Supplementary material for Empirical likelihood meta analysis with publication bias correction under Copas-like selection model

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Abstract This supplementary material is divided into ten sections. Section 1 reviews Lemma 1, Theorems 1 and 2, and Proposition 1 in the main paper. Section 2 proves Lemma 1 of the main paper. Section 3 proves Theorem 1 of the main paper. The proof in Section 3 is built on two lemmas, Lemmas 2 and 3, whose proofs are given in Sections 4-7. Sections 8 and 9 contain proofs of Theorem 2 and Proposition 1 of the main paper. Some tedious technical derivations are postponed to Section 10.

1 Notation and main results in the main paper

Recall that \( \gamma = (\gamma_1, \gamma_2, \rho, \tau, \theta)^\top \). Hereafter we use \( \gamma_{12} \) and \( \gamma_{45} \) to denote \( (\gamma_1, \gamma_2) \) and \( (\tau, \theta) \), respectively. Define

\[
\begin{align*}
    f_1(\theta; s_i; \gamma) &= \Pr(Z_i > 0 | \theta_i^* = \theta_i, s_i^* = s_i) = \Phi(v_i(\gamma)), \\
    f_2(\theta, s_i; \gamma_{45}) &= \Pr(\theta_i^* = \theta_i | s_i^* = s_i) = \frac{1}{\sqrt{2\pi(\tau^2 + s_i^2)}} \exp \left\{ -\frac{(\theta_i - \theta)^2}{2(\tau^2 + s_i^2)} \right\}, \\
    f_3(s_i; \gamma_{12}) &= \Pr(Z_i > 0 | s_i^* = s_i) = \Phi(\gamma_1 + \gamma_2 / s_i).
\end{align*}
\]

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where $\Phi(t)$ is the standard normal distribution function and
\[
v_t(\gamma) = v(\theta, s; \gamma) = \frac{\gamma_1 + (\gamma_2/s_t) + \rho s_t(\theta_1 - \bar{\theta})/(\tau^2 + s_t^2)}{\sqrt{1 - \rho^2 s_t^2/(\tau^2 + s_t^2)}}.
\] (1)

The full log-likelihood is
\[
\ell(N, \alpha, \gamma) = \log \left(\frac{N}{n}\right) + (N - n) \log(1 - \alpha) - \sum_{i=1}^{n} \log[1 + \lambda f_3(s_i; \gamma_{12}) - \alpha] \\
+ \sum_{i=1}^{n} \log f_1(\theta, s; \gamma) + \sum_{i=1}^{n} \log f_2(\theta, s; \gamma_{45}),
\]
\[
= \ell_c (\gamma) + \ell_m (N, \alpha, \gamma_{12}),
\]
where
\[
\ell_c (\gamma) = \sum_{i=1}^{n} \left\{ \log f_1(\theta, s; \gamma) + \log f_2(\theta, s; \gamma_{45}) - \log f_3(s_i; \gamma_{12}) \right\}
\] (2)
is a conditional likelihood and
\[
\ell_m (N, \alpha, \gamma_{12}) = \log \left(\frac{N}{n}\right) + (N - n) \log(1 - \alpha) - \sum_{i=1}^{n} \log[1 + \lambda f_3(s_i; \gamma_{12}) - \alpha] \\
+ \sum_{i=1}^{n} \log f_3(s_i; \gamma_{12})
\] (3)
is a marginal likelihood.

Let $(N_0, \alpha_0, \gamma_0)$ be the truth of $(N, \alpha, \gamma)$ with $\gamma_0 = (\gamma_{10}, \gamma_{20}, \rho_0, \theta_0)$. Throughout the paper, we use $\gamma_{12,0}$ and $\gamma_{45,0}$ to denote the truths of $\gamma_{12}$ and $\gamma_{45}$. Define $A^\otimes = AA^*$ and $A^\oplus = A + A^*$ for a matrix $A$. $E_{12} = (I_k, 0_{k \times 3})^\top$, and $E_{45} = (0_{2 \times 3}, I_2)^\top$ with $I_k$ the $k \times k$ identity matrix. We use $\nabla_\gamma$ to denote the differentiation operator with respect to $\gamma$. Let $\varphi_1 = \mathbb{E}[(f_3(s_i; \gamma_{12,0}))^{-1}]$ and $m_{\varphi_2} = \mathbb{E}\left\{ \nabla_{\gamma_{12}} \log f_3(s_i; \gamma_{12,0}) \right\}$.

Define
\[
V_c = \mathbb{E}\left[ \nabla_{\gamma_{12}}f_1(\theta, s_i; \gamma_{12}) \nabla_{\gamma} \right] \mathbb{E}_{45} \int \nabla_{\gamma_{12}}f_2(t, s_i^*; \gamma_{45,0}) \nabla_{\gamma}f_{3}(t, s_i^*; \gamma_{45,0}) dt E_{45}^\top \\
+ \mathbb{E}\left[ \nabla_{\gamma_{12}}f_1(t, s_i^*; \gamma_{12}) \nabla_{\gamma} \nabla_{\gamma}f_2(t, s_i^*; \gamma_{45,0}) \nabla_{\gamma}f_{3}(t, s_i^*; \gamma_{45,0}) dt E_{45}^\top \right] \mathbb{E}_{12}^\top \\
- \mathbb{E}\left[ \nabla_{\gamma_{12}}f_1(t, s_i^*; \gamma_{12}) \nabla_{\gamma}f_2(t, s_i^*; \gamma_{45,0}) \nabla_{\gamma}f_{3}(t, s_i^*; \gamma_{12,0}) dt E_{45}^\top \right] \mathbb{E}_{12} \right],
\]

\[
V_m = \begin{pmatrix}
\alpha_{0,0} & 1 - \alpha_{0,0} & 0 \\
1 - \alpha_{0,0} & \alpha_{0,0} & m_2^2 \\
(-1 - \alpha_{0,0}) & m_2^2 & \frac{-1 - \alpha_{0,0}}{1 - \alpha_{0,0}} \\
0 & m_2^2 & \frac{-1 - \alpha_{0,0}}{1 - \alpha_{0,0}} \\
\end{pmatrix},
\]
and

\[ V_c(\gamma) = (-1)E \left\{ \nabla \gamma^T f_i(\theta'_i, s'_i; \gamma) \frac{f_i(\theta'_i, s'_i; \gamma)}{f_i(\theta'_i, s'_i; \gamma)} - \left( \nabla \gamma f_i(\theta'_i, s'_i; \gamma) \right) \hat{\gamma}_1 \right\} F_i(\theta'_i, s'_i; \gamma) \]

Furthermore, define

\[ \Omega = F_1^T V_c F_2 + F_1^T \hat{V}_m F_1, \]

where \( F_1 = (I_d, 0_{d \times 3}) \) and \( F_2 = (0_{5 \times 2}, I_5) \).

**Lemma 1** If \( \rho \neq 0 \), the parameter \( \gamma = (\gamma_1, \gamma_2, \rho, \tau, \theta) \) is identifiable.

**Condition 1** The matrix \( V_c(\gamma) \) in (4) is finite and continuous in a neighborhood of \( \gamma_0 \).

**Condition 2** Suppose the Hessian matrix of the function \( H_m \) defined in the supplementary material is finite and continuous in a neighborhood of the origin.

**Theorem 1** Assume Conditions 1 and 2, and that the matrix \( \Omega \) defined in (5) is positive definite. As \( N_0 \to \infty \), the following results hold.

1. \( N_0^{1/2}(N/N_0 - 1, \hat{\alpha} - \alpha_0, (\hat{\sigma} - \gamma_0)^T) \to N(0, \Omega^{-1}) \), where \( \to \) stands for convergence in distribution.
2. \( N_0^{1/2}(\hat{\gamma} - \gamma_0) \to N(0, \sigma^2) \), and \( N_0^{1/2}(\hat{\sigma} - \gamma_0) \to N(0, \Omega^{-1}) \), where \( \sigma^2 \) is the (1, 1) element of \( \Omega^{-1} \) and \( \Omega^{-1} \) is the down-right \( 5 \times 5 \) submatrix of \( \Omega^{-1} \).
3. The likelihood ratio \( R(N_0, \alpha_0, \gamma_0) = 2(\ell(\hat{N}, \hat{\alpha}, \hat{\gamma}) - \ell(N_0, \alpha_0, \gamma_0)) \to \chi^2_2 \).

