Annals of the Institute of Statistical Mathematics (2022) 74:399–433
https://doi.org/10.1007/s10463-021-00800-8

Wigner and Wishart ensembles for sparse Vinberg models

Hideto Nakashima[1] **· Piotr Graczyk**[2]

Received: 10 August 2020 / Revised: 14 April 2021 / Accepted: 14 April 2021 /
Published online: 26 May 2021

Abstract

Vinberg cones and the ambient vector spaces are important in modern statistics of sparse models. The aim of this paper is to study eigenvalue distributions of Gaussian, Wigner and covariance matrices related to growing Vinberg matrices. For Gaussian or Wigner ensembles, we give an explicit formula for the limiting distribution. For Wishart ensembles defined naturally on Vinberg cones, their limiting Stieltjes transforms, support and atom at 0 are described explicitly in terms of the Lambert–Tsallis functions, which are defined by using the Tsallis q-exponential functions.

Keywords Eigenvalue distributions · Covariance matrices · Wigner matrices · Homogeneous cones · Vinberg cones · q-Exponential · Lambert–Tsallis functions

1 Introduction

In modern data analysis, there is a strong need of covariance models with sparsity (see, e.g., Hastie et al., 2015). In mathematical statistics we search for models with some mathematical structure allowing rigorous multivariate and asymptotical analysis. Vinberg matrix models presented in this paper constitute an important class of such sparse models.

Asymptotics of empirical eigenvalue distributions are a classical topic of the random matrix theory (RMT). There are numerous interactions of RMT with important areas of modern multivariate statistics: high-dimensional statistical inference, estimation of large covariance matrices, principal component analysis

✉ Hideto Nakashima
hideto@ism.ac.jp

Piotr Graczyk
piotr.graczyk@univ-angers.fr

[1] The Institute of Statistical Mathematics, 10-3 Midori-cho, Tachikawa, Tokyo 190-8562, Japan

[2] Laboratoire de Mathématiques LAREMA, Université d'Angers, 2 boulevard Lavoisier, 49045 Angers Cedex 01, France

(PCA), time series and many others, see the review papers by Diaconis (2003, Section 2), Johnstone (2007), Paul and Aue (2014), Bun et al. (2017), the book of Yao et al. (2015) and the references therein. RMT is also used in signal processing (including MIMO) and compressed sensing (see Hastie et al., 2015, Chapter 10, for example) in the restricted isometry property (RIP) introduced by Candès and Tao (2005). Fujikoshi and Sakurai (2016) and Bai et al. (2018) used RMT methods to study consistency of the criteria AIC and BIC in estimation of the number of components in PCA. Distribution of the largest eigenvalue of a Wishart matrix was studied in Takayama et al. (2020).

In this paper, we concentrate on proving fundamental theorems of RMT, the Wigner and Marchenko–Pastur-type limit theorems for considered Vinberg models.

High-dimensional spectral asymptotics for growing sparse models seem to have never been studied before and we are convinced that our results will be useful in modern multivariate statistical covariance analysis. A potential perspective of applications of our results to statistical problems is to study estimation of large sparse Vinberg covariance matrices, the number of significative PCA factors and asymptotics of the largest eigenvalue of a sparse Wishart Vinberg matrix in our subsequent researches.

Covariance matrices are defined naturally on Vinberg matrices by a quadratic construction (see Sect. 2.3), thanks to quadratic triangular group actions on positive definite Vinberg matrices (cf. Sect. 2.1).

In Sects. 3 and 4, we provide a complete study of limiting eigenvalue distributions related to Vinberg matrices. The main results are contained in Theorem 5 for the Wigner Ensembles, and in Theorems 18 and 24 and Corollaries 19 and 21 for the Wishart Ensembles of Vinberg matrices. We are able to treat both real and complex matrix ensembles, but in view of statistical applications, we focus on real random matrices.

As a special case of Corollary 19, we provide an elementary and short proof of a result of Dykema and Haagerup (2004, §8) on the asymptotic empirical eigenvalue distribution μ_0 for the covariance of the triangular real Gaussian ensemble. The proof in Dykema and Haagerup (2004) is based on the theory of free probability with involved calculations, and the Stieltjes transform $S_0(z)$ is given implicitly by determining all the moments of μ_0. Later, Cheliotis (2018) mentioned that $S_0(z)$ can be expressed in terms of the Lambert W function.

Our paper contributes to the study of triangular random matrices initiated by Dykema and Haagerup (2004) and continued in Cheliotis (2018), also in the framework of the theory of Muttalib-Borodin biorthogonal ensembles (see Borodin, 1999; Forrester, 2010; Forrester and Wang, 2017; Muttalib, 1995). This is a part of recent developments in the theory of singular values of non-symmetric random matrices (see the survey by Chafaï, 2009). In contrast to Cheliotis (2018), we do not dispose of an explicit formula for the joint eigenvalue density.

The analysis, probability and statistics on homogeneous cones develops intensely in recent years (Andersson and Wojnar, 2004; Graczyk and Ishi, 2014; Graczyk et al., 2019; Ishi, 2014, 2016; Letac and Massam, 2007; Nakashima, 2020; Yamasaki and Nomura, 2015), and Vinberg cones and dual Vinberg cones are basic examples of homogeneous cones (see Sect. 2.1). Our results are a first contribution to the RMT on homogeneous cones.

The main method used in our paper is the *variance profile method* for Gaussian and Wigner matrix ensembles, presented in Sect. 2.4. It was applied first in Shlyakhtenko (1996) in the Gaussian case and developed in Anderson and Zeitouni (2006) in the Wigner case. We use the recent approach of Bordenave (2019). In Theorem 3, we slightly strengthen for our needs the main variance profile result of Bordenave (2019). Theorem 3 will be useful for studying of eigenvalue distributions related to general growing sparse models.

Note that the variance profile methods were also developed directly for Wishart ensembles by Hachem et al. (2005, 2006, 2007, 2008) (cf. Remark 26). The variance profile methods are related to operator-valued free probability theory (Mingo and Speicher, 2017, Chapter 9).

Our expression of a limiting Stieltjes transform for Wishart Ensembles of Vinberg matrices is based on the introduction of Lambert–Tsallis functions $W_{\kappa,\gamma}$; see Sect. 4.1. The Lambert–Tsallis functions are defined by using Tsallis q-exponential functions, now actively studied in Information Geometry (cf. Amari and Ohara, 2011; Zhang et al., 2018).

Outlines of all proofs are given. Technical details are omitted and can be viewed in Supplementary material available from the editor of the journal.

Simulations of histograms of eigenvalues of Vinberg matrices are illustrated by Figs. 1, 2 and 3 in the Wigner case and by Figs. 4, 5 and 6 in the Wishart case.

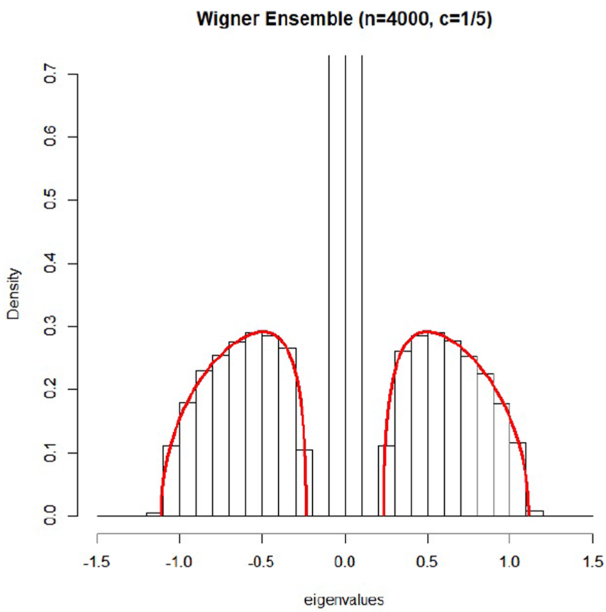


Fig. 1 Simulation for $c = \frac{1}{5}$

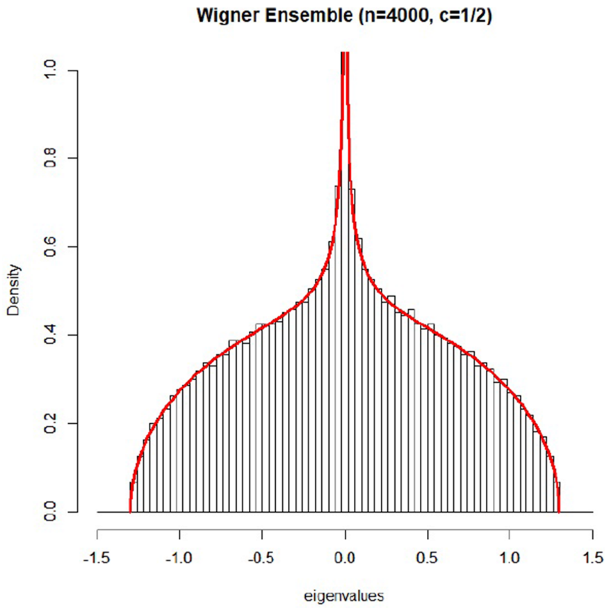


Fig. 2 Simulation for $c = \frac{1}{2}$

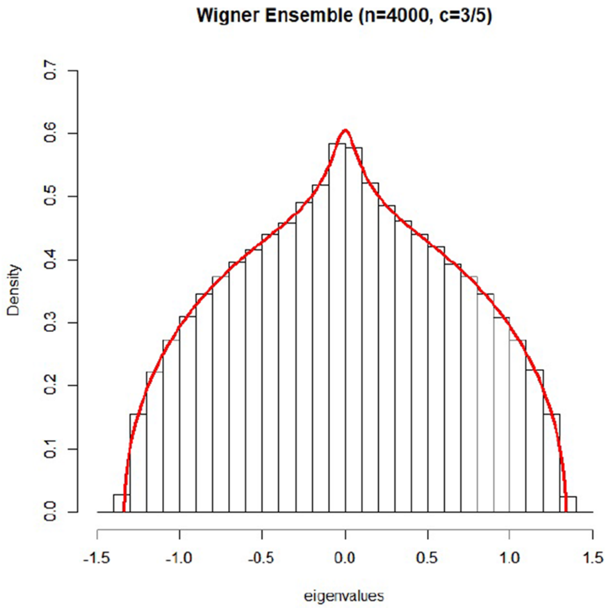


Fig. 3 Simulation for $c = \frac{3}{5}$

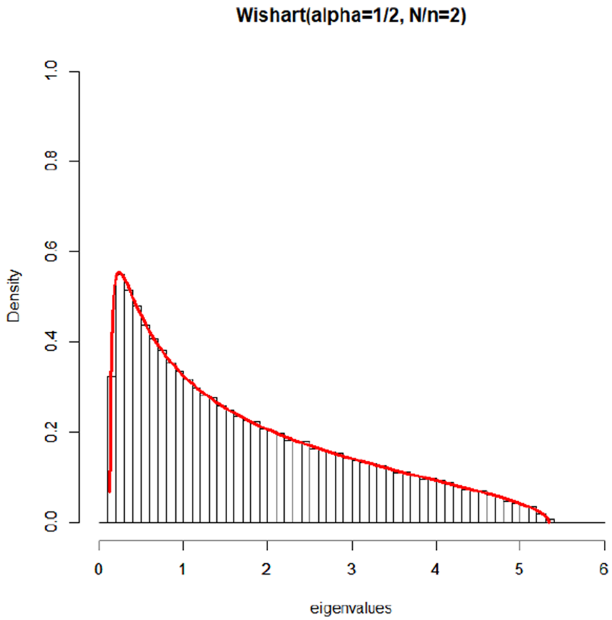


Fig. 4 Simulation for $\alpha = \frac{1}{2}$

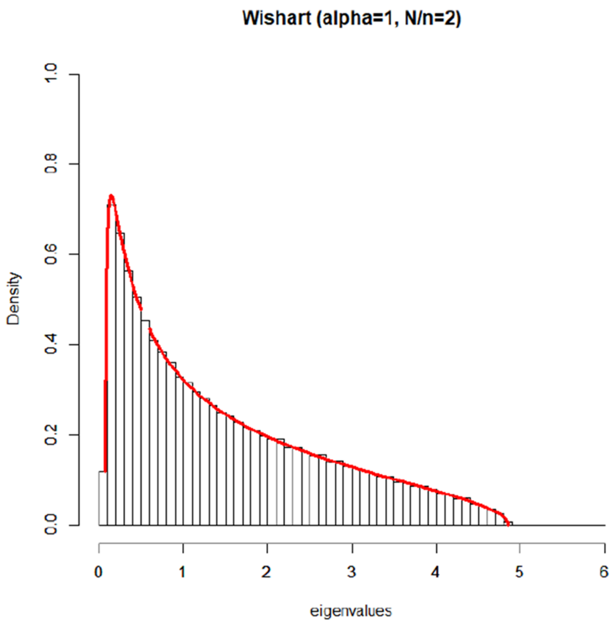


Fig. 5 Simulation for $\alpha = 1$

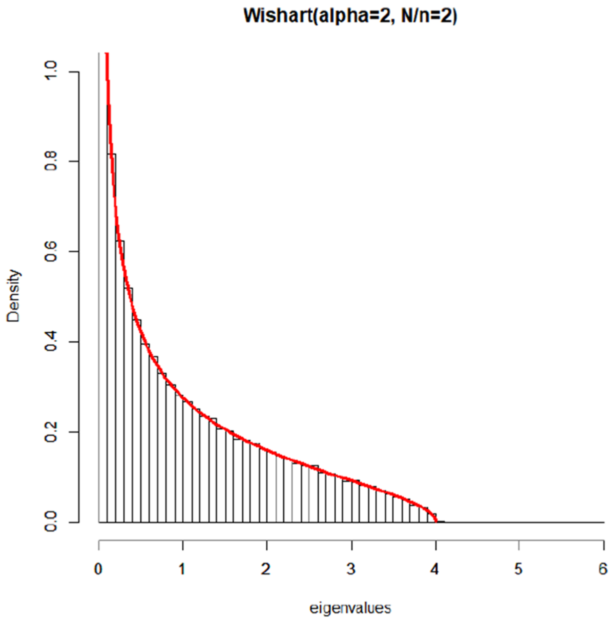


Fig. 6 Simulation for $\alpha = 2$

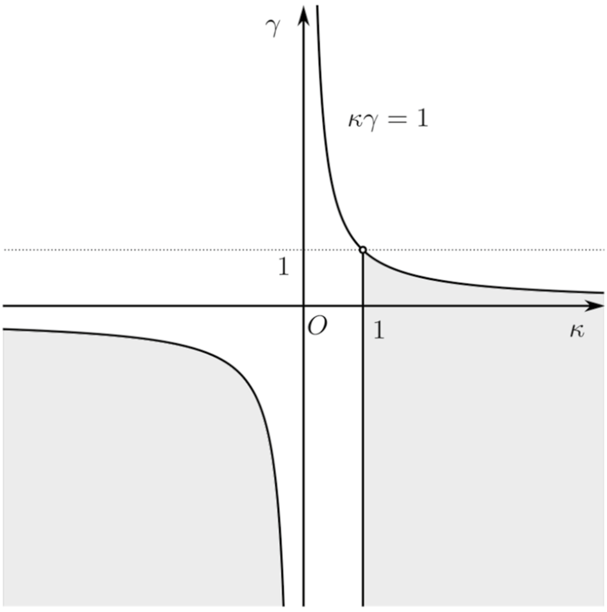


Fig. 7 Region of κ and γ

2 Preliminaries

We begin this paper with recalling the definition of the empirical eigenvalue distribution of a symmetric matrix. Let $\mathrm{Sym}(n, \mathbb{R})$ be the space of symmetric matrices of size n and $\mathrm{Sym}(n, \mathbb{R})^{+}$ the open convex cone of positive definite symmetric matrices in $\mathrm{Sym}(n, \mathbb{R})$. Let $\lambda_1(X) \geq \cdots \geq \lambda_n(X)$ be the ordered eigenvalues of $X \in \mathrm{Sym}(n, \mathbb{R})$ with counting multiplicities. Denote by δ_a the Dirac measure at a. Then, the empirical eigenvalue distribution μ_X of X is defined by $\mu_X = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i(X)}$.

