y \ge p' + m_1 x$, or if (ii) $x \ge p'$ and $0 \le y \le \min\{(x-p')/m_1, cp'\}$, and $\sigma(x,y) = 0$ otherwise.

Then, we can show that $\sigma$ is the variance profile of $V_n$. On the other hand, let us consider a subspace $E'_{\underline{k}} := \{\xi = \binom{\eta}{\zeta} \in E_{\underline{k}};\ \zeta = 0\}$ of $E_{\underline{k}}$, and let $V_n' = \Big\{Y(\xi_n);\ \xi_n \in E'_{\underline{k}}\Big\}$. Then, $\sigma$ is also the variance profile of $V_n'$. Thus, we consider equivalently the limiting eigenvalue distribution of $V_n'$, and that of covariance matrices on $E'_{\underline{k}}$. If $\xi_n = \binom{\eta_n}{0} \in E'_{\underline{k}}$, then $\frac{1}{n} Q_{\underline{k}}(\xi_n) = \frac{1}{n}\begin{pmatrix} \eta_n \eta_n^\top & 0 \\ 0 & 0 \end{pmatrix}$, and thus it is enough to study the limiting eigenvalue distribution of $\frac{1}{n}\eta_n\eta_n^\top$. The variance profile of covariance matrix $\frac{1}{a_n}\eta_n\eta_n^\top$, rescaled by size of $\eta_n\eta_n^\top$, has a trapezoidal form (22)

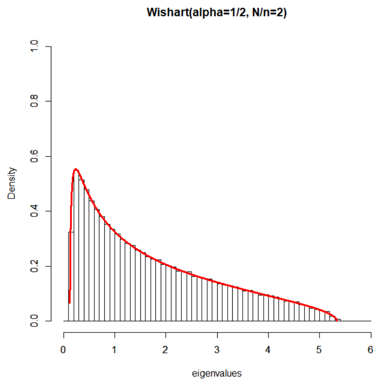



FIGURE 10.
Simulation for $\alpha = 1/2$

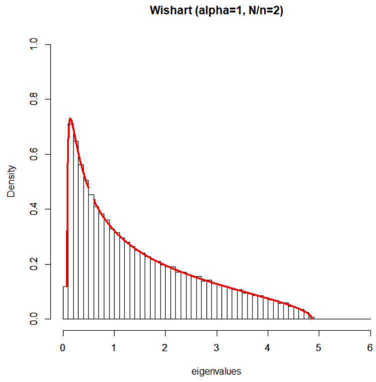



FIGURE 11.
Simulation for $\alpha = 1$

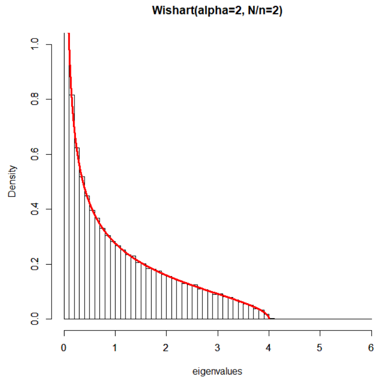



FIGURE 12.
Simulation for $\alpha = 2$

(illustrated by (23)) with parameters $\alpha = m_1$ and $p = \lim_n \frac{a_n}{a_n+N_n} = \frac{c}{c+m_1+m_2(1-c)}$. Applying Proposition 17, we see that the corresponding Stieltjes transform $T_1(z)$ is given by

$$T_1(z) = T_{\kappa,\gamma}(z) \quad \text{with} \quad \kappa = \frac{1}{1-m_1}, \quad \gamma = \frac{2p-1}{p} = \frac{c - m_1 - m_2(1-c)}{c}.$$

In general, for two symmetric matrices $A_i$ $(i = 1, 2)$ of size $n_i$, the Stieltjes transform $S(z)$ of $\mathrm{diag}(A_1,\ A_2)/(n_1+n_2)$ can be described by using the Stieltjes transforms $S_i(z)$ of $A_i/n_i$ $(i = 1, 2)$ as

$$S(z) = S_1\left(\frac{n_1+n_2}{n_1}z\right) + S_2\left(\frac{n_1+n_2}{n_2}z\right) \quad (z \in \mathbb{C}^+). \tag{35}$$

In our situation, we have $(n_1, n_2) = (a_n, b_n)$ and $(A_1, A_2) = (\eta_n\eta_n^\top, 0)$. Hence, we have $S_2(z) = -\frac{1}{z}$ and $S_1(z)$ is the Stieltjes transform of $\eta_n\eta_n^\top/a_n$ so that $\lim_{n\to\infty} S_1(z) = T_1(z)$. Thus, taking the limit $n \to \infty$, we get the limiting Stieltjes transform $T(z)$ corresponding to $E'_{\underline{k}}$, and hence to $E_{\underline{k}}$ by using (35) with $S_1(z) = T_1(z)$ and $S_2(z) = -\frac{1}{z}$, which proves the corollary. $\square$

**Remark 25.** In the Figures 10-12 we present simulations of $\underline{k}$-indexed Wishart ensembles $X_n = Q_{\underline{k}}(\xi_n)$ on the symmetric cone $\mathrm{Sym}^+(n, \mathbb{R})$ (i.e. $c = 1$), for $n = 4000$ and $N = |\underline{k}| = 2n$ with parameters $\alpha = m_1 = 1/2$, 1 and 2, respectively. We have $\gamma = -1$ and $\kappa = 2, \infty, -1$ respectively. The red line is the graph of $d(x)$ generated by the R program from its Stieltjes transform given in Corollary 21. In two first cases, the limiting density $d(x)$ is continuous on $\mathbb{R}$ with compact support contained in $(0, \infty)$. The last case $(\kappa, \gamma) = (-1, -1)$ corresponds to $(\kappa', \gamma') = (1, 0)$ which is the classical Wishart case with $C = 1$. Thus its density explodes to $\infty$ at 0.

**Remark 26.** Let $Y_n$ be a rectangular $n \times p$ i.i.d. matrix with variance profile $\sigma^2(x, y)$, and assume that $\lim_{n\to\infty} p/n = c$. In papers Hachem et al. (2005, 2006, 2008) a functional equation

$$\tau(u, z) = \big(-z + \int_0^1 \sigma^2(u, v)\big(1 + c\int_0^1 \sigma^2(x, v)\tau(x, z)dx\big)^{-1}dv\big)^{-1}$$

is given to get the limiting Stieltjes transform $f(z)$ for the rescaled random matrices $Y_nY_n^*$, as the integral $\int_0^1 \tau(u, z)du$. This equation appears in Girko (1990) in the setting of Gram matrices based on Gaussian fields, cf. (Hachem et al., 2006, Remark 3.1).

However, thanks to symmetry, solving the equations (24) resulting from Theorem 3 is easier than solving the last functional-integral equation for $\tau(u, z)$. Therefore we opted for variance profile method for Gaussian and Wigner ensembles as the main tool of studying Wishart ensembles of Vinberg matrices.

## 5. Complementary remarks

**Remark 27** (Modified Vinberg matrices)**.** Observe that the variance profiles (11) and (22) remain the same when we consider the following two cases (a) and (b) of modified Vinberg matrices. This is due to the fact that different forms of $d$ in the lower right block of the matrix $U$ can be absorbed by the perturbation terms $\delta_{ij}$ in (5). Accordingly, we obtain Theorem 5 for the Wigner Ensembles and Theorems 18 and 24 and Corollaries 19 and 21 for the Wishart Ensembles on the corresponding matrix spaces.
(a) For $s = 0$ or $s \in 2\mathbb{N}+1$, let $\mathcal{U}_n^s$ be the subspace of $\mathrm{Sym}(n,\mathbb{R})$ defined by

$$\mathcal{U}_n^s = \left\{ U = \begin{pmatrix} x & y \\ y^\top & d \end{pmatrix} ; \begin{array}{l} x \in \mathrm{Sym}(a_n,\mathbb{R}),\ y \in \mathrm{Mat}(a_n \times b_n,\mathbb{R}), \\ d \text{ is a } s\text{-diagonal matrix of size } b_n \end{array} \right\}.$$

Here, 0-diagonal matrix means the zero matrix. From the statistical point of view of Gaussian covariance models (Lauritzen (1996)), the space $\mathcal{U}_n^0$ does not apply, because covariance matrices have non-zero diagonal terms and non-zero determinant.
(b) Take $k \in \mathbb{Z}_{>0}$ and assume that each $b_n$ is a multiple of $k$, say $b_n = kc_n$. Let $\mathcal{U}_n^{(k)}$ be a subspace of $\mathrm{Sym}(a_n+b_n,\mathbb{R})$ consisting of matrices $U$ of the form above with $d$ being a block diagonal matrix $d = \mathrm{diag}(d_1,\ldots,d_{c_n})$, where each $d_j$ is a square matrix of size $k$.

**Remark 28** (Matrix Ensembles related to dual cones of $P_n$)**.** In this remark, we consider the dual cone $Q_{G_n}$ of $P_n$, which is realized as a minimal matrix form in the sense of Yamasaki and Nomura (2015) as follows. Let $\mathcal{V}_n$ be a subspace of $\mathrm{Sym}(a_n(b_n+1),\,\mathbb{R})$ defined by

$$\mathcal{V}_n := \left\{ \mathrm{diag}\left( \begin{pmatrix} x & y_1 \\ y_1^\top & d_1 \end{pmatrix}, \ldots, \begin{pmatrix} x & y_{b_n} \\ y_{b_n}^\top & d_{b_n} \end{pmatrix} \right) ; \begin{array}{l} x \in \mathrm{Sym}(a_n,\mathbb{R}), \\ y_1,\ldots,y_{b_n} \in \mathbb{R}^{a_n}, \\ d_1,\ldots,d_{b_n} \in \mathbb{R} \end{array} \right\}. \tag{36}$$

Then, the dual cone $Q_{G_n}$ is described as $Q_{G_n} := \mathcal{V}_n \cap \mathrm{Sym}(a_n(b_n+1),\,\mathbb{R})^+$.

We consider Wigner Ensembles $V_n \in \mathcal{V}_n$ and quadratic Wishart Ensembles $X_n \in Q_{G_n}$ as those in the sense of $\mathrm{Sym}(a_n(b_n+1),\,\mathbb{R})$. Assume that $\lim_{n\to+\infty} a_n = \infty$. By the theory of lower rank perturbation (see Tao (2012, §2.4.1), for example), the study of eigenvalue distributions of these ensembles boils down to the study of the eigenvalue distributions of $x$ and, after suitable normalization, the limiting eigenvalue distributions of $V_n$ and $X_n$ are the same as for $x \in \mathrm{Sym}(a_n,\,\mathbb{R})$.

This essential difference in the Random Matrix Theory for the cones $Q_{G_n}$ and $P_n$ may be explained by a substantial difference between the cones $Q_{G_n}$ and $P_n$ in terms of numbers of *sources* in the sense of Yamasaki and Nomura (2015). In the case $P_n$, there is only one source so that $P_n$ can be realized in a usual matrix form. On the other hand, $Q_{G_n}$ has $b_n$ sources so that $b_n$ copies of a usual matrix form appear.

**Remark 29** (Relation of Vinberg cones to graphical models)**.** Let $G$ be a graph with vertices $V = \{1,2,\ldots,n\}$ and edges $E$. We say that a statistical character $\mathcal{X} = (X_1,\ldots,X_n)$ has the *dependence graph* $G$ when each conditional independence of marginals $X_i$ and $X_j$ with respect to remaining variables corresponds to the absence of the edge $\{i,j\}$ in $E$. Thus the dependence graph $G$ is a tool of encoding of the conditional independence of marginals of $\mathcal{X}$. We say that $\mathcal{X}$ belongs to the graphical model governed by $G$.

Let $\mathcal{U}_G$ be the subspace of $\mathrm{Sym}(n,\mathbb{R})$ containing matrices with $u_{ij} = 0$ if the edge $\{i,j\} \notin E$. Cones $P_G = \mathrm{Sym}(n,\mathbb{R})^+ \cap \mathcal{U}_G$ and their dual cones $Q_G$ are basic objects of graphical model theory. Actually, a Gaussian $n$-dimensional model $N(m,\Sigma)$ is governed by the graph $G$ if and only if the inverse covariance matrix $\Sigma^{-1} \in P_G$ (cf. Lauritzen (1996)).

