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Asymptotic behavior of mean density estimators based on a single observation: the Boolean model case

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Abstract

The mean density estimation of a random closed set in $\mathbb{R}^d$, based on a single observation, is a crucial problem in several application areas. In the case of stationary random sets, a common practice to estimate the mean density is to take the n-dimensional volume fraction with observation window as large as possible. In the present paper, we provide large and moderate deviation results for these estimators when the random closed set Θ_n belongs to the quite general class of stationary Boolean models with Hausdorff dimension $n < d$. Moreover, we establish a central limit theorem and a Berry–Esseen bound for the family of estimators under study. Our findings allow to recover some well-known results in the literature on Boolean models. Finally, we also provide a guideline for the estimation of the mean density of non-stationary Boolean models characterized by high intensity of the underlying Poisson point process.

Keywords Hausdorff measure · Large deviations · Moderate deviations · Point processes · Stochastic geometry

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1 Introduction

Random sets in $\mathbb{R}^d$ with integer Hausdorff dimension $n < d$ are used to model several real situations: Fiber processes, boundaries of germ–grain models, n-facets of random tessellations, and surfaces of full-dimensional random sets represent only few simple examples. We refer the reader to Beneš and Rataj (2004) for a more extensive account. The random set, denoted here by Θ_n, is typically assumed to have locally finite n-dimensional Hausdorff measure $\mathcal{H}^n$, and in many of the examples mentioned above one is usually interested in the estimation of the so-called *mean density* of Θ_n, which is defined as the density of the measure $\mathbb{E}[\mathcal{H}^n(\Theta_n \cap \cdot)]$ on $\mathbb{R}^d$, whenever it exists. Such a density will be here denoted by $\lambda_{\Theta_n}(x)$. Recently, a series of asymptotic results concerning the estimation of λ_{Θ_n} in the non-stationary setting has been provided whenever an i.i.d. random sample for Θ_n is available; moreover, the asymptotic properties of the proposed estimators have been studied when the sample size goes to infinity. See, e.g., Camerlenghi et al. (2014a,b) (2016), Camerlenghi and Villa(2018) and references therein. Actually, in a lot of real problems, the sample size is equal to 1, i.e., only one observation of the random set is available: Our goal is to analyze the theoretical properties of mean density estimators in this noteworthy situation. More precisely, in the stationary setting, a widely used estimator of the mean density is

$$\frac{\mathcal{H}^n(\Theta_n \cap W_r)}{\mathcal{H}^d(W_r)}, \quad W_r \subset \mathbb{R}^d \tag{1}$$

provided that the *observation window* W_r is sufficiently large, i.e., $W_r = rW$ with $r > 0$ sufficiently large and $W \subset \mathbb{R}^d$. A series of papers and results on the estimator (1) are available in the literature; nevertheless, only partial results on large and moderate deviations are known for special kinds of processes: Our goal is to fill this gap in the framework of Boolean models. Indeed, in the present paper, we provide large and moderate deviation principles for random quantities associated with the family of random variables $\{\mathcal{H}^n(\Theta_n \cap rW)\}_{r>0}$ (see Theorems 2 and 3) when Θ_n is a Boolean model (see, for instance, Chiu et al. (2013)). Our theoretical findings generalize known results in the literature for stationary Poisson cluster point processes (Burton and Dehling 1990; Hwang 2000; Jiang et al. 1992) to the case of stationary Boolean models; moreover, as a by-product, our main theorems allow to recover well-known results on consistency and asymptotic normality of suitable estimators of the mean density λ_{Θ_n}, as stated in Corollaries 1 and 2, respectively. We mention that consistency is usually proved exploiting ergodic arguments, whereas normal convergence by means of characteristic functions. See, e.g., Chiu et al. (2013, p. 115), Beneš and Rataj (2004, Theorem 3.50), Pawlas and Beneš (2004, Theorem 3.1, Sect. 4), and Pawlas (2003). Among the several papers dealing with consistency and asymptotic normality of functionals of Boolean models we also mention: Hug et al.(2016) which provides multivariate limit theorems (and the corresponding rates of convergence) for additive, translational invariant, and locally bounded functionals of stationary Boolean models with convex grains; Heinrich and Pawlas(2008) where the authors present convergence theorems for empirical distribution functions of size characteristics of stationary germ–grain models; and Penrose and Yukich (2003) and Penrose and Yukich(2005) which contain results on law of large numbers and normal approximations in geometric probability. The asymptotic normality result that we provide in Corollary 2 is strengthened by a Berry–Esseen bound (Theorem 4) which quantifies the rate of convergence of the family of estimators under study to the limiting normal distribution. From the one hand, our strategy of proof borrows ideas from Heinrich and Schmidt (1985) who focus on shot-noise processes, and, from the other hand, it takes advantage of geometric measure theory. We underline that Berry–Esseen bounds for functionals of Boolean models in the case of full-dimensional random sets, i.e., $n = d$, have been already derived in the literature by Heinrich (2005) and Hug et al. (2016).

We point out that the assumption n strictly less than d is crucial in the proofs of the results we provide throughout the paper. In particular it is worth observing that under such assumption, $\mathcal{H}^n(\Theta_n \cap rW)$ is actually a compound Poisson process if one omits the effects of the observation window rW; large deviation results for compound Poisson processes may be obtained much more straightforwardly (see, for instance, Dembo and Zeitouni (1998)), and they coincide with those associated with the family of random variables $\{\mathcal{H}^n(\Theta_n \cap rW)\}_{r>0}$, provided here. Our main theorems provide the reader with a rigorous proof of this intuition taking into account the border effect of the observation window. Typically this argument does not apply to the d-dimensional case, which has been deeply investigated by Heinrich(2005).

We also stress that the stochastic process we consider here $\{\mathcal{H}^n(\Theta_n \cap rW)\}_{r>0}$ resembles a shot-noise process (see Eq. (9)): Our results on normal convergence of the sequence $\{\mathcal{H}^n(\Theta_n \cap rW)\}_{r>0}$ (Corollary 2), and the Berry–Esseen bound (Theorem 4), are comparable to those for Poisson shot-noise processes of Heinrich and Schmidt (1985, Sect. 3). Indeed, as in the case of shot-noise processes, the moment generating function of $\mathcal{H}^n(\Theta_n \cap rW)$ can be expressed as exponential of an integral which contains all information of the process, and this is relatively easy to use for proving normal convergence as well as large and moderate deviation principles. See also Sect. 5 for a more extensive discussion on the connection with shot-noise processes.

Finally, in this paper we also discuss the case of non-stationary Boolean models, and we provide the statistician with a guideline for the estimation of the mean density in presence of a single observation when the underlying Poisson process has *high* intensity. To the best of our knowledge, nothing is known until now in the case of a single observation for non-stationary random closed sets Θ_n, unless by assuming local stationarity (see, e.g., Capasso and Micheletti (2008), Sect. 4.1). We point out that the proposed estimator of $\lambda_{\Theta_n}(x)$, for a fixed point $x \in \mathbb{R}^d$, in the non-stationary case, is actually the same of the stationary one, by replacing W_r with a ball centered at the point x with a suitable *optimal radius* r depending on the intensity of the underling Poisson point process. It is worth noticing that such a radius r has to be taken equal to infinity if Θ_n is stationary, in accordance with intuition and the results shown in Sect. 3, but this is not obviously the case when $\lambda_{\Theta_n}(x)$ depends on the choice of $x \in \mathbb{R}^d$.

The paper proceeds as follows: In Sect. 2 we recall some preliminaries on Boolean models and large deviations, also setting useful notations. In Sect. 3 we extensively discuss large and moderate deviations for estimators as (1) in the case of stationary Boolean models, whereas in Sect. 4 we propose a guideline for the estimation of the mean density in the non-stationary setting. Section 5 depicts the relevant connections with the existent literature.

2 Preliminaries and notation

Before stating our results concerning the asymptotic behavior of the mean density estimators, in the present section we provide some basics on Boolean models and large deviations, very useful in the sequel.

2.1 Preliminaries on Boolean models

To lighten the presentation we shall use similar notation to previous works (see, e.g., Camerlenghi et al. (2014a), Villa (2014)). For any fixed $n < d$, $\mathcal{H}^n$ is the n-dimensional Hausdorff measure in $\mathbb{R}^d$, dx stands for the Lebesgue measure on $\mathbb{R}^d$, and $\mathcal{B}_{\mathbb{R}^d}$ is the Borel σ-algebra of the d-dimensional Euclidean space $\mathbb{R}^d$. The set $B_r(x)$ and the number b_d will denote the closed ball with center x and radius $r > 0$ and the volume of the unit ball in $\mathbb{R}^d$, respectively. Further, for any $A \subset \mathbb{R}^d$ and $r > 0$, the Minkowski enlargement of A at size r is denoted by $A_{\oplus r} := \{x \in \mathbb{R}^d : \operatorname{dist}(x, A) \leq r\}$, where $\operatorname{dist}(x, A)$ is the Euclidean distance of the point x to the set A; the diameter of the set A will be denoted by $\operatorname{diam}(A) = \sup\{|x - y| : x, y \in A\}$.