If $\{X_n\}_{n=1}^{\infty}$ ($X_n \in \mathrm{Sym}(n, \mathbb{R})$) is a sequence of Gaussian, Wigner or Wishart matrices, and then it is well known that there exists a limit μ of μ_{X_n} as $n \to \infty$, and the sequence of random measures μ_{X_n} converges almost surely weakly to the semi-circle law or the Marchenko–Pastur law, respectively (see for example Bai and Silverstein, 2010; Bordenave, 2019). The limits μ of μ_{X_n}, in the almost sure weak sense, are said to be the "limiting eigenvalue distributions μ of X_n." For simplicity, we will say "i.i.d. matrices" instead of "matrices with independent and identically distributed non-null terms".

2.1 Generalized dual Vinberg cones and Vinberg matrices

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be non-decreasing sequences of positive integers such that $a_n + b_n = n$ and the ratio a_n/n converges to $c \in [0, 1]$. Then, we introduce the matrix space $\mathcal{U}_n$ as a subspace of $\mathrm{Sym}(n, \mathbb{R})$ defined by

$$\mathcal{U}_n := \left\{ U = \begin{pmatrix} x & y \\ y^{\top} & d \end{pmatrix};\ \begin{array}{l} x \in \mathrm{Sym}(a_n, \mathbb{R}),\ y \in \mathrm{Mat}(a_n \times b_n, \mathbb{R}), \\ d \text{ is a diagonal matrix of size } b_n \end{array} \right\},$$

where $\mathrm{Mat}(a_n \times b_n; \mathbb{R})$ denotes the space of $a_n \times b_n$ matrices. Set

$$P_n := \mathcal{U}_n \cap \mathrm{Sym}(n, \mathbb{R})^{+}.$$

Then, P_n is an open convex cone in $\mathcal{U}_n$. Moreover, the cone P_n admits a transitive group action, *i.e.*, P_n is a *homogeneous cone*, since the following triangular group

$$H_n := \left\{ h = \begin{pmatrix} h_1 & y \\ 0 & d \end{pmatrix} \in GL(n, \mathbb{R});\ \begin{array}{l} h_1 \in GL(a_n, \mathbb{R}) \text{ is upper triangular}, \\ y \in \mathrm{Mat}(a_n \times b_n; \mathbb{R}), \\ d : \text{diagonal of size } b_n \end{array} \right\}$$

acts on P_n transitively by the quadratic action $\rho(h)U := hUh^{\top}$ for $h \in H_n$ and $U \in P_n$. This is easily verified by using the Cholesky decomposition (cf. Ishi, 2016, p. 3). For definition and basic properties of homogeneous cones, see Vinberg (1963), Ishi (2014).

If $n = 3$ and $(a_n, b_n) = (1, 2)$, then P_3 is the dual Vinberg cone (see Example 1) so that, in this paper, we call P_n a *generalized dual Vinberg cone* and elements $U \in \mathcal{U}_n$ *Vinberg matrices*. On the other hand, if we set $a_n = n - 1$ and $b_n = 1$, then $\mathcal{U}_n$ is the space $\mathrm{Sym}(n, \mathbb{R})$ of symmetric matrices of size n, and hence our discussion covers the classical results. In what follows, we introduce two kinds of random matrices related to the homogeneous cones P_n, that is, Gaussian and Wigner matrices and Wishart quadratic (covariance) matrices.

2.2 Gaussian and Wigner matrices in $\mathcal{U}_n$

Analogously to the classical Wigner matrices, we say that $U_n = (u_{ij}) \in \mathcal{U}_n$ is a Wigner random matrix if

$$\begin{cases} \bullet \text{ the diagonal terms } (u_{ii}) \text{ are independent of the off-diagonal terms } (u_{ij})_{i<j}, \\ \bullet \text{ the diagonal } u_{ii}'\text{s are centered i.i.d. variables with variance } v' \text{ and fourth moment } M_4', \\ \bullet \text{ the non-nul off-diagonal } u_{ij}'\text{s}, i<j, \text{ are centered i.i.d. variables} \\ \quad \text{with variance } v \text{ and fourth moment } M_4, \end{cases} \tag{1}$$

where v, v', M_4, M_4' are fixed positive real numbers. If the non-null terms u_{ij} are Gaussian, with $v = 1$ and $v' = 2$, the matrices U_n form a Gaussian Orthogonal Ensemble of Vinberg matrices. In Sect. 3, we consider empirical eigenvalue distributions of rescaled Wigner matrices $U_n/\sqrt{n} \in \mathcal{U}_n$.

2.3 Quadratic construction of Wishart (covariance) matrices in $\mathcal{U}_n$

Recall that sample covariance matrices, essential in multivariate statistical analysis, are defined as a quadratic map $\frac{1}{n}VV^\top$ of the observed centered sample vector V. Consequently, Wishart matrices are constructed quadratically both in Random Matrix Theory and in statistics. In this section, we define, by a quadratic construction, Wishart (covariance) matrices in $\mathcal{U}_n$.

We first recall the notion of a direct sum of quadratic maps. Let $Q_i : \mathbb{R}^{m_i} \to \mathbb{R}^m$ $(i = 1, \dots, k)$ be quadratic maps. Then, the direct sum $Q_1 \oplus \cdots \oplus Q_k$ is an $\mathbb{R}^m$-valued quadratic map on $\mathbb{R}^{m_1} \oplus \cdots \oplus \mathbb{R}^{m_k}$ given by

$$Q(x) := Q_1(x_1) + \cdots + Q_k(x_k) \quad \text{where} \quad x = x_1 + \cdots + x_k \quad \left(x_i \in \mathbb{R}^{m_i}\right).$$

If $Q_1 = \cdots = Q_k$, then the direct sum Q is denoted by $Q_1^{\oplus k}$. As showed in Graczyk and Ishi (2014), any homogeneous cone Ω admits a canonical family of the so-called *basic quadratic maps* q_j $(j = 1, \dots, r)$ defined for each j on a suitable finite dimensional vector space E_j and with values in the closure $\overline{\Omega}$ of Ω. The number r is called the rank of Ω and $r = n$ for the cones $\mathcal{U}_n$. Using the basic quadratic maps q_j, one constructs quadratic maps $Q_{\underline{k}}$ for $\underline{k} \in \mathbb{Z}^r_{\geq 0}$ by

$$Q_{\underline{k}} := q_1^{\oplus k_1} \oplus \cdots \oplus q_r^{\oplus k_r},$$

defined on $E_{\underline{k}} := E_1^{\oplus k_1} \oplus \cdots \oplus E_r^{\oplus k_r}$. The maps $Q_{\underline{k}}$ are Ω-positive, *i.e.,* if $\xi \in E_{\underline{k}} \setminus \{\mathbf{0}\}$, then $Q_{\underline{k}}(\xi) \in \overline{\Omega} \setminus \{\mathbf{0}\}$.

In our case $\Omega = P_n$, the basic quadratic maps are given as follows (cf. Graczyk and Ishi, 2014). For $j = 1, \dots, n$, define $E_j \subset \mathbb{R}^n$ by

$$E_j = \left\{ \begin{pmatrix} \boldsymbol{\xi} \\ \mathbf{0} \end{pmatrix} \in \mathbb{R}^n;\ \boldsymbol{\xi} \in \mathbb{R}^j \right\} \quad (j \le a_n),$$
$$E_j = \left\{ \begin{pmatrix} \boldsymbol{\xi} \\ \mathbf{0} \end{pmatrix} + \xi' \boldsymbol{e}_j \in \mathbb{R}^n;\ \boldsymbol{\xi} \in \mathbb{R}^{a_n},\ \xi' \in \mathbb{R} \right\} \quad (j > a_n),$$

where $\boldsymbol{e}_i$ $(i = 1, \dots, n)$ is the vector in $\mathbb{R}^n$ having 1 on the ith position and zeros elsewhere. We note that each E_j corresponds to the jth column of the Lie algebra $\mathfrak{h}_n$ of H_n, that is, we have $\mathfrak{h}_n = \{H = (\xi_1, \dots, \xi_n);\ \xi_j \in E_j\}$. Then, the basic quadratic maps $q_j : E_j \to \mathcal{U}_n$ of the cone P_n are defined by

$$q_j(\xi_j) := \xi_j \xi_j^\top \in \mathcal{U}_n \quad (\xi_j \in E_j).$$

Let $\underline{k} \in \mathbb{Z}^n_{\ge 0}$. Then, $E_{\underline{k}}$ can be viewed as a subspace of $\mathrm{Mat}(n \times (k_1 + \cdots + k_n); \mathbb{R})$ of the form

$$\left\{ \eta = \left(\overbrace{\xi_1^{(1)}, \dots, \xi_1^{(k_1)}}^{k_1}, \dots, \overbrace{\xi_{n-1}^{(1)}, \dots, \xi_{n-1}^{(k_{n-1})}}^{k_{n-1}}, \overbrace{\xi_n^{(1)}, \dots, \xi_n^{(k_n)}}^{k_n} \right);\ \begin{array}{l} \xi_j^{(i)} \in E_j, \\ 1 \le j \le n, \\ 1 \le i \le k_j \end{array} \right\},$$

and then $Q_{\underline{k}}(\eta) = \eta\eta^\top$ for $\eta \in E_{\underline{k}}$. In order to simplify formulas when we apply the so-called variance profile method in Sect. 4, we do not multiply $\frac{1}{2}$ in definition of $Q_{\underline{k}}(\eta)$.

When $\eta \in E_{\underline{k}}$ is an i.i.d. random matrix whose non-null terms have the normal law $N(0, v)$, the law of $Q_{\underline{k}}(\eta)$ is a *Wishart law* $\gamma_{Q_{\underline{k}}, 1/(2v)\mathrm{Id}_n}$ on the cone P_n. For the definition of all Wishart laws on the cone P_n; see Graczyk and Ishi (2014). More generally, in this paper, we consider eigenvalue distributions of rescaled matrix $Q_{\underline{k}}(\eta)/n$ under the assumption that $\eta \in E_{\underline{k}}$ is a centered rectangular i.i.d. matrix whose non-null terms have variance v and finite fourth moments M_4.

We consider two-dimensional multiparameters $\underline{k} = \underline{k}(n) \in \mathbb{Z}^n_{\ge 0}$ of the form

$$\underline{k} = m_1(1, \dots, 1) + m_2(\overbrace{0, \dots, 0}^{a_n}, \overbrace{1, \dots, 1}^{b_n}) \quad (m_1, m_2 \in \mathbb{Z}_{\ge 0}). \tag{2}$$

Example 1 Let $n = 3$, $a_3 = 1$ and $b_3 = 2$. In this case, P_3 is the dual Vinberg cone (cf. Vinberg, 1963, p. 397, Ishi, 2001, §5.2):

$$P_3 = \left\{ x = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & 0 \\ x_{13} & 0 & x_{33} \end{pmatrix};\ x \text{ is positive definite} \right\}.$$

Consider $m_1 = m_2 = 1$, so $\underline{k} = (1, 2, 2)$. Then $E_{\underline{k}} = E_{(1,2,2)}$ can be written as

$$E_{(1,2,2)} = \left\{ \eta = \begin{pmatrix} x & y_{11} & y_{12} & z_{11} & z_{12} \\ 0 & y_{21} & y_{22} & 0 & 0 \\ 0 & 0 & 0 & z_{21} & z_{22} \end{pmatrix}; x, y_{ij}, z_{ij} \in \mathbb{R} \right\},$$

and $Q_{(1,2,2)}(\eta) = \eta\,\eta^{\top}$ is given as

$$Q_{(1,2,2)}(\eta) = \begin{pmatrix} x^2 + y_{11}^2 + y_{12}^2 + z_{11}^2 + z_{12}^2 & y_{11}y_{21} + y_{12}y_{22} & z_{11}z_{21} + z_{12}z_{22} \\ y_{11}y_{21} + y_{12}y_{22} & y_{21}^2 + y_{22}^2 & 0 \\ z_{11}z_{21} + z_{12}z_{22} & 0 & z_{21}^2 + z_{22}^2 \end{pmatrix}.$$

If x, y_{ij}, z_{ij} are $N(0, v)$ i.i.d. Gaussian variables, the random matrix $Q_{(1,2,2)}(\eta)$ has a Wishart law on P_3.

The form (2) of the Wishart multiparameter $\underline{k}$ englobes and generalizes the following cases. In both cases, with rescaling $1/n$, the limiting eigenvalue distribution is known.

(i) The classical Wishart Ensemble $MM^{\top}$ on $\mathrm{Sym}(n, \mathbb{R})^{+}$, where $M = M_{n\times N}$ is an i.i.d. matrix with finite fourth moment M_4, with parameter $C := \lim_n \frac{N}{n} > 0$ (see Anderson et al., 2010; Faraut, 2014) for $(a_n, b_n) = (n-1, 1)$, $m_1 = 0$ and $m_2 \sim Cn$. The limiting eigenvalue distribution is the Marchenko–Pastur law μ_C with parameter C, i.e., denoting $a = \left(\sqrt{C} - 1\right)^2, b = \left(\sqrt{C} + 1\right)^2$ and $[x]_+ := \max(x, 0)$ $(x \in \mathbb{R})$,

$$\mu_C = [1 - C]_+ \delta_0 + \frac{\sqrt{(t-a)(b-t)}}{2\pi t}\chi_{[a,b]}(t)dt.$$

(ii) The Wishart Ensemble related to the Triangular Gaussian Ensemble (Cheliotis, 2018; Dykema and Haagerup, 2004) for $(a_n, b_n) = (n-1, 1)$, $m_1 = 1$ and $m_2 = 0$. When $v = 1$, the limiting eigenvalue distribution, which we call the *Dykema–Haagerup measure* χ_1, is absolutely continuous with respect to Lebesgue measure and has support equal to the interval $[0, e]$. Its density function ϕ is defined on the interval $(0, e]$ by the implicit formula (Dykema and Haagerup, 2004, Theorem 8.9)

$$\phi\left(\frac{\sin x}{x}\exp(x\cot x)\right) = \frac{1}{\pi}\sin x \exp(-x\cot x) \qquad (0 \le x < \pi), \tag{3}$$

with $\phi(0+) = \infty$ and $\phi(e) = 0$. For $v \neq 1$, the limiting measure χ_v has density $\phi(y/v)/v$ on the segment $(0, ve]$.

2.4 Resolvent method for Wigner ensembles with a variance profile σ

Let $\mathbb{C}^{+}$ denote the upper half plane in $\mathbb{C}$. In this paper, the Stieltjes transform $S(z) = S_{\mu}(z)$ of a finite measure or a nonnegative L^1-function μ on $\mathbb{R}$ is defined to be

$$S(z) = \int_{\mathbb{R}} \frac{\mu(dt)}{t - z} \quad (z \in \mathbb{C}^+).$$

In the sequel, we will need the following properties of the Stieltjes transform, which are not difficult to prove.

Proposition 2

1. *Suppose that $S(z)$ is the Stieltjes transform of a finite measure ν on $\mathbb{R}$. If for all $x \in \mathbb{R}$ it holds*

$$\lim_{y \to 0+} \operatorname{Im} S(x + iy) = 0$$

then $S(z) \equiv 0$ and ν is a null measure ($\nu(B) = 0$ for any Borel set B).

2. *Suppose $f \geq 0$ and $f \in L^1(\mathbb{R})$. Let $S(z)$ be the Stieltjes transform of f. If f is continuous at x then*

$$\lim_{y \to 0+} \frac{1}{\pi} \operatorname{Im} S(x + iy) = f(x). \tag{4}$$

If f is continuous on an interval $[a, b]$, $a < b$, the convergence (4) is uniform for $x \in [a, b]$.

