Supplementary Material for

"Simultaneous confidence bands for nonparametric regression with missing covariate data"

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This supplementary material contains some proofs of the lemmas given in Appendices A.1 and A.2.

Proof of Lemma 2. Let $T_n = n^{\delta}$ with $1/(2+\eta) < \delta < 1/3$ which together with Assumption (A5) implies that $T_n n^{-1/2} h^{-1/2} \log^{1/2} n \to 0, T_n^{-(1+\eta)} \ll n^{-1/2} h^{-1/2}, \sum_{n=1}^{\infty} T_n^{-(2+\eta)} < \infty$. We truncate the random error as follows:

$$\begin{aligned} \varepsilon_i &= \varepsilon_{i,1} + \varepsilon_{i,2} + \mu_{in}, \qquad \text{(S.1)} \\ \varepsilon_{i,1} &= \varepsilon_i I\left(|\varepsilon_i| > T_n\right), \varepsilon_{i,2} = \varepsilon_i I\left(|\varepsilon_i| \le T_n\right) - \mu_{in}, \\ \mu_{in} &= \mathrm{E}\left\{\varepsilon_i I\left(|\varepsilon_i| \le T_n\right) \Big| X_i\right\}, \end{aligned}$$

where $I(\cdot)$ is an indicator function. Firstly, note that

$$\begin{aligned} |\mu_{in}| &= \left|\mathrm{E}\left\{\varepsilon_i I\left(|\varepsilon_i| \le T_n\right) \Big| X_i\right\}\right| = \left|\mathrm{E}\left\{\varepsilon_i I\left(|\varepsilon_i| > T_n\right) \Big| X_i\right\}\right| \\ &\le T_n^{-(1+\eta)} \mathrm{E}\left(|\varepsilon_1|^{2+\eta} \Big| X_1\right) \le T_n^{-(1+\eta)} M_\eta, \end{aligned}$$

where M_η is defined in Assumption (A2). It together with Lemma 3 implies that

$$\begin{aligned}
&\sup_{x\in[a_0,b_0]}\left|n^{-1}\sum_{i=1}^{n}\frac{\delta_i}{\pi_i}K_h\left(X_i-x\right)\left(X_i-x\right)^l\mu_{in}\right|\\
&\quad\le \sup_{x\in[a_0,b_0]}\left\{\frac{h^l}{n}\sum_{i=1}^{n}\frac{\delta_i}{\pi_i}K_h\left(X_i-x\right)\left|\mu_{in}\right|\right\}\\
&\quad\le T_n^{-(1+\eta)}M_\eta h^l\sup_{x\in[a_0,b_0]}\left|L_{n,0}\left(x\right)\right|\\
&\quad= O_p\left(T_n^{-(1+\eta)}h^l\right)=o_p\left(n^{-1/2}h^{l-1/2}\right). \qquad \text{(S.2)}
\end{aligned}$$

Next, notice that

$$\sum_{n=1}^{\infty}P\left\{\left|\varepsilon_n\right|>T_n\right\}\le M_\eta\sum_{n=1}^{\infty}T_n^{-(2+\eta)}<\infty.$$

The Borel-Cantelli Lemma implies that

$$P\{\omega\colon \text{there exists } N\left(\omega\right)>0 \text{ such that } \left|\varepsilon_n\left(\omega\right)\right|\le T_n \text{ for } n>N\left(\omega\right)\}=1.$$

Thus,

$$\begin{aligned}
&P\{\omega\colon \text{there exists } N_0\left(\omega\right) \text{ such that } \left|\varepsilon_i\left(\omega\right)\right|\le T_n \text{ for } 1\le i\le n,\ n>N_0\left(\omega\right)\}\\
&\quad=1,
\end{aligned}$$

which concludes that

$$P\left\{\omega\colon \text{there exists } N_0\left(\omega\right) \text{ such that } \varepsilon_{i,1}=0 \text{ for } 1\le i\le n,\ n>N_0\left(\omega\right)\right\}=1.$$

Therefore, for large n,

$$n^{-1}\sum_{i=1}^{n}\frac{\delta_i}{\pi_i}K_h\left(X_i-x\right)\left(X_i-x\right)^l\varepsilon_{i,1}=0 \text{ a.s.} \qquad \text{(S.3)}$$