An important class of graphical models, called *daisy graphs*, is defined as follows. Let $a+b=n$ and let $D(a,b)$ be a graph with vertices $V = \{1,\ldots,n\}$, such that the first $a$ elements form a complete graph and the latter $b$ elements are satellites(petals) of the complete graph, that is, each satellite connects to all elements in the complete graph and does not connect to the other satellites (see Figure 13 below). The double circle around the vertex $a_n$ in Figure 13 indicates the complete graph with $a_n$ vertices.

In high dimensional statistics, it is essential to let the number of observed characters $n$ tend to infinity. From the graphical model theory point of view, the pattern of the growing graphs $G_n$ and of the corresponding cones $P_{G_n}$ should remain the same. This requirement is met by growing daisy graphs $D(a_n,b_n)$ for non-decreasing sequences of positive integers $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ such that $a_n + b_n = n$.

The position of zeros in Wigner and Wishart Vinberg matrices in $\mathcal{U}_n$ considered in this paper is encoded by daisy graphs $D(a_n,b_n)$ by setting $u_{ij} = 0$ whenever $i$ and $j$ are not connected by an edge in $D(a_n,b_n)$. We can also consider the class of generalized daisy graphs $D(a,b,k)$, with $b$ complete satellites of $k$ vertices, so that there are $N = a+kb$ vertices. If all three sequences $a_n, b_n, k_n$ are non-decreasing, the graphs $D(a_n,b_n,k_n)$ form a growing sequence of graphical models. The case when $k_n = k$ is fixed for $n$ large enough corresponds to Remark 27(b).

Mathematical bases of Wishart distributions on matrix cones related to decomposable and homogeneous graphs considered in this paper were laid down by Lauritzen (1996); Letac and Massam (2007); Ishi (2014); Graczyk and Ishi (2014). This paper is a first step towards studying RMT related to growing Gaussian graphical models.

Note that statistical Gaussian graphical models is a different notion from Erdös-Rényi random graphs, which are deeply studied in the RMT (cf. the book of Durrett (2006)).

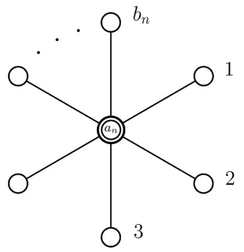



FIGURE 13. Daisy Graph

## 6. APPENDIX

In this Appendix, we give proofs of Theorem S9, Theorem 11 and Proposition 13.

6.1. **Proofs.** By definition, $f_{\kappa,\gamma}(z)$ has a pole at $z=-1/\gamma$ when $\gamma \neq \frac{1}{\kappa}$, and $z=-\kappa$ may be a branch point of $f$. We first assume that $\kappa > 0$. Although the condition on $\kappa$ is $\kappa \geq 1$ when $\kappa$ is positive, we also deal with the case $0<\kappa<1$ in order to apply it to the case $\kappa<0$. We have

$$f'(z) = \frac{\kappa\gamma z^2 + \big(1+\kappa\big)z + \kappa}{\kappa(1+\gamma z)^2}\left(1+\frac{z}{\kappa}\right)^{\kappa-1}.$$

Let $\alpha_1,\ \alpha_2$ be the two solutions of $g(z) := \kappa\gamma z^2 + \big(1+\kappa\big)z+\kappa = 0$. Then, $f'(z)=0$ implies $z=\alpha_i$ $(i=1,2)$ or $z=-\kappa$ if $\kappa>1$.

Set $z=x+yi$. We have

$$\frac{z}{1+\gamma z} = \frac{z(1+\gamma\bar{z})}{|1+\gamma z|^2} = \frac{(x+\gamma x^2+\gamma y^2)+i(y+\gamma xy-\gamma xy)}{(1+\gamma x)^2+\gamma^2 y^2} = \frac{(x+\gamma x^2+\gamma y^2)+iy}{(1+\gamma x)^2+\gamma^2y^2}$$

and

$$\begin{aligned}\left(1+\frac{z}{\kappa}\right)^{\kappa} &= \exp\left(\kappa\left(\log\left|1+\frac{z}{\kappa}\right| + i\mathrm{Arg}\left(1+\frac{z}{\kappa}\right)\right)\right) = \left(\left(1+\frac{x}{\kappa}\right)^2+\frac{y^2}{\kappa^2}\right)^{\frac{\kappa}{2}} e^{i\kappa\theta(x,y)} \\ &= \left(\left(1+\frac{x}{\kappa}\right)^2+\frac{y^2}{\kappa^2}\right)^{\frac{\kappa}{2}}\Big(\cos(\kappa\theta(x,y))+i\sin(\kappa\theta(x,y))\Big),\end{aligned}$$

where

$$\theta(x,y) = \mathrm{Arg}\Big(1+\frac{z}{\kappa}\Big).$$

Here $\mathrm{Arg}(w)$ stands for the principal argument of $w$; $-\pi < \mathrm{Arg}(w) \leq \pi$. Note that we now take the main branch of power function. Thus,

$$\begin{aligned} f(z) &= \frac{\Big((1+x/\kappa)^2+(y/\kappa)^2\Big)^{\frac{\kappa}{2}}}{(1+\gamma x)^2+\gamma y^2}(x+\gamma x^2+\gamma^2y^2+iy)\Big(\cos(\kappa\theta(x,y))+i\sin(\kappa\theta(x,y))\Big) \\ &= \frac{\Big((1+x/\kappa)^2+(y/\kappa)^2\Big)^{\frac{\kappa}{2}}}{(1+\gamma x)^2+\gamma^2y^2}\left(\begin{array}{c}(x+\gamma x^2+\gamma y^2)\cos(\kappa\theta(x,y))-y\sin(\kappa\theta(x,y)) \\ +i\big\{(x+\gamma x^2+\gamma y^2)\sin(\kappa\theta(x,y))+y\cos(\kappa\theta(x,y))\big\}\end{array}\right)\end{aligned} \tag{37}$$

We want to know the inverse image of the real axis, that is, $f^{-1}(\mathbb{R})$.

To do so, we consider the implicit function

$$(x+\gamma x^2+\gamma y^2)\sin(\kappa\theta(x,y))+y\cos(\kappa\theta(x,y)) = 0.$$

If $\sin(\kappa\theta(x,y))=0$, then $\cos(\kappa\theta(x,y))$ does not vanish so that $y$ needs to be zero. Moreover, in this case we also have $x \geq -\kappa$ if $\kappa$ is not integer; otherwise, if $x<-\kappa$ then $\theta(x,y)\to\pi$ as $y\to+0$, but then $\sin(\kappa\pi)\neq 0$ whenever $\kappa\notin\mathbb{Z}$.

Assume that $\sin(\kappa\theta(x,y))\neq 0$. Then the equation can be rewritten as

$$(x+\gamma x^2+\gamma y^2)+y\cot(\kappa\theta(x,y)) = 0. \tag{38}$$

If we change variables by

$$re^{i\theta} = 1+\frac{z}{\kappa}, \quad \text{or equivalently} \quad x=\kappa(r\cos\theta-1), \quad y=\kappa r\sin\theta, \tag{39}$$

then the equation (38) can be written as

$$\begin{aligned} & \kappa(r\cos\theta-1)+\gamma\big\{(\kappa(r\cos\theta-1))^2+(\kappa r\sin\theta)^2\big\}+\kappa r\sin\theta\cot(\kappa\theta)=0 \\ \iff\ & \gamma\kappa^2r^2+\big\{\kappa\cos\theta-2\gamma\kappa^2\cos\theta+\kappa\sin\theta\cot(\kappa\theta)\big\}r+(\gamma\kappa^2-\kappa)=0 \\ \iff\ & \gamma\kappa r^2+\left\{\frac{\sin((\kappa+1)\theta)}{\sin(\kappa\theta)}-2\gamma\kappa\cos\theta\right\}r+\gamma\kappa-1=0.\end{aligned}$$

In the last, we use

$$\cos\theta+\sin\theta\cot(\kappa\theta) = \frac{\cos\theta\sin(\kappa\theta)+\sin\theta\cos(\kappa\theta)}{\sin(\kappa\theta)} = \frac{\sin((\kappa+1)\theta)}{\sin(\kappa\theta)}.$$

Set

$$b(\theta) := \frac{\sin((\kappa+1)\theta)}{\sin(\kappa\theta)} - 2\gamma\kappa\cos\theta. \tag{40}$$

We have

$$\lim_{\theta\to 0} b(\theta) = \frac{\kappa+1}{\kappa} - 2\gamma\kappa =: b(0),$$

and the solution in $r$ of the equation in the case $\theta = 0$

$$0 = \gamma\kappa r^2 + (\frac{\kappa+1}{\kappa} - 2\gamma\kappa)r + \gamma\kappa - 1 = \gamma\kappa(r^2 - 2r + 1) + (1 + \frac{1}{\kappa})r - 1 = \gamma\kappa(r-1)^2 + (1+\frac{1}{\kappa})(r-1) + \frac{1}{\kappa}$$

is given as

$$r = 1 + \frac{-(1+1/\kappa) \pm \sqrt{(1+1/\kappa)^2 - 4\gamma}}{2\gamma\kappa}.$$

Note that these two $r = r_\pm$ correspond in $(x,y)$ coordinates to $\alpha_1, \alpha_2$ because $1 + \frac{x}{\kappa} = r$ and because the equation defining $r_\pm$ can be rewritten as

$$0 = \gamma\kappa(r-1)^2 + (1+\tfrac{1}{\kappa})(r-1) + \tfrac{1}{\kappa} = \gamma\kappa\cdot\frac{x^2}{\kappa^2} + (1+\tfrac{1}{\kappa})\frac{x}{\kappa} + \frac{1}{\kappa} = \frac{\gamma\kappa x^2 + (\kappa+1)x + \kappa}{\kappa^2}. \tag{41}$$

We also note that, if we set $(x,y) = (0,0)$, or equivalently $(r,\theta) = (1,0)$ then

$$x + \gamma x^2 + \gamma y^2 + y\cot(\kappa\theta(x,y)) = \kappa\Big(\gamma\kappa r^2 + b(\theta)r + \gamma\kappa - 1\Big) = 1 > 0.$$

Let $\Omega$ be the connected component of $\{z \in \mathbb{C};\ (x + \gamma x^2 + \gamma y^2) + y\cot(\kappa\theta(x,y)) > 0\}$ including $z = 0$. Let $D = \Omega \cap \mathbb{C}^+$. For $\theta > 0$, the equation

$$\gamma\kappa r^2 + b(\theta)r + \gamma\kappa - 1 = 0 \tag{42}$$

has a (formal) solution

$$r = r_\pm(\theta) = \frac{-b(\theta) \pm \sqrt{b(\theta)^2 - 4\gamma\kappa(\gamma\kappa - 1)}}{2\gamma\kappa}.$$

We want $r$ to be positive real. Set $D(\theta) = b(\theta)^2 - 4a(a-1)$. We have for $\varepsilon = \pm 1$

$$r'_\varepsilon(\theta) = \frac{1}{2a}\left(-b'(\theta) + \varepsilon\frac{2b(\theta)b'(\theta)}{2\sqrt{D(\theta)}}\right) = \frac{-\varepsilon b'(\theta)}{2a}\cdot\frac{-b(\theta) + \varepsilon\sqrt{D(\theta)}}{\sqrt{D(\theta)}} = -\varepsilon b'(\theta)\frac{r_\varepsilon(\theta)}{\sqrt{D(\theta)}}. \tag{43}$$

We shall show that $f_{\kappa,\gamma}$ maps $D \to \mathbb{C}^+$ bijectively, and its main tool is the following Argument Principle (see Ahlfors (1979, Theorem 18, p.152), for example).

**Theorem 30** (Ahlfors (1979, Theorem 18, p.152)). **The argument principle.** *If $f(z)$ is meromorphic in a domain $\Omega$ with the zeros $a_j$ and the poles $b_k$, then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_j n(\gamma, a_j) - \sum_k n(\gamma, b_k)$$

*for every cycle $\gamma$ which is homologous to zero in $\Omega$ and does not pass through any of the zeros or poles. Here, $n(\gamma, a)$ is the winding number of $\gamma$ with respect to $a$.*

We also use the following lemma.