We recall that, given a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, a *random closed set* Θ in the d-dimensional Euclidean space $\mathbb{R}^d$ is a measurable map

$$\Theta : (\Omega, \mathscr{F}) \longrightarrow (\mathbb{F}, \sigma_{\mathbb{F}}),$$

where $\mathbb{F}$ denotes the class of the closed subsets in $\mathbb{R}^d$, and $\sigma_{\mathbb{F}}$ is the σ-algebra generated by the so-called *Fell topology* that is the topology generated by the set system

$$\{\mathscr{F}_G : G \in \mathscr{G}\} \cup \{\mathscr{F}^C : C \in \mathscr{C}\}$$

where $\mathscr{G}$ and $\mathscr{C}$ are the system of the open and compact subsets of $\mathbb{R}^d$, respectively (see, e.g., Matheron(1975)). A wide class of random closed sets in $\mathbb{R}^d$ can be represented as *germ–grain model* by means of marked point processes in $\mathbb{R}^d$ with marks in the class of compact subset of $\mathbb{R}^d$, as follows:

$$\Theta(\omega) = \bigcup_{(x_i, s_i) \in \Psi(\omega)} x_i + Z(s_i), \qquad \omega \in \Omega, \tag{2}$$

where $\Psi = \{(X_i, S_i)\}_{i \in \mathbb{N}}$ is a marked point process in $\mathbb{R}^d$ with marks in a suitable mark space $\mathbf{K}$ so that $Z_i = Z(S_i)$, $i \in \mathbb{N}$, is a random set containing the origin. Here the X_i's are the germs, whereas the $Z_i = Z(S_i)$'s are the grains of the process. Each $Z_i = Z(S_i)$ denotes the shape of the grain centered at the associated germ X_i, and it is conveniently identified by the value of a suitable random parameters $S_i \in \mathbf{K}$, that is, Z is a deterministic function from $\mathbf{K}$ to the family $\mathscr{C}_0$ of compact sets of $\mathbb{R}^d$

containing the origin. For instance, in the very simple case of random balls, $\mathbf{K} = \mathbb{R}^+$ and S stands for the radius of a ball containing the origin; in segment processes in $\mathbb{R}^2$, $\mathbf{K} = \mathbb{R}_+ \times [0, 2\pi]$ and $S = (L, \beta)$ where L and β are the random length and orientation of the segment attached to the origin, respectively. See also Matheron (1975) and Molchanov(2005) for additional details on random closed sets.

Throughout the paper, we shall denote by Θ_n any random closed set in $\mathbb{R}^d$ with integer dimension $0 \leq n < d$, represented as in (2), having locally finite n-dimensional Hausdorff measure $\mathcal{H}^n$. In particular Ψ will be a marked Poisson point process with intensity measure $\Lambda(\mathrm{d}(x,s)) = f(x,s)\mathrm{d}xQ(\mathrm{d}s)$, so that Θ_n is an n-dimensional Boolean model. The function f is called the *intensity* of the process, while Q is a probability measure on $\mathbf{K}$. The expected measure induced by Θ_n defined as $\mathbb{E}[\mu_{\Theta_n}](A) := \mathbb{E}[\mathcal{H}^n(\Theta_n \cap A)]$, $A \in \mathcal{B}_{\mathbb{R}^d}$, turns out to be absolutely continuous with respect to the d-dimensional Hausdorff measure $\mathcal{H}^d$, and its density (i.e., its Radon–Nikodym derivative) with respect to $\mathcal{H}^d$ is called *mean density* of Θ_n, and denoted by $\lambda_{\Theta_n}(\cdot)$. In particular it holds (see (Villa (2014), Proposition 5)):

$$\lambda_{\Theta_n}(x) = \int_{\mathbf{K}} \int_{x-Z(s)} f(y,s)\mathcal{H}^n(\mathrm{d}y)Q(\mathrm{d}s), \tag{3}$$

for $\mathcal{H}^d$-a.e. $x \in \mathbb{R}^d$, where $-Z(s)$ is the reflection of $Z(s)$ at the origin. Finally we remind that, if Φ is a Poisson point process on a general Polish space $\mathbb{X}$, having intensity measure μ, then, for any measurable function $g : \mathbb{X} \to \mathbb{R}$ such that

$$\int_{\mathbb{X}} \min\{|g(x)|, 1\}\mu(\mathrm{d}x) < \infty, \tag{4}$$

it holds

$$\mathbb{E}\Big[\exp\Big\{\vartheta \sum_{x\in\Phi} g(x)\Big\}\Big] = \exp\Big\{\int_{\mathbb{X}} (e^{\vartheta g(x)} - 1)\mu(\mathrm{d}x)\Big\} \tag{5}$$

for any complex number ϑ. Such a result, which can be found in (Kingman (1993), p. 28) will be applied to the point process Ψ in the proof of the two main theorems of the paper.

2.2 Preliminaries on large deviations

The theory of large deviations is concerned with the asymptotic estimation of probabilities of rare events, by giving an asymptotic computation of small probabilities in exponential scale; see Dembo and Zeitouni(1998) as a reference on this topic.

We start with some basic definitions of large deviations on the real line $\mathbb{R}$, which is the case we need in our paper. A rate function on $\mathbb{R}$ is a lower semicontinuous function $I : \mathbb{R} \to [0, \infty]$ or, equivalently a function whose level sets $\{x : I(x) \leq a\}$ are closed; moreover, I is said to be a good rate function if its level sets are compact.

Finally we say that a family of probability measures $\{\mu_r\}_{r>0}$ on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ satisfies the *Large Deviation Principle* (LDP from now on), as $r \to \infty$, with speed v_r and rate function I if the following conditions hold: $v_r \to \infty$; for every $A \in \mathcal{B}_{\mathbb{R}}$ we have

$$\liminf_{r\to+\infty}\frac{1}{v_r}\log\mu_r(A)\geq-\inf_{x\in\mathring{A}}I(x)$$

and

$$\limsup_{r\to+\infty}\frac{1}{v_r}\log\mu_r(A)\leq-\inf_{x\in\bar{A}}I(x),\tag{6}$$

where $\mathring{A}$ and $\bar{A}$ are the interior and the closure of A, respectively. Moreover we say that a family of (real valued) random variables satisfies the LDP if the family probability measures induced by them (on $\mathbb{R}$) does.

An important large deviation tool is the Gärtner–Ellis theorem (see, e.g., Dembo and Zeitouni (1998, Theorem 2.3.6); see also Dembo and Zeitouni(1998, Definition 2.3.5) for the concept of essentially smooth function). Here we recall its statement for real valued random variables.

Theorem 1 (Gärtner–Ellis Theorem) *Let $\{Z_r\}_{r>0}$ be a family of real valued random variables and v_r a speed function. We assume that, for all $t\in\mathbb{R}$,*

$$G(t):=\lim_{r\to+\infty}\frac{1}{v_r}\log\mathbb{E}[e^{v_rtZ_r}]$$

exists as an extended real number. We further suppose that $0\in\mathring{\mathcal{D}}(G)$, where $\mathcal{D}(G):=\{t\in\mathbb{R}:G(t)<+\infty\}$. Then, if G is essentially smooth and lower semicontinuous, $\{Z_r\}_{r>0}$ satisfies the LDP with speed v_r and good rate function G^* defined by

$$G^*(y):=\sup_{t\in\mathbb{R}}\{ty-G(t)\}.$$

The function G^* is called Legendre transform of G. We recall that the function G above is essentially smooth if:

(1) $\mathring{\mathcal{D}}(G)$ is non-empty;
(2) G is differentiable throughout $\mathring{\mathcal{D}}(G)$;
(3) G is steep, i.e., $|G'(t)|$ tends to infinity whenever t converges to a boundary point of $\mathcal{D}(G)$.

In particular it is known (see, e.g., den Hollander (2008, Theorem V.6, p. 54) or Ellis (1985, the comment just after the definition of essentially smooth function, p. 224)) that (3) holds vacuously if the function G is finite (i.e., $\mathcal{D}(G)=\mathbb{R}$) and differentiable.

In this paper we use Theorem 1 to prove our main results (Theorems 2–3), and in all the cases under study the function G turns out to be finite and differentiable everywhere, thus we do not need to check that G is essentially smooth in our proofs. While Theorem 2 concerns a LDP, in Theorem 3 we focus on *Moderate Deviations*, obtaining a class of LDPs with the same rate function; in some sense, these LDPs fill the gap between a central limit theorem and a strong law of large numbers (see Corollaries 2 and 1). For completeness, among the references on large/moderate deviations and central limit theorems, we recall the paper by Bryc(1993) in which, under a suitable regularity condition, an asymptotic normality result can be obtained as a consequence of a LDP as the one in Theorem 2.

3 Stationary Boolean models

Let us consider a Boolean model Θ_n with integer Hausdorff dimension $n < d$ as in (2), where Ψ has intensity measure $\Lambda(\mathrm{d}(x,s)) = \alpha \mathrm{d}x Q(\mathrm{d}s)$, for a certain constant $\alpha > 0$. In this case we can speak about *typical grain* Z_0, that is, all the grains $Z_i = Z(S_i)$ are independent and identically distributed as $Z_0 = Z_0(S)$, where the random parameter $S \in \mathbf{K}$ has distribution Q. Hence, in the rest of the paper, the $Z(S_i)$'s are i.i.d. random variables distributed as Z_0.