Finally, we apply Lemma 1 to deal with $n^{-1}\sum_{i=1}^{n}\frac{\delta_i}{\pi_i}K_h\left(X_i-x\right)\left(X_i-x\right)^l\varepsilon_{i,2}$. Notice that $\mathrm{E}\left(\varepsilon_{i,2}|X_i\right)=0$ a.s. and

$$\begin{aligned}
\mathrm{var}\left(\varepsilon_{i,2}\middle|X_i\right) &= \mathrm{E}\left\{\varepsilon_i^2 I\left(\left|\varepsilon_i\right|\le T_n\right)\middle|X_i\right\}-\left(\mu_{in}\right)^2\\
&= \mathrm{E}\left(\varepsilon_i^2\middle|X_i\right)-\mathrm{E}\left\{\varepsilon_i^2 I\left(\left|\varepsilon_i\right|>T_n\right)\middle|X_i\right\}-\left(\mu_{in}\right)^2\\
&= \sigma^2\left(X_i\right)+U_p\left(T_n^{-\eta}+T_n^{-2(1+\eta)}\right).
\end{aligned}$$

For $k \geq 3$,

$$\mathrm{E}\left(\left|\varepsilon_{i,2}\right|^{k}\middle|X_i\right) = \mathrm{E}\left(\left|\varepsilon_{i,2}\right|^{k-2}\left|\varepsilon_{i,2}\right|^{2}\middle|X_i\right) \leq (2T_n)^{k-2}\,\mathrm{E}\left(\left|\varepsilon_{i,2}\right|^{2}\middle|X_i\right).$$

Let $\xi_{i,l,n}(x) = n^{-1}\frac{\delta_i}{\pi_i}K_h(X_i - x)(X_i - x)^l\varepsilon_{i,2}$. Then one has $\mathrm{E}\,\xi_{i,l,n}(x) = 0$,

$$\begin{aligned}
&\mathrm{E}\,\xi_{i,l,n}^2(x) \leq c_\pi^{-1}n^{-2}\,\mathrm{E}\left(K_h(X_i - x)(X_i - x)^l\varepsilon_{i,2}\right)^2 \\
&= c_\pi^{-1}n^{-2}\,\mathrm{E}\left[K_h^2(X_i - x)(X_i - x)^{2l}\left\{\sigma^2(X_i) + U_p\left(T_n^{-\eta} + T_n^{-2(1+\eta)}\right)\right\}\right] \\
&= c_\pi^{-1}n^{-2}h^{2l-1}\sigma^2(x)f_X(x)\int K^2(v)v^{2l}dv\left\{1 + u(1)\right\},
\end{aligned}$$

and

$$\mathrm{E}\left|\xi_{i,l,n}(x)\right|^k \leq \left(2n^{-1}h^{l-1}c_\pi^{-1}\left\|K\right\|_\infty T_n\right)^{k-2}\mathrm{E}\left|\xi_{i,l,n}(x)\right|^2.$$

Therefore, $\xi_{i,l,n}(x)$ satisfies Cramér's Conditions in Lemma 1 with $c = 2n^{-1}h^{l-1}c_\pi^{-1}\left\|K\right\|_\infty T_n$. Thus, for any fixed $x \in [a_0, b_0]$, large enough $\gamma > 0$, and large enough n,

$$\begin{aligned}
&P\left\{\left|\sum_{i=1}^n \xi_{i,l,n}(x)\right| > \gamma n^{-1/2}h^{l-1/2}\log^{1/2} n\right\} \\
&\leq 2\exp\left\{-\frac{\gamma^2 n^{-1}h^{2l-1}\log n}{4\sum_{i=1}^n \mathrm{E}\,\xi_{i,l,n}^2(x) + 4n^{-1}h^{l-1}c_\pi^{-1}\left\|K\right\|_\infty T_n\gamma n^{-1/2}h^{l-1/2}\log^{1/2} n}\right\} \\
&= 2\exp\left\{-\frac{\gamma^2\log n}{4nh^{1-2l}\sum_{i=1}^n \mathrm{E}\,\xi_{i,l,n}^2(x) + 4c_\pi^{-1}\left\|K\right\|_\infty\gamma T_n n^{-1/2}h^{-1/2}\log^{1/2} n}\right\} \\
&\leq 2n^{-5},
\end{aligned}$$

which holds since $nh^{1-2l}\sum_{i=1}^n \mathrm{E}\,\xi_{i,l,n}^2(x)$ is bounded and $T_n n^{-1/2}h^{-1/2}\log^{1/2} n \to 0$. To further bound $\sum_{i=1}^n \xi_{i,l,n}(x)$ uniformly, we discretize $[a_0, b_0]$ by equally spaced points $u_0 = a_0 < x_1 < \cdots < x_{d_n} = b_0$ with $d_n = n^3 - 1$. Thus, for large n,

$$\begin{aligned}
&P\left\{\max_{j=0}^{d_n}\left|\sum_{i=1}^{n}\xi_{i,l,n}\left(x_j\right)\right| > \gamma n^{-1/2}h^{l-1/2}\log^{1/2} n\right\}\\
&\le \sum_{j=0}^{d_n} P\left\{\left|\sum_{i=1}^{n}\xi_{i,l,n}\left(x_j\right)\right| > \gamma n^{-1/2}h^{l-1/2}\log^{1/2} n\right\}\\
&\le \sum_{j=0}^{d_n} 2n^{-5} = 2\left(d_n+1\right)n^{-5} = 2n^{-2}.
\end{aligned}$$

