Supplementary material for Empirical likelihood meta analysis with publication bias correction under Copas-like selection model

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Abstract This supplementary material is divided into ten sections. Section 1 reviews Lemma 1, Theorems 1 and 2, and Proposition 1 in the main paper. Section 2 proves Lemma 1 of the main paper. Section 3 proves Theorem 1 of the main paper. The proof in Section 3 is built on two lemmas, Lemmas 2 and 3, whose proofs are given in Sections 4-7. Sections 8 and 9 contain proofs of Theorem 2 and Proposition 1 of the main paper. Some tedious technical derivations are postponed to Section 10.

1 Notation and main results in the main paper

Recall that $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \rho, \tau, \theta)^{\top}$. Hereafter we use $\boldsymbol{\gamma}_{12}$ and $\boldsymbol{\gamma}_{45}$ to denote (γ_1, γ_2) and (τ, θ), respectively. Define

$$f_1(\theta_i, s_i; \boldsymbol{\gamma}) = \text{pr}(Z_i > 0 | \theta_i^* = \theta_i, s_i^* = s_i) = \Phi\{v_i(\boldsymbol{\gamma})\},$$

$$f_2(\theta_i, s_i; \boldsymbol{\gamma}_{45}) = \text{pr}(\theta_i^* = \theta_i | s_i^* = s_i) = \frac{1}{\sqrt{2\pi(\tau^2 + s_i^2)}} \exp\left\{-\frac{(\theta_i - \theta)^2}{2(\tau^2 + s_i^2)}\right\},$$

$$f_3(s_i; \boldsymbol{\gamma}_{12}) = \text{pr}(Z_i > 0 | s_i^* = s_i) = \Phi(\gamma_1 + \gamma_2 / s_i),$$

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where $\Phi(t)$ is the standard normal distribution function and

$$v_i(\boldsymbol{\gamma}) = v(\theta_i, s_i; \boldsymbol{\gamma}) = \frac{\gamma_1 + (\gamma_2/s_i) + \rho s_i(\theta_i - \theta)/(\tau^2 + s_i^2)}{\sqrt{1 - \rho^2 s_i^2/(\tau^2 + s_i^2)}}. \tag{1}$$

The full log-likelihood is

$$\begin{aligned}\ell(N, \alpha, \boldsymbol{\gamma}) = \log\binom{N}{n} + (N-n)\log(1-\alpha) - \sum_{i=1}^{n} \log[1 + \lambda\{f_3(s_i; \boldsymbol{\gamma}_{12}) - \alpha\}] \\ + \sum_{i=1}^{n} \log f_1(\theta_i, s_i; \boldsymbol{\gamma}) + \sum_{i=1}^{n} \log f_2(\theta_i, s_i; \boldsymbol{\gamma}_{45}), \\ = \ell_c(\boldsymbol{\gamma}) + \ell_m(N, \alpha, \boldsymbol{\gamma}_{12}),\end{aligned}$$

where

$$l_c(\boldsymbol{\gamma}) = \sum_{i=1}^{n} \{\log f_1(\theta_i, s_i; \boldsymbol{\gamma}) + \log f_2(\theta_i, s_i; \boldsymbol{\gamma}_{45}) - \log f_3(s_i; \boldsymbol{\gamma}_{12})\} \tag{2}$$

is a conditional likelihood and

$$\begin{aligned}\ell_m(N, \alpha, \boldsymbol{\gamma}_{12}) = \log\binom{N}{n} + (N-n)\log(1-\alpha) - \sum_{i=1}^{n} \log[1 + \lambda\{f_3(s_i; \boldsymbol{\gamma}_{12}) - \alpha\}] \\ + \sum_{i=1}^{n} \log f_3(s_i; \boldsymbol{\gamma}_{12}) \end{aligned} \tag{3}$$

is a marginal likelihood.

Let $(N_0, \alpha_0, \boldsymbol{\gamma}_0)$ be the truth of $(N, \alpha, \boldsymbol{\gamma})$ with $\boldsymbol{\gamma}_0 = (\boldsymbol{\gamma}_{10}, \boldsymbol{\gamma}_{20}, \rho_0, \tau_0, \theta_0)$. Throughout the paper, we use $\boldsymbol{\gamma}_{12,0}$ and $\boldsymbol{\gamma}_{45,0}$ to denote the truths of $\boldsymbol{\gamma}_{12}$ and $\boldsymbol{\gamma}_{45}$. Define $\mathbf{A}^{\otimes 2} = \mathbf{A}\mathbf{A}^\top$ and $\mathbf{A}^{\oplus 2} = \mathbf{A} + \mathbf{A}^\top$ for a matrix $\mathbf{A}$, $\mathbf{E}_{12} = (\mathbf{I}_2, \mathbf{0}_{2\times 3})^\top$, and $\mathbf{E}_{45} = (\mathbf{0}_{2\times 3}, \mathbf{I}_2)^\top$ with $\mathbf{I}_k$ the $k \times k$ identity matrix. We use $\nabla_{\boldsymbol{\gamma}}$ to denote the differentiation operator with respect to $\boldsymbol{\gamma}$. Let $\varphi_1 = \mathbb{E}[\{f_3(s_i^*; \boldsymbol{\gamma}_{12,0})\}^{-1}]$ and $\boldsymbol{m}\varphi_2 = \mathbb{E}\left\{\nabla_{\boldsymbol{\gamma}_{12}} \log f_3(s_i^*; \boldsymbol{\gamma}_{12,0})\right\}$. Define

$$\begin{aligned}\mathbf{V}_c = \mathbb{E}\frac{\{\nabla_{\boldsymbol{\gamma}} f_1(\theta_i^*, s_i^*; \boldsymbol{\gamma}_0)\}^{\otimes 2}}{f_1(\theta_i^*, s_i^*; \boldsymbol{\gamma}_0)} + \mathbb{E}\Big[\mathbf{E}_{45}\int \frac{\{\nabla_{\boldsymbol{\gamma}_{45}} f_2(t, s_i^*; \boldsymbol{\gamma}_{45,0})\}^{\otimes 2}}{f_2(t, s_i^*; \boldsymbol{\gamma}_{45,0})} f_1(t, s_i^*; \boldsymbol{\gamma}_0) dt \mathbf{E}_{45}^\top\Big] \\ + \mathbb{E}\Big[\int \nabla_{\boldsymbol{\gamma}} f_1(t, s_i^*; \boldsymbol{\gamma}_0) \nabla_{\boldsymbol{\gamma}_{45}^\top} f_2(t, s_i^*; \boldsymbol{\gamma}_{45,0}) dt \mathbf{E}_{45}^\top\Big]^{\oplus 2} \\ - \mathbb{E}\Big[\mathbf{E}_{12}\frac{\{\nabla_{\boldsymbol{\gamma}_{12}} f_3(s_i^*; \boldsymbol{\gamma}_{12,0})\}^{\otimes 2}}{f_3(s_i^*; \boldsymbol{\gamma}_{12,0})}\mathbf{E}_{12}^\top\Big],\end{aligned}$$

$$\tilde{\mathbf{V}}_m = \begin{pmatrix} \frac{\alpha_0}{1-\alpha_0} & \frac{1}{1-\alpha_0} & \mathbf{0} \\ \frac{1}{1-\alpha_0} & \frac{1-\varphi_1}{(1-\alpha_0)(1-\alpha_0\varphi_1)} & \frac{\boldsymbol{m}\varphi_2^\top}{1-\alpha_0\varphi_1} \\ \mathbf{0} & \frac{\boldsymbol{m}\varphi_2}{1-\alpha_0\varphi_1} & -\frac{\alpha_0 \boldsymbol{m}\varphi_2^{\otimes 2}}{1-\alpha_0\varphi_1} \end{pmatrix},$$

and

$$\begin{aligned}\mathbf{V}_c(\boldsymbol{\gamma}) = (-1)\mathbb{E}\Big[&\frac{\nabla_{\boldsymbol{\gamma}\boldsymbol{\gamma}^\top} f_1(\theta_i^*, s_i^*; \boldsymbol{\gamma})}{f_1(\theta_i^*, s_i^*; \boldsymbol{\gamma})} f_1(\theta_i^*, s_i^*; \boldsymbol{\gamma}_0) - \Big(\frac{\nabla_{\boldsymbol{\gamma}} f_1(\theta_i^*, s_i^*; \boldsymbol{\gamma})}{f_1(\theta_i^*, s_i^*; \boldsymbol{\gamma})}\Big)^{\otimes 2} f_1(\theta_i^*, s_i^*; \boldsymbol{\gamma}_0) \\ &+\mathbf{E}_{45}\Big\{\frac{\nabla_{\boldsymbol{\gamma}_{45}\boldsymbol{\gamma}_{45}^\top} f_2(\theta_i^*, s_i^*; \boldsymbol{\gamma}_{45})}{f_2(\theta_i^*, s_i^*; \boldsymbol{\gamma}_{45})} - \Big(\frac{\nabla_{\boldsymbol{\gamma}_{45}} f_2(\theta_i^*, s_i^*; \boldsymbol{\gamma}_{45})}{f_2(\theta_i^*, s_i^*; \boldsymbol{\gamma}_{45})}\Big)^{\otimes 2}\Big\}\mathbf{E}_{45}^\top f_1(\theta_i^*, s_i^*; \boldsymbol{\gamma}_0) \\ &-\mathbf{E}_{12}\Big\{\frac{\nabla_{\boldsymbol{\gamma}_{12}\boldsymbol{\gamma}_{12}^\top} f_3(s_i^*; \boldsymbol{\gamma}_{12})}{f_3(s_i^*; \boldsymbol{\gamma}_{12})} - \Big(\frac{\nabla_{\boldsymbol{\gamma}_{12}} f_3(s_i^*; \boldsymbol{\gamma}_{12})}{f_3(s_i^*; \boldsymbol{\gamma}_{12})}\Big)^{\otimes 2}\Big\}\mathbf{E}_{12}^\top f_3(s_i^*; \boldsymbol{\gamma}_{12,0})\Big]. \end{aligned} \quad (4)$$

Furthermore, define

$$\boldsymbol{\Omega} = \mathbf{F}_2^\top \mathbf{V}_c \mathbf{F}_2 + \mathbf{F}_1^\top \tilde{\mathbf{V}}_m \mathbf{F}_1, \quad (5)$$

where $\mathbf{F}_1 = (\mathbf{I}_4, \mathbf{0}_{4\times 3})$ and $\mathbf{F}_2 = (\mathbf{0}_{5\times 2}, \mathbf{I}_5)$.

Lemma 1 *If $\rho \neq 0$, the parameter $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \rho, \tau, \theta)$ is identifiable.*

Condition 1 *The matrix $\mathbf{V}_c(\gamma)$ in (4) is finite and continuous in a neighborhood of $\boldsymbol{\gamma}_0$.*

Condition 2 *Suppose the Hessian matrix of the function H_m defined in the supplementary material is finite and continuous in a neighborhood of the origin.*

Theorem 1 *Assume Conditions 1 and 2, and that the matrix $\boldsymbol{\Omega}$ defined in (5) is positive definite. As $N_0 \to \infty$, the following results hold.*

(1) $N_0^{1/2}(\widehat{N}/N_0 - 1,\ \widehat{\alpha} - \alpha_0, (\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0)^\top) \xrightarrow{d} N(\mathbf{0},\ \boldsymbol{\Omega}^{-1})$, where $\xrightarrow{d}$ stands for convergence in distribution.
(2) $N_0^{1/2}(\widehat{N}/N_0 - 1) \xrightarrow{d} N(0,\ \sigma^2)$, and $N_0^{1/2}(\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0) \xrightarrow{d} N(\mathbf{0},\ \mathbf{V}^{-1})$, where σ^2 is the $(1,1)$ element of $\boldsymbol{\Omega}^{-1}$ and $\mathbf{V}^{-1}$ is the down-right 5×5 submatrix of $\boldsymbol{\Omega}^{-1}$.
(3) The likelihood ratio $R(N_0, \alpha_0, \boldsymbol{\gamma}_0) = 2\{\ell(\widehat{N}, \widehat{\alpha}, \widehat{\boldsymbol{\gamma}}) - \ell(N_0, \alpha_0, \boldsymbol{\gamma}_0)\} \xrightarrow{d} \chi_7^2$.