It directly follows by (3) that the mean density λ_{Θ_n} of Θ_n is independent of x and it will be denoted by

$$\lambda_{\Theta_n}(x) \equiv \lambda_{\Theta_n} \equiv \alpha \mathbb{E}_Q[\mathcal{H}^n(Z_0)]. \tag{7}$$

We remind that, as well known in the literature, in the stationary case a good estimator of λ_{Θ_n} is provided by the quantity

$$\widehat{\lambda}^{(r)}_{\Theta_n} := \frac{\mathcal{H}^n(\Theta_n \cap rW)}{|rW|}, \tag{8}$$

where W is a compact convex set containing $B_\varepsilon(0)$ for some $\varepsilon > 0$. So $\{rW\}_{r>0}$ is a *convex averaging sequence* as $r \to \infty$, that is: $\{rW\}_{r>0}$ is an increasing sequence of convex and compact subsets of $\mathbb{R}^d$ such that

$$\sup_{\rho>0}\{B_\rho(x) \subset W_r \text{ for some } x\} \to +\infty \quad \text{as } r \to +\infty,$$

see, e.g., (Chiu et al. (2013), p. 114). Thus, rW may be interpreted as an observation window whose width goes to infinity as $r \to +\infty$, with volume (Lebesgue measure) $|rW| := \mathcal{H}^d(rW)$, see, e.g., (Diggle (1983), p. 34). Therefore a problem of interest is the study of large and moderate deviation principles associated with the family of random variables $\{X_r(W)\}_{r>0}$ where

$$X_r(W) := X(rW) := \mathcal{H}^n(\Theta_n \cap rW)$$

for any $W \in \mathcal{B}(\mathbb{R}^d)$ and $r > 0$. Note that $\mathbb{E}[X_r(W)] = r^d|W|\lambda_{\Theta_n}$, where λ_{Θ_n} is defined in (7); moreover, by (Villa (2014), Lemma 3), in what follows it is useful to recall that

$$\mathcal{H}^n(\Theta_n \cap A) \overset{\text{a.s.}}{=} \sum_{(x_i,s_i)\in\Psi} \mathcal{H}^n((x_i + Z(s_i)) \cap A) \qquad \forall A \in \mathcal{B}_{\mathbb{R}^d} \tag{9}$$

holds for any lower dimensional (i.e., with Hausdorff dimension $n < d$) Boolean model Θ_n in $\mathbb{R}^d$ with intensity measure of the type $\Lambda(\mathrm{d}(x,s)) = f(x,s)\mathrm{d}xQ(\mathrm{d}s)$. In the rest of the section we prove the main results, which concern large and moderate deviation principles for the estimator (8) (see Theorems 2 and 3). These results are useful from both a probabilistic and statistical standpoint, allowing to obtain a strong laws of large numbers for the estimator in (8) (see Corollary 1) as well as its asymptotic normality (see Corollary 2). In Sect. 3.3 we strengthen the result on asymptotic normality proving a Berry–Esseen bound.

3.1 Large deviations

In the present section we state and prove the LDP for the estimators $\widehat{\lambda}^{(r)}_{\Theta_n}$ as in (8) (Theorem 2); moreover, as a by-product, we obtain the well-known strong law of large numbers for $X_r(W)$ in Corollary 1.

Theorem 2 *Assume that* $\mathbb{E}_Q[e^{t\mathcal{H}^n(Z_0)}] < \infty$ *for all* $t > 0$. *Then the family of estimators* $\left\{\frac{X_r(W)}{r^d|W|}\right\}_{r>0}$ *satisfies the LDP with speed function* $v_r = r^d|W|$ *(as* $r \to +\infty$*) and good rate function*

$$J^*(y) = \sup_{t\in\mathbb{R}}\{ty - \alpha\mathbb{E}_Q[e^{t\mathcal{H}^n(Z_0)} - 1]\}.$$

Proof We want to apply Theorem 1. Then we shall show that, for all $t \in \mathbb{R}$, we have

$$\lim_{r\to+\infty} \frac{1}{r^d|W|} \log \mathbb{E}\left[\exp\left\{t\mathcal{H}^n(\Theta_n \cap rW)\right\}\right] = J(t), \tag{10}$$

where

$$J(t) := \alpha\mathbb{E}_Q[e^{t\mathcal{H}^n(Z_0)} - 1]. \tag{11}$$

The case $t = 0$ is immediate. It is useful to remark that, since $\mathbb{E}_Q[e^{t\mathcal{H}^n(Z_0)}] < \infty$ for all $t > 0$, we have $\mathbb{E}_Q[(\mathcal{H}^n(Z_0))^k] < \infty$ for all $k \in \mathbb{N}$. Moreover, by (9), we have

$$\mathbb{E}\left[\exp\left\{t\mathcal{H}^n(\Theta_n \cap rW)\right\}\right] = \mathbb{E}\left[\exp\left\{t\sum_{(x_i,s_i)\in\Psi} \mathcal{H}^n((x_i + Z(s_i)) \cap rW)\right\}\right].$$

Then we can evaluate the last expression (in the right hand side) by using Equation (5) with $g(x, s) = \mathcal{H}^n((x + Z(s)) \cap rW)$, $\vartheta = t$ and $\boldsymbol{\Phi} = \boldsymbol{\Psi}$; note that the integrability condition (4) for g is satisfied; indeed,

$$\begin{aligned}
&\int_{\mathbb{R}^d\times\mathbf{K}} \mathcal{H}^n((x + Z(s)) \cap rW)\Lambda(\mathrm{d}(x, s)) \\
&\quad = \mathbb{E}_Q\left[\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} \mathbf{1}_{x+Z_0}(y)\mathbf{1}_{rW}(y)\mathcal{H}^n(\mathrm{d}y)\right)\alpha\mathrm{d}x\right] \\
&\quad = \mathbb{E}_Q\left[\int_{\mathbb{R}^d} \mathbf{1}_{Z_0}(\xi)\left(\int_{\mathbb{R}^d} \mathbf{1}_{rW}(\xi + x)\mathrm{d}x\right)\alpha\mathcal{H}^n(\mathrm{d}\xi)\right] \\
&\quad = r^d|W|\alpha\mathbb{E}_Q[\mathcal{H}^n(Z_0)] < \infty.
\end{aligned} \tag{12}$$

Therefore we get

$$\log \mathbb{E}\Big[\exp\Big\{t\mathcal{H}^n(\Theta_n \cap rW)\Big\}\Big] \\ = \mathbb{E}_Q\Big[\int_{\mathbb{R}^d}\Big(e^{t\mathcal{H}^n((x+Z_0)\cap rW)} - 1\Big)\alpha \mathrm{d}x\Big]. \tag{13}$$

Now we are ready to study the limit in (10). We start from (13), and we get

$$\begin{aligned}
&\frac{1}{r^d|W|}\log \mathbb{E}\Big[\exp\Big\{t\mathcal{H}^n(\Theta_n \cap rW)\Big\}\Big] \\
&= \frac{1}{r^d|W|}\mathbb{E}_Q\Big[\int_{\mathbb{R}^d}\Big(e^{t\mathcal{H}^n((x+Z_0)\cap rW)} - 1\Big)\frac{\mathcal{H}^n((x+Z_0)\cap rW)}{\mathcal{H}^n((x+Z_0)\cap rW)}\alpha \mathrm{d}x\Big] \\
&= \frac{1}{r^d|W|}\mathbb{E}_Q\Big[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{e^{t\mathcal{H}^n((x+Z_0)\cap rW)} - 1}{\mathcal{H}^n((x+Z_0)\cap rW)}\mathbf{1}_{(x+Z_0)}(y)\mathbf{1}_{rW}(y)\mathcal{H}^n(\mathrm{d}y)\alpha \mathrm{d}x\Big] \\
&= \frac{1}{r^d|W|}\mathbb{E}_Q\Big[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{e^{t\mathcal{H}^n((x+Z_0)\cap rW)} - 1}{\mathcal{H}^n((x+Z_0)\cap rW)}\mathbf{1}_{Z_0}(y-x)\mathbf{1}_{rW}(y)\mathcal{H}^n(\mathrm{d}y)\alpha \mathrm{d}x\Big];
\end{aligned}$$

then, thanks to the change of variables $(x, y) \to (z, \xi) := (y/r, y - x)$, we finally obtain

$$\begin{aligned}
&\frac{1}{r^d|W|}\mathbb{E}_Q\Big[\int_{\mathbb{R}^d}\Big(e^{t\mathcal{H}^n((x+Z_0)\cap rW)} - 1\Big)\alpha \mathrm{d}x\Big] \\
&\quad = \frac{\alpha}{|W|}\mathbb{E}_Q\Big[\int_{\mathbb{R}^{2d}}\frac{e^{t\mathcal{H}^n((r(z-\xi/r)+Z_0)\cap rW)} - 1}{\mathcal{H}^n((r(z-\xi/r)+Z_0)\cap rW)} \\
&\qquad\qquad \times \mathbf{1}_{Z_0}(\xi)\mathbf{1}_W(z)\mathcal{H}^n(\mathrm{d}\xi)\mathrm{d}z\Big].
\end{aligned} \tag{14}$$