Then one has

$$\sum_{n=1}^{\infty} P\left\{\max_{j=0}^{d_n}\left|\sum_{i=1}^{n}\xi_{i,l,n}\left(x_j\right)\right| > \gamma n^{-1/2}h^{l-1/2}\log^{1/2} n\right\} < +\infty.$$

The Borel-Cantelli Lemma implies that

$$\max_{j=0}^{d_n}\left|\sum_{i=1}^{n}\xi_{i,l,n}\left(x_j\right)\right| = O\left(n^{-1/2}h^{l-1/2}\log^{1/2} n\right) \text{ a.s.}$$

Notice that

$$\begin{aligned}
&\sup_{x\in[a_0,b_0]}\left|\sum_{i=1}^{n}\xi_{i,l,n}(x)\right|\\
&\le \max_{j=0}^{d_n}\left|\sum_{i=1}^{n}\xi_{i,l,n}\left(x_j\right)\right| + \max_{j=0}^{d_n-1}\sup_{x\in[x_j,x_{j+1}]}\left|\sum_{i=1}^{n}\xi_{i,l,n}(x) - \sum_{i=1}^{n}\xi_{i,l,n}\left(x_j\right)\right|\\
&= \max_{j=0}^{d_n}\left|\sum_{i=1}^{n}\xi_{i,l,n}\left(x_j\right)\right| + \max_{j=0}^{d_n-1}\sup_{x\in[x_j,x_{j+1}]}\left|n^{-1}\sum_{i=1}^{n}\frac{\delta_i}{\pi_i}\times\right.\\
&\qquad\qquad \left.\left\{K_h\left(X_i-x\right)\left(X_i-x\right)^l - K_h\left(X_i-x_j\right)\left(X_i-x_j\right)^l\right\}\varepsilon_{i,2}\right|\\
&\le \max_{j=0}^{d_n}\left|\sum_{i=1}^{n}\xi_{i,l,n}\left(x_j\right)\right| + \left(2c_\pi^{-1}T_n\right)h^{l-2}\left(\left\|K^{(1)}\right\|_\infty + l\left\|K\right\|_\infty\right)d_n^{-1}\left(b_0-a_0\right)\\
&= O\left(n^{-1/2}h^{l-1/2}\log^{1/2} n\right) \text{ a.s.} \qquad\qquad \text{(S.4)}
\end{aligned}$$

By (S.1)–(S.4), one obtains that

$$\sup_{x\in[a_0,b_0]}\left|n^{-1}\sum_{i=1}^{n}\frac{\delta_i}{\pi_i}K_h\left(X_i-x\right)\left(X_i-x\right)^l\varepsilon_i\right|=O_p\left(n^{-1/2}h^{l-1/2}\log^{1/2}n\right),$$

completing the proof. □

Proof of Lemma 3. Clearly, for $l=0,1,2$ and uniformly for all $x\in[a_0,b_0]$,

$$\begin{aligned}\mathrm{E}\left\{L_{n,l}\left(x\right)\right\}&=\int h^{-1}K\left(\frac{u-x}{h}\right)\left(u-x\right)^l f_X\left(u\right)du\\&=\int K\left(v\right)h^l v^l f_X\left(x+hv\right)dv=h^l f_X\left(x\right)\mu_l\left(K\right)\left\{1+u\left(h\right)\right\}.\end{aligned}$$

Let $\eta_{i,l,n}(x)=n^{-1}\frac{\delta_i}{\pi_i}K_h\left(X_i-x\right)\left(X_i-x\right)^l-\mathrm{E}\left\{n^{-1}\frac{\delta_i}{\pi_i}K_h\left(X_i-x\right)\left(X_i-x\right)^l\right\}$. Then $L_{n,l}\left(x\right)-\mathrm{E}\left\{L_{n,l}\left(x\right)\right\}=\sum_{i=1}^{n}\eta_{i,l,n}\left(x\right)$. It is clear that $\mathrm{E}\,\eta_{i,l,n}\left(x\right)=0$ and

$$\begin{aligned}\mathrm{E}\left\{\eta_{i,l,n}\left(x\right)\right\}^2&\leq n^{-2}\,\mathrm{E}\left\{\frac{1}{\pi_i}K_h^2\left(X_i-x\right)\left(X_i-x\right)^{2l}\right\}\\&\leq n^{-2}c_\pi^{-1}\,\mathrm{E}\left\{K_h^2\left(X_i-x\right)\left(X_i-x\right)^{2l}\right\}\\&=n^{-2}c_\pi^{-1}f_X\left(x\right)h^{2l-1}\int K^2\left(v\right)v^{2l}dv\left\{1+u\left(1\right)\right\}.\end{aligned}$$