Theorem 2 *Assume Condition 1, and that V_c is positive definite. Then as $N_0 \to \infty$, (1) $N_0^{1/2}(\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0) \xrightarrow{d} N(\mathbf{0}, \mathbf{V}_c^{-1})$, and (2) $N_0^{1/2}(\tilde{N}/N_0 - 1) \xrightarrow{d} N(0, \sigma_c^2)$, where $\sigma_c^2 = \varphi_1 - 1 + \boldsymbol{m}\varphi_2^\top \mathbf{E}_{12} \mathbf{V}_c^{-1} \mathbf{E}_{12}^\top \boldsymbol{m}\varphi_2$.*

Proposition 1 *With the symbols used in Theorem 1 and Theorem 2 in the main paper, $\sigma^2 = \sigma_c^2$ and $\mathbf{V} = \mathbf{V}_c$.*

2 Proof of Lemma 1

Denote θ_i by y_i. We note that the observations (y_i, s_i) has the same distribution as (θ_i^*, s_i^*) given $Z_i > 0$, and its density function is

$$\begin{aligned} g(y, s; \boldsymbol{\gamma}) &= \frac{f_1(y, s; \boldsymbol{\gamma}) f_2(y, s; \boldsymbol{\gamma}_{45})}{f_3(y, s; \boldsymbol{\gamma}_{12})} \\ &= \frac{1}{\sqrt{2\pi(\tau^2 + s^2)}} \frac{\exp\left\{-\frac{(y-\theta)^2}{2(\tau^2+s^2)}\right\}}{\Phi(\gamma_1 + \gamma_2/s)} \cdot \Phi\Big\{\frac{\gamma_1 + (\gamma_2/s) + \rho s(y-\theta)/(\tau^2+s^2)}{\sqrt{1 - \rho^2 s^2/(\tau^2+s^2)}}\Big\}. \end{aligned}$$

To prove this lemma, it suffices to show that when $\rho \neq 0$, the fact the equality $g(y, s; \boldsymbol{\gamma}_a) = g(y, s; \boldsymbol{\gamma}_b)$ for all (y, s) satisfying $s > 0$ implies $\boldsymbol{\gamma}_a = \boldsymbol{\gamma}_b$, where $\boldsymbol{\gamma}_a = (\gamma_{1a}, \gamma_{2a}, \rho_a, \tau_a, \theta_a)$ and $\boldsymbol{\gamma}_b = (\gamma_{1b}, \gamma_{2b}, \rho_b, \tau_b, \theta_b)$.

We consider the following special cases: Case I. For fixed y, letting $s \to 0+$, we have

$$g(y, s; \boldsymbol{\gamma}) = \frac{1}{\sqrt{2\pi\tau^2}} \exp\Big\{ -\frac{(y-\theta)^2}{2\tau^2}\Big\}\{1 + o(1)\}.$$

The limit distribution belongs to the normal distribution family, where it is well known that the mean and variance parameters θ and τ ($\tau > 0$) are identifiable. In other words, the equation $g(y, s; \boldsymbol{\gamma}_a) = g(y, s; \boldsymbol{\gamma}_b)$ for all (y, s) such that $s > 0$ implies $(\theta_a = \theta_b)$ and $\tau_a = \tau_b$.

Case II. Because θ is identifiable, we assume $\theta_a = \theta_b = \theta$. For $y = \theta + \xi s$ and any fixed ξ, when s is large, we have

$$g(\theta + \xi s, s; \boldsymbol{\gamma}) = \frac{1}{\sqrt{2\pi}} \frac{e^{-\xi^2/2}}{\Phi(\gamma_1)} \cdot \Phi\Big(\frac{\gamma_1 + \xi\rho}{\sqrt{1-\rho^2}}\Big) \cdot s^{-1}\{1 + o(1)\}.$$

It follows that

$$1 = \lim_{s\to\infty} \frac{g(\theta + \xi s, s; \boldsymbol{\gamma}_a)}{g(\theta + \xi s, s; \boldsymbol{\gamma}_b)} = \frac{\frac{1}{\Phi(\gamma_{1a})} \cdot \Phi\Big(\frac{\gamma_{1a}+\xi\rho_a}{\sqrt{1-\rho_a^2}}\Big)}{\frac{1}{\Phi(\gamma_{1b})} \cdot \Phi\Big(\frac{\gamma_{1b}+\xi\rho_b}{\sqrt{1-\rho_b^2}}\Big)},$$

which means

$$\frac{\Phi(\gamma_{1b})}{\Phi(\gamma_{1a})} = \frac{\cdot\Phi\Big(\frac{\gamma_{1b}+\xi\rho_b}{\sqrt{1-\rho_b^2}}\Big)}{\Phi\Big(\frac{\gamma_{1a}+\xi\rho_a}{\sqrt{1-\rho_a^2}}\Big)}$$

holds for any fixed ξ. When $\rho_a \neq 0$ and $\rho_b \neq 0$, the left-hand side of the above equation does not depend on ξ, while the right-hand side does, this indicates that $\rho_a = \rho_b$ and $\gamma_{1a} = \gamma_{1b}$ must hold simultaneously.

Case III. Because all parameters except γ_2 are identifiable, we assume that $\boldsymbol{\gamma}_a = (\gamma_{1a}, \rho_a, \tau_a, \theta_a) = (\gamma_{1b}, \rho_b, \tau_b, \theta_b) = (\gamma_1, \rho, \tau, \theta)$. Because

$$1 = \frac{g(y, s; \boldsymbol{\gamma}_a)}{g(y, s; \boldsymbol{\gamma}_b)} = \frac{\frac{1}{\Phi(\gamma_1+\gamma_{2a}/s)} \cdot \Phi\Big\{\frac{\gamma_1+(\gamma_{2a}/s)+\rho s(y-\theta)/(\tau^2+s^2)}{\sqrt{1-\rho^2 s^2/(\tau^2+s^2)}}\Big\}}{\frac{1}{\Phi(\gamma_1+\gamma_{2b}/s)} \cdot \Phi\Big\{\frac{\gamma_1+(\gamma_{2b}/s)+\rho s(y-\theta)/(\tau^2+s^2)}{\sqrt{1-\rho^2 s^2/(\tau^2+s^2)}}\Big\}},$$

we have

$$\frac{\Phi(\gamma_1 + \gamma_{2b}/s)}{\Phi(\gamma_1 + \gamma_{2a}/s)} = \frac{\Phi\Big\{\frac{\gamma_1+(\gamma_{2b}/s)+\rho s(y-\theta)/(\tau^2+s^2)}{\sqrt{1-\rho^2 s^2/(\tau^2+s^2)}}\Big\}}{\Phi\Big\{\frac{\gamma_1+(\gamma_{2a}/s)+\rho s(y-\theta)/(\tau^2+s^2)}{\sqrt{1-\rho^2 s^2/(\tau^2+s^2)}}\Big\}}.$$

When $\rho \neq 0$, the left-hand side is independent of y, while the right-hand side is a function of y, indicating that $\gamma_{2a} = \gamma_{2b}$ must hold. This proves Lemma 1.

3 Proof of Theorem 1

With similar arguments to the proof of Lemma 1 of Qin and Lawless (1994), it can be proved that the maximum likelihood estimators(MLE) $(\widehat{N}, \widehat{\alpha}, \widehat{\boldsymbol{\gamma}})$ satisfies $(\widehat{N}/N_0 - 1, \widehat{\alpha} - \alpha_0, \widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0) = O_p(N_0^{-1/2})$ and that the maximum conditional likelihood estimators satisfies $\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0 = O_p(N_0^{-1/2})$. To prove Theorems 1 and 2, and Proposition 1 in the main paper, we begin by studying the behaviors of $\ell(N, \alpha, \boldsymbol{\gamma})$, $l_c(\boldsymbol{\gamma})$, and $\ell_m(N, \alpha, \boldsymbol{\gamma}_{12})$ for $(N, \alpha, \boldsymbol{\gamma})$ satisfying $(N/N_0 - 1, \alpha - \alpha_0, \boldsymbol{\gamma} - \boldsymbol{\gamma}_0) = O_p(N_0^{-1/2})$.

Define

$$\mathbf{u}_c = N_0^{-1/2} \sum_{i=1}^{n} \Big\{ \frac{\nabla_{\boldsymbol{\gamma}} f_1(\theta_i, s_i; \boldsymbol{\gamma}_0)}{f_1(\theta_i, s_i; \boldsymbol{\gamma}_0)} + \mathbf{E}_{45} \frac{\nabla_{\boldsymbol{\gamma}_{45}} f_2(\theta_i, s_i; \boldsymbol{\gamma}_{45,0})}{f_2(\theta_i, s_i; \boldsymbol{\gamma}_{45,0})} - \mathbf{E}_{12} \frac{\nabla_{\boldsymbol{\gamma}_{12}} f_3(s_i; \boldsymbol{\gamma}_{12})}{f_3(s_i; \boldsymbol{\gamma}_{12})} \Big\}.$$

The following lemma discloses the asymptotical behaviors of the conditional likelihood $\ell_c(\boldsymbol{\gamma})$. The results in this lemma are proved in Sections 4 and 5, respectively.

Lemma 2 *Assume Condition 1. Let* $\boldsymbol{\zeta}_c = N_0^{1/2}(\boldsymbol{\gamma} - \boldsymbol{\gamma}_0)$. (I) *For* $\boldsymbol{\gamma} = \boldsymbol{\gamma}_0 + O_p(N_0^{-1/2})$,

$$\ell_c(\boldsymbol{\gamma}) = \ell_c(\boldsymbol{\gamma}_0) + \mathbf{u}_c^\top \boldsymbol{\zeta}_c - \frac{1}{2} \boldsymbol{\zeta}_c^\top \mathbf{V}_c \boldsymbol{\zeta}_c + o_p(\|\boldsymbol{\zeta}_c\|^2).$$

(II) $\mathbb{V}\mathrm{ar}(\mathbf{u}_c) = \mathbf{V}_c$.

Define $\mathbf{u}_{ma} = (u_{m1}, u_{m2}, \mathbf{u}_{m34}^\top)^\top$, where

$$u_{m1} = N_0^{1/2} \frac{n/N_0 - \alpha_0}{1 - \alpha_0}, \quad u_{m2} = -N_0^{-1/2} \frac{N_0 - n}{1 - \alpha_0} + N_0^{-1/2} \sum_{i=1}^{n} \frac{1}{f_3(s_i; \boldsymbol{\gamma}_{12,0})}, \quad \mathbf{u}_{m34} = \mathbf{0}$$

and

$$u_{mb} = -N_0^{-1/2} \sum_{i=1}^{n} \frac{\alpha_0 \{ f_3(s_i; \boldsymbol{\gamma}_{12,0}) - \alpha_0 \}}{f_3(s_i; \boldsymbol{\gamma}_{12,0})}.$$

Define

$$\mathbf{V}_{maa} = \begin{pmatrix} \frac{\alpha_0}{1-\alpha_0} & \frac{1}{1-\alpha_0} & \mathbf{0} \\ \frac{1}{1-\alpha_0} & \frac{1}{1-\alpha_0} - \varphi_1 & \boldsymbol{m}\varphi_2^\top \\ \mathbf{0} & \boldsymbol{m}\varphi_2 & \mathbf{0} \end{pmatrix}, \quad \mathbf{V}_{mab} = \begin{pmatrix} \mathbf{0} \\ -\alpha_0^2 \varphi_1 \\ \alpha_0^2 \boldsymbol{m}\varphi_2 \end{pmatrix}, \quad V_{mbb} = \alpha_0^3 - \alpha_0^4 \varphi_1.$$

The next lemma, which are proved in Sections 6 and 7, studies the asymptotical behaviors of the marginal likelihood $\ell_m(N, \alpha, \boldsymbol{\gamma}_{12})$.

Lemma 3 *Assume Condition 2. Let* $\boldsymbol{\zeta}_m = N_0^{1/2}((N/N_0 - 1), (\alpha - \alpha_0), (\boldsymbol{\gamma}_{12} - \boldsymbol{\gamma}_{12,0}))^\top$. (I) *If* $(N, \alpha, \boldsymbol{\gamma}_{12})$ *satisfies* $(N/N_0 - 1, \alpha - \alpha_0, \boldsymbol{\gamma}_{12} - \boldsymbol{\gamma}_{12,0}) = O_p(N_0^{-1/2})$*, then*

$$\ell_m(N, \alpha, \boldsymbol{\gamma}_{12}) = \ell_m(N_0, \alpha_0, \boldsymbol{\gamma}_{12,0}) + \tilde{\mathbf{u}}_m^\top \boldsymbol{\zeta}_m - \frac{1}{2} \boldsymbol{\zeta}_m^\top \tilde{\mathbf{V}}_m \boldsymbol{\zeta}_m + o_p(\|\boldsymbol{\zeta}_m\|^2)$$

where $\tilde{\mathbf{u}}_m = \mathbf{u}_{ma} - \mathbf{V}_{mab} \mathbf{V}_{mbb}^{-1} \mathbf{u}_{mb}$ *and* $\tilde{\mathbf{V}}_m = \mathbf{V}_{maa} - \mathbf{V}_{mab} \mathbf{V}_{mbb}^{-1} \mathbf{V}_{mba}$. (II) $\mathbb{V}\mathrm{ar}(\tilde{\mathbf{u}}_m) = \tilde{\mathbf{V}}_m$.