**Lemma 31.** *Let $f(z) = u(x,y) + iv(x,y)$ be a holomorphic function. The implicit function $v(x,y) = 0$ has an intersection point at $z = x + yi$ only if $f'(z) = 0$.*

*Proof.* Let $p(t) = (x(t), y(t))$ be a continuous path in $\mathbb{C} \cong \mathbb{R}^2$ satisfying $v\big(p(t)\big) = 0$ for all $t \in [0,1]$. We assume that $(x'(t), y'(t)) \neq (0,0)$. Set

$$g(t) := u(p(t)) = u(x(t), y(t)), \quad h(t) := v(p(t)) = v(x(t), y(t)).$$

Obviously, we have $h'(t) \equiv 0$ for any $t$, and

$$h'(t) = v_x x'(t) + v_y y'(t) = (v_x, v_y)\cdot(x'(t), y'(t)).$$

Assume that $g'(t_0) = 0$ for some point $t_0 \in [0,1]$. Then

$$\begin{aligned} g'(t) &= u_x x'(t) + u_y y'(t) = (u_x, u_y)\cdot(x'(t), y'(t)) \\ &= (v_x, v_y)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\cdot(x'(t), y'(t)) = (v_x, v_y)\cdot(-y'(t), x'(t)), \end{aligned}$$

the condition $g'(t_0) = 0$ implies that the vector $(v_x, v_y)$ is orthogonal both to $(x'(t_0), y'(t_0))$ and $(-y'(t_0), x'(t_0))$, which are non-zero vectors and mutually orthogonal. Such vector is only zero vector in $\mathbb{R}^2$, that is, $(v_x, v_y) = (0,0)$, and hence $(u_x, u_y) = (0,0)$ by Cauchy-Riemann equations. Thus, if $g'(t_0) = 0$ then $p(t_0)$ needs to satisfy $f'(p(t_0)) = 0$. $\square$

Recall that we now assume $\kappa > 0$. Set $a := \kappa\gamma$. We will consider the cases (i) $a < 0$, (ii) $0 < a < 1$ and $\kappa > 1$, and some other exceptional cases. It is usually sufficient to consider $D$ because $\Omega$ has a symmetry with respect to the real axis. For brevity, we set $\theta_0 := \frac{\pi}{\kappa}$ and $\theta_1 := \frac{\pi}{\kappa+1}$. Note that $\theta_0 > \theta_1$.

6.2. **The case of $a = \kappa\gamma < 0$, $\kappa > 0$.** In this case, $\alpha_1 < \alpha_2$ because $(1+\kappa)^2 - 4a\kappa > 0$. Since $a < 0$ we have $\gamma < 0$ and $g(0) = \kappa > 0$, $g(-\kappa) = (a-1)\kappa^2 < 0$, $g(-1/\gamma) = \kappa - 1/\gamma > 0$. This means that

$$-\kappa < \alpha_1 < 0 < -\frac{1}{\gamma} < \alpha_2.$$

Note that $D(0) = (1+1/\kappa)^2 - 4a/\kappa > 0$ and

$$\begin{aligned} D(\theta_1) &= (-2a\cos\theta_1)^2 - 4a(a-1) = 4a^2\cos^2\theta_1 - 4a^2 + 4a \\ &= 4a - 4a^2\sin^2\theta_1 = 4a(1 - a\sin^2\theta_1) < 0. \end{aligned}$$

This implies that there exists a $\theta \in (0,\theta_1)$ such that $D(\theta) = 0$. We denote by $\theta_* \in (0,\theta_1)$ the smallest positive real such that $D(\theta_*) = 0$.

We show now that $D$ is bounded and $D \subset \left\{ z \in \mathbb{C}^+;\ \mathrm{Arg}(1+\frac{z}{\kappa}) \in (0, \frac{\pi}{\kappa+1}) \right\}$

We shall show that $D(\theta)$ is monotonic decreasing in the interval $(0,\theta_*)$. We have

$$\begin{aligned} b'(\theta) &= \frac{(\kappa{+}1)\cos((\kappa+1)\theta)\sin\kappa\theta - \kappa\sin((\kappa{+}1))\theta\cos\kappa\theta}{\sin^2\kappa\theta} + 2a\sin\theta \\ &= \frac{-\kappa\sin\theta + \cos((\kappa+1)\theta)\sin\kappa\theta}{\sin^2\kappa\theta} + 2a\sin\theta \\ &= \frac{-\kappa\sin\theta + \frac{1}{2}(\sin((2\kappa+1)\theta) - \sin\theta)}{\sin^2\kappa\theta} + 2a\sin\theta \\ &= \frac{\sin((2\kappa+1)\theta) - (2\kappa+1)\sin\theta}{2\sin^2\kappa\theta} + 2a\sin\theta. \end{aligned}$$

Note that $2\kappa+1 > 1$ since now we assume that $\kappa > 0$. Let us consider the function

$$H_\alpha(\theta) := \sin\alpha\theta - \alpha\sin\theta \quad \text{for} \quad \alpha > 1. \tag{44}$$

For a small enough $\theta$ we have

$$H_\alpha(\theta) = \alpha\theta - \frac{(\alpha\theta)^3}{6} - \alpha(\theta - \frac{\theta^3}{6}) + o(\theta^3) = -\alpha\frac{\alpha^2-1}{6}\theta^3 + o(\theta^3) < 0$$

and by

$$H'_\alpha(\theta) = \alpha\cos(\alpha\theta) - \alpha\cos\theta = -2\alpha\sin\frac{\alpha+1}{2}\theta\,\sin\frac{\alpha-1}{2}\theta,$$

we see that $H_\alpha$ is decreasing in the interval $(0, 2\pi/(\alpha{+}1))$, and in particular is negative. Therefore, since $a\sin\theta < 0$, $b'(\theta)$ is also negative in the interval $(0,\theta_1)$. This means that $b(\theta)$ is decreasing. Note that $b(0) = 1 + 1/\kappa - 2a > 0$ and the sign $\mathsf{s}$ of $b(\theta_1) = -2a\cos\theta_1$ depends on $\kappa$.

If $\mathsf{s} \geq 0$ then we see that $D'(\theta) = 2b(\theta)b'(\theta) < 0$ so that $D$ is monotonic decreasing. Let us assume that $\mathsf{s} < 0$. In this case, since $b$ is monotonic decreasing, there is a unique $\varphi$ such that $b(\varphi) = 0$. Since $D'(\theta) = 2b(\theta)b'(\theta)$, we need to have $\theta_* < \varphi$. In fact, if not so, then we have $D(\varphi) > 0$ by definition of $\theta_*$. Since $b(\theta)$ is monotonic $b(\theta) < 0$ for any $\theta \in (\varphi,\theta_1)$, we see that $D'(\theta) = 2b(\theta)b'(\theta) > 0$ in the same interval. But it contradicts the fact that $D(\theta_1) < 0$.

Set $\varphi = \theta_1$ when $\mathsf{s} \geq 0$. Therefore, we obtain that $D$ is monotonic decreasing in the interval $(0,\varphi)$ containing $\theta_*$. In particular, $D$ is monotonic decreasing in the interval $(0,\theta_*)$ in both cases, and $D(\theta_* + \delta) < 0$ for small enough $\delta > 0$; more precisely, $\theta_* + \delta < \varphi$. Therefore, $r_\pm$ are defined on $(0,\theta_*]$ and $r_\pm$ are not defined for $\theta \in (\theta_*,\varphi)$. Since $r_+(\theta_*) = r_-(\theta_*)$ by the fact $D(\theta_*) = 0$, the curves $r_+(\theta)$, $\theta \in (0,\theta_*]$ followed by $r_-(\theta_* - \theta)$, $\theta \in (0,\theta_*]$, form a continuous curve going from $\alpha_2$ to $\alpha_1$ in the upper half-plane. Denote it by $r_{+-}$.

Since $r_+ \cdot r_- = 1 - \frac{1}{a} > 0$ and $-(r_+ + r_-) = \frac{b(\theta)}{a} < 0$ for $\theta \in (0,\theta_*)$, Vieta's formulas tell us that two solutions of (42) are both positive. Consequently, $r_+(\theta)$ is increasing while $r_-$ is decreasing by (43).

In order to study the set $\mathcal{S}$, let us consider $f(x)$ for the real $x \in [\alpha_1,\alpha_2]$. By differentiating, we have

$$f'_{\kappa,\gamma}(x) = \frac{\gamma x^2 + (1+1/\kappa)x + 1}{(1+\gamma x)^2}\left(1+\frac{x}{\kappa}\right)^{\kappa-1} = \frac{\gamma(x-\alpha_1)(x-\alpha_2)}{(1+\gamma x)^2}\left(1+\frac{x}{\kappa}\right)^{\kappa-1}.$$

Since $\gamma < 0$, we have

| $x$ | $\alpha_1$ | $\cdots$ | $0$ | $\cdots$ | $-\frac{1}{\gamma}$ | $\cdots$ | $\alpha_2$ |
|---|---|---|---|---|---|---|---|
| $f'$ | | $+$ | | | $\times$ | $+$ | $0$ |
| $f$ | $f(\alpha_1)$ | $\nearrow$ | $0$ | $\nearrow\infty$ | $\times$ | $-\infty\nearrow$ | $f(\alpha_2)$ |

where

$$\lim_{h\to -0} f(-\tfrac{1}{\gamma} + h) = +\infty, \qquad \lim_{h\to +0} f(-\tfrac{1}{\gamma} + h) = -\infty.$$

Here, $\times$ means that $f$ and $f'$ is not defined at that point. See Figure 18.

**Claim.** One has $0 > f(\alpha_1) > f(\alpha_2)$.

*Proof of the claim.* $0 > f(\alpha_1)$ is obvious by the above table. We shall show $f(\alpha_1) > f(\alpha_2)$. By the fact that $\alpha_1\alpha_2 = \frac{1}{\gamma}$, we have

$$\frac{f(\alpha_2)}{f(\alpha_1)} = \frac{\alpha_2(1+\gamma\alpha_1)}{(1+\gamma\alpha_2)\alpha_1} \cdot \left(\frac{1+\alpha_2/\kappa}{1+\alpha_1/\kappa}\right)^{\kappa} = \frac{\alpha_2+1}{\alpha_1+1} \cdot \left(\frac{1+\alpha_2/\kappa}{1+\alpha_1/\kappa}\right)^{\kappa}.$$

Since $1+\gamma\alpha_2 < 0$ and $\alpha_1 < 0$, we have $\alpha_1 + 1 = (1+\gamma\alpha_2)\alpha_1 > 0$. Moreover, the facts that $1+\alpha_1/\kappa > 0$ and $\alpha_2 > \alpha_1$ yield that

$$\frac{\alpha_2+1}{\alpha_1+1} > 1 \quad \text{and} \quad \frac{1+\alpha_2/\kappa}{1+\alpha_1/\kappa} > 1,$$

whence we obtain $\dfrac{f(\alpha_2)}{f(\alpha_1)} > 1$. Since $f(\alpha_2) < 0$ because $\alpha_2 > -\frac{1}{\gamma}$ and $\gamma < 0$, we conclude that $0 > f(\alpha_1) > f(\alpha_2)$. □

Thus, for the case $\kappa > 0$ and $\gamma < 0$ we have (S1) $\mathcal{S} = (f_{\kappa,\gamma}(\alpha_2), f_{\kappa,\gamma}(\alpha_1))$, where $f_{\kappa,\gamma}(\alpha_2) < f_{\kappa,\gamma}(\alpha_1) < 0$.

Now we show that $f_{\kappa,\gamma}\colon D \to \mathbb{C}^+$ is bijective.

We take a path $C = C(t)$ ($t \in [0,1]$) in such a way that by starting from $z = -\frac{1}{\gamma}$, it goes to $z = \alpha_2$ along the real axis, next goes to $z = \alpha_1$ along the curve $r_{+-}$ defined by (38) and connecting $\alpha_2$ and $\alpha_1$ in the upper half plane, and then it goes to $z = -\frac{1}{\gamma}$ along the real axis (see Figure 15). Here, we can assume that $C'(t) \neq 0$ whenever $C(t) \neq \alpha_i$, $i = 1, 2$. Actually, the curve $v(x,y) = 0$ has a tangent line unless $f'$ vanishes. If we take an arc-length parameter $t$, then $C'(t)$ represents the direction of the tangent line at $(x,y) = C(t)$. We note that $C(t)$ describes the boundary of $D$.