Therefore,

$$\sum_{i=1}^{n}\mathrm{E}\left\{\eta_{i,l,n}\left(x\right)\right\}^2\leq n^{-1}c_\pi^{-1}f_X\left(x\right)h^{2l-1}\int K^2\left(v\right)v^{2l}dv\left\{1+u\left(1\right)\right\}.$$

Next, note that

$$\begin{aligned}\left|\eta_{i,l,n}\left(x\right)\right|&=\left|n^{-1}\frac{\delta_i}{\pi_i}K_h\left(X_i-x\right)\left(X_i-x\right)^l-\mathrm{E}\left\{n^{-1}\frac{\delta_i}{\pi_i}K_h\left(X_i-x\right)\left(X_i-x\right)^l\right\}\right|\\&\leq 2c_\pi^{-1}\left\|K\right\|_\infty\frac{h^{l-1}}{n}.\end{aligned}$$

Hence, for $k \geq 3$,

$$\begin{aligned}
\mathrm{E}\left|\eta_{i,l,n}(x)\right|^{k} &= \mathrm{E}\left|\eta_{i,l,n}(x)\right|^{k-2}\left|\eta_{i,l,n}(x)\right|^{2} \\
&\leq \left(2c_{\pi}^{-1}\|K\|_{\infty} n^{-1}h^{l-1}\right)^{k-2}\mathrm{E}\left|\eta_{i,l,n}(x)\right|^{2} \\
&\leq k!\left(2c_{\pi}^{-1}\|K\|_{\infty} n^{-1}h^{l-1}\right)^{k-2}\mathrm{E}\left|\eta_{i,l,n}(x)\right|^{2},
\end{aligned}$$

which implies that $\eta_{i,l,n}(x), i = 1, 2, \ldots, n$, satisfy the Cramér's Conditions in Lemma 1 with $c = 2c_{\pi}^{-1}\|K\|_{\infty} n^{-1}h^{l-1}$. Therefore, using the inequality in Lemma 1 and the discretization method as in the proof of Lemma 2, one can obtain that

$$\sup_{x\in[a_0,b_0]}\left|\sum_{i=1}^{n}\eta_{i,l,n}(x)\right| = O\left(n^{-1/2}h^{l-1/2}\log^{1/2} n\right) \text{ a.s.}$$

Therefore,

$$\begin{aligned}
L_{n,l}(x) &= \mathrm{E}\left\{L_{n,l}(x)\right\} + \sum_{i=1}^{n}\eta_{i,l,n}(x) \\
&= h^{l}f_X(x)\mu_l(K) + u_p\left(h^{l+1}\right) + U_p\left(n^{-1/2}h^{l-1/2}\log^{1/2} n\right).
\end{aligned}$$

The proof is completed. □

Proof of Lemma 4. It is clear that

$$\begin{aligned}
M_{n,l}(x) &= n^{-1}\sum_{i=1}^{n}\frac{\delta_i}{\pi_i}K_h(X_i - x)(X_i - x)^{l}\varepsilon_i \\
&\quad + n^{-1}\sum_{i=1}^{n}\frac{\delta_i}{\pi_i}K_h(X_i - x)(X_i - x)^{l}\left\{m(X_i) - m(x) - m^{(1)}(x)(X_i - x)\right\}.
\end{aligned}$$

When $l = 0$, using Lemma 3, one has

$$\begin{aligned}
&n^{-1}\sum_{i=1}^{n}\frac{\delta_i}{\pi_i}K_h(X_i - x)\left\{m(X_i) - m(x) - m^{(1)}(x)(X_i - x)\right\} \\
&= n^{-1}\sum_{i=1}^{n}\frac{\delta_i}{\pi_i}K_h(X_i - x)\left\{2^{-1}m^{(2)}(x)(X_i - x)^2 + u_p\left(h^2\right)\right\} \\
&= 2^{-1}m^{(2)}(x)L_{n,2} + u_p\left(h^2\right) \\
&= 2^{-1}m^{(2)}(x)h^2 f_X(x)\mu_2(K) + u_p\left(h^2\right),
\end{aligned}$$

which concludes that

$$M_{n,0}(x) = n^{-1}\sum_{i=1}^{n}\frac{\delta_i}{\pi_i}K_h(X_i - x)\varepsilon_i + 2^{-1}m^{(2)}(x)h^2 f_X(x)\mu_2(K) + u_p\left(h^2\right).$$