It can be found that

$$\tilde{\mathbf{V}}_m = \mathbf{V}_{maa} - \mathbf{V}_{mab}\mathbf{V}_{mbb}^{-1}\mathbf{V}_{mba} = \begin{pmatrix} \frac{\alpha_0}{1-\alpha_0} & \frac{1}{1-\alpha_0} & \mathbf{0} \\ \frac{1}{1-\alpha_0} & \frac{1-\varphi_1}{(1-\alpha_0)(1-\alpha_0\varphi_1)} & \frac{\boldsymbol{m}_{\varphi_2}^\top}{1-\alpha_0\varphi_1} \\ \mathbf{0} & \frac{\boldsymbol{m}_{\varphi_2}}{1-\alpha_0\varphi_1} & -\frac{\alpha_0\boldsymbol{m}_{\varphi_2}^{\otimes 2}}{1-\alpha_0\varphi_1} \end{pmatrix}.$$

We note that the matrix $\tilde{\mathbf{V}}_m$ is singular and its rank is at most 3. Hence it is infeasible to obtain a reasonable estimator for N by directly maximizing ℓ_m.

To study the behavior of the full log-likelihood $\ell(N, \alpha, \boldsymbol{\gamma})$, we need to merge the approximates of ℓ_c and ℓ_m. Let $\boldsymbol{\zeta} = N_0^{1/2}((N/N_0 - 1), (\alpha - \alpha_0), (\boldsymbol{\gamma} - \boldsymbol{\gamma}_0)^\top)^\top$, and $\mathbf{F}_1 = (\mathbf{I}_4, \mathbf{0}_{4\times 3})$ and $\mathbf{F}_2 = (\mathbf{0}_{5\times 2}, \mathbf{I}_5)$. Clearly $\boldsymbol{\zeta}_c = \mathbf{F}_2\boldsymbol{\zeta}$ and $\boldsymbol{\zeta}_m = \mathbf{F}_1\boldsymbol{\zeta}$. Since

$$\ell_c(\boldsymbol{\gamma}) = \ell_c(\boldsymbol{\gamma}_0) + \mathbf{u}_c^\top\mathbf{F}_2\boldsymbol{\zeta} - \frac{1}{2}\boldsymbol{\zeta}^\top\mathbf{F}_2^\top\mathbf{V}_c\mathbf{F}_2\boldsymbol{\zeta} + o_p(\|\boldsymbol{\zeta}\|^2),$$

$$\ell_m(N, \alpha, \boldsymbol{\gamma}_{12}) = \ell_m(N_0, \alpha_0, \boldsymbol{\gamma}_{12,0}) + \tilde{\mathbf{u}}_m^\top\mathbf{F}_1\boldsymbol{\zeta} - \frac{1}{2}\boldsymbol{\zeta}^\top\mathbf{F}_1^\top\tilde{\mathbf{V}}_m\mathbf{F}_1\boldsymbol{\zeta} + o_p(\|\boldsymbol{\zeta}\|^2),$$

it follows that

$$\ell(N, \alpha, \boldsymbol{\gamma}) = \ell(N_0, \alpha_0, \boldsymbol{\gamma}_0) + (\mathbf{u}_c^\top\mathbf{F}_2 + \tilde{\mathbf{u}}_m^\top\mathbf{F}_1)\boldsymbol{\zeta} - \frac{1}{2}\boldsymbol{\zeta}^\top\boldsymbol{\Omega}\boldsymbol{\zeta} + o_p(\|\boldsymbol{\zeta}\|^2).$$

Because $\mathbb{E}\{\mathbf{u}_c|(s_i^*, Z_i > 0) : i = 1, 2, \ldots N_0\} = 0$ and $\mathbf{u}_m$ depends only on $\{(s_i^*, Z_i > 0) : i = 1, 2, \ldots N_0\}$, we have

$$\mathbb{C}\text{ov}(\mathbf{u}_c, \tilde{\mathbf{u}}_m) = \mathbb{E}(\mathbf{u}_c\tilde{\mathbf{u}}_m^\top) = \mathbf{0}$$

and

$$\mathbb{V}\text{ar}(\mathbf{u}_c^\top\mathbf{F}_2 + \tilde{\mathbf{u}}_m^\top\mathbf{F}_1) = \mathbf{F}_2^\top\mathbb{V}\text{ar}(\mathbf{u}_c)\mathbf{F}_2 + \mathbf{F}_1^\top\mathbb{V}\text{ar}(\tilde{\mathbf{u}}_m)\mathbf{F}_1 = \mathbf{F}_2^\top\mathbf{V}_c\mathbf{F}_2 + \mathbf{F}_1^\top\tilde{\mathbf{V}}_m\mathbf{F}_1 = \boldsymbol{\Omega},$$

where we have used Lemmas 2 and 3. Therefore the maximum likelihood estimators $(\widehat{N}, \widehat{\alpha}, \widehat{\boldsymbol{\gamma}})$ satisfies

$$\begin{aligned} N_0^{1/2}(\widehat{N}/N_0 - 1,\ \widehat{\alpha} - \alpha_0, (\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0)^\top) &= \boldsymbol{\Omega}^{-1}(\mathbf{u}_c^\top\mathbf{F}_2 + \tilde{\mathbf{u}}_m^\top\mathbf{F}_1) + o_p(1) \\ &\xrightarrow{d} N(\mathbf{0},\ \boldsymbol{\Omega}^{-1}), \end{aligned}$$

since $\boldsymbol{\Omega}$ is positive definite. Accordingly the likelihood ratio

$$R(N_0, \alpha_0, \boldsymbol{\gamma}_0) = 2\{\ell(\widehat{N}, \widehat{\alpha}, \widehat{\boldsymbol{\gamma}}) - \ell(N_0, \alpha_0, \boldsymbol{\gamma}_0)\} \xrightarrow{d} \chi_7^2.$$

This proves results (1) and (3) in Theorem 1 of the main paper. Result (2) is implied by Result (1) of Theorem 1. This finishes proving Theorem 1.

4 Proof of Lemma 2: Result (I)

4.1 Preparations

Rewrite the conditional log-likelihood as

$$l_c(\boldsymbol{\gamma}) = \sum_{i=1}^{n}\{\log f_1(\theta_i, s_i; \boldsymbol{\gamma}) + \log f_2(\theta_i, s_i; \boldsymbol{\gamma}_{45}) - \log f_3(s_i; \boldsymbol{\gamma}_{12})\}. \tag{6}$$

A second-order Taylor expansion of $l_c(\boldsymbol{\gamma})$ at $\boldsymbol{\gamma}_0$ gives

$$l_c(\boldsymbol{\gamma}) = l_c(\boldsymbol{\gamma}_0) + \{\nabla_{\boldsymbol{\gamma}} \ell_c(\boldsymbol{\gamma}_0)\}^{\top}(\boldsymbol{\gamma} - \boldsymbol{\gamma}_0) + \frac{1}{2}(\boldsymbol{\gamma} - \boldsymbol{\gamma}_0)^{\top}\nabla_{\boldsymbol{\gamma}\boldsymbol{\gamma}^{\top}} \ell_c(\boldsymbol{\gamma}_0)(\boldsymbol{\gamma} - \boldsymbol{\gamma}_0) + O(N_0\|(\boldsymbol{\gamma} - \boldsymbol{\gamma}_0)\|^3)$$

Direct calculations give

$$\begin{aligned}
\nabla_{\boldsymbol{\gamma}} l_c(\boldsymbol{\gamma}) &= \sum_{i=1}^{n}\Big\{\frac{\nabla_{\boldsymbol{\gamma}} f_1(\theta_i, s_i; \boldsymbol{\gamma})}{f_1(\theta_i, s_i; \boldsymbol{\gamma})} + \mathbf{E}_{45}\frac{\nabla_{\boldsymbol{\gamma}_{45}} f_2(\theta_i, s_i; \boldsymbol{\gamma}_{45})}{f_2(\theta_i, s_i; \boldsymbol{\gamma}_{45})} - \mathbf{E}_{12}\frac{\nabla_{\boldsymbol{\gamma}_{12}} f_3(s_i; \boldsymbol{\gamma}_{12})}{f_3(s_i; \boldsymbol{\gamma}_{12})}\Big\},\\
\nabla_{\boldsymbol{\gamma}\boldsymbol{\gamma}^{\top}} l_c(\boldsymbol{\gamma}) &= \sum_{i=1}^{n}\Big[\frac{\nabla_{\boldsymbol{\gamma}\boldsymbol{\gamma}^{\top}} f_1(\theta_i, s_i; \boldsymbol{\gamma})}{f_1(\theta_i, s_i; \boldsymbol{\gamma})} - \Big(\frac{\nabla_{\boldsymbol{\gamma}} f_1(\theta_i, s_i; \boldsymbol{\gamma})}{f_1(\theta_i, s_i; \boldsymbol{\gamma})}\Big)^{\otimes 2}\\
&\quad + \mathbf{E}_{45}\Big\{\frac{\nabla_{\boldsymbol{\gamma}_{45}\boldsymbol{\gamma}_{45}^{\top}} f_2(\theta_i, s_i; \boldsymbol{\gamma}_{45})}{f_2(\theta_i, s_i; \boldsymbol{\gamma}_{45})} - \Big(\frac{\nabla_{\boldsymbol{\gamma}_{45}} f_2(\theta_i, s_i; \boldsymbol{\gamma}_{45})}{f_2(\theta_i, s_i; \boldsymbol{\gamma}_{45})}\Big)^{\otimes 2}\Big\}\mathbf{E}_{45}^{\top}\\
&\quad - \mathbf{E}_{12}\Big\{\frac{\nabla_{\boldsymbol{\gamma}_{12}\boldsymbol{\gamma}_{12}^{\top}} f_3(s_i; \boldsymbol{\gamma}_{12})}{f_3(s_i; \boldsymbol{\gamma}_{12})} - \Big(\frac{\nabla_{\boldsymbol{\gamma}_{12}} f_3(s_i; \boldsymbol{\gamma}_{12})}{f_3(s_i; \boldsymbol{\gamma}_{12})}\Big)^{\otimes 2}\Big\}\mathbf{E}_{12}^{\top}\Big].
\end{aligned}$$

To simply the above expressions we need to discuss properties of the data.

Suppose the results of the total N study are (θ_i^*, s_i^*) $(i = 1, 2, \ldots, N)$, each accompanied with a random indicator Z_i. The results (θ_i^*, s_i^*)'s with $Z_i > 0$ are denoted by (θ_i, s_i)'s. Hence $(\theta_i, s_i) \overset{d}{=} \{(\theta_i^*, s_i^*)|Z_i > 0\}$, where $\overset{d}{=}$ means "having the same distribution as". Thus for any function g,

$$\sum_{i=1}^{n} g(\theta_i, s_i) = \sum_{i=1}^{N_0} g(\theta_i^*, s_i^*) I(Z_i > 0),$$

which implies

$$\mathbb{E}\{\sum_{i=1}^{n} g(\theta_i, s_i)|(\theta_i^*, s_i^*), i = 1, \ldots, N_0\} = \sum_{i=1}^{N_0} g(\theta_i^*, s_i^*) f_1(\theta_i^*, s_i^*; \boldsymbol{\gamma}_0), \tag{7}$$

$$\mathbb{E}\{\sum_{i=1}^{n} g(\theta_i, s_i)|s_i^*, i = 1, \ldots, N_0\} = \sum_{i=1}^{N_0} g(\theta_i^*, s_i^*) f_3(s_i^*; \boldsymbol{\gamma}_{12,0}), \tag{8}$$

$$\mathbb{E}\{\sum_{i=1}^{n} g(\theta_i, s_i)\} = N_0\mathbb{E}\{g(\theta_i^*, s_i^*) f_3(s_i^*; \boldsymbol{\gamma}_{12,0})\}. \tag{9}$$

Another important key equality is $\mathbb{E}\{f_1(\theta_i^*, s_i^*; \boldsymbol{\gamma})|s_i^*\} = f_3(s_i^*; \boldsymbol{\gamma}_{12})$ or

$$0 = \int f_1(\theta, s_i^*; \boldsymbol{\gamma}) f_2(\theta, s_i^*; \boldsymbol{\gamma}_{45}) d\theta - f_3(s_i^*; \boldsymbol{\gamma}_{12}), \tag{10}$$

which holds for any parameter value in the parameter space and for any $s_i^* > 0$. Taking derivatives on both sides of (10) once gives

$$\begin{aligned}\mathbf{0} = &\int \nabla_{\boldsymbol{\gamma}} f_1(\theta, s_i^*; \boldsymbol{\gamma}) f_2(\theta, s_i^*; \boldsymbol{\gamma}_{45}) d\theta + \mathbf{E}_{45} \int f_1(\theta, s_i^*; \boldsymbol{\gamma}) \nabla_{\boldsymbol{\gamma}_{45}} f_2(\theta, s_i^*; \boldsymbol{\gamma}_{45}) d\theta \\ &- \mathbf{E}_{12} \nabla_{\boldsymbol{\gamma}_{12}} f_3(s_i^*; \boldsymbol{\gamma}_{12}).\end{aligned} \tag{11}$$