**Theorem 2** Assume Condition 1, and that \( V_c \) is positive definite. Then as \( N_0 \to \infty \),

1. \( N_0^{1/2}(\hat{\gamma} - \gamma_0) \to N(0, V_c^{-1}) \), and
2. \( N_0^{1/2}(\hat{\sigma} - \gamma_0) \to N(0, \sigma_c^2) \), where \( \sigma_c^2 = \varphi_1 - 1 + m \varphi_2 E_{12} V_c^{-1} E_{12} m \).

**Proposition 1** With the symbols used in Theorem 1 and Theorem 2 in the main paper, \( \sigma^2 = \sigma_c^2 \) and \( V = V_c \).

**2 Proof of Lemma 1**

Denote \( \theta_j \) by \( y_j \). We note that the observations \( (y_s, s_j) \) has the same distribution as \( (\theta'_j, s'_j) \) given \( Z_i > 0 \), and its density function is

\[
g(y_s, s_j) = f(y_s, s_j) f(y_s, s_j) / f(y_s, s_j, s_j)
\]

\[
= \frac{1}{\sqrt{2\pi} \sigma^2} \exp \left\{ -\frac{(y_s - \theta_j)^2}{2\sigma^2} \right\} \Phi(\gamma_1 + \gamma_2 / s) \frac{1}{\sqrt{1 - \rho^2 s^2 / (\tau^2 + s^2)}} \Phi(\gamma_1 + \gamma_2 / s) \frac{\sqrt{1 - \rho^2 s^2 / (\tau^2 + s^2)}}{\sqrt{\tau^2 + s^2}}.
\]
To prove this lemma, it suffices to show that when $\rho \neq 0$, the fact the equality $g(y, s; \gamma_a) = g(y, s; \gamma_b)$ for all $(y, s)$ satisfying $s > 0$ implies $\gamma_a = \gamma_b$, where $\gamma_a = (\gamma_{1a}, \gamma_{2a}, \rho_a, \tau_a, \theta_a)$ and $\gamma_b = (\gamma_{1b}, \gamma_{2b}, \rho_b, \tau_b, \theta_b)$.

We consider the following special cases: Case I. For fixed $y$, letting $s \to 0+$, we have

$$g(y, s; \gamma) = \frac{1}{\sqrt{2\pi s^2}} \exp \left\{ -\frac{(y-\theta)^2}{2s^2} \right\} [1 + o(1)].$$

The limit distribution belongs to the normal distribution family, where it is well known that the mean and variance parameters $\theta$ and $\tau$ ($\tau > 0$) are identifiable. In other words, the equation $g(y, s; \gamma_a) = g(y, s; \gamma_b)$ for all $(y, s)$ such that $s > 0$ implies $(\theta_a = \theta_b)$ and $\tau_a = \tau_b$.

Case II. Because $\theta$ is identifiable, we assume $\theta_a = \theta_b = \theta$. For $y = \theta + \xi s$ and any fixed $\xi$, when $s$ is large, we have

$$g(\theta + \xi s, s; \gamma) = \frac{1}{\sqrt{2\pi s^2}} e^{-\xi^2/2} \cdot \Phi \left( \frac{\gamma_1 s + \xi \rho}{\sqrt{1-\rho^2}} \right) \cdot s^{-1} [1 + o(1)].$$

It follows that

$$1 = \lim_{s \to \infty} g(\theta + \xi s, s; \gamma_a) = \lim_{s \to \infty} g(\theta + \xi s, s; \gamma_b) = \frac{1}{\Phi(\gamma_{1a})} \cdot \Phi \left( \frac{\gamma_1 s + \xi \rho}{\sqrt{1-\rho^2}} \right),$$

which means

$$\frac{\Phi(\gamma_{1b})}{\Phi(\gamma_{1a})} = \cdot \Phi \left( \frac{\gamma_1 s + \xi \rho}{\sqrt{1-\rho^2}} \right)$$

holds for any fixed $\xi$. When $\rho_a \neq 0$ and $\rho_b \neq 0$, the left-hand side of the above equation does not depend on $\xi$, while the right-hand side does, this indicates that $\rho_a = \rho_b$ and $\gamma_{1a} = \gamma_{1b}$ must hold simultaneously.

Case III. Because all parameters except $\gamma_2$ are identifiable, we assume that $\gamma_a = (\gamma_{1a}, \rho_a, \tau_a, \theta_a) = (\gamma_{1b}, \rho_b, \tau_b, \theta_b) = (\gamma_1, \rho, \tau, \theta)$. Because

$$1 = \frac{g(y, s; \gamma_a)}{g(y, s; \gamma_b)} = \frac{1}{\Phi(\gamma_{1a} + \gamma_{2b}/s)} \cdot \Phi \left( \frac{\gamma_1 + (\gamma_{2a}/s) + p(y-\theta)/\sqrt{s^2 + \tau^2}}{\sqrt{1-p^2(s^2 + \tau^2)}} \right),$$

we have

$$\frac{\Phi(\gamma_{1} + \gamma_{2a}/s)}{\Phi(\gamma_{1} + \gamma_{2b}/s)} = \frac{\Phi \left( \frac{\gamma_1 + (\gamma_{2a}/s) + p(y-\theta)/\sqrt{s^2 + \tau^2}}{\sqrt{1-p^2(s^2 + \tau^2)}} \right)}{\Phi \left( \frac{\gamma_1 + (\gamma_{2b}/s) + p(y-\theta)/\sqrt{s^2 + \tau^2}}{\sqrt{1-p^2(s^2 + \tau^2)}} \right)}.$$

When $\rho \neq 0$, the left-hand side is independent of $y$, while the right-hand side is a function of $y$, indicating that $\gamma_{2a} = \gamma_{2b}$ must hold. This proves Lemma 1.
3 Proof of Theorem 1

With similar arguments to the proof of Lemma 1 of Qin and Lawless (1994), it can be proved that the maximum likelihood estimators (MLE) \((\hat{N}, \hat{\alpha}, \hat{\gamma})\) satisfies \((\hat{N}/N_0 - 1, \hat{\alpha} - \alpha_0, \hat{\gamma} - \gamma_0) = O_p(N_0^{-1/2})\) and that the maximum conditional likelihood estimators satisfies \(\tilde{\gamma} - \gamma_0 = O_p(N_0^{-1/2})\). To prove Theorems 1 and 2, and Proposition 1 in the main paper, we begin by studying the behaviors of \(\ell(N, \alpha, \gamma), l_c(\gamma), \) and \(\ell_m(N, \alpha, \gamma_{12})\) for \((N, \alpha, \gamma)\) satisfying \((N/N_0 - 1, \alpha - \alpha_0, \gamma - \gamma_0) = O_p(N_0^{-1/2})\).

Define
\[
\mathbf{u}_c = N_0^{-1/2} \sum_{i=1}^n \left( \frac{\nabla \ell_c(\theta_i; s_i; \gamma_0)}{f_1(\theta_i; s_i; \gamma_0)} + E_{45} \frac{\nabla \ell_c, f_2(\theta_i; s_i; \gamma_{45,0})}{f_2(\theta_i; s_i; \gamma_{45,0})} - E_{12} \frac{\nabla \ell_c, f_3(s_i; \gamma_{12})}{f_3(s_i; \gamma_{12})} \right).
\]

The following lemma discloses the asymptotical behaviors of the conditional likelihood \(\ell_c(\gamma)\). The results in this lemma are proved in Sections 4 and 5, respectively.

**Lemma 2** Assume Condition 1. Let \(\zeta_c = N_0^{1/2}(\gamma - \gamma_0)\). (I) For \(\gamma = \gamma_0 + O_p(N_0^{-1/2})\),
\[
\ell_c(\gamma) = \ell_c(\gamma_0) + \mathbf{u}_c^\top \zeta_c + \frac{1}{2} \zeta_c^\top \mathbf{V}_c \zeta_c + o_p(||\zeta_c||^2).
\]

(II) \(\text{Var}(\mathbf{u}_c) = \mathbf{V}_c\).