Similarly, when $l = 1$, one has

$$\begin{aligned}
&n^{-1}\sum_{i=1}^{n}\frac{\delta_i}{\pi_i}K_h(X_i - x)(X_i - x)\left\{m(X_i) - m(x) - m^{(1)}(x)(X_i - x)\right\}\\
&= n^{-1}\sum_{i=1}^{n}\frac{\delta_i}{\pi_i}K_h(X_i - x)(X_i - x)\left\{2^{-1}m^{(2)}(x)(X_i - x)^2 + u_p\left(h^2\right)\right\}\\
&= U_p\left(h^3\right),
\end{aligned}$$

which together with Lemma 2 establishes

$$M_{n,1}(x) = U_p\left(h^3 + n^{-1/2}h^{1/2}\log^{1/2} n\right) = U_p\left(n^{-1/2}h^{1/2}\log^{1/2} n\right).$$

The proof is completed. □

Proof of Lemma 5. Since $\pi(y)$ is assumed to follow a parametric model $\pi(y, \boldsymbol{\alpha})$ and has bounded first order partial derivative with respect to $\boldsymbol{\alpha}$, it is easy to show that $\sup_{y\in\mathbb{R}}|\pi(y) - \hat{\pi}(y)| = \sup_{y\in\mathbb{R}}|\pi(y, \boldsymbol{\alpha}) - \hat{\pi}(y, \hat{\boldsymbol{\alpha}})| = O_p\left(n^{-1/2}\right)$, where $\hat{\boldsymbol{\alpha}}$ is a root-n consistent estimator of $\boldsymbol{\alpha}$. This together with Lemma 3 implies that there exists some constant $d > 0$ such that

$$\begin{aligned}
\sup_{x\in[a_0,b_0]}\left|L_{n,l}(x) - \hat{L}_{n,l}(x)\right| &= \sup_{x\in[a_0,b_0]}\left|n^{-1}\sum_{i=1}^{n}\left(\frac{\delta_i}{\pi_i} - \frac{\delta_i}{\hat{\pi}_i}\right)K_h(X_i - x)(X_i - x)^l\right|\\
&\le dh^l \sup_{1\le i\le n}|\pi_i - \hat{\pi}_i| \sup_{x\in[a_0,b_0]}|L_{n,0}(x)|\\
&= O_p\left(n^{-1/2}h^l\right) = O_p\left(n^{-1/2}\right).
\end{aligned}$$

Next, using Lemma 1 and a method similar to that in the proof of Lemma 2, one can verify that

$$\sup_{x\in[a_0,b_0]}\left|n^{-1}\sum_{i=1}^{n}\frac{\delta_i}{\pi_i}K_h(X_i - x)\left\{|\varepsilon_i| - \mathrm{E}\left(|\varepsilon_i|\,\middle|X_i\right)\right\}\right| = O_p\left(n^{-1/2}h^{-1/2}\log^{1/2} n\right).$$

Meanwhile, it is readily seen that $\max_{1\le i\le n}\left|\frac{\hat{\pi}_i - \pi_i}{\hat{\pi}_i}\right| = O_p(n^{-1/2})$ and that

$$\sup_{x\in[a_0,b_0]}\left\{n^{-1}\sum_{i=1}^{n}\frac{\delta_i}{\pi_i}K_h(X_i - x)\,\mathrm{E}\left(|\varepsilon_i|\,\middle|X_i\right)\right\} = O_p(1).$$