Recall that if μ is a probabilistic measure on $\mathbb{R}$, with Stieltjes transform $S(z)$ and the absolutely continuous part of μ has density f, then (4) holds for almost all x (Lemma 3.2 (iii) of Bordenave, 2019).

We present now the following, slightly strengthened result from the Lecture Notes of Bordenave (2019, §3.2), that will be a main tool of proofs in this paper.

Let $\sigma : [0, 1] \times [0, 1] \to [0, \infty)$ be a bounded Borel measurable symmetric function. For each integer n, we partition the interval [0, 1] into n equal intervals J_i, $i = 1, \ldots, n$. Put $Q_{ij} := J_i \times J_j$, which is a partition of $[0, 1] \times [0, 1]$. We assume that Y_{ij} ($i \leq j$) are independent centered real variables, defined on a common probability space, with variance

$$\mathbb{E} Y_{ij}^2 = \frac{1}{n} \left(\int_{Q_{ij}} \frac{\sigma(x, y)}{|Q_{ij}|} \, dx \, dy + \delta_{ij}(n) \right), \tag{5}$$

for a sequence $\delta_{ij}(n)$. We note that the law of Y_{ij} depends on n. We set $Y_{ji} := Y_{ij}$ and we consider the symmetric matrix $Y_n := (Y_{ij})_{1 \leq i,j \leq n}$. We note that, if σ is continuous, then, up to a perturbation $\delta_{ij}(n)$, the variance of $\sqrt{n} Y_{ij}$ is approximatively $\sigma(i/n, j/n)$, and hence we call σ a variance profile in this paper.

Theorem 3 *Let $\delta_0(n) := \frac{1}{n^2} \sum_{i,j \leq n} |\delta_{ij}(n)|$. Assume (5) and suppose that*

$$\lim_n \delta_0(n) = 0 \quad \text{and} \quad \max_{i,j\le n} \frac{\mathbb{E}(Y_{ij}^4)}{n(\mathbb{E}Y_{ij}^2)^2} = o(1) \quad (Y_{ij} \neq 0). \tag{6}$$

Let μ_{Y_n} be the empirical eigenvalue distribution of Y_n. Then, there exists a probability measure μ_σ depending on σ such that μ_{Y_n} converges weakly to μ_σ almost surely. The Stieltjes transform S_σ of μ_σ is given as follows.

(a) *For each z with $\operatorname{Im} z > \sqrt{\sup \sigma}$, there exists a unique $\mathbb{C}^+$-valued L^1-solution $\eta_z : [0,1] \mapsto \mathbb{C}^+$, of the equation*

$$\eta_z(x) = -\left(z + \int_0^1 \sigma(x,y)\, \eta_z(y)\, dy \right)^{-1} \quad \text{(for almost all } x \in [0,1]), \tag{7}$$

and the function $z \mapsto \eta_z(x)$ extends to an analytic $\mathbb{C}^+$-valued function on $\mathbb{C}^+$, for almost all $x \in [0,1]$. Then,

$$S_\sigma(z) = \int_0^1 \eta_z(x)\, dx.$$

(b) *The function $x \to \eta_z(x)$ is also a solution of (7) for $0 < \operatorname{Im} z \le \sqrt{\sup \sigma}$.*

Proof The proof is the same as the proof of Bordenave (2019, Theorem 3.1), where a stronger assumption $|\delta_{ij}(n)| \le \delta(n)$ is required for some sequence $\delta(n)$ going to 0. It is replaced by the first condition of (6). Detailed analysis of the proof of the approximate fixed point equation in Bordenave (2019, page 42) shows that the second condition of (6) is the weakest assumption on the fourth moments $\mathbb{E}Y_{ij}^4$ ensuring the concentration of the conditional variance related to the Schur complement of the Stieltjes transform of the approximating matrix of Y_n. The property (b) is observed in Bordenave (2019, page 39) by analiticity. ◻

Since now we assume that $\sigma : [0,1]^2 \to [0,+\infty)$ is bounded (not obligatorily by 1 as in Bordenave, 2019). Consequently, the condition on z should be $\operatorname{Im} z > \sqrt{\sup \sigma}$, not $\operatorname{Im} z > 1$. Theorem 3 shows that, to each variance profile function σ, one associates uniquely a Stieltjes transform $S_\sigma(z)$ of a probability measure. For the correspondence between σ and S_σ, the conditions (6) are not needed. We define $S_\sigma(z)$ as the *Stieltjes transform associated to σ.*

Remark 4 A prototype of the variance profile method for Wigner ensembles was given by Anderson and Zeitouni (2006, Theorem 3.2). Theorem 3.1 of Bordenave (2019) and Theorem 3 provide a simple general approach. Special cases of variance profile convergence results for Wigner matrices were studied before, as discussed below in (i) and (ii).

(i) If we set $\sigma(x,y) = 1$ for all x, y, then $\sqrt{n}Y$ is a Wigner ensemble with $v = v' = 1$. Let $S_{\mathrm{sc}}(z)$ be the Stieltjes transform of the semi-circle law on $[-2,2]$. Then, the functions $x \to \eta_z(x)$ do not depend on x (but do on z) and the functional equation (7) gives the equation $S_{\text{sc}}(z) = -(z + S_{\text{sc}}(z))^{-1}$, which is well known from the detailed study of resolvent matrices (see Tao, 2012, §2.4.3).

(ii) The paper Anderson and Zeitouni (2006) deals primarily with a variance profile σ such that $\int \sigma(x, y)\, dy = 1$ for any x, corresponding to a band matrix model. For band matrix ensembles, see also Erdös et al. (2012a, 2012b), Nica et al. (2002), Shlyakhtenko (1996).

3 Wigner ensembles of Vinberg matrices

In this section, we give explicitly the limiting eigenvalue distributions μ for the Wigner matrices $U_n \in \mathcal{U}_n$ defined by (1). Let χ_I denote the indicator function of a subset $I \subset \mathbb{R}$. For a real number a, its cubic root is denoted by $\sqrt[3]{a} \in \mathbb{R}$ and set $[\,a\,]_+ = \max(a, 0)$. We introduce two real numbers α_c, β_c depending on $c \in [0, 1)$ by

$$\alpha_c = \frac{8 + 4c - 13c^2 - \sqrt{c(8-7c)^3}}{8(1-c)}, \quad \beta_c = \frac{8 + 4c - 13c^2 + \sqrt{c(8-7c)^3}}{8(1-c)}. \tag{8}$$

It is clear that $\alpha_0 = \beta_0 = 1$, $\alpha_c < \beta_c$ and $\beta_c > 0$ for all $c \in (0, 1)$. We note that $\alpha_{1/2} = 0$, $\alpha_c < 0$ when $c > 1/2$, $\lim_{c\to 1-} \alpha_c = -\infty$, $\lim_{c\to 1-}(1-c)\alpha_c = -1/4$ and $\lim_{c\to 1-} \beta_c = 4$, so that we set $\beta_1 = 4$. It can be shown that $c \mapsto \alpha_c$ is strictly decreasing and $c \mapsto \beta_c$ is strictly increasing on [0, 1].

Theorem 5 *Let U_n be a Wigner matrix on $\mathcal{U}_n$ defined by (1). Assume that $\lim_{n\to+\infty} a_n/n = c \in (0, 1)$. Then, the limiting eigenvalue distribution μ of the rescaled matrices $U_n/\sqrt{n}$ exists and is given for $c \in (0, 1)$ as*

$$\mu = f_c(t)\, dt + [1 - 2c]_+ \delta_0$$

with

$$f_c(t) := \frac{\sqrt[3]{R_+\left(t/\sqrt{v}; c\right)} - \sqrt[3]{R_-\left(t/\sqrt{v}; c\right)}}{2\sqrt{3}\pi\, t}\, \chi_{[\alpha_c, \beta_c]}\left(\frac{t^2}{v}\right), \tag{9}$$

where, for $x^2 \in [\alpha_c, \beta_c]$,

$$R_\pm(x; c) := x^6 - 3(c+1)x^4 + \frac{3}{2}(5c^2 - 2c + 2)x^2 + (2c-1)^3$$
$$\pm 3c\sqrt{3-3c} \cdot x\sqrt{(x^2 - \alpha_c)(\beta_c - x^2)}.$$

The support of μ is given as

$$\operatorname{supp}\mu = \begin{cases} \left[-\sqrt{v\beta_c}, -\sqrt{v\alpha_c}\right] \cup \{0\} \cup \left[\sqrt{v\alpha_c}, \sqrt{v\beta_c}\right] & (\text{if } c \in (0, \frac{1}{2})) \\ \left[-\sqrt{v\beta_c}, \sqrt{v\beta_c}\right] & (\text{if } c \in [\frac{1}{2}, 1)). \end{cases} \tag{10}$$

If $c = 0$, then $\mu = \delta_0$. If $c = 1$, then μ is the semicircle law on $[-2\sqrt{v}, 2\sqrt{v}]$.

Remark 6 The formula (9) is valid for the extreme cases $c = 0$ or $c = 1$. If $c = 0$ then there is no density and $\mu = \delta_0$. If $c = 1$, then it can be checked that $\sqrt[3]{R_+(x;1)} - \sqrt[3]{R_-(x;1)} = \sqrt{3}x\sqrt{4-x^2}$ so that, for $v = 1$ we get the semicircle law $\mu(dt) = (1/2\pi)\sqrt{4-t^2}\chi_{[-2,2]}(t)dt$ of Wigner (1955).

Remark 7 Note that the limiting measure μ does not depend on the diagonal variance v'. This phenomenon already holds for the classical Wigner ensemble. In terms of the variance profile method, it may be explained by the fact that the variance profile (11) does not depend on v' because the difference $|v - v'|$ on the diagonal is absorbed by the perturbation terms δ_{ii}.

Remark 8 An intuitive explanation of the fact that if $c < \frac{1}{2}$ then μ has an atom at 0 and $\mu((0, \sqrt{v\alpha_c})) = 0$ is that small eigenvalues are strongly attracted by the zero eigenvalue and asymptotically vanish. Note that if $c = 0$, the model is asymptotically diagonal. For the diagonal Wigner matrices, the empirical eigenvalue distribution converges to 0 by the Strong Law of Large Numbers.

Sketch of the proof We first derive the Stieltjes transform of the limiting eigenvalue distribution by applying Theorem 3 to $Y_n = U_n/\sqrt{n}$. Let $U_n = (U_{ij})_{1\le i,j\le n}$, so that $Y_{ij} = (1/\sqrt{n})U_{ij}$. The variance profile is given as

$$\sigma(x,y) = \begin{cases} v & \text{if } (x,y) \in \mathcal{C}, \\ 0 & \text{otherwise,} \end{cases} \qquad \mathcal{C} := \left\{(x,y) \in [0,1]^2;\ \min(x,y) \le c\right\}. \tag{11}$$

Here, the perturbation term $\delta_{ij}(n)$ equals $\delta_{ij}(n) = \mathbb{E}U_{ij}^2 - v\frac{|\mathcal{C}\cap Q_{ij}|}{|Q_{ij}|}$.

We check easily that the conditions (6) are satisfied, since, by (1) and writing $M := \max\{|v - v'|, v', v\}$, we get

$$\delta_0(n) \le \frac{3M}{n} \quad \text{and} \quad \max_{i,j\le n} \frac{\mathbb{E}(Y_{ij}^4)}{n(\mathbb{E}Y_{ij}^2)^2} \le \frac{\max\{M_4, {M_4}'\}}{n\min\{v, v'\}}.$$

Let us fix $z \in \mathbb{C}^+ = \{z \in \mathbb{C};\ \operatorname{Im} z > 0\}$. The functional equation (7) from Theorem 3 becomes

$$\eta_z(x) = \begin{cases} -\left(z + v\int_0^1 \eta_z(y)\,dy\right)^{-1} & (x \le c), \\ -\left(z + v\int_0^c \eta_z(y)\,dy\right)^{-1} & (x > c). \end{cases}$$

Observe that the right-hand sides are independent of x. Integrating both sides of these equations, we obtain the following simultaneous equations

$$B = \frac{-c}{z + vA}, \quad A - B = \frac{c-1}{z + vB}, \tag{12}$$

where $A = \int_0^1 \eta_z(x)\,dx$ and $B = \int_0^c \eta_z(x)\,dx$. Note that A is the desired Stieltjes transform $S(z)$.

If $c = 0$, then we have $A = -1/z$ so that the limiting measure is $\mu = \delta_0$. If $c = 1$ then the equation (7) reduces to the equation $A = -(z + vA)^{-1}$, which corresponds to the Stieltjes transform of the semi-circular law (cf. Tao, 2012, p. 178). Thus we assume $0 < c < 1$ in what follows. Then, the cubic equation for A, resulting from (12) writes

$$v^2 zA^3 + (2vz^2 + v^2(1-2c))A^2 + (z^2 + 2v(1-c))zA + z^2 - vc^2 = 0 \tag{13}$$

and it is an algebraic equation with polynomial coefficients. The last equation (13) is reduced to

$$Y^3 + p\left(z_v\right)Y + q\left(z_v\right) = 0, \tag{14}$$

where we set $z_v := z/\sqrt{v}$,

$$Y = Y(z) := \frac{vA}{z} + \frac{2}{3} - \frac{(2c-1)v}{3z^2}, \tag{15}$$

and the coefficients p, q are given by the following analytical rational functions on $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$

$$p(z) := -\frac{z^4 - 2(c+1)z^2 + (2c-1)^2}{3z^4},$$
$$q(z) := -\frac{2}{27} \cdot \frac{z^6 - 3(c+1)z^4 + \frac{3}{2}(5c^2 - 2c + 2)z^2 + (2c-1)^3}{z^6}.$$

We shall use classical results on the solutions of cubic equations. Let $\mathrm{Disc}(z)$ be the discriminant of the cubic equation (14), that is, $\mathrm{Disc}(z) = (s_1(z) - s_2(z))^2(s_2(z) - s_3(z))^2(s_1(z) - s_3(z))^2$, where $s_i(z)$ are solutions in Y of (14). Then, it is well known that $\mathrm{Disc}(z)$ can be expressed by $p(z)$ and $q(z)$, using α_c, β_c in (8), as (cf. Ronald, 2004)

$$\mathrm{Disc}(z) = -\left(4p(z)^3 + 27q(z)^2\right) = \frac{4c^2(1-c)}{z^{10}}(z^2 - \alpha_c)(z^2 - \beta_c).$$

Let $\mathcal{E} = \left\{z \in \mathbb{C};\, z = 0 \text{ or } \mathrm{Disc}(z_v) = 0\right\}$ be the set of exceptional points of (14). For $z \notin \mathcal{E}$, the equation (14) has three different solutions (cf. Ronald, 2004). Cardano's method and formula (15) imply that, for $z \in \mathbb{C}^+$

$$S(z)=\frac{z(u_+(z)+u_-(z))}{3v}-\frac{2z}{3v}+\frac{2c-1}{3z} \tag{16}$$

with $u_\pm(z) := \left(F_c(z_v) \pm i\,D_c(z_v)\right)^{\frac{1}{3}}$, $F_c(z) := -\frac{27}{2}q(z)$ and

$$D_c(z) := 27\cdot\sqrt{\frac{\mathrm{Disc}(z)}{4\cdot 27}}=\frac{3c\sqrt{3-3c}}{z^5}\sqrt{(z^2-\alpha_c)(z^2-\beta_c)},$$

where convenient branches of the cube and the square roots are chosen, respectively, for $u_\pm(z)$ and $D_c(z)$ to be such that $S(z)$ is a Stieltjes transform of a probability measure. In particular, $S(z)$ is holomorphic on $\mathbb{C}^+$ and

$$u_+(z)\cdot u_-(z)=-3p(z),\quad \text{and}\quad \operatorname{Im} S(z)>0 \quad (z\in\mathbb{C}^+). \tag{17}$$

Note that the branches of the roots may be different on different subregions of $\mathbb{C}^+$ and that $U := (u_+ + u_-)/3$ is a solution of (14). In order to derive the limiting eigenvalue distribution μ from $S(z)$, we will need the following properties of $S(z)$. Set $\mathbb{R}^* := \mathbb{R}\setminus\{0\}$.