We first show that $f_{\kappa,\gamma}$ maps the boundary of $D$ to $\mathbb{R}$ bijectively. We take $t_i$, $i = 1, 2$ as $C(t_i) = \alpha_i$. Note that the sub-curve $C(t)$, $t \in (t_2, t_1)$ describes the curve $r_{+-}(t)$, and $f_{\kappa,\gamma}$ does not have a pole or singular point on $C(t)$, $t \in (t_2, t_1)$. Set $f(z) = u(x,y) + iv(x,y)$. By Lemma 31, the implicit function $v(x,y) = 0$ may have an intersection point only if $f'(x+iy) = 0$, i.e. at $x + iy = \alpha_i$ $(i = 1, 2)$ or at $x + iy = -\kappa$ if $\kappa > 1$. Then, the function $g(t) = u(C(t))$, $t \in [t_2, t_1]$ attains maximum and minimum in the interval because it is a continuous function on a compact set. Moreover, $g'$ never vanishes in $(t_2, t_1)$ by the above argument and by the fact that $f'(C(t)) \neq 0$ for $t \in (t_2, t_1)$. Therefore, $g$ is monotone and hence it takes maximal and minimal values at the endpoints $t = t_2, t_1$. Now we have $f(\alpha_1) > f(\alpha_2)$ by the last claim so that the image of $g$ is $[f(\alpha_2), f(\alpha_1)]$, and the function $g$ is bijective.

We shall show that for any $w_0 \in \mathbb{C}^+$ there exists one and only one $z_0 \in D$ such that $f(z_0) = w_0$. Let us take an $R > 0$ such that $|w_0| < R$. For $\delta > 0$, let $C' = C_\delta$ be a path obtained from $C$ in such a way that the pole $z = -1/\gamma$ is avoided by a semi-circle $-\frac{1}{\gamma} + \delta e^{i\theta}$, $\theta \in (0,\pi)$ of radius $\delta$ (see Figure 16). Denote by $D'$ the domain surrounded by the curve $C'$.

Then, we can choose $\delta > 0$ such that

$$\left| f\big(-\tfrac{1}{\gamma} + \delta e^{i\theta}\big)\right| > R \quad (\text{for all } \theta \in (0,\pi)).$$

In fact, if $z = -\frac{1}{\gamma} + \delta e^{i\theta}$, then we have

$$\big|1+\gamma z\big| = |\gamma|\delta, \quad \big|z\big| = \big|-\tfrac{1}{\gamma} + \delta e^{i\theta}\big| > \frac{1}{2|\gamma|} \quad (\text{if } \delta < \tfrac{1}{2|\gamma|}),$$

and

$$\big|1 + \tfrac{z}{\kappa}\big| = \big|1 - \tfrac{1}{\kappa\gamma} + \tfrac{\delta}{\kappa}e^{i\theta}\big| > \frac{\kappa\gamma - 1}{2\kappa\gamma} \quad (\text{if } \delta < \tfrac{\kappa}{2}\big|1 - \tfrac{1}{\kappa\gamma}\big|),$$

so that

$$\left| f\big(-\tfrac{1}{\gamma} + \delta e^{i\theta}\big)\right| > \frac{1}{2|\gamma|^2}\Big(\frac{\kappa\gamma-1}{2\kappa\gamma}\Big)^{\kappa} \cdot \frac{1}{\delta}.$$

Thus it is enough to take

$$\delta = \min\Big(\tfrac{1}{2|\gamma|^2 R}\big(\tfrac{\kappa\gamma-1}{2\kappa\gamma}\big)^{\kappa},\ \tfrac{1}{2|\gamma|},\ \tfrac{\kappa}{2}\big|1 - \tfrac{1}{\kappa\gamma}\big|\Big).$$

Since $f$ is non-singular on the semi-circle $-\frac{1}{\gamma} + \delta e^{i\theta}$, $\theta \in [0,\pi]$, the curve $\theta \mapsto f(-\frac{1}{\gamma} + \delta e^{i\theta})$ does not have a singular angular point, so that it is homotopic to a large semicircle (with radius larger than $R$) in the upper half-plane (see Figure 17).

Note that

$$\operatorname{Im} f(x+yi) = \frac{\Big((1+x/\kappa)^2 + (y/\kappa)^2\Big)^{\frac{\kappa}{2}}}{(1+\gamma x)^2 + \gamma^2 y^2}\big\{(x + \gamma x^2 + \gamma y^2)\sin(\kappa\theta(x,y)) + y\cos(\kappa\theta(x,y))\big\}.$$

By changing variables as in (39), we have

$$\begin{aligned}\operatorname{Im} f(re^{i\theta}) &= \text{positive factor} \times \sin(\kappa\theta) \cdot (ar^2 + b(\theta)r + a - 1) \\ &= \text{positive factor} \times \sin(\kappa\theta) \cdot a(r - r_-(\theta))(r - r_+(\theta)).\end{aligned}$$

Note that the inside of the path $C$ can be written as $\{re^{i\theta};\ \theta \in (0,\theta_*),\ r \in (r_-(\theta), r_+(\theta))\}$ in $(r,\theta)$ coordinates. Since $a<0$ and $\sin(\kappa\theta)>0$ when $\theta \in (0,\theta_*)$, we see that $\operatorname{Im} f(z) > 0$ if $z$ is inside of the path $C$. In particular, the inside set of the curve $f(C')$ is a bounded domain in $\mathbb{C}^+$ including $w_0$.

Since the winding number of the path $f(C')$ with respect to $w = w_0$ is exactly one, we see that

$$\frac{1}{2\pi i}\int_{C'} \frac{f'(z)}{f(z)-w_0}dz = \frac{1}{2\pi i}\int_{f(C')} \frac{dw}{w-w_0} = 1.$$

By definition of $f$, we see that $f(z)-w_0$ does not have a pole in $D'$. Therefore, by the argument principle, the function $f(z)-w_0$ has only one zero point, say $z_0 \in D' \subset D$. Thus, we obtain $f(z_0) = w_0$, and such $z_0 \in D$ is unique. We conclude that the map $f$ is a bijection from $D$ to the upper half-plane $\mathbb{C}^+$.

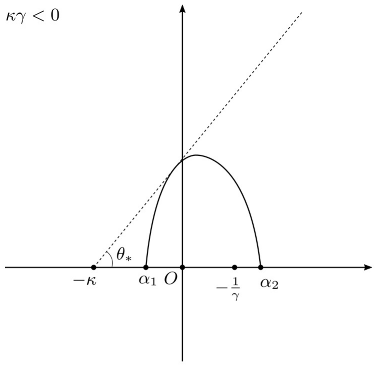



FIGURE 14. The case of (i)

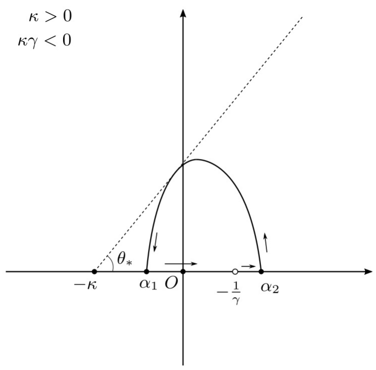



FIGURE 15. The case of (i)

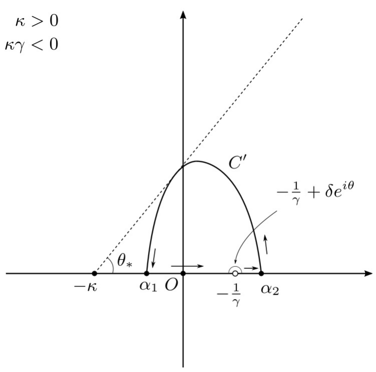



FIGURE 16. Curve $C'$ in case (i)

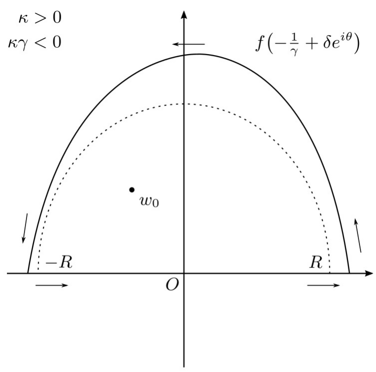



FIGURE 17. Curve $f(C')$ in case (i)

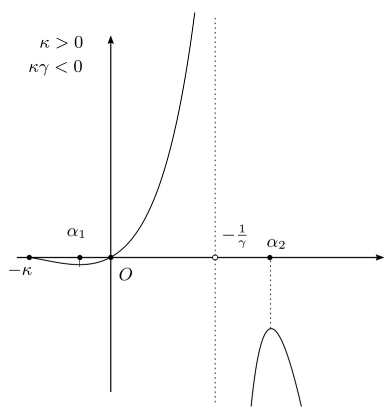



FIGURE 18. $f(x)$ for $x \geq -\kappa$, case (i)

**6.3. The case of** $0 < a = \kappa\gamma < 1$**.** In this case, we have $(1+\kappa)^2 - 4a\kappa = (1+\kappa-2a)^2 + 4a(1-a) > 0$ so that $\alpha_1 < \alpha_2$ are real. Since $0 < a < 1$ we have $\gamma > 0$ and $-1/\gamma < -\kappa$. Since $g(0) = \kappa > 0$, $g(-\kappa) = (a-1)\kappa^2 < 0$ and $g(-1/\gamma) = -1/\gamma + \kappa < 0$, we have

$$\alpha_1 < -\frac{1}{\gamma} < -\kappa < \alpha_2 < 0.$$

Let us prove that $D$ is unbounded and $D \subset \{z \in \mathbb{C}^+;\ \mathrm{Arg}(1+\frac{z}{\kappa}) \in (0, \frac{\pi}{\kappa})\}$.

Since $D(\theta) = b(\theta)^2 + 4a(1-a) > 0$, we always have two real solutions for the equation (42). By $r_+ \cdot r_- = \frac{a-1}{a} < 0$, only one of $r_+$, $r_-$ is a positive solution. Since $|b(\theta)| < \sqrt{D(\theta)}$, we see that

$$r = r_+(\theta) = \frac{\sqrt{D(\theta)} - b(\theta)}{2a}$$

is the only positive real solution of (42). In the same way as in (41) we see that $\lim_{\theta\to+0} r_+(\theta) = \alpha_2$. Recall that $\kappa > 1$.