Now we can apply the dominated convergence theorem to determine the limit of (14) as $r \to \infty$. In order to do this, we observe that

$$\left|\frac{e^{t\mathcal{H}^n((r(z-\xi/r)+Z_0)\cap rW)} - 1}{\mathcal{H}^n((r(z-\xi/r)+Z_0)\cap rW)}\right| \le |t|e^{|t|\mathcal{H}^n((r(z-\xi/r)+Z_0)\cap rW)} \le |t|e^{|t|\mathcal{H}^n(Z_0)}$$

where we have exploited the inequality $|1 - e^w| \le |w|e^{|w|}$ (for any $w \in \mathbb{R}$); moreover, the required integrability condition holds noting that

$$\begin{aligned}
&\mathbb{E}_Q\Big[\int_{\mathbb{R}^{2d}}|t|e^{|t|\mathcal{H}^n(Z_0)}\mathbf{1}_{Z_0}(\xi)\mathbf{1}_W(z)\mathcal{H}^n(\mathrm{d}\xi)\mathrm{d}z\Big] \\
&\quad = |t|\cdot|W|\cdot\mathbb{E}_Q[\mathcal{H}^n(Z_0)e^{|t|\mathcal{H}^n(Z_0)}] \le |t|\cdot|W|\cdot\mathbb{E}_Q[e^{2|t|\mathcal{H}^n(Z_0)}] < +\infty,
\end{aligned}$$

(here we take into account that $\mathbb{E}_Q[e^{t\mathcal{H}^n(Z_0)}] < +\infty$ for all $t > 0$). Then we have to evaluate

$$\lim_{r\to+\infty}\frac{e^{t\mathcal{H}^n((r(z-\xi/r)+Z_0)\cap rW)} - 1}{\mathcal{H}^n((r(z-\xi/r)+Z_0)\cap rW)}\mathbf{1}_{Z_0}(\xi)\mathbf{1}_W(z). \tag{15}$$

We remark that, for all $z \in W$ and Q almost surely, we have

$$\begin{aligned}\mathcal{H}^n((r(z-\xi/r)+Z_0)\cap rW) &= \int_{\mathbb{R}^d} \mathbf{1}_{(rz-\xi+Z_0)}(y)\mathbf{1}_W(y/r)\mathcal{H}^n(\mathrm{d}y)\\ &= \int_{\mathbb{R}^d} \mathbf{1}_{Z_0}(y-rz+\xi)\mathbf{1}_W(y/r)\mathcal{H}^n(\mathrm{d}y)\\ &= \int_{\mathbb{R}^d} \mathbf{1}_{Z_0}(y')\mathbf{1}_W((y'+rz-\xi)/r)\mathcal{H}^n(\mathrm{d}y')\\ &= \int_{\mathbb{R}^d} \mathbf{1}_{Z_0}(y')\mathbf{1}_W(y'/r+z-\xi/r)\mathcal{H}^n(\mathrm{d}y') \to \mathcal{H}^n(Z_0)\ (as\ r\to\infty);\end{aligned}$$

in fact, one has $\mathcal{H}^n(Z_0) < +\infty$ because $\mathbb{E}_Q[\mathcal{H}^n(Z_0)] < +\infty$, and the last limit holds true by an application of the dominated convergence theorem.

Thus, for $z \in W$, the limit in (15) boils down to

$$\lim_{r\to+\infty} \frac{e^{t\mathcal{H}^n((r(z-\xi/r)+Z_0)\cap rW)}-1}{\mathcal{H}^n((r(z-\xi/r)+Z_0)\cap rW)} = \frac{e^{t\mathcal{H}^n(Z_0)}-1}{\mathcal{H}^n(Z_0)}.$$

In conclusion the dominated convergence theorem applied to the integral on the right hand side of (14) yields

$$\begin{aligned}\lim_{r\to+\infty} \frac{1}{r^d|W|}\mathbb{E}_Q\left[\int_{\mathbb{R}^d}\left(e^{t\mathcal{H}^n((x+Z_0)\cap rW)}-1\right)\alpha\mathrm{d}x\right]\\ = \frac{\alpha}{|W|}\mathbb{E}_Q\left[\int_{\mathbb{R}^{2d}} \frac{e^{t\mathcal{H}^n(Z_0)}-1}{\mathcal{H}^n(Z_0)}\mathbf{1}_{Z_0}(\xi)\mathbf{1}_W(z)\mathcal{H}^n(\mathrm{d}\xi)\mathrm{d}z\right]\\ = \alpha\mathbb{E}_Q[e^{t\mathcal{H}^n(Z_0)}-1] = J(t).\end{aligned}$$

Thus the limit (10) is checked and the conclusion follows. □

Now we present a standard consequence of Theorem 2.

Corollary 1 *Under the assumptions of Theorem 2, we get*

$$\frac{X_r(W)}{r^d|W|} \longrightarrow \alpha\mathbb{E}_Q[\mathcal{H}^n(Z_0)] \quad \text{almost surely} \tag{16}$$

as $r \to +\infty$.

Proof We start with some remarks on the convex functions J and J^* in Theorem 2 (and its proof). We can say that $J^*(y)$ uniquely vanishes at the point $y_0 = J'(0) = \alpha\mathbb{E}_Q[\mathcal{H}^n(Z_0)]$. In fact, for every $y > 0$, there exists a unique $t_y \in \mathbb{R}$ such that

$$J^*(y) = t_y y - J(t_y);$$

moreover, $t \mapsto J'(t)$ is strictly increasing and, by a standard argument of convex analysis (see, e.g., Theorem 26.5 in Rockafellar (1970)), its inverse function $y \mapsto t_y$ is the derivative of $J^*(y)$; thus, $J^*(y)$ uniquely attains its minimum at some y such that $t_y = 0$, i.e., if and only if $y = J'(0)$, and we have $J^*(J'(0)) = 0$ because $J(0) = 0$.

Then we take $\delta > 0$, and we consider the closed set

$$C_\delta := \{y \in \mathbb{R} : \ |y - y_0| \geq \delta\}.$$

Then $J^*(C_\delta) := \inf_{y \in C_\delta} J^*(y)$ is positive and, by Theorem 2 and by the upper bound for closed sets (6) with $C = C_\delta$ (obviously with $Z_r = \frac{X_r(W)}{r^d|W|}$ and $v_r = r^d|W|$) we get

$$\limsup_{n \to +\infty} \frac{1}{r^d|W|} \log \mathbb{P}\left(\frac{X_r(W)}{r^d|W|} \in C_\delta\right) \leq -J^*(C_\delta);$$

thus, for every $\eta \in (0, J^*(C_\delta))$, there exists r_0 such that, for every $r > r_0$, we have

$$\mathbb{P}\left(\frac{X_r(W)}{r^d|W|} \in C_\delta\right) \leq e^{-r^d|W|(J^*(C_\delta)-\eta)}.$$

In conclusion (16) holds by a standard application of the first Borel–Cantelli lemma. □

Equation (16) may also be written as

$$\lim_{r \to +\infty} \frac{\mathcal{H}^n(\Theta_n \cap rW)}{|rW|} = \alpha \mathbb{E}_Q[\mathcal{H}^n(Z_0)] \equiv \lambda_{\Theta_n} \quad \text{a.s.} \tag{17}$$

and this clarifies why Corollary 1 is a strong law of large numbers for the estimator $\widehat{\lambda}^{(r)}_{\Theta_n}$ of the mean density λ_{Θ_n}.

Remark 1 Typically the function J in (11) comes up when one applies Theorem 1 to a suitably normalized compound Poisson process; see, e.g., (Dembo and Zeitouni (1998), Exercise 2.3.18). In this regard, we underline that if one omits the border effect of the window rW, i.e., considering the compound Poisson process $\sum_{(x_i,s_i) \in \Psi} \mathcal{H}^n(Z(s_i))\mathbf{1}_{rW}(x_i)$ in place of $X_r(W)$, the LDP stated in Theorem 2 holds again with the same speed $v_r = r^d|W|$ and rate function J^*. We have provided a rigorous proof of this intuition in Theorem 2, which duly takes into account the border effect of the observation window. We finally note that the limit value $\alpha \mathbb{E}_Q[\mathcal{H}^n(Z_0)]$ in Corollary 1 coincides with $J'(0)$.

3.2 Moderate deviations and asymptotic normality

Typically moderate deviations concern a class of LDPs for suitable families of centered random variables and governed by the same quadratic rate function which uniquely vanishes at the origin. More precisely, as we shall see in Theorem 3, for every choice of positive numbers $\{a_r\}_{r>0}$ such that

$$\lim_{r \to \infty} \frac{a_r}{\sqrt{r^d}} = 0 \quad \text{and} \quad \lim_{r \to \infty} a_r = +\infty \tag{18}$$

hold, we consider the centered random variables $\left\{\frac{X_r(W)-\mathbb{E}[X_r(W)]}{a_r\sqrt{r^d}}\right\}_{r>0}$ and we prove the LDP with speed $v_r = a_r^2 \to +\infty$, and the same rate function $\widetilde{J}^*$ which does not depend on the choice of a_r. In some cases we use the terminology moderate deviation principle (MDP for short). One will immediately realize that the statement of Theorem 3 has a close analogy with the analogue results in Hwang (2000), and in Jiang et al. (1992) for the particular case of cluster Poisson processes; this will be clarified in Sect. 5.