Proposition 9 *The limit $S(x)=\lim_{y\to+0} S(x+yi)$ exists for each $x\in\mathbb{R}^*$. The function S is continuous on $\mathbb{R}^*$ and $S(x)$ is a solution of (13) on $\mathbb{R}^*$.*

Sketch of the proof of the proposition It is sufficient to prove it for a solution $U(z)$ of the reduced equation (14) on $\mathbb{C}^+$, such that $U(z)$ is holomorphic on $\mathbb{C}^+$. We apply Theorem X.3.7 of Palka (1991) to a convenient connected and simply connected domain D avoiding the set $\mathcal{E}$. By the discussion of Ahlfors (1979, p. 304), U has at most an ordinary algebraic singularity at a nonzero exceptional point, so $U(z)$ is continuous on $\mathbb{R}^*$. □

Without loss of generality, we suppose $v=1$. We first assume that $x=0$. The detailed local analysis of (16) and (17) that we omit here, shows that

(Z1) if $0<c<\frac{1}{2}$, then $\lim_{y\to+0} y\operatorname{Im} S(yi)=1-2c$, so μ has an atom at 0 with the mass $1-2c<1$,
(Z2) if $c=\frac{1}{2}$, then $\lim_{y\to+0}\operatorname{Im} S(yi)=+\infty$, $\lim_{y\to+0} y\operatorname{Im} S(yi)=0$ so μ does not have an atom at 0,
(Z3) if $\frac{1}{2}<c<1$, then $\lim_{y\to+0}\operatorname{Im} S(yi)=c(2c-1)^{-1/2}=\pi f_c(0)$, so μ does not have an atom at 0.

Next we consider the case $x\neq 0$. Combining the fact that $S(z)$ is an odd function as a function on $\mathbb{C}\setminus\mathbb{R}$ by (16) and the property $S(\bar{z})=\overline{S(z)}$ of the Stieltjes transform, we obtain $\operatorname{Im} S(-x+iy)=\operatorname{Im} S(x+iy)$ so that $\operatorname{Im} S(-x)=\operatorname{Im} S(x)$. Thus we can assume that $x>0$.

Suppose $\mathrm{Disc}(x) \geq 0$. Since the coefficients p, q of (14) are real on $\mathbb{R}^*$, the equation (14) has only real solutions (cf. Ronald, 2004). Therefore, $S(x)$ is real so that the density of μ vanishes at such points.

Next we assume that $\mathrm{Disc}(x) < 0$. By Proposition 9, $S(x)$ is a solution of the cubic equation (13) and $U(x) = (u_+(x) + u_-(x))/3$ is a solution of the reduced equation (14). In particular, the formulas (16) and (17) hold for $S(x)$, with convenient choices of branches of cubic roots and square roots. Consequently, we have

$\left\{F_c(x) + iD_c(x), F_c(x) - iD_c(x)\right\} = \left\{R'_+(x), R'_-(x)\right\}$

as a set, where $R'_\pm(x) := R_\pm(x;c)/x^6 \in \mathbb{R}$. Let $\omega = e^{2i\pi/3}$ denote the cube root of 1 with positive imaginary part. Then, (16) yields that the sum $u_+(x) + u_-(x)$ has the following form

$$u_+(x) + u_-(x) = \omega^{k_+}\sqrt[3]{R'_+(x)} + \omega^{k_-}\sqrt[3]{R'_-(x)} \quad \text{with} \quad k_+, k_- \in \{0, 1, 2\}.$$

By the first condition in (17), as $p(x) \in \mathbb{R}$, we need to have $k_+ + k_- \equiv 0 \bmod 3$, that is, $(k_+, k_-) = (0, 0)$, (1, 2) and (2, 1). Using the fact that $R'_+(x) > R'_-(x)$ when $x > 0$ and $\mathrm{Disc}(x) < 0$, we see that the imaginary part of $u_+(x) + u_-(x)$ and of $\lim_{y\to 0+} S(x+iy)$ is, respectively, nul, positive and negative in these three cases. Since $\mathrm{Im}\, S(z) > 0$, the last case is impossible. Set $h(x) := \mathrm{Im}\left(\omega\sqrt[3]{R'_+(x)} + \omega^2\sqrt[3]{R'_-(x)}\right)$. Notice that h is a strictly positive continuous function on the set $\{x \in \mathbb{R};\, \mathrm{Disc}(x) < 0\}$ and that $\frac{1}{\pi}h(t) = f_c(t)$, the density part of μ in the formula (9). Since the function $\mathrm{Im}\, S$ is continuous on $\mathbb{R}^*$ by Proposition 9, we have $\mathrm{Im}\, S \equiv h$ or $\mathrm{Im}\, S \equiv 0$ on the set $\{x \in \mathbb{R}^*;\, \mathrm{Disc}(x) < 0\}$.

We now show that the latter case is impossible. Note that μ has no atoms different from zero because $S(z)$ is continuous on $\overline{\mathbb{C}^+} \setminus \{0\}$. By Theorem 2.4.3 of Anderson et al. (2010) and by the dominated convergence, we have for closed intervals $[a, b] \subset \mathbb{R}^*$

$$\mu([a,b]) = \frac{1}{\pi}\lim_{y\to 0+}\int_a^b S(x+iy)\,dx = \frac{1}{\pi}\int_a^b \lim_{y\to 0+} S(x+iy)\,dx = 0, \tag{18}$$

so that $\mu(0,\infty) = 0$ and, symmetrically, $\mu(-\infty, 0) = 0$. Since μ is a probability measure, we get $\mu = \delta_0$. This contradicts properties (Z1-3) proven in the case $x = 0$. Thus, we have $\mathrm{Im}\, S \equiv h$ on the set $\{x \in \mathbb{R}^*;\, \mathrm{Disc}(x) \leq 0\}$ and, for $x \in \mathbb{R}^*$, $\lim\limits_{y\to 0+} \dfrac{1}{\pi}\mathrm{Im}\, S(x+iy) = \dfrac{1}{\pi}h(x) = f_c(x)$. Note that f_c has a compact support $\{\mathrm{Disc}(x) \leq 0\}$. For $c \neq \frac{1}{2}$, the function f_c is continuous on $\mathbb{R}$. For $c = \frac{1}{2}$, a detailed analysis shows that $\lim_{x\to 0} f_c(0) = \infty$, with $f_c(x) \sim |x|^{-1/2}$ at $x = 0$ and f_c is continuous on $\mathbb{R}^*$. By property (Z3), if $c > \frac{1}{2}$ then $\lim_{y\to 0+} \mathrm{Im}\, S(iy) = \pi f_c(0)$. When $c \neq 1/2$, Proposition 2.1 implies that $\mu = f_c(t)\,dt + [1-2c]_+\delta_0$. Actually, if $s(z)$ is the Stieltjes transform of $\mu - f_c(t)\,dt - [1-2c]_+\delta_0$, then, using Proposition 2.2, we get $\lim_{y\to 0+} \mathrm{Im}\, s(x+iy) = 0$ for all $x \in \mathbb{R}$. When $c = 1/2$, by Proposition 2.2, we get $\lim_{y\to 0+} \mathrm{Im}\, s(x+iy) = 0$ for all $x \in \mathbb{R}^*$, uniformly on compact intervals $[a, b] \subset \mathbb{R}^*$. Like in (18), we conclude by Theorem 2.4.3 in Anderson et al. (2010) that $\mu = f_c(t)\,dt$. The support formula (10) follows by $\mathrm{supp} f_c = \{\mathrm{Disc}(x) \leq 0\}$. ☐

In the Figs. 1, 2 and 3 we present graphical comparison between simulations for $n = 4000$ and the limiting densities, when $c = 1/5, 1/2, 3/5$.

Remark 10 The Wigner case may be considered in a framework of operator-valued free probability theory by methods of the rectangular free probability (cf. Benaych-Georges, 2009; Mingo and Speicher, 2017, Chapter 9).

4 Wishart ensembles of Vinberg matrices

In this section, we shall consider the quadratic Wishart (covariance) matrices introduced in Sect. 2.3. We first prepare some special functions which we need later. They generalize the Lambert W function appearing (see Cheliotis, 2018) in the case $P_n = \mathrm{Sym}(n, \mathbb{R})^+$ and $\underline{m} = (1, \dots, 1)$.

4.1 Lambert–Tsallis *W* function and Lambert–Tsallis function $W_{\kappa,\gamma}$

For a nonzero real number κ, we set

$$\exp_\kappa(z) := \left(1 + \frac{z}{\kappa}\right)^\kappa \quad (1 + \frac{z}{\kappa} \in \mathbb{C} \setminus \mathbb{R}_{\le 0}), \quad \log^{\langle\kappa\rangle}(z) := \frac{z^\kappa - 1}{\kappa} \quad (z \in \mathbb{C} \setminus \mathbb{R}_{\le 0}),$$

where we take the main branch of the power function when κ is not integer. If $\kappa = \frac{1}{1-q}$, then it is exactly the so-called Tsallis *q-exponential function* and *q-logarithm*, respectively (cf. Amari and Ohara, 2011; Zhang et al., 2018). We have the following relationship between these two functions:

$$\log^{\langle 1/\kappa\rangle} \circ \exp_\kappa(z) = z \quad \left(-\pi < \kappa \mathrm{Arg}\left(1 + \frac{z}{\kappa}\right) < \pi\right). \tag{19}$$

Since $\lim_{\kappa\to\infty} \exp_\kappa(z) = e^z$, we regard $\exp_\infty(z) = e^z$ and $\log^{\langle 0\rangle}(z) = \log(z)$.

For two real numbers κ, γ such that $\gamma \le \frac{1}{\kappa} \le 1$ and $\gamma < 1$ (see Fig. 7), we introduce a holomorphic function $f_{\kappa,\gamma}(z)$, which we call *generalized Tsallis function*, by

$$f_{\kappa,\gamma}(z) := \frac{z}{1 + \gamma z} \exp_\kappa(z) \quad (1 + \frac{z}{\kappa} \in \mathbb{C} \setminus \mathbb{R}_{\le 0}).$$

We note that $\kappa \in (-\infty, 0) \cup [1, +\infty)$. Analogously to Tsallis q-exponential, we also consider $f_{\infty,\gamma}(z) = \frac{ze^z}{1+\gamma z}$ $(z \in \mathbb{C})$. In particular, $f_{\infty,0}(z) = ze^z$.

In our work, it is crucial to consider an inverse function to $f_{\kappa,\gamma}$. A multivariate inverse function of $f_{\infty,0}(z) = ze^z$ is called the Lambert W function and studied in Corless et al. (1996). Hence, we call an inverse function to $f_{\kappa,\gamma}$ the *Lambert–Tsallis W function.*

The function $f_{\kappa,\gamma}(z)$ has the inverse function $w_{\kappa,\gamma}$ in a neighborhood of $z = 0$, because we have $f'_{\kappa,\gamma}(0) = 1 \ne 0$ by

$$f'_{\kappa,\gamma}(z) = \frac{\gamma z^2 + (1 + 1/\kappa)z + 1}{(1+\gamma z)^2}\left(1 + \frac{z}{\kappa}\right)^{\kappa-1}.$$

Let us present some properties of $f_{\kappa,\gamma}$. When $\gamma\kappa \neq 1$, the function $f_{\kappa,\gamma}$ has a pole at $x = -\frac{1}{\gamma}$. By the condition on κ and γ, the function $\gamma z^2 + (1 + 1/\kappa)z + 1$ has two real roots, say $\alpha_1 \leq \alpha_2$, when $\gamma \neq 0$. If $\gamma = 0$, there is only one real root, that we denote $\alpha_2 = -\frac{\kappa}{\kappa+1}$. $f'_{\kappa,\gamma}(z) = 0$ implies $z = \alpha_i$ $(i = 1, 2)$, or $z = -\kappa$ if $\kappa > 1$. For the case $\kappa < 0$, it is convenient to change the variable by a homographic action $z' = \frac{z}{1+\frac{z}{\kappa}}$. Then

$$f_{\kappa,\gamma}(z) = f_{\kappa',\gamma'}(z') \quad \text{where} \quad \kappa' = -\kappa > 0, \quad \gamma' = \gamma - \frac{1}{\kappa}.$$

Since a homographic action by an element in $SL(2,\mathbb{R})$ leaves $\mathbb{C}^+$ invariant, the analysis of the case $\kappa < 0$ reduces to the case $\kappa' > 0$ and $\gamma' \leq 0$.

Let us set $\mathcal{S} := \mathbb{R} \setminus f_{\kappa,\gamma}(\mathbb{R})$. We shall see that $\mathcal{S}$ appears as the slit with respect to the Lambert–Tsallis W function. The set $\mathcal{S}$ has the following possibilities.

(S1) $\mathcal{S} = (f_{\kappa,\gamma}(\alpha_2), f_{\kappa,\gamma}(\alpha_1))$, where $f_{\kappa,\gamma}(\alpha_2) < f_{\kappa,\gamma}(\alpha_1) < 0$. It occurs when $\kappa \in [1, +\infty]$ and $\gamma < 0$, and when $\kappa < 0$ and $\gamma' = \gamma - \frac{1}{\kappa} < 0$.
(S2) $\mathcal{S} = (-\infty, f_{\kappa,\gamma}(\alpha_2))$, where $f_{\kappa,\gamma}(\alpha_2) < 0$. It occurs when $\kappa > 1$ and $\gamma \geq 0$ and when $(\kappa, \gamma) = (1, 0)$.
(S3) $\mathcal{S} = (-\infty, f_{\kappa,\gamma}(\alpha_1))$, where $f_{\kappa,\gamma}(\alpha_1) < 0$. It occurs when $\kappa < 0$ and $\gamma' = \gamma - \frac{1}{\kappa} = 0$.
(S4) $\mathcal{S} = (f_{\kappa,\gamma}(\alpha_1), f_{\kappa,\gamma}(\alpha_2))$, where $f_{\kappa,\gamma}(\alpha_1) < f_{\kappa,\gamma}(\alpha_2) < 0$. It occurs when $\kappa = 1$ and $0 < \gamma \leq 1$.

The cases (S1,S2,S3) are studied in detail in the Supplementary Material. The case (S4) appears in the well-known Wishart Ensemble case.

Theorem 11 *Let $\mathcal{S}$ be an interval or half-line given by* (S1)–(S4) *above, and* $\overline{\mathcal{S}} \subset (-\infty, 0)$ *its closure. Then, there exists a complex domain $\Omega \subset \mathbb{C}$, symmetric with respect to the real axis and containing 0, such that $f_{\kappa,\gamma}$ maps Ω bijectively to $\mathbb{C} \setminus \overline{\mathcal{S}}$. Consequently, the function $w_{\kappa,\gamma}$ can be continued in a unique way to a holomorphic function $W_{\kappa,\gamma}$ defined on $\mathbb{C} \setminus \overline{\mathcal{S}}$. The codomain of $W_{\kappa,\gamma}$ is Ω, that is, $W_{\kappa,\gamma}(\mathbb{C} \setminus \overline{\mathcal{S}}) = \Omega$.*

Proof The proof is based on the properties of $f_{\kappa,\gamma}$ showed in Proposition 13. □

Recall that the main branch of the Lambert W function is holomorphic on $\mathbb{C} \setminus (-\infty, -\frac{1}{e}]$ (see Corless et al., 1996).