Similarly, taking derivatives on both sides twice gives

$$\begin{aligned}\mathbf{0} = &\int \nabla_{\boldsymbol{\gamma}\boldsymbol{\gamma}^\top} f_1(\theta, s_i^*; \boldsymbol{\gamma}) f_2(\theta, s_i^*; \boldsymbol{\gamma}_{45}) d\theta + \left\{ \int \nabla_{\boldsymbol{\gamma}} f_1(\theta, s_i^*; \boldsymbol{\gamma}) \nabla_{\boldsymbol{\gamma}_{45}^\top} f_2(\theta, s_i^*; \boldsymbol{\gamma}_{45}) d\theta \mathbf{E}_{45}^\top \right\}^{\oplus 2} \\ &+ \mathbf{E}_{45} \int f_1(\theta, s_i^*; \boldsymbol{\gamma}) \nabla_{\boldsymbol{\gamma}_{45}\boldsymbol{\gamma}_{45}^\top} f_2(\theta, s_i^*; \boldsymbol{\gamma}_{45}) d\theta \mathbf{E}_{45}^\top - \mathbf{E}_{12} \nabla_{\boldsymbol{\gamma}_{12}\boldsymbol{\gamma}_{12}^\top} f_3(s_i^*; \boldsymbol{\gamma}_{12}) \mathbf{E}_{12}^\top.\end{aligned} \tag{12}$$

4.2 Matrix $\mathbf{V}_c$

It can be verified that the leading term of $-(1/N_0)\nabla_{\boldsymbol{\gamma}\boldsymbol{\gamma}^\top} \ell_c(\boldsymbol{\gamma})$ is equal to $\mathbf{V}_c(\boldsymbol{\gamma})$, which is defined in (4).

When $\boldsymbol{\gamma} = \boldsymbol{\gamma}_0$, by Equations (10) and (12), we have

$$\begin{aligned}\mathbf{V}_c &= \mathbf{V}_c(\boldsymbol{\gamma}_0) \\ &= \mathbb{E} \int \frac{\left(\nabla_{\boldsymbol{\gamma}} f_1(t, s_i^*; \boldsymbol{\gamma}_0)\right)^{\otimes 2}}{f_1(t, s_i^*; \boldsymbol{\gamma}_0)} f_2(t, s_i^*; \boldsymbol{\gamma}_{45,0}) dt \\ &\quad + \mathbf{E}_{45} \mathbb{E} \int \frac{\left(\nabla_{\boldsymbol{\gamma}_{45,0}} f_2(t, s_i^*; \boldsymbol{\gamma}_{45,0})\right)^{\otimes 2}}{f_2(t, s_i^*; \boldsymbol{\gamma}_{45,0})} f_1(t, s_i^*; \boldsymbol{\gamma}_0) dt \mathbf{E}_{45}^\top \\ &\quad - \mathbf{E}_{12} \mathbb{E} \frac{\left(\nabla_{\boldsymbol{\gamma}_{12}} f_3(s_i^*; \boldsymbol{\gamma}_{12,0})\right)^{\otimes 2}}{f_3(s_i^*; \boldsymbol{\gamma}_{12,0})} \mathbf{E}_{12}^\top \\ &\quad + \{\mathbb{E} \int \nabla_{\boldsymbol{\gamma}} f_1(\theta, s_i^*; \boldsymbol{\gamma}_0) \nabla_{\boldsymbol{\gamma}_{45}^\top} f_2(\theta, s_i^*; \boldsymbol{\gamma}_{45,0}) d\theta \mathbf{E}_{45}^\top\}^{\oplus 2}.\end{aligned}$$

Denote

$$\begin{aligned}\mathbf{u}_c &= N_0^{-1/2} \nabla_{\boldsymbol{\gamma}} \ell_c(\boldsymbol{\gamma}_0) \\ &= N_0^{-1/2} \sum_{i=1}^{n} \left\{ \frac{\nabla_{\boldsymbol{\gamma}} f_1(\theta_i, s_i; \boldsymbol{\gamma}_0)}{f_1(\theta_i, s_i; \boldsymbol{\gamma}_0)} + \mathbf{E}_{45} \frac{\nabla_{\boldsymbol{\gamma}_{45}} f_2(\theta_i, s_i; \boldsymbol{\gamma}_{45,0})}{f_2(\theta_i, s_i; \boldsymbol{\gamma}_{45,0})} - \mathbf{E}_{12} \frac{\nabla_{\boldsymbol{\gamma}_{12}} f_3(s_i; \boldsymbol{\gamma}_{12})}{f_3(s_i; \boldsymbol{\gamma}_{12})} \right\} \\ &= N_0^{-1/2} \sum_{i=1}^{N_0} \left\{ \frac{\nabla_{\boldsymbol{\gamma}} f_1(\theta_i, s_i; \boldsymbol{\gamma}_0)}{f_1(\theta_i, s_i; \boldsymbol{\gamma}_0)} + \mathbf{E}_{45} \frac{\nabla_{\boldsymbol{\gamma}_{45}} f_2(\theta_i, s_i; \boldsymbol{\gamma}_{45,0})}{f_2(\theta_i, s_i; \boldsymbol{\gamma}_{45,0})} \right. \\ &\quad \left. - \mathbf{E}_{12} \frac{\nabla_{\boldsymbol{\gamma}_{12}} f_3(s_i; \boldsymbol{\gamma}_{12,0})}{f_3(s_i; \boldsymbol{\gamma}_{12,0})} \right\} I(Z_i > 0).\end{aligned}$$

Then we have

$$
\begin{aligned}
l_c(\boldsymbol{\gamma}) &= l_c(\boldsymbol{\gamma}_0) + \{\nabla_{\boldsymbol{\gamma}} \ell_c(\boldsymbol{\gamma}_0)\}^\top (\boldsymbol{\gamma} - \boldsymbol{\gamma}_0) + \frac{1}{2}(\boldsymbol{\gamma} - \boldsymbol{\gamma}_0)^\top \nabla_{\boldsymbol{\gamma}\boldsymbol{\gamma}^\top} \ell_c(\boldsymbol{\gamma}_0)(\boldsymbol{\gamma} - \boldsymbol{\gamma}_0) \\
&\quad + O(N_0 \|(\boldsymbol{\gamma} - \boldsymbol{\gamma}_0)\|^3) \\
&= l_c(\boldsymbol{\gamma}_0) + \mathbf{u}_c^\top \boldsymbol{\zeta}_c + \frac{1}{2}\boldsymbol{\zeta}_c^\top \mathbf{V}_c \boldsymbol{\zeta}_c + o(N_0 \|\boldsymbol{\zeta}_c\|^2),
\end{aligned}
$$

where $\mathbf{u}_c = N_0^{-1/2} \nabla_{\boldsymbol{\gamma}} \ell_c(\boldsymbol{\gamma}_0)$ and $\mathbf{V}_c = N_0^{-1} \nabla_{\boldsymbol{\gamma}\boldsymbol{\gamma}^\top} \ell_c(\boldsymbol{\gamma}_0)$. This proves Result (I) of Lemma 2.

5 Proof of Lemma 2: Result (II)

It follows directly from Equation (10) that $\mathbb{E}(\mathbf{u}_c) = 0$. The variance of $\mathbf{u}_c$ is

$$
\begin{aligned}
\mathbb{V}\mathrm{ar}(\mathbf{u}_c) &= \frac{1}{N_0} \mathbb{E}\{(\nabla_{\boldsymbol{\gamma}} \ell_c(\boldsymbol{\gamma}_0))^{\otimes 2}\} \\
&= \mathbb{E}\Big[\Big\{\frac{\nabla_{\boldsymbol{\gamma}} f_1(\theta_i^*, s_i^*; \boldsymbol{\gamma}_0)}{f_1(\theta_i^*, s_i^*; \boldsymbol{\gamma}_0)} + \mathbf{E}_{45} \frac{\nabla_{\boldsymbol{\gamma}_{45}} f_2(\theta_i^*, s_i^*; \boldsymbol{\gamma}_{45,0})}{f_2(\theta_i^*, s_i^*; \boldsymbol{\gamma}_{45,0})} - \mathbf{E}_{12} \frac{\nabla_{\boldsymbol{\gamma}_{12}} f_3(s_i^*; \boldsymbol{\gamma}_{12,0})}{f_3(s_i^*; \boldsymbol{\gamma}_{12,0})}\Big\}^{\otimes 2} I(Z_i > 0)\Big] \\
&= \mathbb{E}\Big[\Big\{\frac{\nabla_{\boldsymbol{\gamma}} f_1(\theta_i^*, s_i^*; \boldsymbol{\gamma}_0)}{f_1(\theta_i^*, s_i^*; \boldsymbol{\gamma}_0)}\Big\}^{\otimes 2} I(Z_i > 0)\Big] + \mathbb{E}\Big[\Big\{\mathbf{E}_{45} \frac{\nabla_{\boldsymbol{\gamma}_{45}} f_2(\theta_i^*, s_i^*; \boldsymbol{\gamma}_{45,0})}{f_2(\theta_i^*, s_i^*; \boldsymbol{\gamma}_{45,0})}\Big\}^{\otimes 2} I(Z_i > 0)\Big] \\
&\quad + \mathbb{E}\Big[\Big\{\mathbf{E}_{12} \frac{\nabla_{\boldsymbol{\gamma}_{12,0}} f_3(s_i^*; \boldsymbol{\gamma}_{12,0})}{f_3(s_i^*; \boldsymbol{\gamma}_{12,0})}\Big\}^{\otimes 2} I(Z_i > 0)\Big] \\
&\quad + \mathbb{E}\Big[\Big\{\frac{\nabla_{\boldsymbol{\gamma}} f_1(\theta_i^*, s_i^*; \boldsymbol{\gamma}_0)}{f_1(\theta_i^*, s_i^*; \boldsymbol{\gamma}_0)} \mathbf{E}_{45} \frac{\nabla_{\boldsymbol{\gamma}_{45,0}} f_2(\theta_i^*, s_i^*; \boldsymbol{\gamma}_{45,0})}{f_2(\theta_i^*, s_i^*; \boldsymbol{\gamma}_{45,0})}\Big\} I(Z_i > 0)\Big]^{\oplus 2} \\
&\quad - \mathbb{E}\Big[\Big\{\frac{\nabla_{\boldsymbol{\gamma}} f_1(\theta_i^*, s_i^*; \boldsymbol{\gamma}_0)}{f_1(\theta_i^*, s_i^*; \boldsymbol{\gamma}_0)} \mathbf{E}_{12} \frac{\nabla_{\boldsymbol{\gamma}_{12}} f_3(s_i^*; \boldsymbol{\gamma}_{12,0})}{f_3(s_i^*; \boldsymbol{\gamma}_{12,0})}\Big\} I(Z_i > 0)\Big]^{\oplus 2} \\
&\quad - \mathbb{E}\Big[\Big\{\mathbf{E}_{45} \frac{\nabla_{\boldsymbol{\gamma}_{45}} f_2(\theta_i^*, s_i^*; \boldsymbol{\gamma}_{45,0})}{f_2(\theta_i^*, s_i^*; \boldsymbol{\gamma}_{45,0})} \mathbf{E}_{12} \frac{\nabla_{\boldsymbol{\gamma}_{12}} f_3(s_i^*; \boldsymbol{\gamma}_{12,0})}{f_3(s_i^*; \boldsymbol{\gamma}_{12,0})}\Big\} I(Z_i > 0)\Big]^{\oplus 2}
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbb{E} \frac{\{\nabla_{\boldsymbol{\gamma}} f_1(\theta_i^*, s_i^*; \boldsymbol{\gamma}_0)\}^{\otimes 2}}{f_1(\theta_i^*, s_i^*; \boldsymbol{\gamma}_0)} + \mathbb{E}\Big[\mathbf{E}_{45} \frac{\int \{\nabla_{\boldsymbol{\gamma}_{45}} f_2(\theta, s_i^*; \boldsymbol{\gamma}_{45,0})\}^{\otimes 2}}{f_2(\theta, s_i^*; \boldsymbol{\gamma}_{45,0})} f_1(\theta, s_i^*; \boldsymbol{\gamma}_0) d\theta \mathbf{E}_{45}^\top\Big] \\
&\quad + \mathbb{E}\Big[\mathbf{E}_{12} \frac{\{\nabla_{\boldsymbol{\gamma}_{12}} f_3(s_i^*; \boldsymbol{\gamma}_{12,0})\}^{\otimes 2}}{f_3(s_i^*; \boldsymbol{\gamma}_{12,0})} \mathbf{E}_{12}\Big] \\
&\quad + \mathbb{E}\Big[\mathbf{E}_{45} \int \nabla_{\boldsymbol{\gamma}} f_1(\theta, s_i^*; \boldsymbol{\gamma}_0) \nabla_{\boldsymbol{\gamma}_{45}} f_2(\theta, s_i^*; \boldsymbol{\gamma}_{45,0}) d\theta\Big]^{\oplus 2} \\
&\quad - \mathbb{E}\Big[\mathbf{E}_{12} \int \nabla_{\boldsymbol{\gamma}} f_1(\theta, s_i^*; \boldsymbol{\gamma}_0) f_2(\theta, s_i^*; \boldsymbol{\gamma}_{45,0}) d\theta \frac{\nabla_{\boldsymbol{\gamma}_{12}} f_3(s_i^*; \boldsymbol{\gamma}_{12,0})}{f_3(s_i^*; \boldsymbol{\gamma}_{12,0})}\}\Big]^{\oplus 2} \\
&\quad - \mathbf{E}_{45} \mathbb{E}\Big[\Big\{\int \nabla_{\boldsymbol{\gamma}_{45}} f_2(\theta, s_i^*; \boldsymbol{\gamma}_{45,0}) f_1(\theta, s_i^*; \boldsymbol{\gamma}_0) d\theta \frac{\nabla_{\boldsymbol{\gamma}_{12}} f_3(s_i^*; \boldsymbol{\gamma}_{12,0})}{f_3(s_i^*; \boldsymbol{\gamma}_{12,0})}\Big\} \mathbf{E}_{12}^\top\Big]^{\oplus 2}.
\end{aligned}
$$