We can also say that moderate deviations fill the gap between two asymptotic regimes which can be seen as a particular choices of a_r such that only one condition in (18) holds:

1. the weak convergence to a centered normal distribution (case $a_r = 1$, where only the first condition in (18) holds), stated in Corollary 2 below;
2. the convergence to zero (case $a_r = \sqrt{r^d}$, where only the second condition in (18) holds), which is an immediate consequence of the convergence stated in Corollary 1.

We are now ready to state and prove the MDP.

Theorem 3 *Let $\{a_r\}_{r>0}$ be a family of positive numbers such that (18) holds. Assume that there exists $t_0 > 0$ such that $\mathbb{E}_Q[e^{t_0\mathcal{H}^n(Z_0)}] < +\infty$. Then*

$$\left\{\frac{X_r(W)-\mathbb{E}[X_r(W)]}{a_r\sqrt{r^d}}\right\}_{r>0}$$

satisfies the LDP with speed $v_r = a_r^2 \to +\infty$ and good rate function

$$\widetilde{J}^*(y) = \frac{y^2}{2\alpha|W|\mathbb{E}_Q[\mathcal{H}^n(Z_0)^2]}.$$

Proof We want to apply Theorem 1. Then we shall show that, for all $t \in \mathbb{R}$, we have

$$\lim_{r\to+\infty}\frac{1}{a_r^2}\log\mathbb{E}\left[\exp\left\{a_r^2 t\frac{X_r(W)-\mathbb{E}[X_r(W)]}{a_r\sqrt{r^d}}\right\}\right] = \widetilde{J}(t), \tag{19}$$

where

$$\widetilde{J}(t) := \frac{\alpha|W|t^2\mathbb{E}_Q[\mathcal{H}^n(Z_0)^2]}{2}.$$

We observe that

$$
\begin{aligned}
&\log \mathbb{E}\Big[\exp\Big\{a_r^2 t \frac{X_r(W) - \mathbb{E}[X_r(W)]}{a_r\sqrt{r^d}}\Big\}\Big] \\
&\quad \overset{(7)}{=} \log \mathbb{E}\Big[\exp\Big\{\frac{a_r t}{\sqrt{r^d}} X_r(W)\Big\}\Big] - a_r t\sqrt{r^d}|W|\alpha \mathbb{E}_Q[\mathcal{H}^n(Z_0)] \\
&\quad \overset{(9)}{=} \log \mathbb{E}\Big[\exp\Big\{\frac{a_r}{\sqrt{r^d}} t \sum_{(x_i,s_i)\in\Psi} \mathcal{H}^n((x_i + Z(s_i)) \cap rW)\Big\}\Big] \\
&\qquad - a_r t\sqrt{r^d}|W|\alpha \mathbb{E}_Q[\mathcal{H}^n(Z_0)],
\end{aligned}
$$

where we have also noticed that $\mathbb{E}[X_r(W)] = r^d|W|\lambda_{\Theta_n}$. Now we compute the first term, and for the expected value we use (5) with $g(x,s) = \mathcal{H}^n((x + Z(s)) \cap rW)$, $\vartheta = \frac{a_r}{\sqrt{r^d}} t$ and $\boldsymbol{\Phi} = \boldsymbol{\Psi}$; then,

$$
\begin{aligned}
&\log \mathbb{E}\Big[\exp\Big\{a_r^2 t \frac{X_r(W) - \mathbb{E}[X_r(W)]}{a_r\sqrt{r^d}}\Big\}\Big] \\
&= \mathbb{E}_Q\left[\int_{\mathbb{R}^d} \left(e^{a_r t \mathcal{H}^n((x+Z_0)\cap rW)/\sqrt{r^d}} - 1\right)\alpha \mathrm{d}x\right] - a_r t\sqrt{r^d}|W|\alpha \mathbb{E}_Q[\mathcal{H}^n(Z_0)] \\
&= \mathbb{E}_Q\left[\int_{\mathbb{R}^d} \sum_{k\geq 1} \frac{(a_r t \mathcal{H}^n((x + Z_0) \cap rW)/\sqrt{r^d})^k}{k!} \alpha \mathrm{d}x\right] \\
&\qquad - a_r t\sqrt{r^d}|W|\alpha \mathbb{E}_Q[\mathcal{H}^n(Z_0)],
\end{aligned}
$$

where in the last equality we consider Taylor series expansion of the exponential function.

Moreover, for $k = 1$, we have

$$
\mathbb{E}_Q\left[\int_{\mathbb{R}^d} \frac{a_r}{\sqrt{r^d}} t \mathcal{H}^n((x + Z_0) \cap rW)\alpha \mathrm{d}x\right] \overset{(12)}{=} a_r t\sqrt{r^d}|W|\alpha \mathbb{E}_Q[\mathcal{H}^n(Z_0)]
$$

and, for $k = 2$, a standard application of the dominated convergence theorem implies that

$$
\lim_{r\to+\infty} \frac{1}{2a_r^2} \mathbb{E}_Q\left[\int_{\mathbb{R}^d} \frac{a_r^2}{r^d} t^2 [\mathcal{H}^n((x + Z_0) \cap rW)]^2 \alpha \mathrm{d}x\right] = t^2 \frac{\alpha|W|\mathbb{E}_Q[\mathcal{H}^n(Z_0)^2]}{2}.
$$

Thus (19) holds true, and we complete the proof, if we check that

$$
\lim_{r\to+\infty} \underbrace{\frac{1}{a_r^2} \mathbb{E}_Q\left[\int_{\mathbb{R}^d} \sum_{k\geq 3} \frac{1}{k!} \left[\frac{a_r}{\sqrt{r^d}} t \mathcal{H}^n((x + Z_0) \cap rW)\right]^k \alpha \mathrm{d}x\right]}_{=: R_r} = 0. \tag{20}
$$

In order to do that we proceed as follows. By Fubini–Tonelli theorem and by the change of variable $y \to y' = y - x$ for the last equality we have

$$
\begin{aligned}
|R_r| &\le \frac{1}{a_r^2}\mathbb{E}_Q\Big[\int_{\mathbb{R}^d}\sum_{k\ge3}\frac{a_r^k|t|^k}{(\sqrt{r^d})^k}\frac{1}{k!}(\mathcal{H}^n((x+Z_0)\cap rW))^k\alpha\mathrm{d}x\Big]\\
&= \sum_{k\ge3}\frac{a_r^{k-2}|t|^k}{k!(\sqrt{r^d})^k}\mathbb{E}_Q\Big[\int_{\mathbb{R}^d}(\mathcal{H}^n((x+Z_0)\cap rW))^{k-1}\\
&\qquad\qquad\times\int_{\mathbb{R}^d}\mathbf{1}_{(x+Z_0)\cap rW}(y)\mathcal{H}^n(\mathrm{d}y)\alpha\mathrm{d}x\Big]\\
&= \sum_{k\ge3}\frac{a_r^{k-2}|t|^k}{k!(\sqrt{r^d})^k}\mathbb{E}_Q\Big[\int_{\mathbb{R}^d}(\mathcal{H}^n((x+Z_0)\cap rW))^{k-1}\\
&\qquad\qquad\times\int_{\mathbb{R}^d}\mathbf{1}_{Z_0}(y')\mathbf{1}_{rW}(y'+x)\mathcal{H}^n(\mathrm{d}y')\alpha\mathrm{d}x\Big].
\end{aligned}
$$

Moreover, the inequality $\mathcal{H}^n((x+Z_0)\cap rW) \le \mathcal{H}^n(Z_0)$ trivially holds true; then, we get

$$
\begin{aligned}
|R_r| &\le \sum_{k\ge3}\frac{a_r^{k-2}|t|^k}{k!(\sqrt{r^d})^k}\mathbb{E}_Q\Big[\mathcal{H}^n(Z_0)^{k-1}\\
&\qquad\qquad\times\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathbf{1}_{Z_0}(y')\mathbf{1}_{rW}(y'+x)\mathcal{H}^n(\mathrm{d}y')\alpha\mathrm{d}x\Big]\\
&= \sum_{k\ge3}\frac{\alpha a_r^{k-2}|t|^k}{k!(\sqrt{r^d})^{k-2}}|W|\mathbb{E}_Q[\mathcal{H}^n(Z_0)^k]\\
&= \alpha|W|\sum_{k\ge3}\left(\frac{a_r}{\sqrt{r^d}}\right)^{k-2}\frac{|t|^k}{t_0^k}\frac{1}{k!}\mathbb{E}_Q[(t_0\mathcal{H}^n(Z_0))^k]\\
&= \alpha|W|\left(\frac{|t|}{t_0}\right)^3\frac{a_r}{\sqrt{r^d}}\sum_{k\ge3}\left(\frac{a_r|t|}{\sqrt{r^d}t_0}\right)^{k-3}\frac{\mathbb{E}_Q[(t_0\mathcal{H}^n(Z_0))^k]}{k!}.
\end{aligned}
$$

Now we recall that, by the first condition in (18), there exists $\bar{r}(t) > 0$ such that $\frac{a_r|t|}{\sqrt{r^d}t_0} < 1$ for $r > \bar{r}(t)$; thus, for r large enough, by Fubini–Tonelli theorem we have

$$
\begin{aligned}
|R_r| &\le \alpha|W|\left(\frac{|t|}{t_0}\right)^3\frac{a_r}{\sqrt{r^d}}\sum_{k\ge3}\frac{\mathbb{E}_Q[(t_0\mathcal{H}^n(Z_0))^k]}{k!}\\
&\qquad\qquad\le \alpha|W|\left(\frac{|t|}{t_0}\right)^3\frac{a_r}{\sqrt{r^d}}\underbrace{\mathbb{E}_Q[e^{t_0\mathcal{H}^n(Z_0)}]}_{<+\infty}.
\end{aligned}
$$

In conclusion (20) holds true by the first condition in (18), and the proof is complete. □

Now, as we said at the beginning of this section, we can state a central limit theorem for the family of random variables considered in Theorem 3. In the sequel use the notation $\xrightarrow{\mathfrak{D}}$ for the convergence in law. In particular we write $\xrightarrow{\mathfrak{D}} N(0,\sigma^2)$ when the limit in law is a centered normal distribution with variance σ^2.