Define \(\mathbf{u}_{ma} = (u_{m1}, u_{m2}, \mathbf{u}_{m34})^\top\), where
\[
u_{m1} = N_0^{1/2} \frac{n_0/N_0 - \alpha_0}{1 - \alpha_0}, \quad u_{m2} = -N_0^{-1/2} \frac{N_0 - n}{1 - \alpha_0} + N_0^{-1/2} \sum_{i=1}^n \frac{1}{f_3(s_i; \gamma_{12,0})}, \quad \mathbf{u}_{m34} = \mathbf{0}
\]
and
\[
u_{mb} = N_0^{-1/2} \sum_{i=1}^n \frac{\alpha_0 f_3(s_i; \gamma_{12,0}) - \alpha_0}{f_3(s_i; \gamma_{12,0})}.
\]

Define
\[
\mathbf{V}_{maa} = \begin{pmatrix}
\frac{\alpha_0}{1 - \alpha_0} & \frac{1}{1 - \alpha_0} & 0 \\
\frac{1}{1 - \alpha_0} & 0 & \frac{\varphi_1}{m \varphi_2} \\
0 & \frac{\varphi_1}{m \varphi_2} & 0
\end{pmatrix}, \quad \mathbf{V}_{mba} = \begin{pmatrix}
0 \\
-a_0^2 \varphi_1 \\
a_0^2 \varphi_1
\end{pmatrix}, \quad \mathbf{V}_{mbb} = a_0^3 - a_0^4 \varphi_1.
\]

The next lemma, which are proved in Sections 6 and 7, studies the asymptotical behaviors of the marginal likelihood \(\ell_m(N, \alpha, \gamma_{12})\).

**Lemma 3** Assume Condition 2. Let \(\zeta_m = N_0^{1/2}((N/N_0 - 1, (\alpha - \alpha_0), (\gamma_{12} - \gamma_{12,0}))\). (I) If \((N, \alpha, \gamma_{12})\) satisfies \((N/N_0 - 1, \alpha - \alpha_0, \gamma_{12} - \gamma_{12,0}) = O_p(N_0^{-1/2})\), then
\[
\ell_m(N, \alpha, \gamma_{12}) = \ell_m(N, \alpha_0, \gamma_{12,0}) + \mathbf{u}_m^\top \zeta_m + \frac{1}{2} \zeta_m^\top \mathbf{V}_m \zeta_m + o_p(||\zeta_m||^2)
\]
where \(\mathbf{u}_m = \mathbf{u}_{ma} - \mathbf{V}_{mba}^{-1} \mathbf{u}_{mb}\) and \(\mathbf{V}_m = \mathbf{V}_{maa} - \mathbf{V}_{mba}^{-1} \mathbf{V}_{mbb} \mathbf{V}_{mba}^{-1} \mathbf{V}_{mba}\). (II) \(\text{Var}(\mathbf{u}_m) = \mathbf{V}_m\).
It can be found that

\[ \hat{V}_m = V_{max} - V_{mba}V_{mba}^{-1}V_{mab} = \begin{pmatrix} \frac{m_0}{1 - s_0} & \frac{s_0}{1 - s_0} & 0 \\ \frac{m_1}{1 - s_1} & \frac{s_1}{1 - s_1} & m_2 \\ 0 & 1 - s_2 & 1 - s_2 \end{pmatrix}. \]

We note that the matrix \( \hat{V}_m \) is singular and its rank is at most 3. Hence it is infeasible to obtain a reasonable estimator for \( N \) by directly maximizing \( \ell_m \).

To study the behavior of the full log-likelihood \( \ell(N, \alpha, \gamma) \), we need to merge the approximates of \( \ell_c \) and \( \ell_m \). Let \( \zeta = N^{1/2}(N - N_0 - 1, (\alpha - \alpha_0)(\gamma - \gamma_0)^\top) \), and \( F_1 = (I_2, 0_{4 \times 3}) \) and \( F_2 = (0_{5 \times 2}, I_3) \). Clearly \( \zeta_c = F_2 \zeta \) and \( \zeta_m = F_1 \zeta \). Since

\[ \ell_c(\gamma) = \ell_c(\gamma \gamma_0) + (\bar{u}_c^c_{2c} F_2 - \frac{1}{2} \zeta_c^\top F_2 \zeta_c + o_p(||\zeta||^2)), \]
\[ \ell_m(N, \alpha, \gamma_{12}) = \ell_m(N, \alpha, \gamma_{120}) + \bar{u}_m^{c} F_1 \zeta - \frac{1}{2} \zeta_c^\top F_1 \bar{V}_m F_1 \zeta + o_p(||\zeta||^2), \]

it follows that

\[ \ell(N, \alpha, \gamma) = \ell(N, \alpha, \gamma_0) + (\bar{u}_c^c_{2c} F_2 + \bar{u}_m^c F_1) \zeta - \frac{1}{2} \zeta_c^\top \Omega \zeta + o_p(||\zeta||^2). \]

Because \( E(u_i(s', Z_i > 0) : i = 1, 2, \ldots N_0) = 0 \) and \( u_m \) depends only on \( \{s', Z_i > 0) : i = 1, 2, \ldots N_0 \} \), we have

\[ \text{Cov}(u_i, \bar{u}_m) = E(u_i \bar{u}_m) = 0 \]

and

\[ \text{Var}(u_i F_2 + \bar{u}_m^c F_1) = F_2^\top \text{Var}(u_i) F_2 + F_1^\top \text{Var}(\bar{u}_m) F_1 = F_2^\top \bar{V}_m F_2 + F_1^\top \bar{V}_m F_1 = \Omega, \]

where we have used Lemmas 2 and 3. Therefore the maximum likelihood estimators \((\bar{N}, \bar{\alpha}, \bar{\gamma})\) satisfies

\[ N_0^{1/2}(\bar{N}/N_0 - 1, \bar{\alpha} - \alpha_0, (\bar{\gamma} - \gamma_0)^\top) = \Omega^{-1} (u_i F_2 + \bar{u}_m^c F_1) + o_p(1) \]

\[ \overset{d}{\rightarrow} N(0, \Omega^{-1}), \]

since \( \Omega \) is positive definite. Accordingly the likelihood ratio

\[ R(N_0, \alpha_0, \gamma_0) = 2[\ell(\bar{N}, \bar{\alpha}, \bar{\gamma}) - \ell(N_0, \alpha_0, \gamma_0)] \overset{d}{\rightarrow} \chi^2_j. \]

This proves results (1) and (3) in Theorem 1 of the main paper. Result (2) is implied by Result (1) of Theorem 1. This finishes proving Theorem 1.
4 Proof of Lemma 2: Result (I)

4.1 Preparations

Rewrite the conditional log-likelihood as

\[ l_c(\gamma) = \sum_{i=1}^{n} \log f_1(\theta_i, s_i; \gamma) + \log f_2(\theta_i, s_i; \gamma_{45}) - \log f_3(s_i; \gamma_{12}). \] (6)

A second-order Taylor expansion of \( l_c(\gamma) \) at \( \gamma_0 \) gives

\[ l_c(\gamma) = l_c(\gamma_0) + [\nabla \gamma \ell_c(\gamma_0)]^T (\gamma - \gamma_0) + \frac{1}{2} (\gamma - \gamma_0) \nabla \nabla \gamma \ell_c(\gamma_0)(\gamma - \gamma_0) + O(||\gamma - \gamma_0||^3) \]

Direct calculations give

\[
\nabla \gamma l_c(\gamma) = \sum_{i=1}^{n} \left\{ \nabla \gamma f_1(\theta_i, s_i; \gamma) + E_{45} \nabla \gamma f_2(\theta_i, s_i; \gamma_{45}) - E_{12} \nabla \gamma f_3(s_i; \gamma_{12}) \right\},
\]

\[
\nabla \nabla \gamma l_c(\gamma) = \sum_{i=1}^{n} \left\{ \nabla \nabla \gamma f_1(\theta_i, s_i; \gamma) - \left( \frac{\nabla \gamma f_1(\theta_i, s_i; \gamma)}{f_1(\theta_i, s_i; \gamma)} \right) \right\} + E_{45} \left\{ \nabla \nabla \gamma f_2(\theta_i, s_i; \gamma_{45}) - \left( \frac{\nabla \gamma f_2(\theta_i, s_i; \gamma_{45})}{f_2(\theta_i, s_i; \gamma_{45})} \right) \right\} + E_{12} \left\{ \nabla \nabla \gamma f_3(s_i; \gamma_{12}) - \left( \frac{\nabla \gamma f_3(s_i; \gamma_{12})}{f_3(s_i; \gamma_{12})} \right) \right\}.
\]

To simply the above expressions we need to discuss properties of the data.