Then by Eq (11), we have

$$
\begin{aligned}
\mathbb{V}\mathrm{ar}(\mathbf{u}_c) &= \mathbb{E}\frac{\{\nabla_{\boldsymbol{\gamma}} f_1(\theta_i, s_i^*; \boldsymbol{\gamma}_0)\}^{\otimes 2}}{f_1(\theta_i, s_i^*; \boldsymbol{\gamma}_0)} + \mathbb{E}\Big[\mathbf{E}_{45}\frac{\int\{\nabla_{\boldsymbol{\gamma}_{45,0}} f_2(\theta, s_i^*; \boldsymbol{\gamma}_{45,0})\}^{\otimes 2}}{f_2(\theta, s_i^*; \boldsymbol{\gamma}_{45,0})} f_1(\theta, s_i^*; \boldsymbol{\gamma}_0) d\theta \mathbf{E}_{45}^\top\Big] \\
&\quad - \mathbb{E}\Big[\mathbf{E}_{12}\frac{\{\nabla_{\boldsymbol{\gamma}_{12}} f_3(s_i^*; \boldsymbol{\gamma}_{12,0})\}^{\otimes 2}}{f_3(s_i^*; \boldsymbol{\gamma}_{12,0})}\mathbf{E}_{12}^\top\Big] + \mathbb{E}\Big[\int \nabla_{\boldsymbol{\gamma}} f_1(\theta, s_i^*; \boldsymbol{\gamma}_0) \nabla_{\boldsymbol{\gamma}_{45}^\top} f_2(\theta, s_i^*; \boldsymbol{\gamma}_{45,0}) d\theta \mathbf{E}_{45}^\top\Big]^{\oplus 2} \\
&= \mathbf{V}_c,
\end{aligned}
$$

which is exactly Result (II) of Lemma 2.

6 Proof of Lemma 3: Result (I)

6.1 Re-expression of ℓ_m

Re-express

$$\ell_m(N, \alpha, \gamma_1, \gamma_2) = \min_{\lambda} h_m(N, \alpha, \boldsymbol{\gamma}_{12}, \lambda),$$

where

$$
\begin{aligned}
h_m(N, \alpha, \boldsymbol{\gamma}_{12}, \lambda) = \log\binom{N}{n} + (N-n)\log(1-\alpha) - \sum_{i=1}^{n} \log[1 + \lambda\{f_3(s_i; \boldsymbol{\gamma}_{12}) - \alpha\}] \\
+ \log f_3(s_i; \boldsymbol{\gamma}_{12}).
\end{aligned}
$$

Let $\lambda_0 = 1/\alpha_0$. We further rewrite

$$h_m(N, \alpha, \boldsymbol{\gamma}_{12}, \lambda) = H(\boldsymbol{\zeta}_m)$$

with

$$H_m(\boldsymbol{\zeta}_m) = h_m(N_0 + N_0^{1/2}\zeta_{m1}, \alpha_0 + N_0^{-1/2}\zeta_{m2}, \boldsymbol{\gamma}_{12,0} + N_0^{-1/2}\boldsymbol{\zeta}_{m34}, \lambda_0 + N_0^{-1/2}\zeta_{m5}). \quad (13)$$

We approximate $h_m(N, \alpha, \boldsymbol{\gamma}_{12}, \lambda)$ by first approximating $H_m(\boldsymbol{\zeta}_m)$. A second-order Taylor expansion of $H(\boldsymbol{\zeta}_m)$ at $\mathbf{0}$ gives

$$H_m(\boldsymbol{\zeta}_m) = H_m(\mathbf{0}) + \{\nabla_{\boldsymbol{\zeta}_m} H_m(\mathbf{0})\}^\top \boldsymbol{\zeta}_m + \frac{1}{2}\boldsymbol{\zeta}_m^\top \{\nabla_{\boldsymbol{\zeta}_m \boldsymbol{\zeta}_m^\top} H_m(\mathbf{0})\}\boldsymbol{\zeta}_m + o(\|\boldsymbol{\zeta}_m\|^2).$$

Below we need to derive specific expressions for $\nabla_{\boldsymbol{\zeta}_m} H_m(\mathbf{0})$ and $\nabla_{\boldsymbol{\zeta}_m \boldsymbol{\zeta}_m^\top} H_m(\mathbf{0})$.

6.2 Preparation

The partial derivatives of H_m involves derivatives of the Gamma function. We first study some properties of $\Gamma(s)$, the Gamma function. For any positive integer c, define

$$S_c(N,n) = \frac{d^c \log\Gamma(N+1)}{dN^c} - \frac{d^c \log\Gamma(N-n+1)}{dN^c} = (-1)^{c-1}(c-1)! \sum_{k=N-n+1}^{N} \frac{1}{k^c}.$$

The fact that x^{-1} and x^{-2} are monotone decreasing function implies that

$$\log\{(N+1)/(N+1-n)\} < S_1(N,n) < \log\{N/(N-n)\} \quad \text{and}$$
$$-n/\{N(N-n)\} < S_2(N,n) < -n/\{(N+1)(N+1-n)\}.$$

Since n follows $B(N_0,\alpha_0)$, by central limit theorem, we have $n/N_0 = \alpha_0 + O_p(N_0^{-1/2})$,

$$\begin{aligned} S_1(N,n) &= \log\left(\frac{N_0}{N_0-n}\right) + O_p(N_0^{-1}) \\ &= -\log(1-\alpha_0) + \frac{n/N_0-\alpha_0}{1-\alpha_0} + O_p(N_0^{-1}) \end{aligned}$$

and

$$\begin{aligned} S_2(N,n) &= -\frac{n}{N_0(N_0-n)} + O_p(N_0^{-2}) \\ &= -\frac{\alpha_0}{N_0(1-\alpha_0)} + O_p(N_0^{-2}). \end{aligned}$$

6.3 Approximation of H_m

Specific expressions of the first-two-order derivatives of H_m are derived in the Appendix. Let $\mathbf{u}_m = \nabla_{\boldsymbol{\zeta}_m} H(\mathbf{0})$. Using the properties of the Gamma function, it can be found that

$$\begin{aligned} &u_{m1} = N_0^{1/2}\frac{n/N_0-\alpha_0}{1-\alpha_0}, \quad u_{m2} = -N_0^{-1/2}\frac{N_0-n}{1-\alpha_0} + N_0^{-1/2}\sum_{i=1}^{n}\frac{1}{f_3(s_i;\boldsymbol{\gamma}_{12,0})}, \\ &\mathbf{u}_{m34} = \mathbf{0}, \quad u_{m5} = -N_0^{-1/2}\sum_{i=1}^{n}\frac{\alpha_0\{f_3(s_i;\boldsymbol{\gamma}_{12,0})-\alpha_0\}}{f_3(s_i;\boldsymbol{\gamma}_{12,0})}. \end{aligned}$$

Let $\mathbf{V}_m = \mathbf{V}_m(N_0,\alpha_0,\boldsymbol{\gamma}_{12,0},1/\alpha_0)$ be the leading term of $-\nabla_{\boldsymbol{\zeta}_m\boldsymbol{\zeta}_m} H(\mathbf{0})$. It follows from the second order partial derivatives of H_m that

$$\mathbf{V}_m = \begin{pmatrix} \frac{\alpha_0}{1-\alpha_0} & \frac{1}{1-\alpha_0} & \mathbf{0} & 0 \\ \frac{1}{1-\alpha_0} & \frac{1}{1-\alpha_0}-\varphi_1 & \boldsymbol{m}\boldsymbol{\varphi}_2^\top & -\alpha_0^2\varphi_1 \\ \mathbf{0} & \boldsymbol{m}\boldsymbol{\varphi}_2 & \mathbf{0} & \alpha_0^2\boldsymbol{m}\boldsymbol{\varphi}_2 \\ 0 & -\alpha_0^2\varphi_1 & \alpha_0^2\boldsymbol{m}\boldsymbol{\varphi}_2^\top & \alpha_0^3-\alpha_0^4\varphi_1 \end{pmatrix} = \begin{pmatrix} \mathbf{V}_{maa} & \mathbf{V}_{mab} \\ \mathbf{V}_{mba} & V_{mbb} \end{pmatrix},$$

where $V_{mbb} = \alpha_0^3 - \alpha_0^4\varphi_1$,

$$\mathbf{V}_{maa} = \begin{pmatrix} \frac{\alpha_0}{1-\alpha_0} & \frac{1}{1-\alpha_0} & \mathbf{0} \\ \frac{1}{1-\alpha_0} & \frac{1}{1-\alpha_0} - \varphi_1 & \boldsymbol{m}\boldsymbol{\varphi}_2^\top \\ \mathbf{0} & \boldsymbol{m}\boldsymbol{\varphi}_2 & \mathbf{0} \end{pmatrix} \quad \text{and} \quad \mathbf{V}_{mab} = \begin{pmatrix} \mathbf{0} \\ -\alpha_0^2\varphi_1 \\ \alpha_0^2\boldsymbol{m}\boldsymbol{\varphi}_2 \end{pmatrix}.$$

In summary we have

$$H_m(\boldsymbol{\zeta}_m) = H_m(\mathbf{0}) + \mathbf{u}_m^\top\boldsymbol{\zeta}_m - \frac{1}{2}\boldsymbol{\zeta}_m^\top\mathbf{V}_m\boldsymbol{\zeta}_m + o_p(\|\boldsymbol{\zeta}_m\|^2).$$

6.4 Approximation of ℓ_m

Partition $\boldsymbol{\zeta}_m = (\boldsymbol{\zeta}_{ma}^\top, \zeta_{mb})^\top$ and $\mathbf{u}_m = (\mathbf{u}_{ma}^\top, u_{mb})^\top$. It can be seen that

$$\begin{aligned} H_m(\boldsymbol{\zeta}_m) &= H_m(\mathbf{0}) + \mathbf{u}_m^\top\boldsymbol{\zeta}_m - \frac{1}{2}\boldsymbol{\zeta}_m^\top\mathbf{V}_m\boldsymbol{\zeta}_m + o_p(\|\boldsymbol{\zeta}_m\|^2) \\ &= H(\mathbf{0}) + \mathbf{u}_{ma}^\top\boldsymbol{\zeta}_{ma} + u_{mb}\zeta_{mb} - \frac{1}{2}\boldsymbol{\zeta}_{ma}^\top\mathbf{V}_{maa}\boldsymbol{\zeta}_{ma} - \boldsymbol{\zeta}_{ma}^\top\mathbf{V}_{mab}\zeta_{mb} \\ &\quad -\frac{1}{2}\zeta_{mb}^2V_{mbb} + o_p(\|\boldsymbol{\zeta}_m\|^2). \end{aligned}$$

Setting $\nabla_{\zeta_{mb}}H_m(\boldsymbol{\zeta}_m) = 0$ gives

$$0 = u_{mb} - \boldsymbol{\zeta}_{ma}^\top\mathbf{V}_{mab} - \zeta_{mb}V_{mbb} + o_p(\|\boldsymbol{\zeta}_m\|),$$

which implies

$$\zeta_{mb} = V_{mbb}^{-1}(u_{mb} - \mathbf{V}_{mba}\boldsymbol{\zeta}_{ma}) + o_p(\|\boldsymbol{\zeta}_{ma}\|).$$