Definition 12 The unique holomorphic extension $W_{\kappa,\gamma}$ of $w_{\kappa,\gamma}$ to $\mathbb{C} \setminus \overline{\mathcal{S}}$ is called *the main branch* of Lambert–Tsallis W function. In this paper, we only study and use $W_{\kappa,\gamma}$ among other branches so that we call $W_{\kappa,\gamma}$ the *Lambert–Tsallis function* for short. Note that in our terminology the Lambert–Tsallis W function is multivalued and the Lambert–Tsallis function $W_{\kappa,\gamma}$ is single-valued.

We summarize the basic properties of the Lambert–Tsallis function that we need later.

Proposition 13

(i) *Let $D = \Omega \cap \mathbb{C}^+$. The function $f_{\kappa,\gamma}$ is continuous and injective on the closure $\overline{D}$. Consequently, $W_{\kappa,\gamma}$ extends continuously from $\mathbb{C}^+$ to $\mathbb{C}^+ \cup \mathbb{R}$, and one has $f_{\kappa,\gamma}(\partial\Omega \cap \mathbb{C}^+) = \mathcal{S}$.*

(ii) *The Lambert–Tsallis function $W_{\kappa,\gamma}$ has the following properties.*

(a) *Suppose that $\kappa \geq 1$ and $\gamma < 0$, or $\kappa < 0$ and $\gamma' \leq 0$. In these cases, the set D is bounded. If $\kappa \geq 1$ then $D \subset \left\{z \in \mathbb{C}^+;\, \mathrm{Arg}\left(1 + \frac{z}{\kappa}\right) \in (0, \frac{\pi}{\kappa+1})\right\}$ and $z \in D$ satisfies $\mathrm{Re}\, z > -\kappa$. If $\kappa = \infty$, then $D \subset \left\{z \in \mathbb{C}^+;\, \mathrm{Im}\, z \in (0,\pi)\right\}$. If $\kappa < 0$ then $D \subset \left\{z \in \mathbb{C}^+;\, \mathrm{Arg}\left(\left(1 + \frac{z}{\kappa}\right)^{-1}\right) \in (0, \frac{\pi}{|\kappa|+1})\right\}$. Moreover, $\lim_{|z|\to+\infty} W_{\kappa,\gamma}(z) = -\frac{1}{\gamma}$ (recall that $-\frac{1}{\gamma}$ is a pole of $f_{\kappa,\gamma}$).*

(b) *Suppose $\kappa \in [1,+\infty]$ and $\gamma = 0$. The set $D = \Omega \cap \mathbb{C}^+$ is unbounded and $f_{\kappa,0}(\infty) = \infty$. If $\kappa \in [1,+\infty)$ then $D \subset \left\{z \in \mathbb{C}^+;\, \mathrm{Arg}\left(1 + \frac{z}{\kappa} \in (0, \frac{\pi}{\kappa+1})\right)\right\}$. If $\kappa = \infty$, then $W_{\infty,0}(z)$ is the classical Lambert function, and one has $D \subset \left\{z \in \mathbb{C}^+;\, \mathrm{Im}\, z \in (0,\pi)\right\}$.*

(c) *Suppose $\gamma > 0$. In this case we have $\kappa \in [1, \frac{1}{\gamma}]$. The set $D = \Omega \cap \mathbb{C}^+$ is unbounded and $f_{\kappa,\gamma}(\infty) = \infty$. Moreover, $D = \left\{z \in \mathbb{C}^+;\, \mathrm{Arg}\left(1 + \frac{z}{\kappa}\right) \in (0, \frac{\pi}{\kappa})\right\}$.*

Proof The main tool is the Argument Principle (cf. Ahlfors, 1979, Theorem 18, p. 152). A detailed study of the inverse image $f_{\kappa,\gamma}^{-1}(\mathbb{R})$ is performed. We omit the technical details, provided in Supplementary Material. □

Remark 14 It is worth underlying that we consider the main branch of the complex power function in the Tsallis q-exponential $\exp_\kappa(z)$ appearing inside the generalized Tsallis function $f_{\kappa,\gamma}$. Consequently, the main branch $W_{\kappa,\gamma}$ is the unique one such that $W(0) = 0$. A complete study of all branches of the Lambert–Tsallis W function will be interesting to do. The study of the Lambert–Tsallis function $W_{\kappa,\gamma}$ in the full range of parameters κ, γ is also an interesting open problem. We exclude the case $\kappa\gamma > 1$ with $\kappa > 0$ because we do not need it later. We note that, when $\kappa\gamma > 1$ and $\kappa > 1$ with a condition $(1+\kappa)^2 - 4\gamma\kappa^2 > 0$, then $f_{\kappa,\gamma}$ maps a subregion of $\mathbb{C}^+$ onto $\mathbb{C}^+$.

4.2 Quadratic Wishart matrices

We will now study eigenvalues of Wishart (covariance) matrices in $P_n \subset \mathcal{U}_n$, defined in Sect. 2.3. We apply the approach of Bordenave (2019, Cor.3.5), based on the variance profile method (Theorem 3).

In this subsection, we first consider the case of $a_n = n - 1$ and $b_n = 1$, that is, P_n is the symmetric cone $\mathrm{Sym}(n,\mathbb{R})^+$ of positive definite symmetric matrices of size n. Let $\xi_n = (\xi_{ij})$ be a rectangular matrix of size $n \times N$, where the entries ξ_{ij} are centered i.i.d. variables with variance v and fourth moment M_4. In order to study eigenvalue distributions of $X_n = \xi_n \xi_n^\top$, we equivalently consider Wigner matrices of the form

$$Y_n := \begin{pmatrix} 0 & \xi_n \\ \xi_n^\top & 0 \end{pmatrix} \in \mathrm{Sym}(n+N, \mathbb{R}). \tag{20}$$

If X_n has eigenvalues $\lambda_j \geq 0\ (j = 1, \dots, n)$, then those of Y_n are exactly $\pm\sqrt{\lambda_j}\ (j = 1, \dots, n)$ and zeros with multiplicity $|N - n|$. Let T_n denote the Stieltjes transform of the empirical eigenvalue distribution of rescaled X_n/n and S_n the Stieltjes transform of rescaled $Y_n/\sqrt{n+N}$. Then, it is easy to see that these Stieltjes transforms satisfy, by setting $p_n := \frac{n}{n+N}$ and $q_n = \frac{N}{n+N}$,

$$T_n\left(\frac{z^2}{p_n}\right) = \frac{1}{2z}\left(\frac{1-2p_n}{z} + S_n(z)\right). \tag{21}$$

In order to study eigenvalue distributions of covariance matrices from Sect. 2.3, with parameters $\underline{k}$ as in (2), we introduce a trapezoidal variance profile σ as follows. Let p, α be real numbers such that $0 < p < 1$ and $0 \leq \alpha \leq (1-p)/p$. Then, σ is defined by

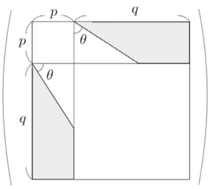
$$\sigma(x, y) := v \quad \text{if } (x, y) \in \mathcal{C}, \qquad \sigma(x, y) := 0 \quad \text{(otherwise)}, \tag{22}$$

where $\mathcal{C}$ is given as

$$\mathcal{C} = \left\{ (x, y) \in [0, 1]^2; \begin{array}{ll} \text{(i)} & x < p \text{ and } y \geq p + \alpha x, \\ \text{(ii)} & x \geq p \text{ and } 0 \leq y \leq \min\{(x-p)/\alpha, p\} \end{array} \right\}.$$

The perturbation term $\delta_{ij}(n)$ in (5) equals $\delta_{ij}(n) = \mathbb{E}U_{ij}^2 - v\frac{|\mathcal{C} \cap Q_{ij}|}{|Q_{ij}|}$. Graphically, $\mathcal{C}$ is of the form below. In particular, the parameter α indicates the slope of the trapezoid which appears in σ.

$\mathcal{C} =$



$$\text{with} \quad \begin{array}{l} p + q = 1, \ p, q > 0 \\ 0 \leq \tan\theta = \alpha \leq \frac{q}{p} \end{array} \tag{23}$$

If $\lim_n p_n = p$, by Theorem 3, this variance profile determines the limiting distribution of empirical eigenvalue distributions of the Wigner matrices Y_n in (20). Recall that, to a variance profile σ, Theorem 3 associates the Stieltjes transform $S_\sigma(z)$. It will be determined in Theorem 15. Analogously, to a variance profile σ of ξ_n, we associate the "covariance Stieltjes transform" $T_\sigma(z)$ of the corresponding covariance matrices $Q_{\underline{k}}(\xi_n) = \xi_n \xi_n^\top$. The covariance Stieltjes transform $T_\sigma(z)$ is related to $S_\sigma(z)$ by the formula (21). It will be determined in Proposition 17.

Theorem 15 *Let σ be a variance profile given in (22), and set $\kappa := 1/(1-\alpha)$ and $\gamma := (2p-1)/p = 1-(q/p)$. Then, the Stieltjes transform $S_\sigma(z)$ associated to σ is given as*

$$S_\sigma(z) = -\frac{2p}{zW_{\kappa,\gamma}\left(-\frac{vp}{z^2}\right)} + \frac{1-2p}{z} - \frac{2z}{v} \quad (z \in \mathbb{C}^+),$$

where $W_{\kappa,\gamma}$ is the Lambert–Tsallis function defined in Sect. 4.1.

Proof We use Theorem 3. Let $z \in \mathbb{C}^+$ with $\operatorname{Im} z > \sqrt{v}$. By (7), we have

$$\eta_z(x) = \begin{cases} -\left(z + v\displaystyle\int_{p+\alpha x}^{1} \eta_z(y)\,dy\right)^{-1} & (0 \le x \le p), \\ -\left(z + v\displaystyle\int_{0}^{\alpha^{-1}(x-p)} \eta_z(y)\,dy\right)^{-1} & (p < x \le p+\alpha p), \\ -\left(z + v\displaystyle\int_{0}^{p} \eta_z(y)\,dy\right)^{-1} & (p+\alpha p < x \le 1). \end{cases}$$

For z fixed, we set

$$a(t) := \eta_z(t), \quad t \in [0,p], \qquad b(t) := \eta_z(p+\alpha t), \quad t \in (0,p].$$

By differentiating both sides in the above equations, we obtain a system

$$\begin{cases} a'(t) = -v\alpha a(t)^2 b(t), \\ b'(t) = va(t)b(t)^2, \end{cases} \tag{24}$$

with initial data $a(p) = -\left(z + v\int_{p+\alpha p}^{1} \eta_z(y)\,dy\right)^{-1}$, $b(0+) = -\frac{1}{z}$. Note that the third line of definition of η_z ensures that η_z is constant on the interval $[p+\alpha p, 1]$ so that we have

$$a(p) = -(z + v(1-p-\alpha p)b(p))^{-1}. \tag{25}$$

By the unicity part of Theorem 3 holding for $\eta_z(x) \in \mathbb{C}^+$, it is enough to show that (24) is satisfied by

$$a(t) = -zw(z)X(t)^{\alpha\kappa}, \quad b(t) = -\frac{1}{z} \cdot X(t)^{-\kappa},$$

where we set $w(z) := -\frac{1}{vp}W_{\kappa,\gamma}\left(-\frac{vp}{z^2}\right)$ and $X(t) := 1 - \frac{vw(z)}{\kappa}\,t$, and that $a(t), b(t) \in \mathbb{C}^+$ for $\operatorname{Im} z$ big enough. Here, we choose the main branches for complex power functions. If $\alpha = 1$ then

$$a(t) = -zw(z)e^{-vw(z)t}, \quad b(t) = -\frac{1}{z} \cdot e^{vw(z)t}.$$

The crucial part of the proof is to show that $a(t)$ satisfies the initial data condition. We only give a proof for this in the case of $\alpha \neq 1$. Set $w = w(z)$ and $X = X(p)$ for brevity. Since $f_{\kappa,\gamma}(-vpw(z)) = -\frac{vp}{z^2}$, we have

$$\begin{aligned}
\frac{wX^{\kappa}}{1+v(1-2p)w} = \frac{1}{z^2} &\iff wz^2X^{\kappa} = 1 + v(1-2p)w \\
&\iff wz^2X^{\kappa} = 1 - \frac{vwp}{\kappa} - (p+\alpha p-1)vw \\
[1em] &\iff X = z^2wX^{\kappa} + (p+\alpha p-1)vw \\
&\iff 1 = zwX^{\kappa-1}\left(z + (p+\alpha p-1)\frac{v}{z}\cdot X^{-\kappa}\right) \\
&\iff -zwX^{\kappa-1} = -\left(z + \frac{v(p+\alpha p-1)}{z}\cdot X^{-\kappa}\right)^{-1}.
\end{aligned}$$

In the second and third equivalences, we use the formulas $\kappa = 1/(1-\alpha)$ and $X = 1 - \frac{vwp}{\kappa}$. Since $a(p) = -zwX^{\alpha\kappa} = -zwX^{\kappa-1}$ by $\alpha\kappa = \kappa - 1$, we see that

$$a(p) = -\left(z + v\cdot\frac{p+\alpha p-1}{zX^{\kappa}}\right)^{-1}.$$

By (25) and by definition of $b(t)$, we conclude that $a(t)$ satisfies the initial condition. We omitted other details of the proof. □

Remark 16 We call the parameter κ of Lambert–Tsallis functions the *angle parameter* since it depends only on the angle of the trapeze in (23). If $\kappa = 1$, then we have $\alpha = 0$ so that the trapeze reduces to a rectangle. If $\alpha = q/p$, i.e., $\kappa = p/(p-q) = 1/\gamma$, then the trapeze reduces to a triangle. On the other hand, the parameter $\gamma = \frac{2p-1}{p} = 1 - C$ depends directly on the shape parameter $C = q/p$. We call γ the *shape parameter* of the Lambert–Tsallis function. Note that the geometric condition $0 \leq \alpha \leq \frac{p}{q}$ is equivalent to the condition $\frac{1}{\kappa} \geq \gamma$. The formula $\gamma = 1 - \frac{q}{p}$ shows that $\gamma \in (-\infty, 1)$, and hence we have $\kappa \in [1, \frac{1}{\gamma}]$ if $0 \leq \gamma < 1$, and $\kappa \in [1, \infty] \cup (-\infty, \frac{1}{\gamma}]$ if $\gamma < 0$.

Now we give the covariance Stieltjes transform $T_{\sigma}(z)$ for the profile σ. We note that, corresponding to a probability measure μ, there exists the so-called R-transform $R(z)$ which plays an important role in the field of free probability (cf. Mingo and Speicher, 2017, Chapter 3). It satisfies a relation $R(z) = S^{-1}(-z) - 1/z$ where $S(z)$ is the Stieltjes transform of μ. We also give the corresponding R-transform R_{σ} related to σ in a view of future studies.

Proposition 17 *Let σ be a variance profile defined in (22) with parameters p and α. Set $\kappa := \frac{1}{1-\alpha}$ and $\gamma := \frac{2p-1}{p} = 1 - \frac{q}{p}$. Then, the covariance Stieltjes transform $T_{\sigma}(z)$ corresponding to the profile σ is described as*

$$T_\sigma(z) = T_{\kappa,\gamma}(z) := -\frac{1}{v} - \frac{1}{zW_{\kappa,\gamma}\left(-\frac{v}{z}\right)} - \frac{\gamma}{z} = \frac{\exp_\kappa\left(W_{\kappa,\gamma}(-v/z)\right) - 1}{v} \tag{26}$$

for $z \in \mathbb{C}^+$, and its R-transform $R_\sigma(z)$ is given as

$$R_\sigma(z) = -\frac{1}{z} - \frac{v\gamma}{1 - vz} - \frac{v}{(1 - vz)\log^{\langle 1/\kappa\rangle}(1 - vz)} \quad (1 - vz \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}).$$

Proof The first equality of the formula for $T_\sigma(z)$ is given by the formula (21), and the second by the definition of the Lambert–Tsallis function. To prove the formula of R-transforms, we use the fact that $-\pi < \kappa \mathrm{Arg}\left(1 + \frac{W(z)}{\kappa}\right) < \pi$ for any $z \in \mathbb{C}^+$ coming by Proposition 13 (ii) and we use relation (19). □

Recall that Ω denotes the codomain of $W_{\kappa,\gamma}$. By Proposition 13, for each $x \in \mathcal{S}$, there are exactly two solutions of $f_{\kappa,\gamma}(z) = x$ in $z \in \partial\Omega$, which are conjugate complex numbers, denoted by $K_+(x)$, $K_-(x)$, such that $\mathrm{Im}\, K_+(x) > 0$. Recall that $\alpha_1 \leq \alpha_2$ are zero-points of the function $\gamma z^2 + (1 + 1/\kappa)z + 1$. Then, we have the following theorem.