Suppose the results of the total \( N \) study are \((\theta_i', s_i') (i = 1, 2, \ldots, N)\), each accompanied with a random indicator \( Z_i \). The results \((\theta_i', s_i')\)'s with \( Z_i > 0 \) are denoted by \((\theta_i, s_i)\)'s. Hence \((\theta_i, s_i) \overset{\text{d}}{=} (\theta_i', s_i')|Z_i > 0\), where \( \overset{\text{d}}{=} \) means "having the same distribution as". Thus for any function \( g \),

\[ \sum_{i=1}^{n} g(\theta_i, s_i) = \sum_{i=1}^{N_0} g(\theta_i', s_i')I(Z_i > 0), \]

which implies

\[ \mathbb{E} \left[ \sum_{i=1}^{n} g(\theta_i, s_i)(\theta_i', s_i'), i = 1, \ldots, N_0 \right] = \sum_{i=1}^{N_0} g(\theta_i', s_i')f_1(\theta_i', s_i'; \gamma_0), \] (7)

\[ \mathbb{E} \left[ \sum_{i=1}^{n} g(\theta_i, s_i)|s_i', i = 1, \ldots, N_0 \right] = \sum_{i=1}^{N_0} g(\theta_i', s_i')f_3(s_i'; \gamma_{12}), \] (8)

\[ \mathbb{E} \left[ \sum_{i=1}^{n} g(\theta_i, s_i) \right] = N_0 \mathbb{E} [g(\theta_i', s_i')f_3(s_i'; \gamma_{12})]. \] (9)
Another important key equality is \( E\{f_1(\theta^*, s^*_t; \gamma)|s^*_t\} = f_3(s^*_t; \gamma_{12}) \) or

\[
0 = \int f_1(\theta, s^*_t; \gamma)f_2(\theta, s^*_t; \gamma_{45})d\theta - f_3(s^*_t; \gamma_{12}).
\]

which holds for any parameter value in the parameter space and for any \( s^*_t > 0 \). Taking derivatives on both sides of (10) once gives

\[
0 = \int \nabla f_1(\theta, s^*_t; \gamma)f_2(\theta, s^*_t; \gamma_{45})d\theta + E_{45} \int f_1(\theta, s^*_t; \gamma)\nabla f_2(\theta, s^*_t; \gamma_{45})d\theta
- E_{12} \nabla f_3(s^*_t; \gamma_{12}).
\]

Similarly, taking derivatives on both sides twice gives

\[
0 = \int \nabla^2 f_1(\theta, s^*_t; \gamma)f_2(\theta, s^*_t; \gamma_{45})d\theta + \left\{ \int \nabla f_1(\theta, s^*_t; \gamma)\nabla f_2(\theta, s^*_t; \gamma_{45})d\theta \right\} E_{45}^\top
+ E_{45} \int f_1(\theta, s^*_t; \gamma)\nabla f_2(\theta, s^*_t; \gamma_{45})d\theta E_{45} - E_{12} \nabla f_3(s^*_t; \gamma_{12})E_{12}.
\]

4.2 Matrix \( V_c \)

It can be verified that the leading term of \(-1/N_0\nabla f(\gamma; \ell)\) is equal to \( V_c(\gamma) \), which is defined in (4).

When \( \gamma = \gamma_0 \), by Equations (10) and (12), we have

\[
V_c = V_c(\gamma_0)
\]

\[
= \mathbb{E} \int \left( \frac{\nabla f_1(t, s^*_t; \gamma_0)}{f_1(t, s^*_t; \gamma_0)} \right)^\top f_2(t, s^*_t; \gamma_{45,0})dt
+ E_{45} \mathbb{E} \int \left( \frac{\nabla f_2(t, s^*_t; \gamma_{45,0})}{f_2(t, s^*_t; \gamma_{45,0})} \right)^\top f_1(t, s^*_t; \gamma_0)dE_{45}^\top
- E_{12} \mathbb{E} \left( \frac{\nabla f_3(s^*_t; \gamma_{12,0})}{f_3(s^*_t; \gamma_{12,0})} \right) E_{12}^\top
+ \left( \mathbb{E} \int \nabla f_1(t, s^*_t; \gamma_0)\nabla f_2(t, s^*_t; \gamma_{45,0})d\theta \right) E_{45}^\top.
\]

Denote

\[
u_c = N_0^{-1/2}\nabla f(\ell, \gamma_0)
\]

\[
= N_0^{-1/2} \sum_{i=1}^n \left\{ \frac{\nabla f_1(\theta, s_i; \gamma_0)}{f_1(\theta, s_i; \gamma_0)} + E_{12} \frac{\nabla f_2(\theta, s_i; \gamma_{45,0})}{f_2(\theta, s_i; \gamma_{45,0})} - E_{12} \frac{\nabla f_3(s_i; \gamma_{12})}{f_3(s_i; \gamma_{12})} \right\}
\]

\[
= N_0^{-1/2} \sum_{i=1}^N \left\{ \frac{\nabla f_1(\theta, s_i; \gamma_0)}{f_1(\theta, s_i; \gamma_0)} + E_{12} \frac{\nabla f_2(\theta, s_i; \gamma_{45,0})}{f_2(\theta, s_i; \gamma_{45,0})}
- E_{12} \frac{\nabla f_3(s_i; \gamma_{12,0})}{f_3(s_i; \gamma_{12,0})} \right\} I(Z_i > 0).
\]
Then we have
\[
I_c(y) = I_c(y_0) + [\nabla y \cdot f_c(y_0)]^T (y - y_0) + \frac{1}{2} (y - y_0)^T \nabla y \cdot \ell_c(y_0)(y - y_0)
+ o(N_0\|y - y_0\|^2)
= I_c(y_0) + u_c^T \xi_c + \frac{1}{2} \xi_c^T \nabla y \cdot \ell_c(y_0) + o(N_0\|\xi_c\|^2).
\]

where \(u_c = N_0^{-1/2} \nabla y \cdot \ell_c(y_0)\) and \(V_c = N_0^{-1} \nabla y \cdot \ell_c(y_0)\). This proves Result (I) of Lemma 2.

5 Proof of Lemma 2: Result (II)

It follows directly from Equation (10) that \(E(u_c) = 0\). The variance of \(u_c\) is
\[
\text{Var}(u_c) = \frac{1}{N_0} E((\nabla y \cdot \ell_c(y_0))^2)
= E\left[\left[\frac{\nabla y 
abla y \cdot f_2 \theta_s y \theta_s y \theta_s}{f_2 \theta_s y \theta_s y \theta_s} - E_{12} \frac{\nabla y \cdot f_2 \theta_s y \theta_s y \theta_s}{f_3 \theta_s y \theta_s y \theta_s} \right]^2 I(Z > 0)\right]
= E\left[\frac{\nabla y \cdot f_2 \theta_s y \theta_s y \theta_s}{f_2 \theta_s y \theta_s y \theta_s} \right]^2 I(Z > 0)
+ E\left[\frac{\nabla y \cdot f_2 \theta_s y \theta_s y \theta_s}{f_2 \theta_s y \theta_s y \theta_s} \right]^2 I(Z > 0)
- E\left[\frac{\nabla y \cdot f_2 \theta_s y \theta_s y \theta_s}{f_2 \theta_s y \theta_s y \theta_s} \right]^2 I(Z > 0)
- E\left[\frac{\nabla y \cdot f_2 \theta_s y \theta_s y \theta_s}{f_2 \theta_s y \theta_s y \theta_s} \right]^2 I(Z > 0)
= E\left[\frac{\nabla y \cdot f_2 \theta_s y \theta_s y \theta_s}{f_2 \theta_s y \theta_s y \theta_s} \right]^2 I(Z > 0)
+ E\left[\frac{\nabla y \cdot f_2 \theta_s y \theta_s y \theta_s}{f_2 \theta_s y \theta_s y \theta_s} \right]^2 I(Z > 0)
- E\left[\frac{\nabla y \cdot f_2 \theta_s y \theta_s y \theta_s}{f_2 \theta_s y \theta_s y \theta_s} \right]^2 I(Z > 0)
- E\left[\frac{\nabla y \cdot f_2 \theta_s y \theta_s y \theta_s}{f_2 \theta_s y \theta_s y \theta_s} \right]^2 I(Z > 0).
\]
Then by Eq (11), we have

\[ \text{Var}(\mathbf{u}_c) = \mathbb{E} \left[ \frac{\nabla \gamma f_1(\theta_i, s_i^*; \gamma_0)}{f_1(\theta_i, s_i^*; \gamma_0)} \right]^2 f_2(\theta_i, s_i^*; \gamma_{45,0}) d\theta_{45} \]

\[ - \mathbb{E} \left[ \frac{\nabla \gamma f_3(s_i^*; \gamma_{12,0})}{f_3(s_i^*; \gamma_{12,0})} \right]^2 E_{12} \]  

\[ = \mathbf{V}_c, \]

which is exactly Result (II) of Lemma 2.