Corollary 2 *As $r \to +\infty$, we have*

$$\frac{X_r(W) - \mathbb{E}[X_r(W)]}{\sqrt{r^d}} \xrightarrow{\mathfrak{D}} N(0, \alpha|W|\mathbb{E}_Q[\mathcal{H}^n(Z_0)^2]).$$

Proof The computations in the proof of Theorem 3 still work well if $a_r = 1$, even if the second condition in (18) fails. So we have

$$\lim_{r\to\infty} \mathbb{E}\left[\exp\left\{t(X_r(W) - \mathbb{E}[X_r(W)])/\sqrt{r^d}\right\}\right] = \exp\left\{\frac{1}{2}t^2\alpha|W|\mathbb{E}_Q[\mathcal{H}^n(Z_0)^2]\right\}$$

for all $t \in \mathbb{R}$, where the limit is the moment generating function of the centered normal distributed random variables with variance $\alpha|W|\mathbb{E}_Q[\mathcal{H}^n(Z_0)^2]$. This completes the proof. $\square$

Some remarks are in order.

Remark 2 Along similar lines to those of Remark 1, one may consider the compound Poisson process $\sum_{(x_i,s_i)\in\Psi} \mathcal{H}^n(Z(s_i))\mathbf{1}_{rW}(x_i)$ in place of $X_r(W)$; hence, the MDP in Theorem 3 and the asymptotic normality result in Corollary 2 still hold as a by-product of the available results on compound Poisson processes. Our proofs have the merit to take into account the border effect of rW providing a rigorous derivation of MDP for the random processes under study. If we refer to the function J in (11), we also point out that the asymptotic variance $\alpha|W|\mathbb{E}_Q[\mathcal{H}^n(Z_0)^2]$ in Corollary 2, appearing in the denominator of the rate function $\widetilde{J}^*$ in Theorem 3, coincides with $J''(0)|W|$.

Remark 3 Finally we explain how Corollary 2 implies the asymptotic normality result for the estimator (8) of the mean density. Recall that

$$\widehat{\lambda}^{(r)}_{\Theta_n} = \frac{\mathcal{H}^n(\Theta_n \cap rW)}{|rW|}\left(= \frac{X_r(W)}{r^d|W|}\right)$$

is the estimator of the mean density λ_{Θ_n}. So we can say that Corollary 2 concerns the asymptotic normality of this estimator. Indeed, the convergence in law stated in that theorem can be rephrased as follows:

$$\sqrt{|W|r^d}(\widehat{\lambda}^{(r)}_{\Theta_n} - \lambda_{\Theta_n}) \xrightarrow{\mathfrak{D}} N(0, \alpha\mathbb{E}_Q[\mathcal{H}^n(Z_0)^2]),$$

with $r \to +\infty$.

3.3 Rate of convergence: a Berry–Esseen bound

In Corollary 2 we proved the convergence in law of $(X_r(W) - \mathbb{E}[X_r(W)])/\sqrt{r^d}$ to a centered normal random variable with variance $\alpha|W|\mathbb{E}_Q[\mathcal{H}^n(Z_0)^2]$; here we would like to estimate the rate of this convergence through a Berry–Esseen bound. More precisely, if we denote by $F_r(\cdot)$ the c.d.f. of the random variable $(X_r(W) - \mathbb{E}[X_r(W)])/\sqrt{r^d}$ and by $F(\cdot)$ the c.d.f. of the limiting Gaussian random variable with mean 0 and variance $\alpha|W|\mathbb{E}_Q[\mathcal{H}^n(Z_0)^2]$, respectively, we aim to determine an upper bound for the Kolmogorov distance

$$D(r) := \sup_{x\in\mathbb{R}} |F_r(x) - F(x)|.$$

In order to do this, we apply the Berry–Esseen inequality (Loève1977), along similar lines as (Heinrich and Schmidt1985). Our bound relies on the geometric property of the observation window W and on the additional assumption of uniform boundedness of the diameter of the typical grain Z_0, say $\mathrm{diam}(Z_0)$, usually fulfilled in real applications. First of all we remind that the set W is compact and convex in $\mathbb{R}^d$; as a consequence its boundary ∂W is a countably $\mathcal{H}^{d-1}$-rectifiable compact set satisfying the condition

$$\eta(B_r(x))) \geq \gamma r^{d-1} \quad \forall x \in \partial W,\ \forall r \in (0,1) \tag{21}$$

for some $\gamma > 0$ and some Radon measure η absolutely continuous with respect to $\mathcal{H}^{d-1}$ in $\mathbb{R}^d$. Refer to (Ambrosio et al.2008) for the details on this standard result in geometric measure theory, which is also related to the existence of the Minkowski content of a set (Ambrosio et al. 2000). We are able to prove the following inequality.

Theorem 4 *Assume that there exists* $t_0 > 0$ *such that* $\mathbb{E}_Q[e^{t_0\mathcal{H}^n(Z_0)}] < +\infty$ *and that* $\mathrm{diam}(Z_0) \leq K$ *for some constant* $K > 0$. *Then, for any* $a \in (0,1)$, *we have*

$$\begin{aligned} D(r) \leq \frac{1}{r}\frac{C(\gamma)}{|W|\pi(1-a)} + \frac{1}{\sqrt{r^d}}\Bigg(& \frac{24}{\pi^{3/2}\ell(a)\sqrt{2\alpha|W|\mathbb{E}_Q[\mathcal{H}^n(Z_0)^2]}} \\ & + \frac{\sqrt{2}\mathbb{E}_Q[e^{t_0\mathcal{H}^n(Z_0)}]}{\sqrt{\pi\alpha|W|}t_0^3(1-a)^{3/2}(\mathbb{E}_Q[\mathcal{H}^n(Z_0)^2])^{3/2}}\Bigg) \end{aligned} \tag{22}$$

where $r > \max\{K/2, 2C(\gamma)/(a|W|)\}$, *whereas* $\ell(a)$ *and* $C(\gamma)$ *are the constants defined as*

$$\ell(a) := \min\left\{ t_0, \frac{t_0^3 a\mathbb{E}_Q[\mathcal{H}^n(Z_0)^2]}{4\mathbb{E}_Q[e^{t_0\mathcal{H}^n(Z_0)}]} \right\}$$

and $C(\gamma) := 2^{d-1}4^d b_d K/\gamma$, *with* γ *the constant which appears in* (21).

Proof In the first part of the proof, we bound the difference between the characteristic function of $(X_r(W) - \mathbb{E}[X_r(W)])/\sqrt{r^d}$, denoted as φ_r, and the characteristic function of the limiting Gaussian random variable, denoted as φ; in the last part of the proof, we apply the Berry–Esseen inequality (Loève1977, p. 297), along with the determined upper bound involving the two characteristic functions, to find (22).

Proceeding along similar lines as in the proof of Theorem 3 we easily realize that the logarithm of the characteristic function of $(X_r(W) - \mathbb{E}[X_r(W)])/\sqrt{r^d}$ equals

$$\log \varphi_r(t) = \mathbb{E}_Q \int_{\mathbb{R}^d} \sum_{k\geq 2} \frac{(\mathrm{i}t\mathcal{H}^n((x+Z_0)\cap rW))^k}{(\sqrt{r^d})^k k!} \alpha \mathrm{d}x, \tag{23}$$

where i denotes the imaginary unit. We now focus on the terms in (23) for $k = 2$ and $k \geq 3$ separately: For $k = 2$, straightforward calculations lead to

$$\begin{aligned} \mathbb{E}_Q \int_{\mathbb{R}^d} \frac{\mathrm{i}^2 t^2 \mathcal{H}^n((x+Z_0)\cap rW))^2}{r^d 2} \alpha \mathrm{d}x = -\frac{\alpha t^2}{2} \mathbb{E}_Q \int_{\mathbb{R}^{2d}} \prod_{i=1}^{2} \mathbf{1}_{Z_0}(\xi_i) \\ \times \int_{\mathbb{R}^d} \mathbf{1}_W(y)\mathbf{1}_W\Big(y + \frac{\xi_2-\xi_1}{r}\Big)\mathrm{d}y\mathcal{H}^n(\mathrm{d}\xi_1)\mathcal{H}^n(\mathrm{d}\xi_2); \end{aligned} \tag{24}$$

for $k \geq 3$ we follow the same lines of the proof of Theorem 3 (see the upper bound for $|R_r|$) and, by taking into account that $|\mathrm{i}| = 1$, we obtain