Theorem 18 *Let σ be a trapezoidal variance profile defined by (22). Let μ_σ be the probability measure corresponding to the associated covariance Stieltjes transform T_σ given by (26). Then, the density function d_σ of μ_σ is given as*

$$d_\sigma(x) = \begin{cases} \dfrac{1}{2\pi x i}\left(\dfrac{1}{K_-(-\frac{v}{x})} - \dfrac{1}{K_+(-\frac{v}{x})}\right) & (\text{if } -\frac{v}{x} \in \mathcal{S}), \\ 0 & (\text{if } -\frac{v}{x} \in \mathbb{R} \setminus \mathcal{S}). \end{cases} \tag{27}$$

Moreover, one has the following possibilities.

1. *In the case $p < q$ and $\frac{q}{p} \neq \alpha$ (i.e., $\kappa \geq 1$ and $\gamma < 0$, or $\kappa < 0$ and $\gamma' < 0$), the measure μ_σ is absolutely continuous and its density $d_\sigma(x)$ is continuous on $\mathbb{R}$. In particular, μ_σ has no atoms. Its support is given as*

$$\mathrm{supp}\mu_\sigma = \left[-\frac{v}{f_{\kappa,\gamma}(\alpha_2)}, -\frac{v}{f_{\kappa,\gamma}(\alpha_1)}\right] = \left[\frac{v}{\alpha_2^2}\left(1 + \frac{\alpha_2}{\kappa}\right)^{1-\kappa}, \frac{v}{\alpha_1^2}\left(1 + \frac{\alpha_1}{\kappa}\right)^{1-\kappa}\right]. \tag{28}$$

2. *In the case $p = q = \frac{1}{2}$ or $\frac{q}{p} = \alpha$ (i.e., $\kappa \geq 1$ and $\gamma = 0$, or $\kappa < 0$ and $\gamma' = 0$), the measure μ_σ is absolutely continuous. Its density d_σ is continuous on $\mathbb{R}^*$ and $\lim_{x \to +0} d_\sigma(x) = +\infty$. In particular, μ_σ has no atoms. Let $\alpha_0 := \alpha_2$ if $\kappa \geq 1$ and $\alpha_0 := \alpha_1 = -1$ if $\kappa < 0$. The support of μ_σ is given as*

$$\operatorname{supp}\mu_\sigma = \left[0, -\frac{v}{f_{\kappa,\gamma}(\alpha_0)}\right] = \left[0, \frac{v}{\alpha_0^2}\left(1+\frac{\alpha_0}{\kappa}\right)^{1-\kappa}\right]. \tag{29}$$

When $\kappa = \infty$, the measure μ_σ is the Dykema–Haagerup measure χ_v with support $[0, ve]$.

3. *In the case $p > q$ (i.e., $\kappa \geq 1$ and $0 < \gamma < 1$), we have $\mu_\sigma = d_\sigma(x)dx + (1-\frac{q}{p})\delta_0$. The measure μ_σ has an atom at $x = 0$ with mass $1-\frac{q}{p}$. Recall that $\kappa \in [1, 1/\gamma]$. When $\kappa > 1$, the support of μ_σ is given by (29). The function d_σ is continuous on $\mathbb{R}^*$ and $\lim_{x\to+0} d_\sigma(x) = +\infty$. For $\kappa = 1$ and $-\infty < \gamma < 1$, the measure μ_σ is the Marchenko–Pastur law μ_C with parameter $C = \frac{q}{p} = 1-\gamma \in (0,1)$ and* $\operatorname{supp} d_\sigma = \left[v(1-\sqrt{C})^2, v(1+\sqrt{C})^2\right]$.

Proof We use Proposition 17. Let $z = x + yi$. By Proposition 13 (i) and the fact that $W_{\kappa,\gamma}(z) = 0$ only if $z = 0$, we see that $l(x) := \lim_{y\to+0} \operatorname{Im} T_\sigma(x+iy)$ exists when $x \neq 0$ and that $l(x) = 0$ when $-v/x \notin \mathcal{S}$.

Assume that $x \neq 0$ and $-v/x \in \mathcal{S}$. Set $a(x) + ib(x) := \lim_{y\to 0+} W_{\kappa,\gamma}(-v/z)$. Since the function $f_{\kappa,\gamma}$ is continuous and injective on the closure $\overline{D} \subset \mathbb{C}^+$, the function $a + ib$ is continuous. By Proposition 13 (i), we have $b(x) > 0$ and $a(x) + ib(x) = K_+(-\frac{v}{x})$. Since $\overline{\mathcal{S}} \subset (-\infty, 0)$ by Theorem 11, we have $-v/x < 0$, that is, $x > 0$. Thus, we obtain for $-v/x \in \mathcal{S}$ with $x \neq 0$

$$\begin{aligned} l(x) &= \lim_{y\to 0+} \operatorname{Im} T_\sigma(x+yi) = \operatorname{Im}\left(-\frac{1}{v} - \frac{1}{x(a(x)+ib(x))} - \frac{\gamma}{x}\right) \\ &= -\frac{1}{2xi}\left(\frac{1}{K_+(-\frac{v}{x})} - \frac{1}{K_-(-\frac{v}{x})}\right) = \frac{b(x)}{x(a(x)^2+b(x)^2)} > 0, \end{aligned} \tag{30}$$

and thus $l(x)$ is a continuous function on $\mathbb{R}^*$. Therefore, $x \in \mathbb{R}^*$ is included in the support of μ_σ if and only if $-v/x \in \overline{\mathcal{S}}$. By (4), we have $d_\sigma(x) = \frac{1}{\pi} l(x)$, so that we obtain (27).

Let us consider the case (S1). In this case, since $\mathcal{S} = (f_{\kappa,\gamma}(\alpha_2), f_{\kappa,\gamma}(\alpha_1))$ and $f_{\kappa,\gamma}(\alpha_1) < 0$, we see that the condition $x \in \operatorname{supp}\mu$ is equivalent to

$$f_{\kappa,\gamma}(\alpha_2) \leq -\frac{v}{x} \leq f_{\kappa,\gamma}(\alpha_1) < 0 \iff -\frac{v}{f_{\kappa,\gamma}(\alpha_2)} \leq x \leq -\frac{v}{f_{\kappa,\gamma}(\alpha_1)}.$$

Recall that $\alpha_i, i = 1, 2$ are the real solutions of the equation $\gamma z^2 + (1 + 1/\kappa)z + 1 = 0$. For a solution α of this equation, we have by $1 + \alpha/\kappa = -\alpha(1 + \gamma\alpha)$

$$f_{\kappa,\gamma}(\alpha) = \frac{\alpha}{1+\gamma\alpha}\left(1+\frac{\alpha}{\kappa}\right)^\kappa = -\alpha^2\left(1+\frac{\alpha}{\kappa}\right)^{\kappa-1},$$

so that we arrive at the assertion 1 of the theorem. The argument for other two cases is similar, and hence we omit it.

Next we consider the case $x = 0$. We present the case $\kappa \in [1, +\infty)$ and $\gamma = 0$. For $z \in \mathbb{C}^+$, let us set $re^{i\theta} = 1 + \frac{W_{\kappa,0}(-v/z)}{\kappa}$ $(r > 0,\ \theta \in (0, \pi))$. By Proposition 13 (ii-b), the set $D = \Omega \cap \mathbb{C}^+$ is unbounded and $f_{\kappa,0}(\infty) = \infty$. Consequently, if $z \to 0$ in $\mathbb{C}^+$, or equivalently $-v/z \to \infty$ in $\mathbb{C}^+$, then we have $W_{\kappa,0}(-v/z) \to \infty$ and $r \to +\infty$. Again by Proposition 13 (ii-b), we see that $\theta \in (0, \frac{\pi}{\kappa+1})$ so that $\sin \kappa\theta > 0$ when $z = -v/(iy) \in \mathbb{C}^+$, and thus

$$\begin{aligned}\operatorname{Im} T_\sigma(z) =&\operatorname{Im} \frac{\exp_\kappa\left(W_{\kappa,0}(-v/z)\right) - 1}{v} = \operatorname{Im} \frac{(re^{i\theta})^\kappa - 1}{v}\\ =&\operatorname{Im} \frac{r^\kappa \cos \kappa\theta - 1 + ir^\kappa \sin \kappa\theta}{v} = \frac{r^\kappa \sin \kappa\theta}{v} \to +\infty \quad (y \to +0).\end{aligned}$$

On the other hand, μ_σ does not have an atom at $x = 0$ because we have by $W_{\kappa,0}(-v/z) \to \infty$ and by $\gamma = 0$

$$yT_\sigma(iy) = -\frac{y}{v} - \frac{1}{iW_{\kappa,0}(-v/(yi))} - \frac{\gamma}{i} \to \gamma i = 0 \quad (y \to +0).$$

The proofs for other cases are similar, and hence we omit them.

The absolute continuity of μ_σ follows from Proposition 2, by considering $\mu_0 := \mu_\sigma - d_\sigma(x)dx$, or, in the case with atom at $x = 0$, of $\mu_0 := \mu_\sigma - d_\sigma(x)dx - \gamma\delta_0$ and using the fact that the Stieltjes transform $S_0(z)$ of μ_0 satisfies $\lim_{y\to 0+} \operatorname{Im} S_0(x + iy) = 0$ for all $x \in \mathbb{R}$. The argument is similar as in the proof of Theorem 5. □

In the following corollary, we give a real implicit equation for the density d_σ analogous to the Dykema–Haagerup equation (3). To do so, we introduce the following notation

$$e_\kappa(z) := \left|\exp_\kappa(z)\right| \geq 0, \quad \theta_\kappa(z) = \kappa \operatorname{Arg}\left(1 + \frac{z}{\kappa}\right) \quad (z \in \mathbb{C}^+).$$

If $\kappa = \infty$, we set $e_\kappa(z) := e^{\operatorname{Re} z}$ and $\theta_\kappa(z) := \operatorname{Im} z$. Then, we have $\exp_\kappa(z) = e_\kappa(z)\left(\cos\left(\theta_\kappa(z)\right) + i \sin\left(\theta_\kappa(z)\right)\right)$.

Corollary 19

(i) *Suppose $v = 1$ for simplicity. For two real numbers κ, γ such that $\gamma \leq \frac{1}{\kappa} \leq 1$ and $\gamma < 1$, the density d_σ of the limiting law μ_σ satisfies the equation*

$$d_\sigma\left(\frac{\sin\left(\theta_\kappa(z)\right)}{b} \cdot \frac{1 + \gamma a - \gamma b \cot\left(\theta_\kappa(z)\right)}{e_\kappa(z)}\right) = \frac{e_\kappa(z)}{\pi} \cdot \sin\left(\theta_\kappa(z)\right) \tag{31}$$

for $z = a + bi \in \partial\Omega \cap \mathbb{C}^+$. In particular, when $(\kappa, \gamma) = (\infty, 0)$, the density d_σ satisfies the equation (3) with $b = x$ and $a = -x \cot x$ ($x \in [0, \pi)$).

(ii) *If $\kappa \in [1, \infty]$ and $\gamma < 0$, then the correspondence $a \mapsto b = b(a)$ is unique for each $z = a + bi \in \partial\Omega \cap \mathbb{C}^+$. Then, $a \in [\alpha_1, \alpha_2]$. The same is true for $\kappa = \infty$ and $\gamma = 0$ with $a \in [-1, +\infty)$.*

Proof

(i) Let $z = a + bi \in \partial D \cap \mathbb{C}^+$. Then, it satisfies $f_{\kappa,\gamma}(z) \in \mathcal{S}$. Suppose $f_{\kappa,\gamma}(z) = -\frac{1}{x}$, and set $X = a + \gamma a^2 + \gamma b^2$ and $Y = |1 + \gamma z|^2 = (1 + \gamma a)^2 + (\gamma b)^2$. Notice that $X^2 + b^2 = (a^2 + b^2)Y$. The equation $f_{\kappa,\gamma}(z) = -\frac{1}{x}$ means that

$$\frac{e_\kappa(z)}{Y}\left(X\cos\left(\theta_\kappa(z)\right) - b\sin\left(\theta_\kappa(z)\right)\right) = -\frac{1}{x}, \tag{32}$$

$$X\sin\left(\theta_\kappa(z)\right) + b\cos\left(\theta_\kappa(z)\right) = 0. \tag{33}$$

The latter one (33) yields that $\cos\left(\theta_\kappa(z)\right) = -\frac{\sin(\theta_\kappa(z))}{b}X$ so that

$$-\frac{1}{x} = -\frac{e_\kappa(z)}{Y}\cdot\frac{\sin\left(\theta_\kappa(z)\right)}{b}(X^2 + b^2) \iff \frac{1}{x}\cdot\frac{b}{a^2 + b^2} = e_\kappa(z)\sin\left(\theta_\kappa(z)\right).$$

On the other hand, (33) can be written as $X = -b\cot\left(\theta_\kappa(z)\right)$, and using this expression together with (32), we obtain

$$-\frac{1}{x} = \frac{e_\kappa(z)}{Y}\left(-b\cot\left(\theta_\kappa(z)\right)\cos\left(\theta_\kappa(z)\right) - b\sin\left(\theta_\kappa(z)\right)\right) = -\frac{b}{\sin\left(\theta_\kappa(z)\right)}\cdot\frac{e_\kappa(z)}{Y}$$

and hence

$$x = \frac{\sin\left(\theta_\kappa(z)\right)}{b}\cdot Y\left(e_\kappa(z)\right)^{-1}.$$

It is easy to check that we have $Y = 1 + \gamma a + \gamma X$. By (30), the density can be described as $d_\sigma(x) = \frac{1}{\pi x}\cdot\frac{b}{a^2+b^2}$ so that we obtain the formula (31).

(ii) We shall show the part (ii) for $\kappa \in (1, \infty)$ and $\gamma < 0$. The other cases can be done by a similar way. Let $z = a + bi \in D = \Omega \cap \mathbb{C}^+$. Set $\theta(a, b) = \mathrm{Arctan}\frac{b}{\kappa+a}$ for $a > -\kappa$ and $b > 0$. By Proposition 13 (ii-a), we see that $\mathrm{Re}\left(1 + \frac{z}{\kappa}\right) = 1 + \frac{a}{\kappa} > 0$ and hence $\theta_\kappa(a + ib) = \kappa\theta(a, b)$. Note that $\frac{\partial}{\partial b}\theta_\kappa(a + ib) = \kappa\cdot\frac{\kappa+a}{(\kappa+a)^2+b^2}$. For given $a > -\kappa$, set $g(y; a) := y\cot(\theta_\kappa(a + iy))$. Let $y_0 > 0$ satisfy $\theta(a, y_0) = \frac{\pi}{\kappa+1}$. Then, we can show that $g(y; a)$ is monotonic decreasing for $y \in (0, y_0)$.