6 Proof of Lemma 3: Result (I)

6.1 Re-expression of \( \ell_m \)

Re-express

\[ \ell_m(N, \alpha, \gamma_1, \gamma_2) = \min_{\lambda} h_m(N, \alpha, \gamma_{12}, \lambda), \]

where

\[ h_m(N, \alpha, \gamma_{12}, \lambda) = \log \left( \frac{N}{n} \right) + (N - n) \log(1 - \alpha) - \sum_{i=1}^{n} \log \left[ 1 + \lambda \left( f_3(s_i^*; \gamma_{12}) - \alpha \right) \right] + \log f_3(s_i^*; \gamma_{12}). \]

Let \( \lambda_0 = 1/\alpha_0 \). We further rewrite

\[ h_m(N, \alpha, \gamma_{12}, \lambda) = H(\zeta_m) \]

with

\[ H_m(\zeta_m) = h_m(N_0 + N_0^{1/2} \zeta_m, \alpha_0 + N_0^{-1/2} \zeta_m, \gamma_{12,0} + N_0^{-1/2} \zeta_{m34}, \lambda_0 + N_0^{-1/2} \zeta_{m5}). \]

We approximate \( h_m(N, \alpha, \gamma_{12}, \lambda) \) by first approximating \( H_m(\zeta_m) \). A second-order Taylor expansion of \( H(\zeta_m) \) at \( 0 \) gives

\[ H_m(\zeta_m) = H_m(0) + \nabla \zeta_m H_m(0) + \frac{1}{2} \zeta_m \nabla^2 \zeta_m \zeta_m H_m(0) + o(\|\zeta_m\|^2). \]

Below we need to derive specific expressions for \( \nabla \zeta_m H_m(0) \) and \( \nabla \zeta_m \zeta_m H_m(0) \).
6.2 Preparation

The partial derivatives of \( H_m \) involves derivatives of the Gamma function. We first study some properties of \( \Gamma(s) \), the Gamma function. For any positive integer \( c \), define

\[
S_c(N, n) = \frac{d^c \log \Gamma(N + 1)}{dN^c} - \frac{d^c \log \Gamma(N - n + 1)}{dN^c} = (-1)^{c+1} (c-1)! \sum_{k=N-n+1}^{N} \frac{1}{k^c}.
\]

The fact that \( x^{-1} \) and \( x^{-2} \) are monotone decreasing function implies that

\[
\log((N + 1)/(N + 1 - n)) < S_1(N, n) < \log(N/(N - n)) \quad \text{and} \quad -n/(N(N - n)) < S_2(N, n) < -n/((N + 1)(N + 1 - n))
\]

Since \( n \) follows \( B(N_0, \alpha_0) \), by central limit theorem, we have \( n/N_0 = \alpha_0 + O_p(N_0^{-1/2}) \),

\[
S_1(N, n) = \log \left( \frac{N_0}{N_0 - n} \right) + O_p(N_0^{-1}) = \log(1 - \alpha_0) + \frac{n/N_0 - \alpha_0}{1 - \alpha_0} + O_p(N_0^{-1})
\]

and

\[
S_2(N, n) = -\frac{n}{N_0(N_0 - n)} + O_p(N_0^{-2}) = -\frac{\alpha_0}{N_0(1 - \alpha_0)} + O_p(N_0^{-2}).
\]

6.3 Approximation of \( H_m \)

Specific expressions of the first-two-order derivatives of \( H_m \) are derived in the Appendix. Let \( u_m = \nabla_{s_m} H(0) \). Using the properties of the Gamma function, it can be found that

\[
u_{m1} = N_0^{1/2} \frac{n/N_0 - \alpha_0}{1 - \alpha_0}, \quad u_{m2} = -N_0^{-1/2} N_0 - n \frac{1}{1 - \alpha_0} + N_0^{1/2} \sum_{i=1}^{n} \frac{1}{f_3(s_i; \gamma_{12,0})},
\]

\[
u_{m3} = 0, \quad u_{m5} = -N_0^{-1/2} \sum_{i=1}^{n} \frac{\alpha_0 (f_3(s_i; \gamma_{12,0}) - \alpha_0)}{f_3(s_i; \gamma_{12,0})}.
\]

Let \( V_m = V_m(N_0, \alpha_0, \gamma_{12,0}, 1/\alpha_0) \) be the leading term of \(-\nabla_{s_m} H(0)\). It follows from the second order partial derivatives of \( H_m \) that

\[
V_m = \begin{pmatrix}
\frac{\alpha_0}{1 - \alpha_0} & \frac{1}{1 - \alpha_0} & 0 & 0 & -\alpha_0 \phi_1 \\
\frac{1}{1 - \alpha_0} & \frac{\alpha_0}{1 - \alpha_0} & \frac{\phi_1}{\phi_2} & -\alpha_0 \phi_1 & m \phi_2 \\
0 & \frac{\phi_1}{\phi_2} & \frac{\phi_1}{\phi_2} & 0 & \frac{\alpha_0}{\phi_2} \phi_2 \\
0 & -\alpha_0 \phi_1 & \frac{\alpha_0}{\phi_2} \phi_2 & \frac{\alpha_0}{\phi_2} \phi_2 & \frac{\alpha_0}{\phi_2} \phi_2
\end{pmatrix} = (V_{maa} \ V_{mab} \ V_{mab} \ V_{mab}),
\]
where $V_{mbb} = a_0^2 - a_0^3 \varphi_1$, 

$$V_{maa} = \begin{pmatrix} \frac{a_0}{1-a_0} & \frac{1}{1-a_0} \varphi_1 & 0 \\ \frac{1}{1-a_0} & \frac{1}{1-a_0} - \varphi_1 & m_2^2 \\ 0 & m_2^2 & 0 \end{pmatrix} \quad \text{and} \quad V_{mab} = \begin{pmatrix} 0 \\ -a_0^3 \varphi_1 \\ a_0^3 m_2^2 \end{pmatrix}.$$

In summary we have

$$H_m(\xi_m) = H_m(0) + u_m^T \xi_m - \frac{1}{2} \xi_m^T V_m \xi_m + \alpha_p(||\xi_m||^2).$$