Therefore,

$$
\begin{aligned}
&\sup_{x\in[a_0,b_0]}\left|M_{n,l}(x)-\hat{M}_{n,l}(x)\right| \\
&\leq \sup_{x\in[a_0,b_0]}\left|n^{-1}\sum_{i=1}^{n}\left(\frac{\delta_i}{\pi_i}-\frac{\delta_i}{\hat{\pi}_i}\right)K_h(X_i-x)(X_i-x)^l\varepsilon_i\right| \\
&\quad+\sup_{x\in[a_0,b_0]}\left|n^{-1}\sum_{i=1}^{n}\left(\frac{\delta_i}{\pi_i}-\frac{\delta_i}{\hat{\pi}_i}\right)K_h(X_i-x)(X_i-x)^l\right. \\
&\qquad\qquad\left.\times\left\{m(X_i)-m(x)-m^{(1)}(x)(X_i-x)\right\}\right| \\
&\leq \max_{1\leq i\leq n}\left|\frac{\hat{\pi}_i-\pi_i}{\hat{\pi}_i}\right|\sup_{x\in[a_0,b_0]}\left|\frac{h^l}{n}\sum_{i=1}^{n}\frac{\delta_i}{\pi_i}K_h(X_i-x)\left\{|\varepsilon_i|-\mathrm{E}\left(|\varepsilon_i|\,\Big|X_i\right)\right\}\right| \\
&\quad+\max_{1\leq i\leq n}\left|\frac{\hat{\pi}_i-\pi_i}{\hat{\pi}_i}\right|\sup_{x\in[a_0,b_0]}\left\{\frac{h^l}{n}\sum_{i=1}^{n}\frac{\delta_i}{\pi_i}K_h(X_i-x)\,\mathrm{E}\left(|\varepsilon_i|\,\Big|X_i\right)\right\} \\
&\quad+\sup_{x\in[a_0,b_0]}\left|n^{-1}\sum_{i=1}^{n}\frac{\delta_i(\pi_i-\hat{\pi}_i)}{\pi_i\hat{\pi}_i}K_h(X_i-x)(X_i-x)^l\right. \\
&\qquad\qquad\left.\times\left\{2^{-1}m^{(2)}(x)(X_i-x)^2+u_p\left(h^2\right)\right\}\right| \\
&=O_p\left(n^{-1/2}\right).
\end{aligned}
$$

The proof is completed. □

Proof of Lemma 6(a) By integration by parts, one has

$$
\begin{aligned}
&\zeta_{2n_0}(x)-\zeta_{3n_0}(x) \\
&=h^{1/2}\{s_n(x)\}^{-1/2}\int\int_{|\varepsilon|\leq\kappa_n}\frac{K_h(u-x)\varepsilon}{\pi(m(u)+\varepsilon)}d\{Z_{n_0}(u,\varepsilon)-B_{n_0}(T(u,\varepsilon))\} \\
&=h^{-1/2}\{s_n(x)\}^{-1/2}\int_{-1}^{1}\int_{|\varepsilon|\leq\kappa_n} \\
&\qquad\qquad\frac{K(v)\varepsilon}{\pi(m(x+hv)+\varepsilon)}d\{Z_{n_0}(x+hv,\varepsilon)-B_{n_0}(T(x+hv,\varepsilon))\} \\
&=h^{-1/2}\{s_n(x)\}^{-1/2}\int_{-1}^{1}\int_{|\varepsilon|\leq\kappa_n}
\end{aligned}
$$

$$\{Z_{n_0}(x+hv,\varepsilon)-B_{n_0}(T(x+hv,\varepsilon))\}\, d\frac{K(v)\,\varepsilon}{\pi(m(x+hv)+\varepsilon)}$$

$$+\,h^{-1/2}\{s_n(x)\}^{-1/2}\int_{-1}^{1}$$

$$\{Z_{n_0}(x+hv,\kappa_n)-B_{n_0}(T(x+hv,\kappa_n))\}\, d\frac{K(v)\,\kappa_n}{\pi(m(x+hv)+\kappa_n)}$$

$$+\,h^{-1/2}\{s_n(x)\}^{-1/2}\int_{-1}^{1}$$

$$\{Z_{n_0}(x+hv,-\kappa_n)-B_{n_0}(T(x+hv,-\kappa_n))\}\, d\frac{K(v)\,\kappa_n}{\pi(m(x+hv)-\kappa_n)}.$$

For the first term, by (13) and (16), one has

$$\sup_{x\in[a_0,b_0]}\left|h^{-1/2}\{s_n(x)\}^{-1/2}\int_{-1}^{1}\int_{|\varepsilon|\le\kappa_n}\{Z_{n_0}(x+hv,\varepsilon)-B_{n_0}(T(x+hv,\varepsilon))\}\right.$$
$$\left. d\frac{K(v)\,\varepsilon}{\pi(m(x+hv)+\varepsilon)}\right| = O_p\left(\kappa_n^2 n^{-1/2}h^{-1/2}\log^2 n\right) = o_p\left(\log^{-1/2} n\right).$$