Putting this ζ_{mb} back into $H_m(\boldsymbol{\zeta}_m)$, we have

$$\begin{aligned} \ell_m = \min_{\zeta_{mb}} H_m(\boldsymbol{\zeta}_m) &= H(0) + \frac{1}{2}V_{mbb}^{-1}u_{mb}^2 - \frac{1}{2}\boldsymbol{\zeta}_{ma}^\top(\mathbf{V}_{maa} - \mathbf{V}_{mab}V_{mbb}^{-1}\mathbf{V}_{mba})\boldsymbol{\zeta}_{ma} \\ &\quad +\boldsymbol{\zeta}_{ma}^\top(\mathbf{u}_{ma} - \mathbf{V}_{mab}V_{mbb}^{-1}u_{mb}) + o_p(\|\boldsymbol{\zeta}_{ma}\|^2). \end{aligned}$$

7 Proof of Lemma 3: Result (II)

Denote $\boldsymbol{\Sigma} = \mathbb{V}\mathrm{ar}(\mathbf{u}_m)$. We have derived in the Appendix that

$$\boldsymbol{\Sigma} = \begin{pmatrix} \frac{\alpha_0}{1-\alpha_0} & \frac{1}{1-\alpha_0} & 0 & 0 \\ \frac{1}{1-\alpha_0} & \varphi_1 + \frac{1}{1-\alpha_0} & 0 & \alpha_0^2\varphi_1 - \alpha_0 \\ 0 & 0 & 0 & 0 \\ 0 & \alpha_0^2\varphi_1 - \alpha_0 & 0 & \alpha_0^4\varphi_1 - \alpha_0^3 \end{pmatrix} \equiv \begin{pmatrix} \boldsymbol{\Sigma}_{aa} & \boldsymbol{\Sigma}_{ab} \\ \boldsymbol{\Sigma}_{ba} & \Sigma_{bb} \end{pmatrix}.$$

We first calculate

$$\begin{aligned} \mathbb{V}\mathrm{ar}(\mathbf{u}_{ma} - \mathbf{V}_{mab}V_{mbb}^{-1}u_{mb}) &= \boldsymbol{\Sigma}_{aa} - \mathbf{V}_{mab}V_{mbb}^{-1}\boldsymbol{\Sigma}_{ba} - \boldsymbol{\Sigma}_{ab}V_{mbb}^{-1}\mathbf{V}_{mba} \\ &\quad +\mathbf{V}_{mab}V_{mbb}^{-1}\Sigma_{bb}V_{mbb}^{-1}\mathbf{V}_{mba}. \end{aligned}$$

It follows from $\Sigma_{bb} = -V_{mbb} = \alpha_0^4\varphi_1 - \alpha_0^3$ that

$$\mathbf{V}_{mab}V_{mbb}^{-1}\mathbf{\Sigma}_{ba} = -\frac{1}{\alpha_0^4\varphi_1 - \alpha_0^3}\begin{pmatrix} 0 \\ -\alpha_0^2\varphi_1 \\ \alpha_0^2\boldsymbol{m}\varphi_2 \end{pmatrix}(0, \alpha_0^2\varphi_1 - \alpha_0, \mathbf{0}) = \begin{pmatrix} 0 & 0 & \mathbf{0} \\ 0 & \varphi_1 & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{m}\varphi_2 & \mathbf{0} \end{pmatrix}$$

and

$$\mathbf{V}_{mab}V_{mbb}^{-1}\Sigma_{bb}V_{mbb}^{-1}\mathbf{V}_{mba} = \mathbf{V}_{mab}\Sigma_{bb}^{-1}\mathbf{V}_{mba} = \Sigma_{bb}^{-1}\begin{pmatrix} 0 \\ -\alpha_0^2\varphi_1 \\ \alpha_0^2\boldsymbol{m}\varphi_2 \end{pmatrix}\Big(0, -\alpha_0^2\varphi_1, \alpha_0^2\boldsymbol{m}\varphi_2^\top\Big)$$

$$= \Sigma_{bb}^{-1}\alpha_0^4\begin{pmatrix} 0 & 0 & \mathbf{0} \\ 0 & \varphi_1^2 & -\boldsymbol{m}\varphi_2^\top\varphi_1 \\ \mathbf{0} & -\boldsymbol{m}\varphi_2\varphi_1 & \boldsymbol{m}\varphi_2^{\otimes 2} \end{pmatrix}.$$

With these equalities, we finally arrive at

$$\mathbb{V}\mathrm{ar}(\mathbf{u}_{ma} - \mathbf{V}_{mab}V_{mbb}^{-1}u_{mb}) = \begin{pmatrix} \frac{\alpha_0}{1-\alpha_0} & \frac{1}{1-\alpha_0} & \mathbf{0} \\ \frac{1}{1-\alpha_0} & -\varphi_1 + \frac{1}{1-\alpha_0} & \boldsymbol{m}\varphi_2^\top \\ \mathbf{0} & \boldsymbol{m}\varphi_2 & \mathbf{0} \end{pmatrix}$$

$$+\Sigma_{bb}^{-1}\alpha_0^4\begin{pmatrix} 0 & 0 & \mathbf{0} \\ 0 & \varphi_1^2 & -\boldsymbol{m}\varphi_2^\top\varphi_1 \\ \mathbf{0} & -\boldsymbol{m}\varphi_2\varphi_1 & \boldsymbol{m}\varphi_2^{\otimes 2} \end{pmatrix}.$$

In the meantime, it can be seen that

$$\begin{aligned}
&\mathbf{V}_{maa} - \mathbf{V}_{mab}V_{mbb}^{-1}\mathbf{V}_{mba} \\
&= \begin{pmatrix} \frac{\alpha_0}{1-\alpha_0} & \frac{1}{1-\alpha_0} & \mathbf{0} \\ \frac{1}{1-\alpha_0} & -\varphi_1 + \frac{1}{1-\alpha_0} & \boldsymbol{m}\varphi_2^\top \\ \mathbf{0} & \boldsymbol{m}\varphi_2 & \mathbf{0} \end{pmatrix} + \Sigma_{bb}^{-1}\begin{pmatrix} 0 \\ -\alpha_0^2\varphi_1 \\ \alpha_0^2\boldsymbol{m}\varphi_2 \end{pmatrix}\Big(0, -\alpha_0^2\varphi_1, \alpha_0^2\boldsymbol{m}\varphi_2^\top\Big) \\
&= \begin{pmatrix} \frac{\alpha_0}{1-\alpha_0} & \frac{1}{1-\alpha_0} & \mathbf{0} \\ \frac{1}{1-\alpha_0} & -\varphi_1 + \frac{1}{1-\alpha_0} & \boldsymbol{m}\varphi_2^\top \\ \mathbf{0} & \boldsymbol{m}\varphi_2 & \mathbf{0} \end{pmatrix} + \Sigma_{bb}^{-1}\alpha_0^4\begin{pmatrix} 0 & 0 & \mathbf{0} \\ 0 & \varphi_1^2 & -\boldsymbol{m}\varphi_2^\top\varphi_1 \\ \mathbf{0} & -\boldsymbol{m}\varphi_2\varphi_1 & \boldsymbol{m}\varphi_2^{\otimes 2} \end{pmatrix}.
\end{aligned}$$

This proves

$$\mathbb{V}\mathrm{ar}(\mathbf{u}_{ma} - \mathbf{V}_{mab}V_{m55}^{-1}u_5) = \mathbf{V}_{maa} - \mathbf{V}_{mab}V_{m55}^{-1}\mathbf{V}_{mba}.$$

8 Proof of Theorem 2

Denote $\tilde{\boldsymbol{\zeta}}_c = N_0^{1/2}(\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0)$. Lemma 2 implies $\tilde{\boldsymbol{\zeta}}_c = N_0^{1/2}(\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0) = \mathbf{V}_c^{-1}\mathbf{u}_c + o_p(1) \xrightarrow{d} N(\mathbf{0}, \mathbf{V}_c^{-1})$, where

$$\begin{aligned}
\mathbf{u}_c = N_0^{1/2}\frac{1}{N_0}\sum_{i=1}^{N_0}\Big\{&\frac{\nabla_{\boldsymbol{\gamma}} f_1(\theta_i^*, s_i^*; \boldsymbol{\gamma}_0)}{f_1(\theta_i^*, s_i^*; \boldsymbol{\gamma}_0)} + \mathbf{E}_{45}\frac{\nabla_{\boldsymbol{\gamma}_{45}} f_2(\theta_i^*, s_i^*; \boldsymbol{\gamma}_{45,0})}{f_2(\theta_i^*, s_i^*; \boldsymbol{\gamma}_{45,0})} \\
&-\mathbf{E}_{12}\frac{\nabla_{\boldsymbol{\gamma}_{12}} f_3(s_i^*; \boldsymbol{\gamma}_{12})}{f_3(s_i^*; \boldsymbol{\gamma}_{12})}\Big\} I(Z_i > 0).
\end{aligned}$$

By the first-order Taylor expansion and the weak law of large numbers, we have

$$N_0^{1/2}(\tilde{N}/N_0 - 1) = t - \boldsymbol{m\varphi}_2^\top \mathbf{E}_{12}^\top \cdot N_0^{1/2}(\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0) + o_p(1) = t - \boldsymbol{m\varphi}_2^\top \mathbf{E}_{12}^\top \mathbf{V}_c^{-1}\mathbf{u}_c + o_p(1)$$

with $t = N_0^{1/2} \frac{1}{N_0} \sum_{i=1}^{N_0} \left\{ \frac{I(Z_i>0)}{f_3(s_i^*;\boldsymbol{\gamma}_{12,0})} - 1 \right\}$. Direct calculations give

$$\mathbb{V}\mathrm{ar}(t) = \mathbb{E}\Big\{ \frac{I(Z_i > 0)}{f_3(s_i^*;\boldsymbol{\gamma}_{12,0})} - 1 \Big\}^2 = \mathbb{E}\frac{I(Z_i > 0)}{f_3^2(s_i^*;\boldsymbol{\gamma}_{12,0})} - 1 = \varphi_1 - 1.$$

Similarly

$$\begin{aligned}
\mathbb{C}\mathrm{ov}(t, \mathbf{u}_c) &= \mathbb{E}\Big[\frac{I(Z_i > 0)}{f_3(s_i^*;\boldsymbol{\gamma}_{12,0})} \Big\{ \frac{\nabla_{\boldsymbol{\gamma}} f_1(\theta_i^*, s_i^*;\boldsymbol{\gamma}_0)}{f_1(\theta_i^*, s_i^*;\boldsymbol{\gamma}_0)} + \mathbf{E}_{45}\frac{\nabla_{\boldsymbol{\gamma}_{45}} f_2(\theta_i^*, s_i^*;\boldsymbol{\gamma}_{45,0})}{f_2(\theta_i^*, s_i^*;\boldsymbol{\gamma}_{45,0})} \\
&\quad - \mathbf{E}_{12}\frac{\nabla_{\boldsymbol{\gamma}_{12}} f_3(s_i^*;\boldsymbol{\gamma}_{12})}{f_3(s_i^*;\boldsymbol{\gamma}_{12})} \Big\} \Big] \\
&= \mathbb{E}\Big\{ \frac{\nabla_{\boldsymbol{\gamma}} f_1(\theta_i^*, s_i^*;\boldsymbol{\gamma}_0)}{f_3(s_i^*;\boldsymbol{\gamma}_{12,0})} + \mathbf{E}_{45}\frac{f_1(\theta_i^*, s_i^*;\boldsymbol{\gamma}_0)\nabla_{\boldsymbol{\gamma}_{45}} f_2(\theta_i^*, s_i^*;\boldsymbol{\gamma}_{45,0})}{f_2(\theta_i^*, s_i^*;\boldsymbol{\gamma}_{45,0}) f_3(s_i^*;\boldsymbol{\gamma}_{12,0})} \\
&\quad - \mathbf{E}_{12}\frac{\nabla_{\boldsymbol{\gamma}_{12}} f_3(s_i^*;\boldsymbol{\gamma}_{12})}{f_3(s_i^*;\boldsymbol{\gamma}_{12})} \Big\} \\
&= \mathbf{0},
\end{aligned}$$

where we have used (11). Therefore

$$\begin{aligned}
\sigma_c^2 &= \mathbb{V}\mathrm{ar}(t - \boldsymbol{m\varphi}_2^\top \mathbf{E}_{12}\mathbf{V}_c^{-1}\mathbf{u}_c) \\
&= \mathbb{V}\mathrm{ar}(t) + \mathbb{V}\mathrm{ar}(\boldsymbol{m\varphi}_2^\top \mathbf{E}_{12}\mathbf{V}_c^{-1}\mathbf{u}_c) \\
&= \varphi_1 - 1 + \boldsymbol{m\varphi}_2^\top \mathbf{E}_{12}\mathbf{V}_c^{-1}\mathbf{E}_{12}^\top \boldsymbol{m\varphi}_2.
\end{aligned}$$

9 Proof of Proposition 1 of the main paper

To prove this proposition, we need to derive specific forms of σ^2 and $\mathbf{V}^{-1}$, which both are part of $\boldsymbol{\Omega}^{-1}$. We shall re-express $\boldsymbol{\Omega}$ by an easy-going form and then derive σ^2 and $\mathbf{V}^{-1}$, respectively.