6.4 Approximation of $\ell_m$ 

Partition $\xi_m = (\xi_m^r, \xi_m^b)^T$ and $u_m = (u_m^a, u_m^b)^T$. It can be seen that

$$H_m(\xi_m) = H_m(0) + u_m^T \xi_m - \frac{1}{2} \xi_m^r V_m \xi_m + \alpha_p(||\xi_m||^2)$$

$$= H(0) + u_m^a \xi_m^a + u_m^b \xi_m^b - \frac{1}{2} \xi_m^a V_{maa} \xi_m^a - \xi_m^a V_{mab} \xi_m^b$$

$$- \frac{1}{2} \xi_m^b V_{mbb} + \alpha_p(||\xi_m||^2).$$

Setting $\nabla \ell_m H_m(\xi_m) = 0$ gives

$$0 = u_m^a - \xi_m^a V_{mab} - \xi_m^b V_{mbb} + \alpha_p(||\xi_m||),$$

which implies

$$\xi_m^b = V_{mbb}^{-1}(u_m^a - V_{mab} \xi_m^a) + \alpha_p(||\xi_m||).$$

Putting this $\xi_m^b$ back into $H_m(\xi_m)$, we have

$$\ell_m = \min_{\xi_m} H_m(\xi_m) = H(0) + \frac{1}{2} V_{mbb}^{-1} a_0^2 - \frac{1}{2} \xi_m^a (V_{maa} - V_{mab} V_{mbb}^{-1} V_{mab}) \xi_m^a$$

$$+ \xi_m^a (u_m^a - V_{mab} V_{mbb}^{-1} u_m^b) + \alpha_p(||\xi_m||^2).$$

7 Proof of Lemma 3: Result (II) 

Denote $\Sigma = \text{Var}(u_m)$. We have derived in the Appendix that

$$\Sigma = \begin{pmatrix} \frac{a_0}{1-a_0} & \frac{1}{1-a_0} \varphi_1 + \frac{1}{1-a_0} a_0^2 \varphi_1 - a_0 \\ \frac{1}{1-a_0} & \frac{1}{1-a_0} + \frac{1}{1-a_0} a_0^3 \varphi_1 - a_0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{pmatrix}.$$

We first calculate

$$\text{Var}(u_m^a - V_{mab} V_{mbb}^{-1} u_m^b) = \Sigma_{aa} - V_{mab} V_{mbb}^{-1} \Sigma_{ba} - \Sigma_{ab} V_{mbb}^{-1} V_{mab} + V_{mab} V_{mbb}^{-1} V_{mab}.$$
It follows from \( \Sigma_{bb} = -V_{mhb} = a_0^b \varphi_1 - a_0^b \) that
\[
V_{mab} V_{mhb}^{-1} \Sigma_{bb} = -\frac{1}{a_0^b \varphi_1 - a_0^b} \begin{pmatrix} 0 & -a_0^b \varphi_1 & \Sigma_{bb}^{-1} \begin{pmatrix} 0 \\ a_0^b \varphi_1 \\ a_0^b \varphi_2 \end{pmatrix} (0, \varphi_1, 0) = \begin{pmatrix} 0 & \varphi_1 & 0 \\ 0 & -m \varphi_2 & 0 \end{pmatrix} \end{pmatrix}
\]
and
\[
V_{mab} V_{mhb}^{-1} \Sigma_{bb} V_{mha} = V_{mab} V_{mhb}^{-1} \Sigma_{bb} = \Sigma_{bb}^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -m \varphi_2 \varphi_1 \end{pmatrix}.
\]
With these equalities, we finally arrive at
\[
\forall \text{Var}(u_{mb} - V_{mab} V_{mhb}^{-1} u_{mb}) = \begin{pmatrix} a_0^b \\ 1 \\ 1 \\ -\varphi_1 + \frac{1}{a_0^b} \Theta \varphi_2 \\ 0 \end{pmatrix} \sum_{bb}^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} a_0^b \\ 1 \\ 1 \\ -\varphi_1 + \frac{1}{a_0^b} \Theta \varphi_2 \\ 0 \end{pmatrix} \sum_{bb}^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
\]
In the meantime, it can be seen that
\[
V_{mba} - V_{mab} V_{mhb}^{-1} V_{mba} = \begin{pmatrix} a_0^b \\ 1 \\ 1 \\ -\varphi_1 + \frac{1}{a_0^b} \Theta \varphi_2 \\ 0 \end{pmatrix} \sum_{bb}^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} a_0^b \\ 1 \\ 1 \\ -\varphi_1 + \frac{1}{a_0^b} \Theta \varphi_2 \\ 0 \end{pmatrix} \sum_{bb}^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
\]
This proves
\[
\forall \text{Var}(u_{mb} - V_{mab} V_{mhb}^{-1} u_{mb}) = V_{mba} - V_{mab} V_{mhb}^{-1} V_{mba}.
\]

8 Proof of Theorem 2

Denote \( \xi = N_{0}^{1/2} (\tilde{\varphi} - \gamma_0) \). Lemma 2 implies \( \xi = N_{0}^{1/2} (\tilde{\varphi} - \gamma_0) = V_{c}^{-1} u_{c} + o_p(1) \rightarrow N(0, V_{c}^{-1}) \), where
\[
\begin{align*}
\begin{pmatrix} u_{c} \\ \xi \end{pmatrix} = N_{0}^{1/2} \frac{1}{N_0} \sum_{i=1}^{N_0} \begin{pmatrix} \nabla \varphi f_i (\theta_i, s_i; \gamma_0) \\ f_1 (\theta_i, s_i; \gamma_0) \end{pmatrix} + E_{45} \frac{\nabla \varphi f_i (\theta_i, s_i; \gamma_{45,0})}{f_2 (\theta_i, s_i; \gamma_{45,0})} \\
- E_{12} \frac{\nabla \varphi f_i (s_i; \gamma_{12})}{f_3 (s_i; \gamma_{12})} I(Z_i > 0).
\end{pmatrix}
\end{align*}
\]
By the first-order Taylor expansion and the weak law of large numbers, we have
\[ N_0^{1/2}(N/N_0 - 1) = t - m\varphi_2 E_{12}^T \cdot N_0^{1/2}(\bar{\gamma} - \gamma_0) + o_p(1) = t - m\varphi_2 E_{12}^T V_c^{-1}u_c + o_p(1) \]
with \( t = N_0^{1/2} \sum_{t=1}^{N_0} \left( \frac{\bar{Z}_t - \bar{\gamma}^s}{\sqrt{N}} \right)^2 \). Direct calculations give
\[ \text{Var}(t) = \frac{\bar{I}(Z_0 > 0)}{f_3(s_1^*, \gamma_{12,0})} - 1 = \frac{\bar{I}(Z_0 > 0)}{f_3(s_1^*, \gamma_{12,0})} - 1 = \varphi_1 - 1. \]
Similarly
\[
\text{Cov}(t, u_c) = \mathbb{E} \left[ \frac{I(Z_0 > 0)}{f_3(s_1^*, \gamma_{12,0})} \left( \nabla \gamma f_1(\theta', s_1^*, \gamma_0) + E_{45} \nabla \gamma \theta f_3(\theta', s_1^*, \gamma_{12,0}) \right) \right] \\
= \mathbb{E} \left[ \frac{\nabla \gamma f_1(\theta', s_1^*, \gamma_0)}{f_3(s_1^*, \gamma_{12,0})} + E_{45} \frac{f_1(\theta', s_1^*, \gamma_0) \nabla \gamma \theta f_3(\theta', s_1^*, \gamma_{12,0})}{f_3(\theta', s_1^*, \gamma_{12,0})} \right] \\
= 0,
\]
where we have used (11). Therefore
\[
\sigma_c^2 = \text{Var}(t - m\varphi_2 E_{12}^T V_c^{-1}u_c) = \text{Var}(t) + \text{Var}(m\varphi_2 E_{12}^T V_c^{-1}u_c) = \varphi_1 - 1 + m\varphi_2 E_{12}^T V_c^{-1}E_{12} m\varphi_2.
\]

9 Proof of Proposition 1 of the main paper

To prove this proposition, we need to derive specific forms of \( \sigma^2 \) and \( V^{-1} \), which both are part of \( \Omega^{-1} \). We shall re-express \( \Omega \) by an easy-going form and then derive \( \sigma^2 \) and \( V^{-1} \), respectively.