Set $h(y) = h(y; a) := a + \gamma a^2 + \gamma y^2 + g(y)$ for the fixed $a > -\kappa$. Recall that $h(y; a) = 0$ if and only if $z = a + iy \in \partial D \cap \mathbb{C}^+$. As $\gamma < 0$, we see that the function $h(y) := a + \gamma a^2 + \gamma y^2 + g(y)$ is decreasing on $y \in (0, y_0)$ for each fixed $a > -\kappa$. Since $\cot(\theta_\kappa(a + iy_0)) = -\frac{\kappa+a}{y_0}$, we see that $h(y_0; a) < 0$. By the fact that $\lim_{y\to+0} g(y; a) = 1 + \frac{a}{\kappa}$, we have $\lim_{y\to+0} h(y; a) = \gamma(a - \alpha_1)(a - \alpha_2)$. Since h is monotonic decreasing on $y \in (0, y_0)$, if $a \in (\alpha_1, \alpha_2)$ then $\lim_{y\to+0} h(y; a) > 0$ so that there exists a unique solution $y = b$ of $h(y; a) = 0$ in $y \in (0, y_0)$ for each $a \in (\alpha_1, \alpha_2)$ by the intermediate value theorem, whereas if $\lim_{y\to+0} h(y; a) \leq 0$ then there is no solution of $h(y;a)=0$ in $y \in (0, y_0)$. Therefore, the correspondence $a \mapsto b=b(a)$ is unique for each $z=a+bi \in \partial\Omega \cap \mathbb{C}^+$. □

Remark 20 Corollary 19 (ii) enables us to write the density d_σ with one real parameter in a way similar to Dykema and Haagerup (2004, Theorem 8.9), see formula (3). In particular, in the case (a), we obtain the formula

$$d_\sigma\left(\frac{\sin b(a)}{b(a)}(1+\gamma a-\gamma b(a)\cot b(a))\, e^{-a}\right)=\frac{1}{\pi}\cdot e^a \sin b(a) \quad (a \in [\alpha_1, \alpha_2]).$$

A natural conjecture that we always have a 1-1 correspondence $a \to b$ or $b \to a$ is not confirmed by numerical generation of the domain Ω. We have a counterexample in the case $\kappa, \gamma < 0$.

4.3 Applications to Wishart ensembles of Vinberg matrices

Now we apply Theorem 18 to the covariance matrix $X_n = Q_{\underline{k}}(\xi_n) \in P_n$ in two situations. The first (Corollary 21) is the case when P_n is the symmetric cone $\mathrm{Sym}(n,\mathbb{R})^+$ with $\underline{k}$ of the form (34) below. The second situation (Theorem 24) is the general case when $P_n \subset \mathcal{U}_n$ is a dual Vinberg cone with $\underline{k}$ of the form (2). This case contains the first one, that we present separately because of the importance of the symmetric cone $\mathrm{Sym}(n,\mathbb{R})^+$.

Let us assume that $\underline{k}=\underline{k}(n)=(k_1,\dots,k_n)$ in (2) is of the form

$$\underline{k}=m_1(1,\dots,1,1)+m_2(n)(0,\dots,0,1), \quad \lim_n \frac{m_2(n)}{n}=m, \tag{34}$$

where $m_1 \in \mathbb{Z}_{\geq 0}$ is a fixed nonnegative integer and $m \in \mathbb{R}_{\geq 0}$ is a nonnegative real such that $m_1+m>0$. Set $N := k_1+\cdots+k_n = m_1 n + m_2(n)$. We note that the case $m_1=0$ corresponds to the classical Wishart ensembles, and if $m_1 \geq 1$ then we have $N \geq n$.

Corollary 21 *Let $\underline{k}$ be as in (34). Suppose that $\xi_n \in E_{\underline{k}}$ is an i.i.d. matrix with finite fourth moments. Let $X_n = \xi_n \xi_n^\top$ and μ_n the empirical eigenvalue distribution of X_n/n. Then, there exists a limiting eigenvalue distribution $\mu=\lim_n \mu_n$. The Stieltjes transform $T(z)$ of μ is given by formula (26)*

$$T(z)=T_{\kappa,\gamma}(z)=\frac{\exp_\kappa\left(W_{\kappa,\gamma}(-v/z)\right)-1}{v} \text{ with } \kappa=\frac{1}{1-m_1},\ \gamma=1-m-m_1.$$

The measure μ is absolutely continuous and has no atoms. If $m_1=0$ then the measure μ is the Marchenko–Pastur law with parameter $C=m$. The case $(m_1,m)=(1,0)$ corresponds to the Dykema–Haagerup measure χ_v. If $m=0$ then the density d is continuous on $\mathbb{R}^*$ and $\lim_{x\to+0} d(x)=+\infty$. When $m_1 \geq 2$ then the support of μ is $[0, v m_1^{m_1/(m_1-1)}]$. Otherwise, for $m_1, m > 0$, the density $d(x)$ of μ is continuous on $\mathbb{R}$, and its support equals $[A(\alpha_2), A(\alpha_1)]$ where $A(\alpha_i) := v\alpha_i^{-2}(1+(1-m_1)\alpha_i)^{m_1/(m_1-1)}$ and $\alpha_1<\alpha_2$ are roots of the function $(1-m_1-m)x^2+(2-m_1)x+1$.

Proof We use Theorem 3. It is enough to show that the matrix Y_n in (20) has the variance profile σ in (22) and that the conditions (6) are satisfied. Since we have for n large enough

$$\left|\delta_0(n)\right| \leq \frac{1}{n^2} \cdot 2v(m_1 + m + 1)n = \frac{2v(m_1 + m + 1)}{n} \to 0 \quad (n \to \infty)$$

and if $\mathbb{E}|Y_{ij}|^2 \neq 0$ then

$$\frac{\mathbb{E}(Y_{ij}^4)}{(n+N)(\mathbb{E}Y_{ij}^2)^2} = \frac{M_4}{v(n+N)} \to 0 \quad (n \to \infty),$$

we can easily check the conditions (6). Thus, we can apply Theorem 18. Consider $m_1 \geq 2$. Then $\kappa < 0$. When $m = 0$, then we have $\gamma' = \gamma - \frac{1}{\kappa} = 0$ so that we apply Theorem 18.2. We have $\alpha = -1, 1 - \frac{1}{\kappa} = m_1$ and $1 - \kappa = \frac{m_1^{\kappa}}{m_1 - 1}$. By (29), the support is given by $\mathrm{supp}\mu = \left[0, \frac{v}{\alpha^2}\left(1 + \frac{\alpha}{\kappa}\right)^{1-\kappa}\right] = \left[0, vm_1^{m_1/(m_1-1)}\right]$. When $m > 0$, we have $\gamma' < 0$ so that we apply Theorem 18.1. The support of μ is given by the formula (28), where $\alpha_1 \leq \alpha_2$ are roots of the function $\gamma x^2 + (1 + 1/\kappa)x + 1$. □

Remark 22 If $m = 0$, our results contain those of Claeys and Romano (2014, Section 4.5.1) and Cheliotis (2018, Theorem 4 and (12)). The result on the limiting densities of biorthogonal ensembles in Cheliotis (2018) can be reproduced from Corollary 21. In fact, our random matrices $Q_{\underline{k}}(\xi_n)$ essentially correspond to those considered in Cheliotis (2018) through adjusting parameters $m_1 = \theta - 1$ and $m_2(n) = b - 1$ (not depending on n), where θ and b are parameters used in that paper.

Remark 23 Until now, we assumed that $m_1 \in \mathbb{Z}_{\geq 0}$ and hence the parameter α of the variance profile σ needs to be also an integer. However, we can take a sequence $\{\underline{k}(n)\}_{n=1}^{\infty}$ so that the corresponding α is an arbitrary given positive real number. In fact, when $\alpha > 0$ is given, we consider a right triangle with lengths 1 and α. For an arbitrary n, we cover the triangle by $1/n \times 1/n$ squares as in the figure. To each $j = 1, \ldots, n$, we associate an integer $k_j(n)$ such that $\frac{k_j(n)}{n} \leq \frac{j}{n}\alpha < \frac{k_j(n)+1}{n}$, or equivalently $k_j(n) \leq j\alpha < k_j(n) + 1$, and we set $k(n) = (k_1(n), \ldots, k_n(n))$. Note that this condition is independent of n so that $k_j(m) = k_j(n)$ when $m \geq n \geq j$, and hence $\{E_{\underline{k}(n)}\}_n$ is a sequence of vector spaces such that $E_{\underline{k}(n)} \subset E_{\underline{k}(n+1)}$.

Let us return to the quadratic Wishart case for general P_n with parameter $\underline{k}$ as in (2) such that $m_1, m_2 \in \mathbb{Z}_{\geq 0}$ with $m_1 + m_2 > 0$ are fixed. Note that $m_2(n)$ in the previous discussion is now $m_2(n) = m_2 b_n$. Set $N_n := m_1 n + m_2 b_n$. Let $E_{\underline{k}}$ be a subspace of $\mathrm{Mat}(n \times N_n, \mathbb{R})$ consisting of matrices of the form

$$\xi = \begin{pmatrix} \eta \\ \zeta \end{pmatrix}, \quad \eta = (\eta_{ij}) \in \mathrm{Mat}(a_n \times N_n, \mathbb{R}),\ \zeta = (\zeta_{ij}) \in \mathrm{Mat}(b_n \times N_n, \mathbb{R}),$$

such that $\eta_{ij} = 0$ if $j \leq (m_1 - 1)i$ and $\zeta_{ij} = 0$ if $M(i,j) \notin \{1, 2, \dots, m_1 + m_2\}$, respectively, where $M(i,j) := j - m_1 a_n - (m_1 + m_2)(i - 1)$.

Theorem 24 *Let $\{P_n\}_n$ be a sequence of generalized dual Vinberg cones such that $\lim_{n\to\infty} a_n/n = c \in (0, 1]$. Let $\underline{k}$ be a vector as in (2) such that m_1, m_2 are fixed. Set $\kappa := 1/(1 - m_1)$ and $\gamma := 1 - \big(m_1 + m_2(1 - c)\big)/c$. Then, the Stieltjes transform $T(z)$ of the limiting eigenvalue distribution of $Q_{\underline{k}}(\xi_n)/n$ with i.i.d. matrices $\xi_n \in E_{\underline{k}}$ is given for $z \in \mathbb{C}^+$ as*

$$T(z) = -\frac{1}{v} - \frac{c}{zW_{\kappa,\gamma}(-\frac{cv}{z})} - \frac{c\gamma + 1 - c}{z} = \frac{\exp_\kappa\big(W_{\kappa,\gamma}(-vc/z)\big) - 1}{v} - \frac{1 - c}{z}.$$

The properties of absolute continuity and support of the limiting measure can be derived analogously to those obtained in Theorem 18 for $c = 1$.

Proof We construct a variance profile σ from $E_{\underline{k}}$ likely to (22). We embed the rectangular matrix $\xi_n \in E_{\underline{k}}$ in a square matrix $Y(\xi_n) = \begin{pmatrix} 0 & \xi_n \\ \xi_n^\top & 0 \end{pmatrix}$, and set $V_n = \left\{ Y(\xi_n);\, \xi_n \in E_{\underline{k}} \right\}$. Set $p' = \lim_{n\to\infty} \dfrac{n}{n + N_n} = \dfrac{1}{1 + m_1 + m_2(1 - c)}$. Let σ be a function $[0, 1] \times [0, 1] \to \mathbb{R}_{\geq 0}$ defined by $\sigma(x, y) = v$ if (i) $x < cp'$ and $y \geq p' + m_1 x$, or if (ii) $x \geq p'$ and $0 \leq y \leq \min\{(x - p')/m_1, cp'\}$, and $\sigma(x, y) = 0$ otherwise.

Then, we can show that σ is the variance profile of V_n. On the other hand, let us consider a subspace $E'_{\underline{k}} := \left\{ \xi = \begin{pmatrix} \eta \\ \zeta \end{pmatrix} \in E_{\underline{k}};\, \zeta = 0 \right\}$ of $E_{\underline{k}}$, and let $V'_n = \left\{ Y(\xi_n);\, \xi_n \in E'_{\underline{k}} \right\}$. Then, σ is also the variance profile of V'_n. Thus, we consider equivalently the limiting eigenvalue distribution of V'_n, and that of covariance matrices on $E'_{\underline{k}}$. If $\xi_n = \begin{pmatrix} \eta_n \\ 0 \in E'_{\underline{k}} \end{pmatrix}$, then $\frac{1}{n} Q_{\underline{k}}(\xi_n) = \frac{1}{n}\begin{pmatrix} \eta_n \eta_n^\top & 0 \\ 0 & 0 \end{pmatrix}$, and thus it is enough to study the limiting eigenvalue distribution of $\frac{1}{n}\eta_n\eta_n^\top$. The variance profile of covariance matrix $\frac{1}{a_n}\eta_n\eta_n^\top$, rescaled by size of $\eta_n\eta_n^\top$, has a trapezoidal form (22) (illustrated by (23)) with parameters $\alpha = m_1$ and $p = \lim_n \frac{a_n}{a_n + N_n} = \frac{c}{c + m_1 + m_2(1 - c)}$. Applying Proposition 17, we see that the corresponding Stieltjes transform $T_1(z)$ is given by

$$T_1(z) = T_{\kappa,\gamma}(z) \quad \text{with} \quad \kappa = \frac{1}{1 - m_1}, \quad \gamma = \frac{2p - 1}{p} = \frac{c - m_1 - m_2(1 - c)}{c}.$$

In general, for two symmetric matrices A_i $(i = 1, 2)$ of size n_i, the Stieltjes transform $S(z)$ of $\mathrm{diag}(A_1, A_2)/(n_1 + n_2)$ can be described by using the Stieltjes transforms $S_i(z)$ of A_i/n_i $(i = 1, 2)$ as

$$S(z) = S_1\left(\frac{n_1 + n_2}{n_1} z\right) + S_2\left(\frac{n_1 + n_2}{n_2} z\right) \quad (z \in \mathbb{C}^+). \tag{35}$$

In our situation, we have $(n_1, n_2) = (a_n, b_n)$ and $(A_1, A_2) = (\eta_n\eta_n^\top, 0)$. Hence, we have $S_2(z) = -\frac{1}{z}$ and $S_1(z)$ is the Stieltjes transform of $\eta_n\eta_n^\top/a_n$ so that

$\lim_{n\to\infty} S_1(z) = T_1(z)$. Thus, taking the limit $n \to \infty$, we get the limiting Stieltjes transform $T(z)$ corresponding to $E'_{\underline{k}}$, and hence to $E_{\underline{k}}$ by using (35) with $S_1(z) = T_1(z)$ and $S_2(z) = -\frac{1}{z}$, which proves the corollary. □

Remark 25 In the Figs. 4, 5 and 6 we present simulations of $\underline{k}$-indexed Wish-art ensembles $X_n = Q_{\underline{k}}(\xi_n)$ on the symmetric cone $\mathrm{Sym}(n,\mathbb{R})^+$ (i.e., $c = 1$), for $n = 4000$ and $N = |\underline{k}| = 2n$ with parameters $\alpha = m_1 = 1/2$, 1 and 2, respectively. We have $\gamma = -1$ and $\kappa = 2, \infty, -1$ respectively. The red line is the graph of $d(x)$ generated by the R program from its Stieltjes transform given in Corollary 21. In two first cases, the limiting density $d(x)$ is continuous on $\mathbb{R}$ with compact support contained in $(0,\infty)$. The last case $(\kappa,\gamma) = (-1,-1)$ corresponds to $(\kappa',\gamma') = (1,0)$ which is the classical Wishart case with $C = 1$. Thus its density explodes to ∞ at 0.