Denote $a = \alpha_0/(1 - \alpha_0\varphi_1)$, $b = \boldsymbol{m\varphi}_2^\top \mathbf{E}_{12}\mathbf{V}_c^{-1}\mathbf{E}_{12}^\top \boldsymbol{m\varphi}_2$, $\mathbf{D} = (\mathbf{0}, a\alpha_0^{-1}\mathbf{E}_{12}^\top \boldsymbol{m\varphi}_2)$ and

$$\mathbf{C} = \begin{pmatrix} \frac{\alpha_0}{1-\alpha_0} & \frac{1}{1-\alpha_0} \\ \frac{1}{1-\alpha_0} & \frac{1-\varphi_1}{(1-\alpha_0)(1-\alpha_0\varphi_1)} \end{pmatrix}.$$

Then we can write

$$\boldsymbol{\Omega} = \begin{pmatrix} \mathbf{C} & \mathbf{D}^\top \\ \mathbf{D} & \mathbf{V}_c - a\mathbf{E}_{12}^\top \boldsymbol{m\varphi}_2 \boldsymbol{m\varphi}_2^\top \mathbf{E}_{12} \end{pmatrix}.$$

Denote the left-up block and the right-down block of $\boldsymbol{\Omega}^{-1}$ by $\boldsymbol{\Omega}^{11}$ and $\boldsymbol{\Omega}^{22}$, respectively. Then $\mathbf{V}^{-1} = \boldsymbol{\Omega}^{22}$ and σ^2 is the (1, 1)-element of $\boldsymbol{\Omega}^{11}$.

By the inverse formula of 2×2 matrix, we have

$$\begin{aligned}\mathbf{V}^{-1} &= \mathbf{\Omega}^{22} = (\mathbf{V}_c - \mathbf{\Delta})^{-1},\\ \mathbf{\Omega}^{11} &= \{\mathbf{C} - \mathbf{D}^\top(\mathbf{V}_c - a\mathbf{E}_{12}^\top \boldsymbol{m\varphi}_2 \boldsymbol{m\varphi}_2^\top \mathbf{E}_{12})^{-1}\mathbf{D}\}^{-1},\end{aligned}$$

where $\mathbf{\Delta} = a\mathbf{E}_{12}^\top \boldsymbol{m\varphi}_2 \boldsymbol{m\varphi}_2^\top \mathbf{E}_{12} + \mathbf{D}\mathbf{C}^{-1}\mathbf{D}^\top$.

Proof We first prove $\mathbf{V} = \mathbf{V}_c$, which is equivalent to showing $\mathbf{\Delta} = \mathbf{0}$. It is easy to see that

$$\begin{aligned}\mathbf{C}^{-1} &= \frac{1}{\frac{\alpha_0(1-\varphi_1)}{(1-\alpha_0)^2(1-\alpha_0\varphi_1)} - \frac{1}{(1-\alpha_0)^2}} \begin{pmatrix} \frac{1-\varphi_1}{(1-\alpha_0)(1-\alpha_0\varphi_1)} & -\frac{1}{1-\alpha_0} \\ -\frac{1}{1-\alpha_0} & \frac{\alpha_0}{1-\alpha_0} \end{pmatrix}\\ &= -(1-\alpha_0)(1-\alpha_0\varphi_1) \begin{pmatrix} \frac{1-\varphi_1}{(1-\alpha_0)(1-\alpha_0\varphi_1)} & -\frac{1}{1-\alpha_0} \\ -\frac{1}{1-\alpha_0} & \frac{\alpha_0}{1-\alpha_0} \end{pmatrix}\\ &= -\begin{pmatrix} 1-\varphi_1 & -(1-\alpha_0\varphi_1) \\ -(1-\alpha_0\varphi_1) & \alpha_0(1-\alpha_0\varphi_1) \end{pmatrix}.\end{aligned}$$

Consequently we have

$$\begin{aligned}\mathbf{\Delta} &= a\mathbf{E}_{12}^\top \boldsymbol{m\varphi}_2 \boldsymbol{m\varphi}_2^\top \mathbf{E}_{12} - (\mathbf{0}, a\alpha_0^{-1}\mathbf{E}_{12}^\top \boldsymbol{m\varphi}_2) \begin{pmatrix} 1-\varphi_1 & -(1-\alpha_0\varphi_1) \\ -(1-\alpha_0\varphi_1) & \alpha_0(1-\alpha_0\varphi_1) \end{pmatrix} (\mathbf{0}, a\alpha_0^{-1}\mathbf{E}_{12}^\top \boldsymbol{m\varphi}_2)^\top\\ &= a\mathbf{E}_{12}^\top \boldsymbol{m\varphi}_2 \boldsymbol{m\varphi}_2^\top \mathbf{E}_{12} - a^2\alpha_0^{-2}\mathbf{E}_{12}^\top \boldsymbol{m\varphi}_2 \boldsymbol{m\varphi}_2^\top \mathbf{E}_{12} \times \alpha_0(1-\alpha_0\varphi_1)\\ &= \mathbf{0},\end{aligned}$$

where the last equation holds because $a = \alpha_0/(1-\alpha_0\varphi_1)$. This proves $\mathbf{V} = \mathbf{V}_c$.

Next we prove $\sigma^2 = \sigma_c^2$. Since

$$(\mathbf{V}_c - a\mathbf{E}_{12}^\top \boldsymbol{m\varphi}_2 \boldsymbol{m\varphi}_2^\top \mathbf{E}_{12})^{-1} = \mathbf{V}_c^{-1} + \frac{\mathbf{V}_c^{-1}\mathbf{E}_{12}^\top \boldsymbol{m\varphi}_2 \boldsymbol{m\varphi}_2^\top \mathbf{E}_{12}\mathbf{V}_c^{-1}}{a^{-1} - b},$$

we further have

$$(\mathbf{\Omega}^{11})^{-1} = \mathbf{C} - \mathbf{D}^\top\mathbf{V}_c^{-1}\mathbf{D} - \frac{\mathbf{D}^\top\mathbf{V}_c^{-1}\mathbf{E}_{12}^\top \boldsymbol{m\varphi}_2 \boldsymbol{m\varphi}_2^\top \mathbf{E}_{12}\mathbf{V}_c^{-1}\mathbf{D}}{a^{-1} - b}.$$

Because $\mathbf{D}^\top\mathbf{V}_c^{-1}\mathbf{E}_{12}^\top \boldsymbol{m\varphi}_2 = a\alpha_0^{-1}b\mathbf{e}_2$ with $\mathbf{e}_2 = (0,1)^\top$ and $\mathbf{D}^\top\mathbf{V}_c^{-1}\mathbf{D} = a^2\alpha_0^{-2}b\mathbf{e}_2\mathbf{e}_2^\top$, it follows that

$$\begin{aligned}(\mathbf{\Omega}^{11})^{-1} &= \mathbf{C} - a^2\alpha_0^{-2}b\mathbf{e}_2\mathbf{e}_2^\top - \frac{\alpha_0^{-2}a^2b^2\mathbf{e}_2\mathbf{e}_2^\top}{a^{-1} - b}\\ &= \mathbf{C} - \frac{a\alpha_0^{-2}b}{a^{-1} - b}\mathbf{e}_2\mathbf{e}_2^\top\\ &= \begin{pmatrix} \frac{\alpha_0}{1-\alpha_0} & \frac{1}{1-\alpha_0} \\ \frac{1}{1-\alpha_0} & \frac{1-\varphi_1}{(1-\alpha_0)(1-\alpha_0\varphi_1)} - \frac{a\alpha_0^{-2}b}{a^{-1}-b} \end{pmatrix}.\end{aligned}$$

After some algebra, we find that the $(2, 2)$-element of the above matrix can be re-written as

$$\begin{aligned}
\frac{1-\varphi_1}{(1-\alpha_0)(1-\alpha_0\varphi_1)} - \frac{a\alpha_0^{-2}b}{a^{-1}-b} &= \frac{1-\varphi_1}{(1-\alpha_0)(1-\alpha_0\varphi_1)} - \frac{\frac{\alpha_0}{1-\alpha_0\varphi_1}\alpha_0^{-2}b}{\frac{1-\alpha_0\varphi_1}{\alpha_0}-b} \\
&= \frac{1-\varphi_1}{(1-\alpha_0)(1-\alpha_0\varphi_1)} - \frac{\frac{1}{1-\alpha_0\varphi_1}b}{1-\alpha_0\varphi_1-\alpha_0 b} \\
&= \frac{1-\varphi_1}{(1-\alpha_0)(1-\alpha_0\varphi_1)} - \frac{b}{(1-\alpha_0\varphi_1)^2-(1-\alpha_0\varphi_1)\alpha_0 b} \\
&= \frac{1-\varphi_1}{(1-\alpha_0\varphi_1)(1-\alpha_0)} - \frac{b}{(1-\alpha_0\varphi_1)\{1-\alpha_0(\varphi_1+b)\}} \\
&= \frac{(1-\varphi_1)\{1-\alpha_0(\varphi_1+b)\}-b(1-\alpha_0)}{(1-\alpha_0\varphi_1)(1-\alpha_0)\{1-\alpha_0(\varphi_1+b)\}}.
\end{aligned}$$

Thus

$$(\mathbf{\Omega}^{11})^{-1} = \frac{1}{1-\alpha_0}\begin{pmatrix} \alpha_0 & 1 \\ 1 & \frac{(1-\varphi_1)\{1-\alpha_0(\varphi_1+b)\}-b(1-\alpha_0)}{(1-\alpha_0\varphi_1)\{1-\alpha_0(\varphi_1+b)\}} \end{pmatrix}.$$

Since σ_2^2 is the $(1, 1)$-element of $\mathbf{\Omega}^{11}$, we have

$$\begin{aligned}
\sigma_2^2 &= (1-\alpha_0)\frac{\frac{(1-\varphi_1)\{1-\alpha_0(\varphi_1+b)\}-b(1-\alpha_0)}{(1-\alpha_0\varphi_1)\{1-\alpha_0(\varphi_1+b)\}}}{\frac{(1-\varphi_1)\{1-\alpha_0(\varphi_1+b)\}-b(1-\alpha_0)}{(1-\alpha_0\varphi_1)\{1-\alpha_0(\varphi_1+b)\}}\alpha_0 - 1} \\
&= (1-\alpha_0)\frac{(1-\varphi_1)\{1-\alpha_0(\varphi_1+b)\}-b(1-\alpha_0)}{(1-\varphi_1)\{1-\alpha_0(\varphi_1+b)\}\alpha_0 - b(1-\alpha_0)\alpha_0 - (1-\alpha_0\varphi_1)\{1-\alpha_0(\varphi_1+b)\}} \\
&= (1-\alpha_0)\frac{(1-\varphi_1)\{1-\alpha_0(\varphi_1+b)\}-b(1-\alpha_0)}{(1-\alpha_0)(\alpha_0\varphi_1-1)} \\
&= \frac{(1-\varphi_1)\{1-\alpha_0(\varphi_1+b)\}-b(1-\alpha_0)}{\alpha_0\varphi_1-1} \\
&= \frac{(1-\varphi_1)(1-\alpha_0\varphi_1)-(1-\varphi_1)\alpha_0 b - b(1-\alpha_0)}{\alpha_0\varphi_1-1} \\
&= \varphi_1 - 1 - \frac{(1-\varphi_1)\alpha_0+(1-\alpha_0)}{\alpha_0\varphi_1-1}\times b \\
&= \varphi_1 - 1 + b = \sigma_1^2.
\end{aligned}$$

This proves $\sigma^2 = \sigma_c^2$ and hence also proves Proposition 1.