Denote \( a = \alpha_0/(1 - \alpha_0 \varphi_1), b = m\varphi_2 E_{12}^T V_c^{-1} E_{12}^T m\varphi_2, D = (0, \alpha_0^{-1} E_{12}^T m\varphi_2) \) and
\[
C = \left( \begin{array}{ccc}
\alpha_0 & \frac{1}{1-\alpha_0} & \frac{1}{1-\alpha_0} \\
\frac{1}{1-\alpha_0} & \frac{1}{1-\alpha_0} & \frac{1}{1-\alpha_0} \\
\frac{1}{1-\alpha_0} & \frac{1}{1-\alpha_0} & (1-\alpha_0)/(1-\alpha_0 \varphi_1) 
\end{array} \right).
\]

Then we can write
\[
\Omega = \left( \begin{array}{ccc}
C & D^T V_c^{-1} & m\varphi_2 E_{12}^T \\
D V_c^{-1} & D^T V_c^{-1} & m\varphi_2 E_{12}^T \\
-m\varphi_2 E_{12}^T & -m\varphi_2 E_{12}^T & \frac{1}{1-\alpha_0} 
\end{array} \right).
\]
Denote the left-up block and the right-down block of \( \Omega^{-1} \) by \( \Omega^{11} \) and \( \Omega^{22} \), respectively. Then \( V^{-1} = \Omega^{22} \) and \( \sigma^2 \) is the (1, 1)-element of \( \Omega^{11} \).
By the inverse formula of $2 \times 2$ matrix, we have
\[
V^{-1} = \Omega^{22} = (V_c - \Lambda)^{-1},
\]
where $\Lambda = aE_{12}^\top \phi \phi_{2}^\top E_{12} + DC^{-1}D'$.
\[\Omega^{11} = [C - D'(V_c - aE_{12}^\top \phi \phi_{2}^\top E_{12})^{-1}]^{-1},\]

where $\Delta = aE_{12}^\top \phi \phi_{2}^\top E_{12}$.

**Proof**

We first prove $V = V_c$, which is equivalent to showing $\Lambda = 0$. It is easy to see that
\[
C^{-1} = \frac{1}{a_0 (1 - \phi_1)} \left( \frac{1 - \phi_1}{1 - a_0} \frac{1 - o_0}{1 + a_0} - \frac{1 - o_0}{1 - a_0} \right) = -(1 - a_0)(1 - a_0 \phi_1) \left( \frac{1 - \phi_1}{1 - a_0} \frac{1 - o_0}{1 + a_0} - \frac{1 - o_0}{1 - a_0} \right) = -\left( \frac{1 - \phi_1}{1 - a_0 \phi_1} \frac{1 - a_0 \phi_1}{a_0 (1 - a_0 \phi_1)} \right).
\]

Consequently we have
\[
\Lambda = aE_{12}^\top \phi \phi_{2}^\top E_{12} - (0, a a_0^{-1} E_{12}^\top \phi \phi_{2}) \left( \frac{1 - \phi_1}{1 - a_0 \phi_1} \frac{1 - a_0 \phi_1}{a_0 (1 - a_0 \phi_1)} \right) (0, a a_0^{-1} E_{12}^\top \phi \phi_{2})^\top = aE_{12}^\top \phi \phi_{2}^\top E_{12} - 2 a_0^{-2} E_{12}^\top \phi \phi_{2} \phi \phi_{2} E_{12} \phi \phi_{2} \phi \phi_{2} = 0,
\]
where the last equation holds because $a = a_0/(1 - a_0 \phi_1)$. This proves $V = V_c$.

Next we prove $\sigma^2 = \sigma^2$. Since
\[
(V_c - aE_{12}^\top \phi \phi_{2} \phi \phi_{2} E_{12})^{-1} = V_c^{-1} + \frac{V_c^{-1} E_{12}^\top \phi \phi_{2} \phi \phi_{2} E_{12} V_c^{-1}}{\sigma^2 - b},
\]
we further have
\[
(\Omega^{11})^{-1} = C - D'V_c^{-1}D = \frac{D'V_c^{-1} E_{12}^\top \phi \phi_{2} \phi \phi_{2} E_{12} V_c^{-1}}{\sigma^2 - b}.
\]
Because $D' V_c^{-1} E_{12}^\top \phi \phi_{2} = a a_0^{-1} b e_2$ with $e_2 = (0, 1)^\top$ and $D'V_c^{-1} D = a^2 a_0^{-2} b e_2 e_2^\top$, it follows that
\[
(\Omega^{11})^{-1} = C - a^2 a_0^{-2} b e_2 e_2^\top = \frac{a^2 a_0^{-2} b e_2 e_2^\top}{\sigma^2 - b} = C - \frac{aa_0^{-2} b}{\sigma^2 - b} e_2 e_2^\top = \frac{a_0^{-1}}{\sigma^2 - b} \left( \frac{1}{1 - a_0} \frac{1 - \phi_1}{1 - a_0 \phi_1} \right).
After some algebra, we find that the $(2, 2)$-element of the above matrix can be rewritten as

\[
\frac{1 - \varphi_1}{(1 - \alpha_0)(1 - a_0\varphi_1)} - \frac{a_0^{-2}b}{a^2 - b} = \frac{1 - \varphi_1}{(1 - \alpha_0)(1 - a_0\varphi_1)} - \frac{\frac{a_0}{\alpha_0}a_0^{-2}b}{\frac{1}{1 - a_0\varphi_1} - b}
\]

\[
= \frac{1 - \varphi_1}{(1 - \alpha_0)(1 - a_0\varphi_1)} - \frac{\frac{1}{1 - a_0\varphi_1}b}{1 - a_0\varphi_1 - a_0b}
\]

\[
= \frac{1 - \varphi_1}{(1 - \alpha_0)(1 - a_0\varphi_1)} - \frac{(1 - \alpha_0\varphi_1)^2 - (1 - \alpha_0\varphi_1)a_0b}{b}
\]

\[
= \frac{1 - \varphi_1}{(1 - \alpha_0\varphi_1)(1 - a_0)} - \frac{(1 - a_0\varphi_1)(1 - a_0)(1 - a_0(\varphi_1 + b))}{b(1 - a_0)}
\]

Thus

\[
(O^{11})^{-1} = \frac{1}{1 - a_0} \left( \frac{a_0}{1 - \varphi_1(1 - a_0(\varphi_1 + b)) - b(1 - a_0)} \right).
\]

Since $\sigma_2^2$ is the $(1, 1)$-element of $O^{11}$, we have

\[
\sigma_2^2 = (1 - \alpha_0)\frac{(1 - \varphi_1)(1 - a_0(\varphi_1 + b)) - b(1 - a_0)}{(1 - a_0)(1 - a_0(\varphi_1 + b))} = 1
\]

\[
= \frac{(1 - \alpha_0)(1 - \varphi_1)(1 - a_0(\varphi_1 + b)) - b(1 - a_0) - (1 - a_0)\alpha_0}{(1 - \alpha_0)(1 - a_0(\varphi_1 + b))}
\]

\[
= \frac{(1 - \varphi_1)(1 - a_0(\varphi_1 + b)) - b(1 - a_0)}{(1 - a_0)(1 - a_0(\varphi_1 + b))}
\]

\[
= \frac{\alpha_0\varphi_1 - 1}{\alpha_0\varphi_1 - 1}
\]

\[
= \varphi_1 - 1 - \frac{(1 - \varphi_1)a_0 + (1 - a_0)}{a_0\varphi_1 - 1} \times b
\]

\[
= \varphi_1 - 1 + b = \sigma_1^2.
\]

This proves $\sigma^2 = \sigma_2^2$ and hence also proves Proposition 1.
10 Other technical derivations

10.1 Derivatives of $H_m$

The first-order partial derivatives of $H_m$ are

$$\nabla_{\zeta} H_m(\zeta_m) = N_0^{1/2} \nabla_h h_m = N_0^{1/2} S_1(N, n) + N_0^{1/2} \log(1 - \alpha),$$

$$\nabla_{\zeta} H_m(\zeta_m) = N_0^{-1/2} \nabla_a h_m = -N_0^{-1/2} \frac{N - n}{1 - \alpha} + N_0^{-1/2} \sum_{i=1}^{n} \frac{\lambda}{1 + \lambda f_3(s_i; \gamma_{12}) - \alpha},$$

$$\nabla_{\zeta} H_m(\zeta_m) = N_0^{-1/2} \nabla_{\zeta} f_3(s_i; \gamma_{12}) = -N_0^{-1/2} \sum_{i=1}^{n} \frac{\lambda}{1 + \lambda f_3(s_i; \gamma_{12}) - \alpha}.$$ 