Similarly,

$$\sup_{x\in[a_0,b_0]}\left|h^{-1/2}\{s_n(x)\}^{-1/2}\int_{-1}^{1}\{Z_{n_0}(x+hv,\kappa_n)-B_{n_0}(T(x+hv,\kappa_n))\}\right.$$
$$\left. d\frac{K(v)\,\kappa_n}{\pi(m(x+hv)+\kappa_n)}\right| = O_p\left(\kappa_n n^{-1/2}h^{-1/2}\log^2 n\right) = o_p\left(\log^{-1/2} n\right),$$

and

$$\sup_{x\in[a_0,b_0]}\left|h^{-1/2}\{s_n(x)\}^{-1/2}\int_{-1}^{1}\{Z_{n_0}(x+hv,-\kappa_n)-B_{n_0}(T(x+hv,-\kappa_n))\}\right.$$
$$\left. d\frac{K(v)\,\kappa_n}{\pi(m(x+hv)-\kappa_n)}\right| = O_p\left(\kappa_n n^{-1/2}h^{-1/2}\log^2 n\right) = o_p\left(\log^{-1/2} n\right),$$

completing the proof. □

Proof of Lemma 6(b) It is clear that the Jacobian of the transformation

T is $f_{X,\varepsilon|\delta=1}(x,\varepsilon)$. Then one has

$$
\begin{aligned}
&\left|\zeta_{3n_0}(x)-\zeta_{4n_0}(x)\right| = h^{1/2}\left\{s_n(x)\right\}^{-1/2}\times \\
&\qquad \left|\int\int_{|\varepsilon|\le\kappa_n}\frac{1}{\pi\left(m(u)+\varepsilon\right)}K_h(u-x)\,\varepsilon d\left\{B_{n_0}\left(T(u,\varepsilon)\right)-W_{n_0}\left(T(u,\varepsilon)\right)\right\}\right| \\
&= \left|h^{-1/2}\left\{s_n(x)\right\}^{-1/2}\int\int_{|\varepsilon|\le\kappa_n}\frac{1}{\pi\left(m(u)+\varepsilon\right)}K\left(\frac{u-x}{h}\right)\varepsilon f_{X,\varepsilon|\delta=1}(u,\varepsilon)\,dud\varepsilon\right| \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times\left|W_{n_0}(1,1)\right| \\
&= \left|h^{1/2}\left\{s_n(x)\right\}^{-1/2}\int_{|\varepsilon|\le\kappa_n}\frac{1}{\pi\left(m(x)+\varepsilon\right)}\varepsilon f_{X,\varepsilon|\delta=1}(x,\varepsilon)\,d\varepsilon\left\{1+u(1)\right\}\right| \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times\left|W_{n_0}(1,1)\right| \\
&= U_p\left(h^{1/2}\right) = u_p\left(\log^{-1/2} n\right).
\end{aligned}
$$

The proof is completed. □

Proof of Lemma 6(c) Note that conditional on $\Delta_n = n_0$,

$$
\begin{aligned}
&\left|\zeta_{5n_0}(x)-\zeta_{6n_0}(x)\right| \\
&= \left|h^{1/2}\int\left[\left\{s_n(u)\right\}^{1/2}\left\{s_n(x)\right\}^{-1/2}-1\right]K_h(u-x)\,dW(u)\right| \\
&= \left|h^{-1/2}\int_{-1}^{1}\left[\left\{s_n(x+hv)\right\}^{1/2}\left\{s_n(x)\right\}^{-1/2}-1\right]K(v)\,dW(x+hv)\right| \\
&= \left|-h^{-1/2}\int_{-1}^{1}W(x+hv)\frac{\partial}{\partial v}\left\{\left(s_n^{1/2}(x+hv)\,s_n^{-1/2}(x)-1\right)K(v)\right\}dv\right| \\
&\le \left|h^{-1/2}\int_{-1}^{1}W(x+hv)\left\{s_n^{1/2}(x+hv)\,s_n^{-1/2}(x)-1\right\}\frac{\partial}{\partial v}K(v)\,dv\right| \\
&\quad + \left|h^{-1/2}\int_{-1}^{1}W(x+hv)K(v)\frac{\partial}{\partial v}\left\{s_n^{1/2}(x+hv)\,s_n^{-1/2}(x)-1\right\}dv\right|. \quad \text{(S.5)}
\end{aligned}
$$

By the definition of $s(x)$ and Assumptions (A1), (A2), and (A4), it is easy to see that $s(x)$, $s^{-1}(x)$, and $s^{(1)}(x)$ are uniformly bounded. Meanwhile, note that