Remark 26 Let Y_n be a rectangular $n \times N$ i.i.d. matrix with variance profile $\sigma(x,y)$, and assume that $\lim_{n\to\infty} N/n = c$. In papers Hachem et al. (2005, 2006, 2008), a functional equation $\tau(u,z) = \left(-z + \int_0^1 \sigma(u,v)\left(1 + c\int_0^1 \sigma(x,v)\tau(x,z)dx\right)^{-1} dv\right)^{-1}$ is given to get the limiting Stieltjes transform $f(z)$ for the rescaled random matrices $\frac{1}{n}Y_nY_n^*$, as the integral $\int_0^1 \tau(u,z)du$. This equation appears in Girko (1990) in the setting of Gram matrices based on Gaussian fields (cf. Hachem et al., 2006, Remark 3.1).

However, thanks to symmetry, solving the equations (24) resulting from Theorem 3 is easier than solving the last functional-integral equation for $\tau(u,z)$. Therefore we opted for variance profile method for Gaussian and Wigner ensembles as the main tool of studying Wishart ensembles of Vinberg matrices.

5 Complementary remarks

Remark 27 (*Modified Vinberg matrices*) Observe that the variance profiles (11) and (22) remain the same when we consider the following two cases (a) and (b) of modified Vinberg matrices. This is due to the fact that different forms of d in the lower right block of the matrix U can be absorbed by the perturbation terms δ_{ij} in (5). Accordingly, we obtain Theorem 5 for the Wigner Ensembles and Theorems 18 and 24 and Corollaries 19 and 21 for the Wishart Ensembles on the corresponding matrix spaces.

(a) For $s = 0$ or $s \in 2\mathbb{N}+1$, let $\mathcal{U}_n^s$ be the subspace of $\mathrm{Sym}(n,\mathbb{R})$ defined by

$$\mathcal{U}_n^s = \left\{ U = \begin{pmatrix} x & y \\ y^\top & d \end{pmatrix};\ \begin{array}{l} x \in \mathrm{Sym}(a_n,\mathbb{R}),\ y \in \mathrm{Mat}(a_n \times b_n,\mathbb{R}), \\ d \text{ is a } s\text{-diagonal matrix of size } b_n \end{array} \right\}.$$

Here, 0-diagonal matrix means the zero matrix. From the statistical point of view of Gaussian covariance models (Lauritzen, 1996), the space $\mathcal{U}_n^0$ does not apply, because covariance matrices have nonzero diagonal terms and nonzero determinant.

(b) Take $k \in \mathbb{Z}_{>0}$ and assume that each b_n is a multiple of k, say $b_n = kc_n$. Let $\mathcal{U}_n^{(k)}$ be a subspace of $\mathrm{Sym}(a_n + b_n, \mathbb{R})$ consisting of matrices U of the form above with d being a block diagonal matrix $d = \mathrm{diag}(d_1, \dots, d_{c_n})$, where each d_j is a square matrix of size k.

Remark 28 (*Matrix ensembles related to dual cones of P_n*) In this remark, we consider the dual cone Q_n of P_n, which is realized as a minimal matrix form in the sense of Yamasaki and Nomura (2015) as follows. Let $\mathcal{V}_n$ be a subspace of $\mathrm{Sym}(a_n(b_n + 1), \mathbb{R})$ defined by

$$\mathcal{V}_n := \left\{ \mathrm{diag}\left(\begin{pmatrix} x & y_1 \\ y_1^\top & d_1 \end{pmatrix}, \dots, \begin{pmatrix} x & y_{b_n} \\ y_{b_n}^\top & d_{b_n} \end{pmatrix} \right); \begin{array}{l} x \in \mathrm{Sym}(a_n, \mathbb{R}), \\ y_1, \dots, y_{b_n} \in \mathbb{R}^{a_n}, \\ d_1, \dots, d_{b_n} \in \mathbb{R} \end{array} \right\}. \tag{36}$$

Then, the dual cone Q_n is described as $Q_n := \mathcal{V}_n \cap \mathrm{Sym}(a_n(b_n + 1), \mathbb{R})^+$.

We consider Wigner Ensembles $V_n \in \mathcal{V}_n$ and quadratic Wishart Ensembles $X_n \in Q_n$ as those in the sense of $\mathrm{Sym}(a_n(b_n + 1), \mathbb{R})$. Assume that $\lim_{n\to+\infty} a_n = \infty$. By the theory of lower rank perturbation (see Tao, 2012, §2.4.1, for example), the study of eigenvalue distributions of these ensembles boils down to the study of the eigenvalue distributions of x and, after suitable normalization, the limiting eigenvalue distributions of V_n and X_n are the same as for $x \in \mathrm{Sym}(a_n, \mathbb{R})$.

This essential difference in the Random Matrix Theory for the cones Q_n and P_n may be explained by a substantial difference between the cones Q_n and P_n in terms of numbers of *sources* in the sense of Yamasaki and Nomura (2015). In the case P_n, there is only one source so that P_n can be realized in a usual matrix form. On the other hand, Q_n has b_n sources so that b_n copies of a usual matrix form appear.

Remark 29 (*Relation of Vinberg cones to graphical models*) Daisy graphs provide natural classes of Gaussian decomposable graphical models (cf. Lauritzen, 1996; Maathuis et al., 2018). They are defined as follows. Let $a + b = n$ and let $D(a, b)$ be a graph with vertices $V = \{1, \dots, n\}$, such that the first a elements form a complete graph and the latter b elements are satellites (petals) of the complete graph, that is, each satellite connects to all elements in the complete graph and does not connect to the other satellites.

The position of zeros in Wigner and Wishart Vinberg matrices in $\mathcal{U}_n$ considered in this paper is encoded by daisy graphs $D(a_n, b_n)$ by setting $u_{ij} = 0$ whenever i and j are not connected by an edge in $D(a_n, b_n)$. We can also consider the class of generalized daisy graphs $D(a, b, k)$, with b complete satellites of k vertices, so that there are $N = a + kb$ vertices. If all three sequences a_n, b_n, k_n are non-decreasing, the graphs $D(a_n, b_n, k_n)$ form a growing sequence of graphical models. The case when $k_n = k$ is fixed for n large enough corresponds to Remark 27(b).

Mathematical bases of Wishart distributions on matrix cones related to decomposable and homogeneous graphs considered in this paper were laid down by

Lauritzen (1996), Letac and Massam (2007), Ishi (2014), Graczyk and Ishi (2014). This paper is a first step towards studying RMT related to growing Gaussian graphical models.

Note that statistical Gaussian graphical models is a different notion from Erdös–Rényi random graphs, which are deeply studied in the RMT (cf. the book of Durrett, 2006).

Supplementary Information The online version contains supplementary material available at https://doi.org/10.1007/s10463-021-00800-8.

Acknowledgements The authors would like to express their sincere gratitude to Professor H. Ishi for strong encouragement and invaluable comments on this work. The authors are very grateful to Professor J. Najim for significant methodological and bibliographical indications for this work and to Professor C. Bordenave for discussions on Theorem 3. The authors thank the Scientific Committee of the CIRM Luminy 2020 Conference "Mathematical Methods of Modern Statistics" (MMMS 2) for giving the possibility to present this work. We thank numerous participants of MMMS 2 for their comments and remarks. We thank the referees and the associated editor for suggestions and corrections that improved significantly the paper. This research was carried out while the first author spent the winter semester of 2018 at Laboratoire de Mathématiques LAREMA under the support of Grant-in-Aid for JSPS fellows (2018J00379).

References

Ahlfors, L. (1979). Complex analysis, an introduction to the theory of analytic functions of one complex variable. *International series in pure and applied mathematics*, 3rd ed. New York: McGraw-Hill.

Amari, S., Ohara, A. (2011). Geometry of *q*-exponential family of probability distributions. *Entropy, 13*(6), 1170–1185.

Anderson, G. W., Guionnet, A., Zeitouni, O. (2010). *An introduction to random matrices*. London: Cambridge University Press.

Anderson, G. W., Zeitouni, O. (2006). A CLT for a band matrix model. *Probability Theory and Related Fields, 134,* 283–338.

Andersson, S. A., Wojnar, G. G. (2004). Wishart distributions on homogeneous cones. *Journal of Theoretical Probability, 17,* 781–818.

Bai, Z., Silverstein, J. W. (2010). Spectral analysis of large dimensional random matrices. *Springer series in statistics*, 2nd ed. New York: Springer.

Bai, Z., Choi, K., Fujikoshi, Y. (2018). Consistency of AIC and BIC in estimating the number of significant components in high-dimensional principal component analysis. *The Annals of Statistics, 46*(3), 1050–1076.

Benaych-Georges, F. (2009). Rectangular random matrices, related convolution. *Probability Theory and Related Fields, 144*(3), 471–515.

Bordenave, C. (2019). *Lecture notes on random matrix theory*. https://www.math.univ-toulouse.fr/~bordenave/IMPA-RMT.pdf

Borodin, A. (1999). Biorthogonal ensembles. *Nuclear Physics B, 536*(3), 704–732.

Bun, J., Bouchaud, J. P., Potters, M. (2017). Cleaning large correlation matrices: Tools from random matrix theory. *Physics Reports, 666,* 1–109.

Candès, E., Tao, T. (2005). Decoding by linear programming. *IEEE Transactions on Information Theory, 51*(12), 4203–4215.

Chafaï, D. (2009). *Singular values of random matrices*, Lecture Notes at Université Paris-Est Marne-la-Vallée. http://djalil.chafai.net/docs/sing.pdf

Cheliotis, D. (2018). Triangular random matrices and biorthogonal ensembles. *Statistics & Probability Letters, 134,* 36–44.

Claeys, T., Romano, S. (2014). Biorthogonal ensembles with two-particle interactions. *Nonlinearity, 27*(10), 2419–2443.
Corless, R. M., Gonnet, G. H., Hare, D. E. G., Jeffrey, D. J., Knuth, D. E. (1996). On the Lambert *W* function. *Advances in Computational Mathematics, 5*(4), 329–359.
Diaconis, P. (2003). Patterns in eigenvalues: The 70th Josiah Willard Gibbs lecture. *Bulletin of the American Mathematical Society, 40,* 155–178.
Durrett, R. (2006). *Random graph dynamics, Cambridge series in statistical and probabilistic mathematics* (Vol. 20). London: Cambridge University Press.
Dykema, K., Haagerup, U. (2004). DT-operator and decomposability of Voiculescu's circular operator. *American Journal of Mathematics, 126,* 121–189.
Erdös, L., Yau, H.-T., Yin, J. (2012a). Rigidity of eigenvalues of generalized Wigner matrices. *Advances in Mathematics, 229,* 1435–1515.
Erdös, L., Yau, H.-T., Yin, J. (2012b). Bulk universality for generalized Wigner matrices. *Probability Theory and Related Fields, 154,* 341–407.
Faraut, J. (2014). Logarithmic potential theory, orthognal polynomials. In P. Graczyk & A. Hassairi (Eds.), *Modern methods of multivariate statistics* (Vol. 82, pp. 1–67). Paris: Hermann.
Forrester, P. J. (2010). *Log-gases and random matrices*. Princeton: Princeton University Press.
Forrester, P. J., Wang, D. (2017). Muttalib–Borodin ensembles in random matrix theory-realisations and correlation functions. *Electronic Journal of Probability, 22*(54), 1–43.
Fujikoshi, Y., Sakurai, T. (2016). High-dimensional consistency of rank estimation criteria in multivariate linear model. *Journal of Multivariate Analysis, 149,* 199–212.
Girko, V. L. (1990). *Theory of random determinants*. London: Kluwer.
Graczyk, P., Ishi, H. (2014). Riesz measures and Wishart laws associated to quadratic maps. *Journal of the Mathematical Society of Japan, 66,* 317–348.
Graczyk, P., Ishi, H., Kołodziejek, B. (2019). Wishart laws and variance function on homogeneous cones. *Probability and Mathematical Statistics, 39*(2), 337–360.
Hachem, W., Loubaton, P., Najim, J. (2005). The empirical eigenvalue distribution of a Gram matrix: From independence to stationarity. *Markov Processes and Related Fields, 11,* 629–648.
Hachem, W., Loubaton, P., Najim, J. (2006). The empirical distribution of the eigenvalues of a Gram matrix with a given variance profile. *Annales de l'I.H.P Probabilités et Statistiques, 42,* 649–670.
Hachem, W., Loubaton, P., Najim, J. (2007). Deterministic equivalents for certain functionals of large random matrices. *The Annals of Applied Probability, 17,* 875–930.
Hachem, W., Loubaton, P., Najim, J. (2008). A CLT for information-theoretic statistics of Gram random matrices with a given variance profile. *The Annals of Applied Probability, 18,* 2071–2130.
Hastie, T., Tibshirani, R., Wainwright, M. (2015). *Statistical learning with sparsity*: *The Lasso and generalizations*. London: Chapman and Hall/CRC.
Ishi, H. (2001). Basic relative invariants associated to homogeneous cones and applications. *Journal of Lie Theory, 11,* 155–171.
Ishi, H. (2014). Homogeneous cones and their applications to statistics. In P. Graczyk & A. Hassairi (Eds.), *Modern methods of multivariate statistics* (Vol. 82, pp. 135–154). Paris: Hermann.
Ishi, H. (2016). Explicit formula of Koszul–Vinberg characteristic functions for a wide class of regular convex cones. *Entropy, 18,* 383. https://doi.org/10.3390/e18110383.
Johnstone, I. M. (2007). High dimensional statistical inference and random matrices. In *International congress of mathematicians* (Vol. I, pp. 307–333), Zurich.
Lauritzen, S. L. (1996). *Graphical models*. Oxford: Oxford University Press.
Letac, G., Massam, H. (2007). Wishart distributions for decomposable graphs. *The Annals of Statistics, 35,* 1278–1323.
Maathuis, M., Drton, M., Lauritzen, S., Wainwright, M. (Eds.). (2018). *Handbook of graphical models*. London: Chapman and Hall/CRC.
Mingo, J. A., Speicher, R. (2017). *Free probability and random matrices, fields institute monographs* (Vol. 35). New York: Springer.
Muttalib, K. A. (1995). Random matrix models with additional interactions. *Journal of Physics A*: *Mathematical and General, 28*(5), L159–L164.
Nakashima, H. (2020). Functional equations of zeta functions associated with homogeneous cones. *Tohoku Mathematical Journal, 72,* 349–378.
Nica, A., Shlyyakhtenko, D., Speicher, R. (2002). Operator-valued distribution I. *International Mathematics Research Notices, 29,* 1509–1538.

Palka, Bruce P. (1991). *An introduction to complex function theory, Undergraduate texts in mathematics*. New York: Springer.

Paul, D., Aue, A. (2014). Random matrix theory in statistics: A review. *Journal of Statistical Planning and Inference, 150,* 1–29.

Ronald, I. S. (2004). *Integers, polynomials, and rings*. New York: Springer.

Shlyakhtenko, D. (1996). Random Gaussian band matrices and freeness with amalgamation. *International Mathematics Research Notices, 20,* 1013–1025.

Takayama, N., Jiu, L., Kuriki, S., Zhang, Y. (2020). Computation of the expected Euler characteristic for the largest eigenvalue of a real non-central Wishart matrix. *Journal of Multivariate Analysis, 179,* 1–18.

Tao, T. (2012). *Topics in random matrix theory, Graduate studies in mathematics* (Vol. 132). Providence: American Mathematical Society.

Vinberg, E. B. (1963). The theory of convex homogeneous cones. *Transactions of Moscow Mathematical Society, 12,* 340–403.

Wigner, E. P. (1955). Characteristic vectors of bordered matrices with infinite dimensions. *Annals of Mathematics, 62,* 548–564.

Yamasaki, T., Nomura, T. (2015). Realization of homogeneous cones through oriented graphs. *Kyushu Journal of Mathematics, 69,* 11–48.

Yao, J., Zheng, S., Bai, Z. (2015). *Large sample covariance matrices and high-dimensional data analysis*. London: Cambridge University Press.

Zhang, F. D., Ng, H. K. T., Shi, Y. M. (2018). Information geometry on the curved q-exponential family with application to survival data analysis. *Physica A*: *Statistical Mechanics and its Applications*, *512,* 788–802.