10 Other technical derivations

10.1 Derivatives of H_m

The first-order partial derivatives of H_m are

$$\nabla_{\zeta_{m1}} H_m(\boldsymbol{\zeta}_m) = N_0^{1/2}\nabla_N h_m = N_0^{1/2} S_1(N,n) + N_0^{1/2}\log(1-\alpha),$$

$$\nabla_{\zeta_{m2}} H_m(\boldsymbol{\zeta}_m) = N_0^{-1/2}\nabla_\alpha h_m = -N_0^{-1/2}\frac{N-n}{1-\alpha} + N_0^{-1/2}\sum_{i=1}^n \frac{\lambda}{1+\lambda\{f_3(s_i;\boldsymbol{\gamma}_{12})-\alpha\}},$$

$$\nabla_{\boldsymbol{\zeta}_{m34}} H_m(\boldsymbol{\zeta}_m) = N_0^{-1/2}\nabla_{\boldsymbol{\gamma}_{12}} h_m = -N_0^{-1/2}\sum_{i=1}^n \frac{\lambda\nabla_{\boldsymbol{\gamma}_{12}} f_3(s_i;\boldsymbol{\gamma}_{12})}{1+\lambda\{f_3(s_i;\boldsymbol{\gamma}_{12})-\alpha\}} + N_0^{-1/2}\sum_{i=1}^n \frac{\nabla_{\boldsymbol{\gamma}_{12}} f_3(s_i;\boldsymbol{\gamma}_{12})}{f_3(s_i;\boldsymbol{\gamma}_{12})},$$

$$\nabla_{\zeta_{m5}} H_m(\boldsymbol{\zeta}_m) = N_0^{-1/2}\nabla_\lambda h_m = -N_0^{-1/2}\sum_{i=1}^n \frac{f_3(s_i;\boldsymbol{\gamma}_{12})-\alpha}{1+\lambda\{f_3(s_i;\boldsymbol{\gamma}_{12})-\alpha\}}.$$

Similarly, its second-order partial derivatives are

$$\nabla_{\zeta_{m1}\zeta_{m1}} H_m(\boldsymbol{\zeta}_m) = N_0 S_2(N,n), \quad \nabla_{\zeta_{m1}\zeta_{m2}} H_m(\boldsymbol{\zeta}_m) = -\frac{1}{1-\alpha}$$

$$\nabla_{\zeta_{m1}\boldsymbol{\zeta}_{m34}} H_m(\boldsymbol{\zeta}_m) = \mathbf{0}, \quad \nabla_{\zeta_{m1}\zeta_{m5}} H_m(\boldsymbol{\zeta}_m) = 0,$$

$$\nabla_{\zeta_{m2}\zeta_{m2}} H_m(\boldsymbol{\zeta}_m) = -N_0^{-1}\frac{N-n}{(1-\alpha)^2} + N_0^{-1}\sum_{i=1}^n \frac{\lambda^2}{[1+\lambda\{f_3(s_i;\boldsymbol{\gamma}_{12})-\alpha\}]^2},$$

$$\nabla_{\zeta_{m2}\boldsymbol{\zeta}_{m34}} H_m(\boldsymbol{\zeta}_m) = -N_0^{-1}\sum_{i=1}^n \frac{\lambda^2\nabla_{\boldsymbol{\gamma}_{12}} f_3(s_i;\boldsymbol{\gamma}_{12})}{[1+\lambda\{f_3(s_i;\boldsymbol{\gamma}_{12})-\alpha\}]^2},$$

$$\nabla_{\zeta_{m2}\zeta_{m5}} H_m(\boldsymbol{\zeta}_m) = N_0^{-1}\sum_{i=1}^n \frac{1}{[1+\lambda\{f_3(s_i;\boldsymbol{\gamma}_{12})-\alpha\}]^2},$$

$$\nabla_{\boldsymbol{\zeta}_{m34}\boldsymbol{\zeta}_{m34}} H_m(\boldsymbol{\zeta}_m) = -N_0^{-1}\sum_{i=1}^n \frac{\lambda\nabla_{\boldsymbol{\gamma}_{12}\boldsymbol{\gamma}_{12}} f_3(s_i;\boldsymbol{\gamma}_{12})}{1+\lambda\{f_3(s_i;\boldsymbol{\gamma}_{12})-\alpha\}} + N_0^{-1}\sum_{i=1}^n \frac{\lambda^2\{\nabla_{\boldsymbol{\gamma}_{12}} f_3(s_i;\boldsymbol{\gamma}_{12})\}^{\otimes 2}}{[1+\lambda\{f_3(s_i;\boldsymbol{\gamma}_{12})-\alpha\}]^2} + N_0^{-1}\sum_{i=1}^n \frac{\nabla_{\boldsymbol{\gamma}_{12}\boldsymbol{\gamma}_{12}} f_3(s_i;\boldsymbol{\gamma}_{12})}{f_3(s_i;\boldsymbol{\gamma}_{12})} - N_0^{-1}\sum_{i=1}^n \frac{\{\nabla_{\boldsymbol{\gamma}_{12}} f_3(s_i;\boldsymbol{\gamma}_{12})\}^{\otimes 2}}{\{f_3(s_i;\boldsymbol{\gamma}_{12})\}^2},$$

$$\nabla_{\boldsymbol{\zeta}_{m34}\zeta_{m5}} H_m(\boldsymbol{\zeta}_m) = -N_0^{-1}\sum_{i=1}^n \frac{\nabla_{\boldsymbol{\gamma}_{12}} f_3(s_i;\boldsymbol{\gamma}_{12})}{[1+\lambda\{f_3(s_i;\boldsymbol{\gamma}_{12})-\alpha\}]^2},$$

$$\nabla_{\zeta_{m5}\zeta_{m5}} H_m(\boldsymbol{\zeta}_m) = N_0^{-1}\sum_{i=1}^n \frac{\{f_3(s_i;\boldsymbol{\gamma}_{12})-\alpha\}^2}{[1+\lambda\{f_3(s_i;\boldsymbol{\gamma}_{12})-\alpha\}]^2}.$$

10.2 Variance of $\mathbf{u}_m$

We now calculate the elements of $\boldsymbol{\Sigma} = \mathbb{V}\text{ar}(\mathbf{u}_m)$. Direct calculations give

$$
\begin{aligned}
\sigma_{11} &= \mathbb{V}\text{ar}\left(N_0^{1/2}\frac{n/N_0-\alpha_0}{1-\alpha_0}\right) = \frac{\alpha_0}{1-\alpha_0},\\
\sigma_{12} &= \mathbb{E}\Big[N_0^{1/2}\frac{n/N_0-\alpha_0}{1-\alpha_0}\Big\{N_0^{-1/2}\sum_{i=1}^{n}\frac{1}{f_3(s_i;\boldsymbol{\gamma}_{12,0})} - N_0^{-1/2}\frac{N_0-n}{1-\alpha_0}\Big\}\Big]\\
&= \mathbb{E}\Big[\frac{n/N_0-\alpha_0}{1-\alpha_0}\Big\{\sum_{i=1}^{n}\frac{1}{f_3(s_i;\boldsymbol{\gamma}_{12,0})} - \frac{N_0-n}{1-\alpha_0}\Big\}\Big]\\
&= \mathbb{E}\Big[\frac{I(Z_i>0)-\alpha_0}{1-\alpha_0}\Big\{\frac{I(Z_i>0)}{f_3(s_i^*;\boldsymbol{\gamma}_{12,0})} - \frac{I(Z_i<0)}{1-\alpha_0}\Big\}\Big]\\
&= \mathbb{E}\Big[\frac{I(Z_i>0)}{f_3(s_i^*;\boldsymbol{\gamma}_{12,0})}\Big] + \mathbb{E}\Big[\frac{\alpha_0 I(Z_i<0)}{(1-\alpha_0)^2}\Big]\\
&= 1 + \frac{\alpha_0}{1-\alpha_0} = \frac{1}{1-\alpha_0},\\
\sigma_{1,34} &= \mathbf{0},\\
\sigma_{15} &= -\mathbb{E}\Big[N_0^{1/2}\frac{n/N_0-\alpha_0}{1-\alpha_0}N_0^{-1/2}\sum_{i=1}^{n}\frac{\alpha_0\{f_3(s_i;\boldsymbol{\gamma}_{12,0})-\alpha_0\}}{f_3(s_i;\boldsymbol{\gamma}_{12,0})}\Big]\\
&= -\mathbb{E}\Big[\frac{n/N_0-\alpha_0}{1-\alpha_0}\sum_{i=1}^{n}\frac{\alpha_0\{f_3(s_i;\boldsymbol{\gamma}_{12,0})-\alpha_0\}}{f_3(s_i;\boldsymbol{\gamma}_{12,0})}\Big]\\
&= -\mathbb{E}\Big[\frac{I(Z_i>0)-\alpha_0}{1-\alpha_0}\frac{\alpha_0\{f_3(s_i^*;\boldsymbol{\gamma}_{12,0})-\alpha_0\}}{f_3(s_i^*;\boldsymbol{\gamma}_{12,0})}I(Z_i>0)\Big]\\
&= -\mathbb{E}\Big[\frac{\alpha_0\{f_3(s_i^*;\boldsymbol{\gamma}_{12,0})-\alpha_0\}}{f_3(s_i^*;\boldsymbol{\gamma}_{12,0})}I(Z_i>0)\Big]\\
&= 0,\\
\sigma_{22} &= \mathbb{E}\left\{\frac{I(Z_i>0)}{f_3(s_i^*;\boldsymbol{\gamma}_{12,0})} - \frac{I(Z_i\le 0)}{1-\alpha_0}\right\}^2\\
&= \mathbb{E}\left\{\frac{I(Z_i>0)}{f_3(s_i^*;\boldsymbol{\gamma}_{12,0})}\right\}^2 + \mathbb{E}\left\{\frac{I(Z_i\le 0)}{1-\alpha_0}\right\}^2\\
&= \mathbb{E}\left\{\frac{1}{f_3(s_i^*;\boldsymbol{\gamma}_{12,0})}\right\} + \frac{1}{1-\alpha_0}\\
&= \varphi_1 + \frac{1}{1-\alpha_0},
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{2,34} &= \mathbf{0}, \\
\sigma_{25} &= \mathbb{E}\Big[\Big\{N_0^{-1/2}\frac{N-n}{1-\alpha_0} - N_0^{-1/2}\sum_{i=1}^{n}\frac{1}{f_3(s_i;\boldsymbol{\gamma}_{12,0})}\Big\}\cdot N_0^{-1/2}\sum_{i=1}^{n}\frac{\alpha_0\{f_3(s_i;\boldsymbol{\gamma}_{12,0})-\alpha_0\}}{f_3(s_i;\boldsymbol{\gamma}_{12,0})}\Big] \\
&= \mathbb{E}\Big[\Big\{\frac{I(Z_i\le 0)}{1-\alpha_0} - \frac{I(Z_i>0)}{f_3(s_i^*;\boldsymbol{\gamma}_{12,0})}\Big\}\frac{I(Z_i>0)\alpha_0\{f_3(s_i^*;\boldsymbol{\gamma}_{12,0})-\alpha_0\}}{f_3(s_i^*;\boldsymbol{\gamma}_{12,0})}\Big] \\
&= -\mathbb{E}\Big[\frac{I(Z_i>0)}{f_3(s_i^*;\boldsymbol{\gamma}_{12,0})}\frac{\alpha_0\{f_3(s_i^*;\boldsymbol{\gamma}_{12,0})-\alpha_0\}}{f_3(s_i^*;\boldsymbol{\gamma}_{12,0})}\Big] \\
&= \mathbb{E}\Big\{\frac{\alpha_0^2}{f_3(s_i^*;\boldsymbol{\gamma}_{12,0})}\Big\} - \alpha_0 \\
&= \alpha_0^2\varphi_1 - \alpha_0, \\
\sigma_{34,34} &= \sigma_{34,5} = \mathbf{0}, \\
\sigma_{55} &= \mathbb{E}\Big[-N_0^{-1/2}\sum_{i=1}^{n}\frac{\alpha_0\{f_3(s_i;\boldsymbol{\gamma}_{12,0})-\alpha_0\}}{f_3(s_i;\boldsymbol{\gamma}_{12,0})}\Big]^2 \\
&= \mathbb{E}\Big[\frac{\alpha_0 I(Z_i>0)\{f_3(s_i^*;\boldsymbol{\gamma}_{12,0})-\alpha_0\}}{f_3(s_i^*;\boldsymbol{\gamma}_{12,0})}\Big]^2 \\
&= \mathbb{E}\Big[\frac{\alpha_0^2\{f_3(s_i^*;\boldsymbol{\gamma}_{12,0})-\alpha_0\}^2}{f_3(s_i^*;\boldsymbol{\gamma}_{12,0})}\Big] \\
&= \alpha_0^4\varphi_1 - \alpha_0^3.
\end{aligned}
$$

In summary we have

$$
\boldsymbol{\Sigma} = \begin{pmatrix} \frac{\alpha_0}{1-\alpha_0} & \frac{1}{1-\alpha_0} & \mathbf{0} & 0 \\ \frac{1}{1-\alpha_0} & \varphi_1 + \frac{1}{1-\alpha} & \mathbf{0} & \alpha_0^2\varphi_1 - \alpha_0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & \alpha_0^2\varphi_1 - \alpha_0 & \mathbf{0} & \alpha_0^4\varphi_1 - \alpha_0^3 \end{pmatrix}.
$$

References

J. Qin and J. Lawless. Empirical likelihood and general estimating equations. *Annals of Statistics*, 22: 300–325, 1994.