$$\begin{aligned}\sup_{x\in[a_0,b_0]}|s(x)-s_n(x)| &= \sup_{x\in[a_0,b_0]}\int_{|\varepsilon|>\kappa_n}\frac{\varepsilon^2}{\pi^2(m(x)+\varepsilon)}f_{X,\varepsilon|\delta=1}(x,\varepsilon)\,d\varepsilon \\ &\le c_\pi^{-2}\sup_{x\in[a_0,b_0]}\int_{|\varepsilon|>\kappa_n}\varepsilon^2 f_{X,\varepsilon|\delta=1}(x,\varepsilon)\,d\varepsilon \\ &= c_\pi^{-2}\sup_{x\in[a_0,b_0]}\int_{|\varepsilon|>\kappa_n}\varepsilon^2 f_{\varepsilon|X,\delta=1}(\varepsilon|x)\,f_{X|\delta=1}(x)\,d\varepsilon \\ &\le c_\pi^{-2}\kappa_n^{-\eta}\sup_{x\in[a_0,b_0]}\left|f_{X|\delta=1}(x)\right| M_\eta/P(\delta=1) = O\left(h^2\log^{-1}n\right), \quad \text{(S.6)}\end{aligned}$$

which holds due to (16) and Assumptions (A1), (A2), and (A4). Hence one obtains that $s_n(x)$, $s_n^{-1}(x)$, and $s_n^{(1)}(x)$ are uniformly bounded which implies

$$\sup_{x\in[a_0,b_0],v\in[-1,1]}\left|\left\{s_n^{1/2}(x+hv)\,s_n^{-1/2}(x)-1\right\}\right| = O(h),$$

and

$$\sup_{x\in[a_0,b_0],v\in[-1,1]}\left|\frac{\partial}{\partial v}\left\{s_n^{1/2}(x+hv)\,s_n^{-1/2}(x)-1\right\}\right| = O(h).$$

These equations together with (S.5) conclude

$$\sup_{x\in[a_0,b_0]}\left|\zeta_{5n_0}(x)-\zeta_{6n_0}(x)\right| = O_p\left(h^{1/2}\right) = o_p\left(\log^{-1/2}n\right),$$

completing the proof. □

Proof of Lemma 7 Clearly, conditional on $\Delta_n = n_0$, $\zeta_{4n_0}(x)$ is a Gaussian process satisfying $\mathrm{E}\left\{\zeta_{4n_0}(x)\middle|\Delta_n = n_0\right\} = 0$ and

$$
\begin{aligned}
&\mathrm{E}\left\{\zeta_{4n_0}(x)\,\zeta_{4n_0}(x')\,|\Delta_n = n_0\right\} = \mathrm{E}\Bigg[h\{s_n(x)\}^{-1/2}\{s_n(x')\}^{-1/2}\int\!\!\int_{|\varepsilon|\le\kappa_n} \\
&\quad \frac{K_h(u-x)\,\varepsilon}{\pi(m(u)+\varepsilon)} dW_{n_0}(T(u,\varepsilon)) \int\!\!\int_{|\varepsilon|\le\kappa_n} \frac{K_h(u-x')\,\varepsilon}{\pi(m(u)+\varepsilon)} dW_{n_0}(T(u,\varepsilon))\Bigg] \\
&= h\{s_n(x)\}^{-1/2}\{s_n(x')\}^{-1/2}\int\!\!\int_{|\varepsilon|\le\kappa_n} \frac{\varepsilon^2}{\pi^2(m(u)+\varepsilon)} K_h(u-x)\,K_h(u-x') \\
&\qquad\qquad\qquad \times f_{X,\varepsilon|\delta=1}(u,\varepsilon)\,du\,d\varepsilon \\
&= h\{s_n(x)\}^{-1/2}\{s_n(x')\}^{-1/2}\int K_h(u-x)\,K_h(u-x')\,s_n(u)\,du.
\end{aligned}
$$

Next, notice that conditional on $\Delta_n = n_0$, $\zeta_{5n_0}(x)$ is also a Gaussian process with mean 0 and covariance function $\mathrm{E}\left\{\zeta_{5n_0}(x)\,\zeta_{5n_0}(x')\,\middle|\,\Delta_n = n_0\right\}$ equal to

$$
\begin{aligned}
&h\{s_n(x)\}^{-1/2}\{s_n(x')\}^{-1/2} \\
&\quad \times \mathrm{E}\left[\int\{s_n(u)\}^{1/2}K_h(u-x)\,dW(u)\int\{s_n(u)\}^{1/2}K_h(u-x')\,dW(u)\right] \\
&= h\{s_n(x)\}^{-1/2}\{s_n(x')\}^{-1/2}\int K_h(u-x)\,K_h(u-x')\,s_n(u)\,du \\
&= \mathrm{E}\left\{\zeta_{4n_0}(x)\,\zeta_{4n_0}(x')\,\middle|\,\Delta_n = n_0\right\}.
\end{aligned}
$$

Thus, $\zeta_{4n_0}(x)$ and $\zeta_{5n_0}(x)$ have the same asymptotic distribution, completing the proof. □