Similarly, its second-order partial derivatives are

$$\nabla_{\zeta} \nabla_{\zeta} H_m(\zeta_m) = \nabla_{\zeta} \nabla_{\zeta} H_m(\zeta_m) = 0.$$ 

$$\nabla_{\zeta} \nabla_{\zeta} H_m(\zeta_m) = 0.$$ 

$$\nabla_{\zeta} \nabla_{\zeta} H_m(\zeta_m) = -N_0^{-1} \frac{N - n}{(1 - \alpha)^2} + N_0^{-1} \sum_{i=1}^{n} \frac{\lambda^2}{[1 + \lambda f_3(s_i; \gamma_{12}) - \alpha]^2},$$

$$\nabla_{\zeta} \nabla_{\zeta} H_m(\zeta_m) = -N_0^{-1} \sum_{i=1}^{n} \frac{\lambda^2 \nabla_{\zeta} f_3(s_i; \gamma_{12})}{[1 + \lambda f_3(s_i; \gamma_{12}) - \alpha]^2},$$

$$\nabla_{\zeta} \nabla_{\zeta} H_m(\zeta_m) = N_0^{-1} \sum_{i=1}^{n} \frac{1}{[1 + \lambda f_3(s_i; \gamma_{12}) - \alpha]^2}.$$ 

$$\nabla_{\zeta} \nabla_{\zeta} H_m(\zeta_m) = -N_0^{-1} \sum_{i=1}^{n} \frac{\lambda^2 \nabla_{\zeta} f_3(s_i; \gamma_{12})}{[1 + \lambda f_3(s_i; \gamma_{12}) - \alpha]^2},$$

$$+ N_0^{-1} \sum_{i=1}^{n} \frac{\nabla_{\zeta} f_3(s_i; \gamma_{12})}{f_3(s_i; \gamma_{12})} - N_0^{-1} \sum_{i=1}^{n} \frac{[\nabla_{\zeta} f_3(s_i; \gamma_{12})]^2}{f_3(s_i; \gamma_{12})^2},$$

$$\nabla_{\zeta} \nabla_{\zeta} H_m(\zeta_m) = -N_0^{-1} \sum_{i=1}^{n} \frac{\nabla_{\zeta} f_3(s_i; \gamma_{12})}{[1 + \lambda f_3(s_i; \gamma_{12}) - \alpha]^2},$$

$$\nabla_{\zeta} \nabla_{\zeta} H_m(\zeta_m) = N_0^{-1} \sum_{i=1}^{n} \frac{[f_3(s_i; \gamma_{12}) - \alpha]^2}{[1 + \lambda f_3(s_i; \gamma_{12}) - \alpha]^2}.$$
10.2 Variance of $u_m$

We now calculate the elements of $\Sigma = \text{Var}(u_m)$. Direct calculations give

$$\sigma_{11} = \text{Var} \left( N_0^{1/2} \frac{n/N_0 - a_0}{1 - a_0} \right) = \frac{a_0}{1 - a_0},$$

$$\sigma_{12} = \mathbb{E} \left[ N_0^{1/2} \frac{n/N_0 - a_0}{1 - a_0} \frac{1}{f_3(s_i; \gamma_{12, 0})} \left( N_0^{-1/2} N_0 - \frac{n}{1 - a_0} \right) \right]$$

$$= \mathbb{E} \left[ \frac{n/N_0 - a_0}{1 - a_0} \left( \sum_{i=1}^{n} \frac{1}{f_3(s_i; \gamma_{12, 0})} - \frac{N_0 - n}{1 - a_0} \right) \right]$$

$$= \mathbb{E} \left[ \frac{I(Z_i > 0) - a_0}{1 - a_0} \left( \frac{I(Z_i > 0)}{f_3(s_i^*; \gamma_{12, 0})} - \frac{I(Z_i < 0)}{1 - a_0} \right) \right]$$

$$= \mathbb{E} \left[ \frac{I(Z_i > 0)}{f_3(s_i^*; \gamma_{12, 0})} + \mathbb{E} \left[ \frac{a_0 I(Z_i < 0)}{(1 - a_0)^2} \right] \right]$$

$$= 1 + \frac{a_0}{1 - a_0} \frac{1}{1 - a_0}.$$

$$\sigma_{1,34} = 0,$$

$$\sigma_{15} = -\mathbb{E} \left[ N_0^{1/2} \frac{n/N_0 - a_0}{1 - a_0} N_0^{-1/2} \sum_{i=1}^{n} \frac{a_0 f_3(s_i; \gamma_{12, 0}) - a_0}{f_3(s_i; \gamma_{12, 0})} \right]$$

$$= -\mathbb{E} \left[ \frac{n/N_0 - a_0}{1 - a_0} \sum_{i=1}^{n} \frac{a_0 f_3(s_i; \gamma_{12, 0}) - a_0}{f_3(s_i; \gamma_{12, 0})} \right]$$

$$= -\mathbb{E} \left[ \frac{I(Z_i > 0) - a_0}{1 - a_0} \frac{a_0 f_3(s_i^*; \gamma_{12, 0}) - a_0}{f_3(s_i^*; \gamma_{12, 0})} I(Z_i > 0) \right]$$

$$= 0,$$

$$\sigma_{22} = \mathbb{E} \left( \frac{I(Z_i > 0)}{f_3(s_i^*; \gamma_{12, 0})} \frac{I(Z_i \leq 0)}{1 - a_0} \right)^2$$

$$= \mathbb{E} \left( \frac{I(Z_i > 0)}{f_3(s_i^*; \gamma_{12, 0})} \right)^2 + \mathbb{E} \left( \frac{I(Z_i \leq 0)}{1 - a_0} \right)^2$$

$$= \mathbb{E} \left( \frac{1}{f_3(s_i^*; \gamma_{12, 0})} \right)^2 + \frac{1}{1 - a_0}$$

$$= \varphi_1 + \frac{1}{1 - a_0}.$$
\( \sigma_{2,34} = 0, \)

\[
\sigma_{25} = \mathbb{E}\left\{ N_0^{-1/2} \left[ \frac{N - n}{1 - \alpha_0} - N_0^{-1/2} \sum_{i=1}^n \frac{1}{f_3(s_i; \gamma_{12,0})} \right] \cdot N_0^{1/2} \sum_{i=1}^n \frac{\alpha_0[f_3(s_i; \gamma_{12,0}) - \alpha_0]}{f_3(s_i; \gamma_{12,0})} \right\}
\]

\[
= \mathbb{E}\left\{ \left[ I(Z_i \leq 0) - \frac{I(Z_i > 0)}{f_3(s_i^*; \gamma_{12,0})} \right] \frac{I(Z_i > 0)\alpha_0[f_3(s_i^*; \gamma_{12,0}) - \alpha_0]}{f_3(s_i^*; \gamma_{12,0})} \right\}
\]

\[
= -\mathbb{E}\left\{ \frac{\alpha_0^2}{f_3(s_i^*; \gamma_{12,0})} - \alpha_0 \right\}
\]

\[
= \alpha_0^2 \varphi_1 - \alpha_0.
\]

\( \sigma_{34,34} = \sigma_{34,5} = 0, \)

\[
\sigma_{55} = \mathbb{E}\left\{ -N_0^{-1/2} \sum_{i=1}^n \frac{\alpha_0[f_3(s_i; \gamma_{12,0}) - \alpha_0]}{f_3(s_i; \gamma_{12,0})} \right\}^2
\]

\[
= \mathbb{E}\left\{ \alpha_0 I(Z_i > 0)\frac{f_3(s_i^*; \gamma_{12,0}) - \alpha_0}{f_3(s_i^*; \gamma_{12,0})} \right\}^2
\]

\[
= \mathbb{E}\left\{ \frac{\alpha_0^2[f_3(s_i^*; \gamma_{12,0}) - \alpha_0]^2}{f_3(s_i^*; \gamma_{12,0})} \right\}
\]

\[
= \alpha_0^4 \varphi_1 - \alpha_0^3.
\]

In summary we have

\[
\Sigma = \begin{pmatrix}
\alpha_0 & \frac{1}{1 - \alpha_0} & 0 & 0 \\
\frac{1}{1 - \alpha_0} & \varphi_1 + \frac{1}{1 - \alpha_0} & \alpha_0^2 \varphi_1 - \alpha_0 \\
0 & 0 & 0 & 0 \\
0 & \alpha_0^2 \varphi_1 - \alpha_0 & \alpha_0^4 \varphi_1 - \alpha_0^3
\end{pmatrix}.
\]

References