$$\Big|\mathbb{E}_Q \int_{\mathbb{R}^d} \sum_{k\geq 3} \frac{(\mathrm{i}t\mathcal{H}^n((x+Z_0)\cap rW))^k}{(\sqrt{r^d})^k k!} \alpha \mathrm{d}x\Big| \leq |W|\alpha \frac{|t|^3}{t_0^3} \frac{\mathbb{E}_Q[e^{t_0\mathcal{H}^n(Z_0)}]}{\sqrt{r^d}} \tag{25}$$

for all $|t| \leq \sqrt{r^d} t_0$. Hence we get the following bound for the difference between the logarithms of the characteristic functions φ_r and φ:

$$
\begin{aligned}
|\log \varphi_r(t) - \log \varphi(t)| &= \left| \log \varphi_r(t) + \alpha |W| \mathbb{E}_Q[\mathcal{H}^n(Z_0)^2] \frac{t^2}{2} \right| \\
&\overset{(25)}{\leq} |W| \alpha \frac{|t|^3}{t_0^3} \frac{\mathbb{E}_Q[e^{t_0 \mathcal{H}^n(Z_0)}]}{\sqrt{r^d}} \\
&\quad + \left| \alpha |W| \mathbb{E}_Q[\mathcal{H}^n(Z_0)^2] \frac{t^2}{2} - \mathbb{E}_Q \int_{\mathbb{R}^d} \frac{t^2 \mathcal{H}^n((x + Z_0) \cap rW))^2}{r^d 2} \alpha \mathrm{d}x \right| \\
&\overset{(24)}{\leq} |W| \alpha \frac{|t|^3}{t_0^3} \frac{\mathbb{E}_Q[e^{t_0 \mathcal{H}^n(Z_0)}]}{\sqrt{r^d}} \\
&\quad + \frac{\alpha t^2}{2} \mathbb{E}_Q \int_{\mathbb{R}^{3d}} \mathbf{1}_W(y) \left| 1 - \mathbf{1}_W\left(y + \frac{\xi_2 - \xi_1}{r} \right) \right| \mathrm{d}y \prod_{i=1}^{2} \mathbf{1}_{Z_0}(\xi_i) \mathcal{H}^n(\mathrm{d}\xi_i) \\
&= |W| \alpha \frac{|t|^3}{t_0^3} \frac{\mathbb{E}_Q[e^{t_0 \mathcal{H}^n(Z_0)}]}{\sqrt{r^d}} \\
&\quad + \frac{\alpha t^2}{2} \mathbb{E}_Q \int_{\mathbb{R}^{3d}} \mathbf{1}_W(y) \mathbf{1}_{W^c}\left(y + \frac{\xi_2 - \xi_1}{r} \right) \mathrm{d}y \prod_{i=1}^{2} \mathbf{1}_{Z_0}(\xi_i) \mathcal{H}^n(\mathrm{d}\xi_i)
\end{aligned}
$$

for all $|t| \leq \sqrt{r^d} t_0$. By observing that

$$
y \in \left(W^c - \frac{\xi_2 - \xi_1}{r} \right) \subseteq W^c_{\oplus \mathrm{diam}(Z_0)/r} \subseteq W^c_{\oplus K/r},
$$

we get

$$
\begin{aligned}
&|\log \varphi_r(t) - \log \varphi(t)| \\
&\leq |W| \alpha \frac{|t|^3}{t_0^3} \frac{\mathbb{E}_Q[e^{t_0 \mathcal{H}^n(Z_0)}]}{\sqrt{r^d}} + \frac{\alpha t^2}{2} \mathbb{E}_Q \int_{\mathbb{R}^{3d}} \mathbf{1}_W(y) \mathbf{1}_{W^c_{\oplus K/r}}(y) \mathrm{d}y \prod_{i=1}^{2} \mathbf{1}_{Z_0}(\xi_i) \mathcal{H}^n(\mathrm{d}\xi_i) \\
&= |W| \alpha \frac{|t|^3}{t_0^3} \frac{\mathbb{E}_Q[e^{t_0 \mathcal{H}^n(Z_0)}]}{\sqrt{r^d}} + \frac{\alpha t^2}{2} \mathbb{E}_Q[\mathcal{H}^n(Z_0)^2] \mathcal{H}^d\left(W^c_{\oplus K/r} \setminus W^c \right) \\
&\leq |W| \alpha \frac{|t|^3}{t_0^3} \frac{\mathbb{E}_Q[e^{t_0 \mathcal{H}^n(Z_0)}]}{\sqrt{r^d}} + \frac{\alpha t^2}{2} \mathbb{E}_Q[\mathcal{H}^n(Z_0)^2] \mathcal{H}^d\left((\partial W)_{\oplus K/r} \right)
\end{aligned}
$$

for all $|t| \leq \sqrt{r^d} t_0$. An application of Ambrosio et al. (2009, Lemma 7) gives

$$
|\log \varphi_r(t) - \log \varphi(t)| \leq |W| \alpha \frac{|t|^3}{t_0^3} \frac{\mathbb{E}_Q[e^{t_0 \mathcal{H}^n(Z_0)}]}{\sqrt{r^d}} + \frac{\alpha t^2}{2r} \mathbb{E}_Q[\mathcal{H}^n(Z_0)^2] C(\gamma) \tag{26}
$$

for all $r > K/2$ and $|t| \leq \sqrt{r^d} t_0$, where $C(\gamma) = 2^{d-1} 4^d b_d K/\gamma$, with γ the constant appearing in (21).

Now we use the elementary inequality $|e^z - 1| \leq |z| e^{|z|}$ to bound the difference between the characteristic functions:

$$
\begin{aligned}
|\varphi_r(t) - \varphi(t)| &= \varphi(t)\left|\frac{\varphi_r(t)}{\varphi(t)} - 1\right| = \varphi(t)\left|e^{\log\varphi_r(t) - \log\varphi(t)} - 1\right| \\
&\leq e^{\log\varphi(t)}|\log\varphi_r(t) - \log\varphi(t)|e^{|\log\varphi_r(t) - \log\varphi(t)|} \\
&\overset{(26)}{\leq} \left(|W|\alpha\frac{|t|^3}{t_0^3}\frac{\mathbb{E}_Q[e^{t_0\mathcal{H}^n(Z_0)}]}{\sqrt{r^d}} + \frac{\alpha t^2}{2r}\mathbb{E}_Q[\mathcal{H}^n(Z_0)^2]C(\gamma)\right) \\
&\qquad \times \exp\left\{|\log\varphi_r(t) - \log\varphi(t)| - t^2\alpha|W|\mathbb{E}_Q[\mathcal{H}^n(Z_0)^2]/2\right\}.
\end{aligned} \tag{27}
$$

By the fact that $r \geq 2C(\gamma)/(a|W|)$ and assuming

$$
|t| \leq \frac{t_0^3\sqrt{r^d}a\mathbb{E}_Q[\mathcal{H}^n(Z_0)^2]}{4\mathbb{E}_Q[e^{t_0\mathcal{H}^n(Z_0)}]}
$$

one can use again Eq. (26) to see that

$$
|\log\varphi_r(t) - \log\varphi(t)| \leq a\frac{t^2|W|\alpha\mathbb{E}_Q[\mathcal{H}^n(Z_0)^2]}{2}.
$$

The last inequality can be used to bound (27):

$$
\begin{aligned}
|\varphi_r(t) - \varphi(t)| \leq &\left(|W|\alpha\frac{|t|^3}{t_0^3}\frac{\mathbb{E}_Q[e^{t_0\mathcal{H}^n(Z_0)}]}{\sqrt{r^d}} + \frac{\alpha t^2}{2r}\mathbb{E}_Q[\mathcal{H}^n(Z_0)^2]C(\gamma)\right) \\
&\times \exp\left\{-t^2(1-a)\alpha|W|\mathbb{E}_Q[\mathcal{H}^n(Z_0)^2]/2\right\},
\end{aligned} \tag{28}
$$

whose validity is guaranteed for all t satisfying

$$
|t| \leq \min\left\{\sqrt{r^d}t_0, \frac{t_0^3\sqrt{r^d}a\mathbb{E}_Q[\mathcal{H}^n(Z_0)^2]}{4\mathbb{E}_Q[e^{t_0\mathcal{H}^n(Z_0)}]}\right\} = \sqrt{r^d}\ell(a)
$$

where $\ell(a)$ is the constant defined in the statement of the theorem. We can now apply the Berry–Esseen inequality (Loève 1977, p. 297) to obtain

$$
\begin{aligned}
D(r) &\leq \frac{2}{\pi}\int_0^{\sqrt{r^d}\ell(a)}\frac{|\varphi_r(t) - \varphi(t)|}{t}\mathrm{d}t + \frac{24}{\pi\sqrt{r^d}\ell(a)}\sup_{x\in\mathbb{R}}|F'(x)| \\
&\leq \frac{2}{\pi}\int_0^{\sqrt{r^d}\ell(a)}\frac{|\varphi_r(t) - \varphi(t)|}{t}\mathrm{d}t + \frac{24}{\pi\sqrt{r^d}\ell(a)\sqrt{2\pi\alpha|W|\mathbb{E}_Q[\mathcal{H}^n(Z_0)^2]}}.
\end{aligned}
$$

Thus the desired estimate of $D(r)$ in (22) now follows by substituting (28) in the previous inequality and by computing the integral over the whole positive real line $\mathbb{R}^+$ with standard calculations. □

We remark that Theorem 4 provides us with a rate of convergence of $(X_r(W) - \mathbb{E}[X_r(W)])/\sqrt{r^d}$ to the limiting centered Gaussian distribution with variance $\alpha|W|\mathbb{E}_Q[\mathcal{H}^n(Z_0)^2]$: The Kolmogorov distance $D(r)$ goes to zero at the rate $\max\{1/r, 1/\sqrt{r^d}\}$ as $r \to +\infty$. It is worth pointing out that one can try to optimize numerically the constant a appearing in (22) to obtain the best bound.