We use a calculation from Section 5.2. Now we show that $b'(\theta)$ is negative on the interval $(\theta_1, \theta_0)$ ($\theta_0 = \pi/\kappa$ and $\theta_1 = \pi/(\kappa+1)$). Recall that

$$b(\theta) = \cos\theta + \sin\theta\ \cot(\kappa\theta) - 2a\cos\theta = (1-2a)\cos\theta + \sin\theta\ \cot(\kappa\theta).$$

Using this expression, we have

$$\begin{aligned} b'(\theta) &= (2a-1)\sin\theta + \cos\theta\cot(\kappa\theta) + (\sin\theta)\big(-(1+\cot^2(\kappa\theta))\cdot\kappa\big) \\ &= (2a-1-\kappa)\sin\theta + \cos\theta\cot(\kappa\theta) - \kappa\sin\theta\cot^2(\kappa\theta) \\ &= (2a-1-\kappa)\sin\theta + \{\cos\theta\cot(\kappa\theta) - \sin\theta\cot^2(\kappa\theta)\} - (\kappa-1)\sin\theta\cot^2(\kappa\theta) \\ &= (2a-1-\kappa)\sin\theta + \frac{\cos\theta\sin(\kappa\theta) - \sin\theta\cos(\kappa\theta)}{\sin(\kappa\theta)}\cdot\cot(\kappa\theta) - (\kappa-1)\sin\theta\cot^2(\kappa\theta) \\ &= (2a-1-\kappa)\sin\theta + \frac{\sin((\kappa-1)\theta)}{\sin(\kappa\theta)}\cdot\cot(\kappa\theta) - (\kappa-1)\sin\theta\cot^2(\kappa\theta). \end{aligned}$$

Let us assume that $\theta \in (\theta_1, \theta_0)$. Then, since the assumption $\kappa > 1$ yields that

$$0 < \frac{\pi}{\kappa+1} < \theta < \frac{\pi}{\kappa} < \pi, \quad \frac{\pi}{2} < \frac{\kappa\pi}{\kappa+1} < \kappa\theta < \pi, \quad 0 < \frac{(\kappa-1)\pi}{\kappa+1} < (\kappa-1)\theta < \frac{(\kappa-1)\pi}{\kappa} < \pi,$$

we see that for $\theta \in (\theta_1, \theta_0)$

$$\sin\theta > 0, \quad \sin((\kappa-1)\theta) > 0, \quad \sin(\kappa\theta) > 0, \quad \cos(\kappa\theta) < 0, \quad \cot(\kappa\theta) < 0.$$

Since $2a - 1 - \kappa < 0$ and $\kappa - 1 > 0$ by $a < 1$ and $\kappa > 1$, we arrive at

$$b'(\theta)\Big(= (-)\times(+) + (+)\times(-) - (+)\times(+)\times(+)\Big) < 0 \quad (\theta \in (\theta_1, \theta_2)).$$

Thus $b(\theta)$ is decreasing on the interval $(\theta_1, \theta_0)$ and since $\sin\theta > 0$ for $\theta \in (\theta_1, \theta_0)$, we have

$$\lim_{\theta\to\theta_0-0} b(\theta) = -\infty.$$

Recall that $D'(\theta) = 2b(\theta)b'(\theta)$. Since we have $b(\theta_1) = -2a\cos\theta_1 < 0$ and $b$ is decreasing, we see that $b < 0$ on the interval $(\theta_1, \theta_0)$. Accordingly, $D(\theta)$ and $r_+(\theta)$ are increasing when $\theta \in (\theta_1, \theta_0)$ by (43). Since $\lim_{\theta\to\theta_0-0} r_+(\theta) = +\infty$, the solution of (42) has an asymptotic line with gradient $\theta = \theta_2 = \frac{\pi}{\kappa}$ in $(r, \theta)$ coordinates. It corresponds to the line $x\sin\theta_0 - y\cos\theta_2 = A$ with a suitable constant $A$. Let us determine $A$. Since $x = \kappa(r(\theta)\cos\theta - 1)$ and $y = \kappa r(\theta)\sin\theta$, we have

$$\begin{aligned} x\sin\theta_0 - y\cos\theta_0 &= \kappa\big\{\sin\theta_0(r(\theta)\cos\theta - 1) - \cos\theta_0 r(\theta)\sin\theta\big\} \\ &= \kappa\big\{r(\theta)(\cos\theta\sin\theta_0 - \sin\theta\cos\theta_0) - \sin\theta_0\big\} \\ &= -\kappa\big\{r(\theta)\sin(\theta-\theta_0) + \sin\theta_0\big\}. \end{aligned}$$

Next, we estimate $r(\theta)$ as $\theta \to \theta_0 - 0$. Since $\sin\kappa\theta \to +0$ as $\theta \to \theta_0 - 0$ (i.e. $\sin(\kappa\theta) = o(\theta - \theta_0)$), we have

$$(\sin(\kappa\theta))b(\theta) = \sin((\kappa+1)\theta) + \varepsilon\sin((\kappa+1)\theta) + \varepsilon$$

and

$$\begin{aligned} (\sin(\kappa\theta))^2 D(\theta) &= (\sin(\kappa\theta)\,b(\theta))^2 + 4a(1-a)(\sin(\kappa\theta))^2 \\ &= (\sin((\kappa+1)\theta) + \varepsilon)^2 + 4a(1-a)(\sin(\kappa\theta))^2 \\ &= (\sin((\kappa+1)\theta))^2 + \varepsilon, \end{aligned}$$

where $\varepsilon = o(\theta - \theta_0)$. Therefore, since $\sin((\kappa+1)\theta) < 0$ when $\theta_1 < \theta < \theta_0$, we obtain

$$\sin(\kappa\theta)\,r(\theta) = \frac{\sqrt{(\sin(\kappa\theta))^2 D(\theta)} - \sin(\kappa\theta)\,b(\theta)}{2a} = \frac{|\sin((\kappa+1)\theta)| - \sin((\kappa+1)\theta) + \varepsilon}{2a} = \frac{-\sin((\kappa+1)\theta)}{a} + \varepsilon.$$

Moreover, if $\theta < \theta_0$ is enough close to $\theta_0$, then

$$\sin((\kappa+1)\theta) = \sin((\kappa+1)\theta_0) + \varepsilon = \sin\Big(\pi + \frac{\pi}{\kappa}\Big) + \varepsilon = -\sin\theta_0 + \varepsilon.$$

This tells us that

$$x\sin\theta_0 - y\cos\theta_0 = -\kappa\left\{\frac{\sin(\theta-\theta_0)}{\sin\kappa\theta}\cdot\left(\frac{\sin\theta_0}{a}+\varepsilon\right)+\sin\theta_0\right\}.$$

Since $\sin(\kappa\theta)=\sin(\pi-\kappa\theta)=-\sin\bigl(\kappa(\theta-\theta_0)\bigr)$, we see that

$$\lim_{\theta\to\theta_0-0}\frac{\sin(\theta-\theta_0)}{\sin\kappa\theta}=\lim_{\theta\to\theta_0-0}-\frac{\sin(\theta-\theta_0)}{\sin\bigl(\kappa(\theta-\theta_0)\bigr)}=\lim_{\theta\to\theta_0-0}-\frac{\theta-\theta_0}{\kappa(\theta-\theta_0)}=-\frac{1}{\kappa},$$

and hence

$$\lim_{\theta\to\theta_0-0}-\kappa\left\{\frac{\sin(\theta-\theta_0)}{\sin\kappa\theta}\cdot\left(\frac{\sin\theta_0}{a}+\varepsilon\right)+\sin\theta_0\right\}=-\kappa\left(-\frac{1}{\kappa}\cdot\frac{\sin\theta_0}{a}+\sin\theta_0\right)=\left(\frac{1}{a}-\kappa\right)\sin\theta_0.$$

This means that $A=\frac{1}{a}-\kappa$ and hence the solution of (42) has an asymptotic line $x\sin\theta_0-y\cos\theta_0=(\frac{1}{a}-\kappa)\sin\theta_0$, or $y=\tan\theta_0(x+\kappa-\frac{1}{a})$.

If $1<\kappa\le 2$, then the asymptotic line is in the second quadrant. If $\kappa>2$, the asymptotic line enters the first quadrant. This is a reason why we need the assumption $\kappa>1$. In fact, if $\kappa<1$ then its asymptotic line is in the third quadrant (if we extend $f$ by analytic continuation) and so we cannot conclude that $f$ maps $\mathbb{C}^+$ onto $\mathbb{C}^+$.

In order to determine the set $\mathcal{S}$, let us consider $f(x)$ for real $x\in[\alpha_2,+\infty)$. Note that $\gamma>0$. In this case, we have

| $x$ | $\alpha_2$ | $\cdots$ | $0$ | $\cdots$ | $+\infty$ |
|---|---|---|---|---|---|
| $f'$ | $0$ | | $+$ | | |
| $f$ | $f(\alpha_2)$ | $\nearrow$ | $0$ | $\nearrow$ | $+\infty$ |

$$\lim_{x\to+\infty}f(x)=+\infty.$$

See Figure 23. Thus, if $\kappa>1$ and $\gamma>0$ then we have (S2) $\mathcal{S}=(-\infty,f_{\kappa,\gamma}(\alpha_2))$, where $f_{\kappa,\gamma}(\alpha_2)<0$.

Now we show that $f_{\kappa,\gamma}\colon D\to\mathbb{C}^+$ is bijective.

We take a path $C=C(t)$, $t\in(0,1]$ in such a way that by starting from $z=\infty$, it goes to $z=\alpha_2$ along the curve $r_+$ defined by (38) in the upper half plane, and then goes to $z=\infty$ along the real axis (see Figure 20). Here, we can assume that $C'(t)\neq 0$ whenever $C(t)\neq\alpha_i$, $i=1,2$. Actually, the curve $v(x,y)=0$ has a tangent line unless $f'$ vanishes. If we take an arc-length parameter $t$, then $C'(t)$ represents the direction of the tangent line at $(x,y)=C(t)$. We note that $C(t)$ describes the boundary of $D$.

We first show that $f_{\kappa,\gamma}$ maps the boundary of $D$ onto $\mathbb{R}$ bijectively. We take $t_2$ such that $C(t_2)=\alpha_2$. Then, the subcurve $C(t)$, $t\in(0,t_2)$ describes the curve $r=r_+(\theta)$, $\theta\in(0,\theta_0)$. Let us see that $f(z)$ ($z\in\mathbb{C}$) diverges when $|z|\to+\infty$. In fact, let $1+\frac{z}{\kappa}=Le^{i\theta}$ with $L>1$. Since $L=\left|1+\frac{z}{\kappa}\right|\le 1+\frac{|z|}{\kappa}$, we have

$$\frac{|z|}{\kappa}\ge L-1,\quad\text{and hence}\quad\frac{1}{|z|}\le\frac{1}{\kappa(L-1)}\le\frac{1}{L-1}\quad(\text{because }\kappa>1).$$

If we take $L$ big enough so that $\frac{1}{L-1}<\gamma$, then

$$\left|\frac{1}{z}+\gamma\right|\le\frac{1}{|z|}+\gamma\le\frac{1}{L-1}+\gamma\le 2\gamma,\quad\text{or}\quad\frac{1}{|(1/z)+\gamma|}\ge\frac{1}{2\gamma},$$

and hence

$$\bigl|f(z)\bigr|=\left|\frac{1}{(1/z)+\gamma}\right|\cdot\left|1+\frac{z}{\kappa}\right|^{\kappa}\ge\frac{L^{\kappa}}{2\gamma}\longrightarrow+\infty\quad(\text{as }L\to+\infty)\tag{45}$$

Therefore, $f(z)$ diverges when $|z|\to+\infty$. We now consider the limit $|z|\to\infty$ along to the path $C(t)$ as $t\to+0$. Recall that $C(t)$ has an asymptotic line $y=(\tan\theta_0)(x+\kappa-\frac{1}{a})$. If $z$ with $1+\frac{z}{\kappa}=Le^{i\theta}$ is on the curve $C(t)$, $t\in(0,t_2)$, and goes to $\infty$ under the condition $\theta\to\theta_0-0$ (that is, we consider the limit along the curve $C(t)$), then we have

$$\left(1+\frac{z}{\kappa}\right)^{\kappa}=L^{\kappa}\cdot e^{i\theta\kappa}\longrightarrow-\infty,\qquad\frac{z}{1+\gamma z}=\frac{1}{(1/z)+\gamma}\longrightarrow\frac{1}{\gamma},$$

and thus the function $g(t):=f(C(t))$ satisfies

$$\lim_{t\to+0}g(t)=\lim_{t\to+0}f\bigl(C(t)\bigr)=-\infty.$$

This shows that if $0<t<t_2$ (recall that $C(t_2)=\alpha_2$), then $g(t)<f(\alpha_2)$ and $g(t)=f(C(t))$ is monotonic increasing. If not so, it leads to a contradiction by Lemma 31, using the fact that $C$ does not include a singular point except for $z=\alpha_2$. Finally we see that $g(t)=f(C(t))$, $t\in(0,1)$ is monotonic from $-\infty$ to $+\infty$.

We shall show that for any $w_0\in\mathbb{C}^+$ there exists one and only one $z_0\in D$ such that $f(z_0)=w_0$. Let us take an $R>0$ such that $|w_0|<R$. For $L>0$, let $\Gamma_L$ be the the circle $-\kappa+Le^{i\theta}$ of origin $z=-\kappa$ with radius $L$. Let $L-\kappa$ and $z_L$ be two distinct intersection points of $C$ and $\Gamma_L$. Let $C':=C_L$ be a closed path obtained from $C$ by connecting $L-\kappa$ and $z_L$ via the arc $A$ of $\Gamma_L$ included in the upper half plane, see Figure 21.

Since $f$ is non-singular on the arc $A$, the curve $f(A)$ does not have a singular point so that it is homotopic to a large semi-circle (whose radius is larger than $R$) in the upper half plane. Note that the domain $D$ that we consider is given in $(r,\theta)$ coordinates as $\{(r,\theta);\ \theta\in(0,\theta_0),\ r>r_+(\theta)\}$. Since

$$\begin{aligned}\operatorname{Im}f(re^{i\theta})&=\text{positive factor}\times\sin(\kappa\theta)\cdot(ar^2+b(\theta)r+a-1)\\&=\text{positive factor}\times\sin(\kappa\theta)\cdot a(r-r_-(\theta))(r-r_+(\theta))\end{aligned}$$

and since $a > 0$ and $\sin(\kappa\theta) > 0$ for $\theta \in (0, \theta_0)$, we see that $\operatorname{Im} f(re^{i\theta})$ is positive on the domain $D$. (see Figure 22). In particular, the inside set $f(D')$ of the curve $f(C')$ is a bounded domain including $w_0 \in \mathbb{C}^+$. Since the winding number of the path $f(C')$ about $w = w_0$ is exactly one, we see that

$$\frac{1}{2\pi i}\int_{C'} \frac{f'(z)}{f(z) - w_0} dz = \frac{1}{2\pi i}\int_{f(C')} \frac{dw}{w - w_0} = 1.$$

We know by definition of $f$ that $f$ does not have a pole on $D'$. Therefore, by the argument principle, the function $f(z) - w_0$ has the only one zero point, say $z_0 \in D'$. Then, we obtain $f(z_0) = w_0$, and such $z_0$ is unique. We conclude that the map $f$ is bijection from the interior set $D$ of $C$ to the upper half plane.

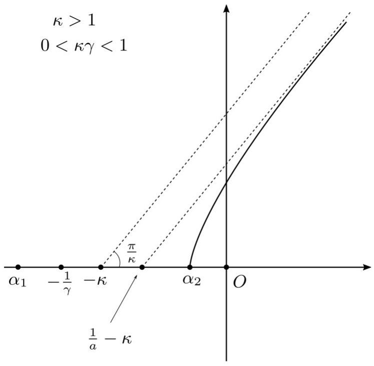



FIGURE 19. The case of (ii),when $\kappa > 2$

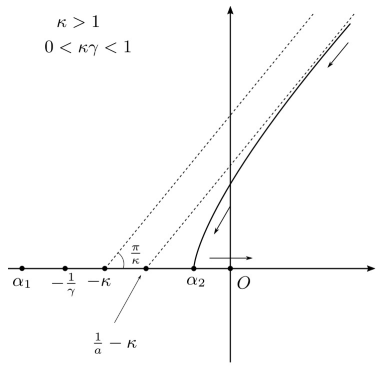



FIGURE 20. The case of (ii), when $\kappa > 2$

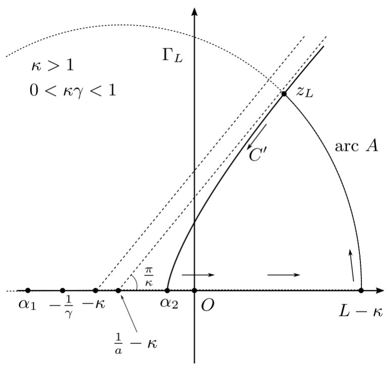



FIGURE 21. Curve $C'$ in case (ii)

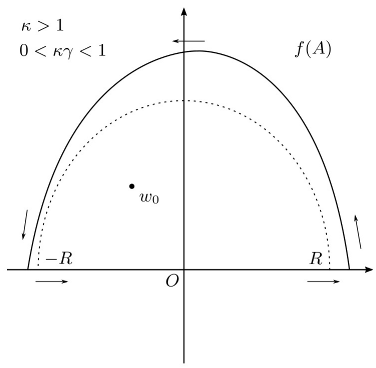



FIGURE 22. Curve $f(C')$ in case (ii)

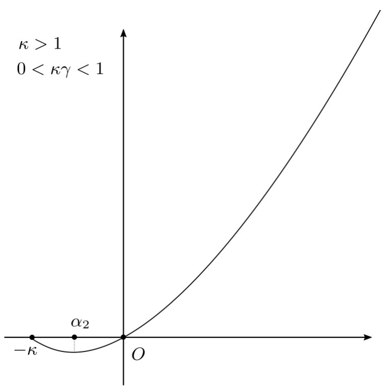



FIGURE 23. $f(x)$ for $x \geq -\kappa$, case (ii)

6.4. **Extremal cases ($\kappa = \pm\infty$, $\gamma = 0$, $\kappa\gamma = 1$).**

6.4.1. *Case $\gamma = 0$, $\kappa > 0$.* In this case, we have by (37)

$$f(z) = z\Big(1+\frac{z}{\kappa}\Big)^{\kappa} = \left(\Big(1+\frac{x}{\kappa}\Big)^2 + \frac{y^2}{\kappa^2}\right)^{\frac{\kappa}{2}} (x+iy)\Big(\cos(\kappa\theta(x,y)) + i\sin(\kappa\theta(x,y))\Big)$$
$$= \left(\Big(1+\frac{x}{\kappa}\Big)^2 + \frac{y^2}{\kappa^2}\right)^{\frac{\kappa}{2}} \begin{pmatrix} x\cos(\kappa\theta(x,y)) - y\sin(\kappa\theta(x,y)) \\ +i(x\sin(\kappa\theta(x,y)) + y\cos(\kappa\theta(x,y))) \end{pmatrix}$$

and

$$f'(z) = \frac{(\kappa+1)z+\kappa}{\kappa}\left(1+\frac{z}{\kappa}\right)^{\kappa-1}.$$

Note that if $f'(z) = 0$ then $z = -\kappa/(\kappa+1)$ (set $\alpha_2 = -\kappa/(\kappa+1)$) or $z = -\kappa$ if $\kappa > 1$, and

$$-\kappa < -\frac{\kappa}{\kappa+1} < 0.$$

We show that $D$ is unbounded and $D \subset \Big\{z \in \mathbb{C}^+;\ \mathrm{Arg}(1+\frac{z}{\kappa}) \in (0, \frac{\pi}{\kappa+1})\Big\}$.

Let us consider the curve $\mathrm{Im}\, f(z) = 0$, that is,

$$x\sin(\kappa\theta(x,y)) + y\cos(\kappa\theta(x,y)) = 0.$$

If $\sin(\kappa\theta(x,y)) = 0$, then $\cos(\kappa\theta(x,y))$ does not vanish so that $y$ needs to be zero, and in this case we also have $x \geq -\kappa$ (if $\kappa$ is not integer). This is because if $x < -\kappa$ then $\theta(x,y) \to \pi$ as $y \to +0$, but then $\sin(\kappa\pi) \neq 0$ whenever $\kappa$ is not integer. Assume that $\sin(\kappa\theta(x,y)) \neq 0$, and change variables by $re^{i\theta} = 1 + z/\kappa$. Then, we have

$$0 = \kappa(r\cos\theta - 1)\cdot\sin(\kappa\theta) + \kappa r\sin\theta\cdot\cos(\kappa\theta) = r(\cos\theta\sin(\kappa\theta) + \sin\theta\cos(\kappa\theta)) - \sin(\kappa\theta) = r\sin((\kappa+1)\theta) - \sin(\kappa\theta),$$

whence

$$r = r(\theta) = \frac{\sin(\kappa\theta)}{\sin((\kappa+1)\theta)}.$$

Since $\sin(\kappa\theta)$ and $\sin((\kappa+1)\theta)$ are both positive in the interval $(0, \frac{\pi}{\kappa+1})$, and since $\lim_{\theta\to\frac{\pi}{\kappa+1}-0} \sin((\kappa+1)\theta) = 0$, we see that

$$\lim_{\theta\to\frac{\pi}{\kappa+1}-0} r(\theta) = +\infty,$$

thus it has an asymptotic line with slope $\tan\frac{\pi}{\kappa+1}$. Let $\theta_1 = \frac{\pi}{\kappa+1}$. Note that $\kappa\theta_1 = \pi - \theta_1$ so that $\cot(\kappa\theta_1) = -\cot\theta_1$. Let $y = (\tan\frac{\pi}{\kappa+1})x + A$. Then, $A$ needs to satisfy

$$x + ((\tan\theta_1)x + A)\cot(\kappa\theta_1) = 0 \iff x - (x + A\cot\theta_1) = 0,$$

so that $A = 0$. Thus, there is an asymptotic line $y = (\tan\theta_1)x$.

In order to study the set $\mathcal{S}$, we consider $f(x)$ for real ($x \in (\alpha, +\infty)$. In this case, we have

| $x$ | $\alpha$ | $\cdots$ | $0$ | $\cdots$ | $+\infty$ |
|---|---|---|---|---|---|
| $f'$ | $0$ | | $+$ | | |
| $f$ | $f(\alpha)$ | $\nearrow$ | $0$ | $\nearrow$ | $+\infty$ |

$$\lim_{x\to+\infty} f(x) = +\infty.$$

Thus in this case we have (S2) $\mathcal{S} = (-\infty, f_{\kappa,\gamma}(\alpha_2))$, where $f_{\kappa,\gamma}(\alpha_2) < 0$.

We can confirm it directly. Since we have $x = -y\cot(\kappa\theta)$, we have by the change of variables $1 + z/\kappa = re^{i\theta}$

$$\mathrm{Re}\, f(z) = \left(\Big(1+\frac{x}{\kappa}\Big)^2 + \frac{y^2}{\kappa^2}\right)^{\frac{\kappa}{2}} (x\cos(\kappa\theta(x,y)) - y\sin(\kappa\theta(x,y)))$$
$$= r(\theta)^{\kappa}(-y\cot(\kappa\theta)\cos(\kappa\theta) - y\sin(\kappa\theta)) = -\frac{r(\theta)^{\kappa} y}{\sin(\kappa\theta)}$$
$$= -\frac{\kappa\sin\theta}{\sin(\kappa\theta)} r(\theta)^{\kappa+1} \qquad \text{(because } y = \kappa r(\theta)\sin(\theta)\text{)}.$$

Thus, when $\theta \in (0, \frac{\pi}{\kappa+1})$, we have $\frac{\kappa\sin\theta}{\sin(\kappa\theta)} > 0$ so that

$$\lim_{\theta\to\frac{\pi}{\kappa+1}-0} f(r(\theta)e^{i\theta}) = -\infty.$$

In order to show that $f_{\kappa,0}\colon D \to \mathbb{C}^+$ is bijective, note that $r(\theta) = 1/b(\theta)$ (where $b(\theta)$ is as in (40) for $\gamma = 0$) and $b(\theta)$ is monotonic decreasing, so that $r(\theta)$ is an increasing function. The discussion of bijectivity of $f_{\kappa,0} : D \to \mathbb{C}^+$ is similar to the case (ii) in Section 6.3.

6.4.2. *Case $\kappa\gamma = 1$, $\kappa > 1$.* In this case, we have

$$f_{\kappa,1/\kappa}(z) = \frac{z}{1+\frac{z}{\kappa}}\Big(1+\frac{z}{\kappa}\Big)^{\kappa} = z\Big(1+\frac{z}{\kappa}\Big)^{\kappa-1} = \frac{\kappa}{\kappa-1}\cdot\frac{\kappa-1}{\kappa}z\left(1+\frac{\frac{\kappa-1}{\kappa}z}{\kappa-1}\right)^{\kappa-1} = \frac{\kappa}{\kappa-1}\cdot f_{\kappa-1,0}\Big(\frac{\kappa-1}{\kappa}z\Big),$$

and hence we can use the result in the case $\gamma = 0$ (since $\kappa - 1 > 0$).

6.4.3. *Case $\kappa = +\infty$.* In this case, we have

$$\begin{aligned} f(z) &= \frac{z}{1+\gamma z}e^z = \frac{(x+\gamma x^2+\gamma y^2)+iy}{(1+\gamma x)^2+\gamma^2 y^2}\cdot e^x(\cos y + i\sin y) \\ &= \frac{e^x}{(1+\gamma x)^2+\gamma^2y^2}\left\{\begin{array}{l}(x+\gamma x^2+\gamma y^2)\cos y - y\sin y \\ +i\big((x+\gamma x^2+\gamma y^2)\sin y + y\cos y\big)\end{array}\right\}. \end{aligned}$$

If $\gamma = 0$ then $f(z) = ze^z$ and we are in the well-known Lambert case (see next subsection). So we assume that $\gamma \neq 0$.

We will show that $D$ is bounded and $D \subset \{z \in \mathbb{C}^+;\ \operatorname{Im} z \in (0,\pi)\}$. We have

$$f'(z) = \frac{\gamma z^2+z+1}{(1+\gamma z)^2}e^z,$$

and $f'(z) = 0$ implies

$$z = \frac{-1\pm\sqrt{1-4\gamma}}{2\gamma}.$$

Note that $\kappa = \infty$ means $\alpha = 1$ so that $\gamma = \frac{p-q}{p} = 1 - \frac{q}{p} \le 0$ by the assumption $\alpha \le \frac{q}{p}$. Thus we consider only the case $\gamma \le 0$. Let us consider the curve $\operatorname{Im} f(z) = 0$, that is,

$$(x+\gamma x^2+\gamma y^2)\sin y + y\cos y = 0.$$

If $\sin y = 0$, then $y = 0$. Assume that $\sin y \neq 0$. Then

$$x + \gamma x^2 + \gamma y^2 + y\cot y = 0,$$

and this equation can be solved in $x$ in such a way that

$$x^2 + \frac{x}{\gamma} + y^2 + \frac{y\cot y}{\gamma} = 0 \iff \Big(x+\frac{1}{2\gamma}\Big)^2 - \frac{1}{4\gamma^2} = -y^2 - \frac{y\cot y}{\gamma} \iff \Big(x+\frac{1}{2\gamma}\Big)^2 = \frac{1}{4\gamma^2} - y^2 - \frac{y\cot y}{\gamma}.$$

Let us consider the function

$$h(y) := \frac{1}{4\gamma^2} - y^2 - \frac{y\cot y}{\gamma} = \frac{1}{4\gamma^2} - \Big(y + \frac{\cot y}{2\gamma}\Big)^2 + \frac{\cot^2 y}{4\gamma^2} = \frac{1}{4\gamma^2\sin^2 y} - \Big(y+\frac{\cot y}{2\gamma}\Big)^2 = \frac{1-\big(2\gamma y\sin y+\cos y\big)^2}{4\gamma^2\sin^2 y}.$$

Note that

$$h(0) = \lim_{y\to 0} h(y) = \frac{1}{4\gamma^2} - \frac{1}{\gamma}\lim_{y\to 0}\frac{y}{\sin y} = \frac{1-4\gamma}{4\gamma^2} \ge 0.$$

In order to solve the equation in $x$, the function $h(y)$ needs to be non-negative, and it is equivalent to the condition that the absolute value of the function $g(y) := \cos y + 2\gamma y\sin y$ is less than or equal to 1. We will show that $g(y)$ is monotonic decreasing in some interval. At first, we observe that $g(0) = 1$ and for $y$ small enough

$$g(y) = \Big(1 - \frac{y^2}{2} + \frac{y^4}{4!}\Big) + 2\gamma y\Big(y - \frac{y^3}{6}\Big) + o(y^4) = 1 - \frac{1-4\gamma}{2}y^2 + \frac{1-8\gamma}{4!}y^4 + o(y^4).$$

If $1-4\gamma \ge 0$, $g$ takes a maximal value at $y=0$ (if $\gamma = 1/4$ then $1-8\gamma = -1 < 0$). Its derivative is

$$g'(y) = -\sin y + 2\gamma(\sin y + y\cos y) = -(1-2\gamma)\sin y + 2\gamma y\cos y = -(1-2\gamma)\Big(\frac{2\gamma}{2\gamma-1}y + \tan y\Big)\cos y.$$

Here we have $-1 \le c := \frac{2\gamma}{2\gamma-1} < 1$ by

$$1-4\gamma \ge 0 \iff 1-2\gamma \ge 2\gamma \quad \text{and} \quad 2\gamma - 1 < 2\gamma.$$

If $\cos y = 0$ then we have $g'(y) \neq 0$ so that $g'(y) = 0$ implies $cy + \tan y = 0$. Since $-1 \le c < 1$, it follows (by derivation) that $cy + \tan y$ is increasing. Thus, we have a unique solution $y_*$ of $cy + \tan y = 0$ in the interval $y_* \in (\pi/2, \pi)$. Note that since $1-2\gamma > 0$, we have $g'(y) < 0$ for $y \in (0, \pi/2)$. Moreover, since for $\frac{\pi}{2} < y < y_* < \frac{3}{2}\pi$ we have $\cos y < 0$ and $cy + \tan y < 0$ ($\lim_{y\to\pi/2+0}\tan y = -\infty$ and $cy+\tan y$ is increasing), we see that $g'(y)$ is also negative for $y \in (\pi/2, y_*)$.

Since we now assume that $\gamma < 0$, we have

$$g(y_*) = \cos y_* + 2\gamma\sin y_* \cdot \Big(\frac{1-2\gamma}{2\gamma}\tan y_*\Big) = \frac{\cos^2 y_* + (1-2\gamma)\sin^2 y_*}{\cos y_*} = \frac{1-2\gamma\sin^2 y_*}{\cos y_*} < -1,$$

so that there exists one and only one $y_0$ in $(0, y_*)$ such that $g(y_0) = -1$ and $g(y_0+\varepsilon) < -1$ for $\varepsilon \in (0, y_* - y_0)$. We have proved that $h(y)$ is non-negative on $y \in [0, y_0]$, and $h(y_0+\varepsilon) < 0$ for any $\varepsilon \in (0, y_*-y_0)$. Therefore, in this interval, we can take a square root of $h(y)$, and we can solve the equation in $x$ as

$$x = x_\pm(y) = -\frac{1}{2\gamma} \pm \sqrt{h(y)} \quad (y \in [0, y_0]).$$

Since $h(y_0) = 0$, these two paths $(x_\pm(y), y)$ form a continuous curve connecting $x_+(0)$ and $x_-(0)$. By construction, it is obvious that the curve $(x_\pm(y), y)$ is in $\mathbb{C}^+$.

Now we study the set $\mathcal{S}$ Let us consider $f(x)$ for real $x$. Since $\gamma < 0$ and $\gamma(-\frac{1}{\gamma})^2 + (-\frac{1}{\gamma}) + 1 = 1 > 0$, we have the following variation table of $f(x)$:

| $x$ | $-\infty$ | $\cdots$ | $\alpha_1$ | $\cdots$ | $0$ | $\cdots$ | $-\frac{1}{\gamma}$ | $\cdots$ | $\alpha_2$ | $\cdots$ | $+\infty$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $f'$ | | $-$ | $0$ | | $+$ | | $\times$ | $+$ | $0$ | $-$ | |
| $f$ | $0$ | $\searrow$ | $f(\alpha_1)$ | $\nearrow$ | $0$ | $\nearrow$ | $\times$ | $\nearrow$ | $f(\alpha_2)$ | $\searrow$ | $-\infty$ |

Since $\gamma\alpha_i + 1 = -\frac{1}{\alpha_i}$, we see that $f(\alpha_i) = -\alpha_i^2 e^{\alpha_i} < 0$. By $\alpha_1\alpha_2 = \frac{1}{\gamma}$, we have

$$\frac{f(\alpha_2)}{f(\alpha_1)} = \frac{\alpha_2(1+\gamma\alpha_1)}{\alpha_1(1+\gamma\alpha_2)} e^{\alpha_2-\alpha_1} = \frac{\alpha_2+1}{\alpha_1+1} e^{\alpha_2-\alpha_1} > 1,$$

whence $f(\alpha_2) < f(\alpha_1) < 0$. Thus, we have (S2) $\mathcal{S} = (f_{\kappa,\gamma}(\alpha_2), f_{\kappa,\gamma}(\alpha_1))$, where $f_{\kappa,\gamma}(\alpha_2) < f_{\kappa,\gamma}(\alpha_1) < 0$.

The discussion of bijectivity of $f : D \to \mathbb{C}^+$ is similar to the case (ii) in Section 6.3.

6.4.4. *Case* $(\kappa,\gamma) = (\infty, 0)$. This case corresponds to the classical Lambert function. Although the detailed analysis of the classical Lambert $W$ function is found in Corless et al. (1996), we give it here for the completeness. Let $f(z) = ze^z$. Set $z = x + yi$ and compute $\operatorname{Re} f$ and $\operatorname{Im} f$.

$$\begin{aligned} f(z) &= (x+yi)e^{x+yi} = e^x(x+yi)(\cos y + i\sin y) \\ &= e^x\{(x\cos y - y\sin y) + i(x\sin y + y\cos y)\}. \end{aligned}$$

Assume that $\operatorname{Im} f(z) = 0$. Then, we have

$$x\sin y + y\cos y = 0.$$

Obviously, real numbers $z = x + 0i$ satisfy this equation. Assume that $y \neq 0$. Then, we see that $\sin y \neq 0$. Otherwise, $\cos y$ needs to be equal to zero but it is impossible. Thus we have

$$x = -y\frac{\cos y}{\sin y} = -y\cot y.$$

We show that $D$ is unbounded and $D \subset \{z \in \mathbb{C}^+;\ \operatorname{Im} z \in (0,\pi)\}$.

Set $g(y) = -y\cot y$. It is defined on $\mathbb{R} \setminus \{n\pi;\ n \in \mathbb{Z}\}$. Note that

$$\lim_{y\to 0} g(y) = -1.$$

We have

$$g'(y) = -\cot y + y(1+\cot^2 y) = -\frac{\cos y}{\sin y} + \frac{y}{\sin^2 y} = \frac{-\sin y\cos y + y}{\sin^2 y} = \frac{2y - \sin 2y}{2\sin^2 y}.$$

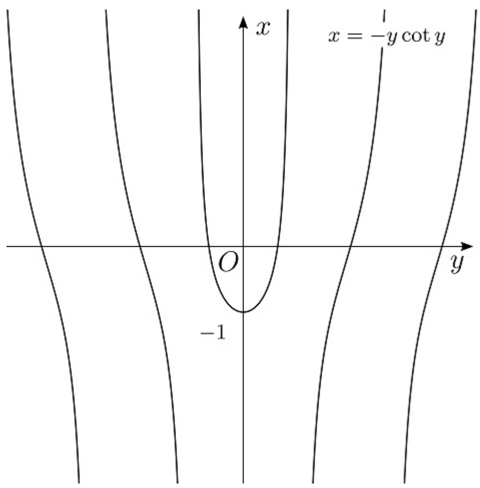
Thus

$$g'(y) = 0 \quad\Rightarrow\quad y = 0, \qquad g'(y) > 0 \quad\Rightarrow\quad y > 0, \qquad g'(y) < 0 \quad\Rightarrow\quad y < 0$$

and

$$\lim_{h\to+0} g(n\pi + h) = \begin{cases} -\infty & (n > 0) \\ +\infty & (n < 0) \end{cases} \qquad \lim_{h\to-0} g(n\pi + h) = \begin{cases} +\infty & (n > 0) \\ -\infty & (n < 0) \end{cases}$$

Thus we have $D = \{z = x + yi \in \mathbb{C}^+;\ 0 < y < \pi,\ x > -y\cot y\}$. The graph of $x = -y\cot y$ is as follows.




Now we describe the set $\mathcal{S}$. We shall consider the value $f(z)$ for $z$ being on the path $p(y) = g(y) + iy$ ($y \in [0,\pi)$). We have

$$f(p(y)) = e^x(x\cos y - y\sin y) = e^{g(y)}(-y\cot y\cos y - y\sin y) = -\frac{ye^{g(y)}}{\sin y}.$$

Since $\lim_{y\to 0} g(y) = -1$, we have $\lim_{y\to+0} f(p(y)) = -e^{-1} = -\frac{1}{e}$. By differentiating both sides in $y$, we see that

$$\begin{aligned}
\frac{d}{dy} f(p(y)) &= -e^{g(y)} \left( \frac{\sin y - y\cos y}{\sin^2 y} + g'(y) \cdot \frac{y}{\sin y} \right) \\
&= -e^{g(y)} \left( \frac{1}{\sin y} - \frac{y\cos y}{\sin^2 y} - \frac{y\cos y}{\sin^2 y} + \frac{y^2}{\sin^3 y} \right) \\
&= -\frac{e^{g(y)}}{\sin y} \left( 1 - \frac{2y\cos y}{\sin y} + \frac{y^2}{\sin^2 y} \right) \\
&= -\frac{e^{g(y)}}{\sin y} \left( \left( \frac{y}{\sin y} - \cos y \right)^2 - \cos^2 y + 1 \right) \\
&= -\frac{e^{g(y)}}{\sin y} \left( \left( \frac{y}{\sin y} - \cos y \right)^2 + \sin^2 y \right) < 0.
\end{aligned}$$

Thus, $f(p(y))$ is decreasing for $y \in [0, \pi)$. Moreover, we have

$$\lim_{y\to\pi-0} f(p(y)) = -\infty \qquad (\because \lim_{y\to\pi-0} g(y) = +\infty \text{ and } \lim_{y\to\pi-0} \frac{1}{\sin y} = +\infty).$$

Thus, we have (S2) $\mathcal{S} = (-\infty, -\frac{1}{e}) \subset \mathbb{R}_{<0}$.

The discussion of bijectivity of $f_{0,\infty} : D \to \mathbb{C}^+$ is similar to the case (ii) in Section 6.3.

6.5. **The case of $\kappa < 0$.** Recall the homographic (linear fractional) action of $SL(2,\mathbb{R})$ on $\mathbb{C}$. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ and $z \in \mathbb{C}^+$, we set

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az+b}{cz+d}.$$

Let $\kappa = -\kappa'$ with positive $\kappa' > 0$. Consider the transformation

$$1 + \frac{z'}{\kappa'} = \left(1 + \frac{z}{\kappa}\right)^{-1}.$$

Then, it can be written as

$$z' = \begin{pmatrix} 1 & 0 \\ 1/\kappa & 1 \end{pmatrix} \cdot z = \frac{z}{1 + z/\kappa} \quad \Longleftrightarrow \quad z = \begin{pmatrix} 1 & 0 \\ -1/\kappa & 1 \end{pmatrix} \cdot z' = \frac{z'}{1 - z'/\kappa}.$$

Note that since $\begin{pmatrix} 1 & 0 \\ 1/\kappa & 1 \end{pmatrix} \in SL(2,\mathbb{R})$, it maps $\mathbb{C}^+$ to $\mathbb{C}^+$ bijectively. Then, since

$$\begin{aligned}
\frac{z}{1+\gamma z} = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \cdot z &= \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/\kappa & 1 \end{pmatrix} \cdot z' = \begin{pmatrix} 1 & 0 \\ \gamma - 1/\kappa & 1 \end{pmatrix} \cdot z' \\
&= \frac{z'}{1 + (\gamma - 1/\kappa) z'} = \frac{z'}{1 + (\gamma + 1/\kappa') z'}
\end{aligned}$$

and

$$\left(1 + \frac{z}{\kappa}\right)^{\kappa} = \left( \left(1 + \frac{z}{\kappa}\right)^{-1} \right)^{-\kappa} = \left(1 + \frac{z'}{\kappa'}\right)^{\kappa'}$$

(recall that we are taking the main branch so that $\log z = -\log(z^{-1})$), we obtain

$$f_{\gamma,\kappa}(z) = \frac{z}{1+\gamma z} \left(1 + \frac{z}{\kappa}\right)^{\kappa} = \frac{z'}{1 + (\gamma + 1/\kappa') z'} \left(1 + \frac{z'}{\kappa'}\right)^{\kappa'} = f_{\gamma + 1/\kappa', \kappa'}(z').$$

Set $\gamma' = \gamma + 1/\kappa'$. Since we now assume that $\frac{1}{\kappa} - \gamma \geq 0$, we have

$$\frac{1}{\kappa} - \gamma \geq 0 \iff 1 \leq \kappa\gamma \iff \gamma'\kappa' \leq 0,$$

and hence by the homographic action, the case $\kappa < 0$ reduces to the case $\kappa' > 0$ and $\kappa'\gamma' \leq 0$.

We will show that $D$ is bounded and $D \subset \left\{ z \in \mathbb{C};\ \mathrm{Arg}(1 + \frac{z}{\kappa})^{-1} \in (0, \frac{\pi}{\kappa+1}) \right\}$. Let $\rho$ denote the inverse transformation of $z' = \frac{z}{1+z/\kappa}$, that is, $\rho(z') = \frac{z'}{1+z'/\kappa'}$. We know by Section 6.2 that $D' = \rho^{-1}(D)$ is bounded and included in the domain $\left\{ z' \in \mathbb{C}^+;\ \mathrm{Arg}(1 + \frac{z'}{\kappa'}) \in (0, \frac{\pi}{\kappa+1}) \right\}$ (see Figure 24). The line $p(t) = -\kappa' + te^{i\theta_*} = \kappa + te^{i\theta_*}$ is mapped by $\rho$ to the line

$$\rho(p(t)) = \frac{\kappa + te^{i\theta_*}}{1 - (\kappa + te^{i\theta_*})/\kappa} = \frac{\kappa + te^{i\theta_*}}{-te^{i\theta_*}/\kappa} = -\kappa - \frac{\kappa^2}{t} e^{-i\theta_*}.$$

By $\rho$, the point $z = \kappa = -\kappa'$ transforms to $z = \infty$, and this point is not included in $\overline{D}$. Consequently, $\Omega = \rho(\Omega')$ is bounded and included in $\left\{ z \in \mathbb{C};\ \mathrm{Arg}(1 + \frac{z}{\kappa})^{-1} \in (0, \frac{\pi}{\kappa+1}) \right\}$ (see Figure 25).

Now we determine the set $\mathcal{S}$.

If $\gamma' < 0$, then $\alpha'_i$ transform to $\alpha_i$ for each $i = 1, 2$, and we have $\mathcal{S} = (f_{\kappa',\gamma'}(\alpha'_2), f_{\kappa',\gamma'}(\alpha'_1)) = (f_{\kappa,\gamma}(\alpha_2), f_{\kappa,\gamma}(\alpha_1))$. Next we consider the case $\gamma' = 0$. In this case, the intersection point $\alpha'$ of $\operatorname{Im} f_{\kappa',\gamma'} = 0$ is given as $\alpha' = -\frac{\kappa'}{\kappa'+1}$. Let $p(t)$, $t \in [0, 1)$ be the path of $\partial D \cap \mathbb{C}^+$ such that $p(0) = \alpha'$. Since $\rho(\infty) = -\kappa$, we see that $\rho(r(t))$, $t \in [0, 1)$ is a path connecting $\alpha = \rho(\alpha') = -1$ and $-\kappa$. In particular, $D$ is bounded. Then, we have $\mathcal{S} = (f_{\kappa,\gamma}(-\kappa), f_{\kappa,\gamma}(\alpha)) = (f_{\kappa',\gamma'}(\infty), f_{\kappa',\gamma'}(\alpha')) = (-\infty, f_{\kappa',\gamma'}(\alpha'))$. We note that the solution of the equation $\gamma z^2 + (1 + 1/\kappa)z + 1 = 0$ with the condition $\gamma = 1/\kappa$ is given as $z = -1, -\kappa$. Since $-1 < -\kappa$, we have $\alpha_1 = -1$ so that (S3) $\mathcal{S} = (-\infty, f_{\kappa,\gamma}(\alpha_1))$, where $f_{\kappa,\gamma}(\alpha_1) < 0$.

The fact that $f_{\kappa,\gamma} \colon D \to \mathbb{C}^+$ is bijective comes from the result for $\kappa' > 0$ and from the fact that homographic transformations are bijective.

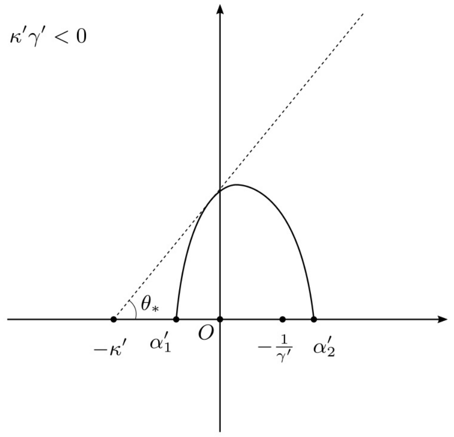



FIGURE 24

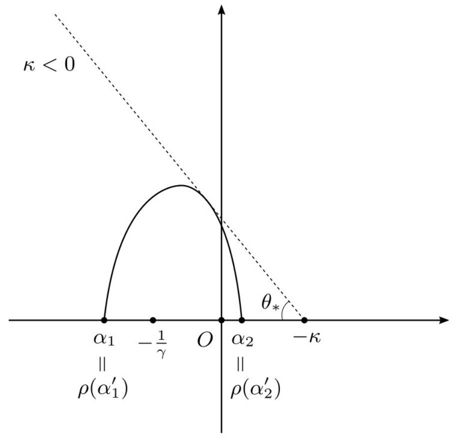



FIGURE 25

6.6. **The domain $\Omega$ of definition of $W_{\kappa,\gamma}$.** In the previous section, we showed that the function $W_{\kappa,\gamma}$ is well defined on $\mathbb{C}^+$. Recall that $\Omega$ is defined on p.35 before (42). We have $\Omega = \{z = x + yi \in \mathbb{C};\ z \in D \text{ or } \bar{z} \in D\} \cup (\mathrm{Cl}(D) \cap \mathbb{R})$. Then, $\Omega$ is a symmetric domain $\overline{\Omega} = \Omega$ (here the bar means complex conjugate). Let $\Omega^+ = D$. By the Schwarz reflection principle (Ahlfors (1979, Theorem 24, p. 172)), we see that $f = f_{\kappa,\gamma}$ is analytically continued to the domain $\Omega$ and $f(\bar{z}) = \overline{f(z)}$ $(z \in \Omega)$. Hence, $f = f_{\kappa,\gamma}$ maps $\overline{D}$ onto $\mathbb{C}^-$, and moreover, if we set $\mathcal{S} = \mathbb{R} \setminus f(\mathbb{R})$, then $f$ maps $\Omega$ onto $\mathbb{C} \setminus \mathcal{S}$ and this correspondence is one-to-one ($D$ is mapped one-to-one to $\mathbb{C}^+$, and so $\overline{D}$ is mapped onto one-to-one $\mathbb{C}^-$. We have verified that $\Omega \cap \mathbb{R}$ is mapped one-to-one onto $f(\mathbb{R})$ in Sections 6.2 and 6.3). Thus, $W_{\kappa,\gamma}$ is well defined on $\mathbb{C} \setminus \mathcal{S}$. We can also verify it directly from (38).

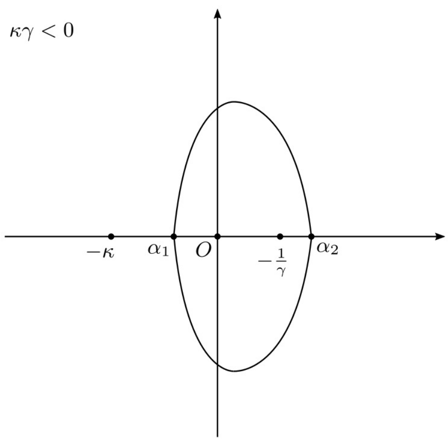



FIGURE 26. Case (i)

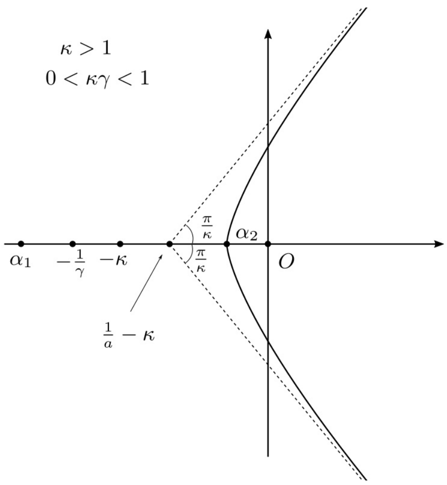



FIGURE 27. Case (ii)

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