4 Discussion on the non-stationary case

In Sect. 3 we have focused on stationary Boolean models, proving asymptotic results for the estimator $\widehat{\lambda}^{(r)}_{\Theta_n}$ of the mean density λ_{Θ_n} based on a single realization of Θ_n as in (2). More precisely, as r grows to infinity, we have been able to determine a strong law of large numbers (Corollary 1), also referred to as ergodic-type (or strong consistency) result, and an asymptotic normality result (Remark 3) for the estimator $\widehat{\lambda}^{(r)}_{\Theta_n}$. As a consequence of this asymptotic theory, the estimator $\widehat{\lambda}^{(r)}_{\Theta_n} = \frac{X_r(W)}{r^d|W|}$ of λ_{Θ_n} has good statistical properties, when the Boolean model is stationary and the observation window W is sufficiently large. The non-stationary case is much more involved and the determination of good estimators of the mean density based on a single realization of Θ_n is still an open question. In the present section we want to provide a guideline on how to choose the estimator of the mean density $\lambda_{\Theta_n}(x)$, for a fixed $x \in \mathbb{R}^d$, based on a single observation in the non-stationary case. We shall do this by taking advantage of the infinite divisibility property of the Poisson random variable, together with already known results in Camerlenghi et al.(2014a) for the estimation of the mean density $\lambda_{\Theta_n}(x)$ based on a sample of *big* size N.

Our basic idea consists in writing the Boolean model as the superposition of i.i.d. Boolean models with rescaled intensity measure. Thus, by using similar notation as in the previous sections, let $\Theta^{(N)}_n$ be an n-dimensional Boolean model in $\mathbb{R}^d$ defined as in (2), whose intensity depends on $N \geq 1$, which is assumed to grow to infinity. More precisely

$$\Theta^{(N)}_n = \bigcup_{(x_i,s_i)\in\Psi^{(N)}} x_i + Z(s_i),$$

and, for a certain $N \geq 1$, we assume that the intensity f_N of the associated marked Poisson point processes of $\Psi^{(N)}$ is of the type

$$f_N(x, s) = N\widetilde{f}(x, s).$$

It is well known that the superposition of independent Poisson point process is still a Poisson point process, whose intensity measure is the sum of the intensity measures of the involved processes. Thus it easily follows that, for any fixed N, $\Theta^{(N)}_n$ equals $\bigcup_{i=1}^N \widetilde{\Theta}^{(i)}_n$ in distribution, where $\{\widetilde{\Theta}^{(i)}_n\}_i$ is a sequence of i.i.d. Boolean models distributed as $\widetilde{\Theta}_n := \bigcup_{(x_i,s_i)\in\widetilde{\Psi}} x_i + Z(s_i)$, where $\widetilde{\Psi}$ is a marked Poisson point process in $\mathbb{R}^d$ with marks in $\mathbf{K}$ and intensity measure $\widetilde{\Lambda}(\mathrm{d}(x,s)) = \widetilde{f}(x,s)\mathrm{d}xQ(\mathrm{d}s)$.

Now, for any $x \in \mathbb{R}^d$, let us consider the sequence $\{L_N(x)\}_{N\geq 1}$ of random variables defined as

$$L_N(x) := \frac{\mathcal{H}^n(\Theta_n^{(N)} \cap B_{r_N}(x))}{N b_d r_N^d},$$

with $\lim_{N\to\infty} r_N = 0$. We expect that the higher f_N is (equivalently N), much more is the information about $\Theta_n^{(N)}$ contained in a neighborhood of x, so that $NL_N(x)$ might provide a good approximation of $\lambda_{\Theta_n^{(N)}}(x)$.

We observe that $L_N(x)$ coincides $\mathbb{P}$-a.s. with the so-called *natural estimator* of the mean density $\lambda_{\widetilde{\Theta}_n}(x)$ of the random closed set $\widetilde{\Theta}_n$ introduced in Camerlenghi et al. (2014a), and defined by

$$\widehat{\lambda}^{\nu,N}_{\widetilde{\Theta}_n}(x) := \frac{\sum_{i=1}^N \mathcal{H}^n(\widetilde{\Theta}_n^{(i)} \cap B_{r_N}(x))}{N b_d r_N^d}; \tag{29}$$

indeed, one has

$$\mathcal{H}^n(\Theta_n^{(N)} \cap B_{r_N}(x)) \overset{\text{a.s.}}{=} \sum_{i=1}^N \mathcal{H}^n(\widetilde{\Theta}_n^{(i)} \cap B_{r_N}(x)).$$

The last equality is a consequence of Lemma 3 in Villa (2014) and the fact that the event that different grains of $\Theta_n^{(N)}$ (and so of the $\widetilde{\Theta}_n^{(i)}$'s) overlap in a subset of $\mathbb{R}^d$ of positive $\mathcal{H}^n$-measure, has null probability. The estimator $\widehat{\lambda}^{\nu,N}_{\widetilde{\Theta}_n}(x)$, called *natural estimator*, can be seen as a particular case of a more general class of *kernel-type estimators* (denoted by $\widehat{\lambda}^{\kappa,N}_{\widetilde{\Theta}_n}(x)$, introduced in Camerlenghi et al. (2014a) as well, and further studied in Camerlenghi and Villa(2018)), by choosing as particular kernel κ on $\mathbb{R}^d$ the function $\kappa(z) := \frac{1}{b_d}\mathbf{1}_{B_1(0)}(z)$. Hence all the results proved in Camerlenghi and Villa (2018) may be applied to $\{L_N(x)\}_{N\geq 1}$, provided that suitable regularity assumptions (generally fulfilled in real applications) are satisfied. It is then possible to state LDP and MDP for the sequence $\{L_N(x)\}_{N\geq 1}$. Moreover, one can also use the results proved in Camerlenghi et al. (2014a) to determine the *best value* of $\bar{r}_N$ in (29) which minimizes the asymptotic mean squared error of $\widehat{\lambda}^{\nu,N}_{\widetilde{\Theta}_n}(x)$ and hence providing *best estimate* of $\lambda_{\Theta_n^{(N)}}(x)$. This suggests that a good estimate of $\lambda_{\Theta_n^{(N)}}(x) = N\lambda_{\widetilde{\Theta}_n}(x)$ based on a single observation of the Boolean model $\Theta_n^{(N)}$, with N sufficiently *big*, is given by

$$\widehat{\lambda}^{\bar{r}}_{\Theta_n^{(N)}}(x) := \frac{\mathcal{H}^n(\Theta_n^{(N)} \cap B_{\bar{r}_N}(x))}{b_d \bar{r}_N^d}$$

with $\bar{r}_N$ as in (Camerlenghi et al. (2014a), Eq. (17)). It can be easily observed that whenever f_N is constant, then $\bar{r} = +\infty$ (see (Camerlenghi et al. (2014a), Sect. 3.3.3) for a detailed discussion), and so $\widehat{\lambda}^{\bar{r}}_{\Theta_n^{(N)}}(x)$ is in accordance with Eq. (17).

5 Connections with the existing literature

In this section we illustrate the relationship of our theoretical findings with some known results in the literature. First of all we show how our results provide a generalization of asymptotic properties of Poisson point processes. More precisely, let $\boldsymbol{\Phi}$ be a stationary Poisson point process in $\mathbb{R}^d$ with intensity α. This corresponds to consider a trivial Boolean model Θ_0 of dimension $n = 0$ as in (2) with $\mathbf{K} = \mathbb{R}^d$ and deterministic typical grain $Z_0 = \{0\}$. Indeed, with these choices, we have that $\boldsymbol{\Psi}(\cdot \times \mathbf{K}) = \boldsymbol{\Phi}$, and $Q(\mathrm{d}s) = \delta_0(s)\mathrm{d}s$, where δ_0 is the Dirac delta function at 0. Hence the random variable $X_r(W)$ under study boils down to

$$X_r(W) = \boldsymbol{\Phi}(rW) = \sum_{x \in \boldsymbol{\Phi}} \mathbf{1}_{rW}(x)$$

whose mean coincides with $\mathbb{E}[X_r(W)] = \alpha|rW|$, and, by Equation (3), the mean density is simply the intensity of the Poisson process, i.e., $\lambda_{\Theta_0} \equiv \alpha$. From Corollary 1 we get that

$$\hat{\boldsymbol{\Phi}}(rW) := \frac{\boldsymbol{\Phi}(rW)}{|rW|} \xrightarrow{\text{a.s.}} \alpha, \quad \text{as } r \to +\infty,$$

where $\hat{\boldsymbol{\Phi}}(W, r)$ is also known as the Berman–Diggle estimator (see, e.g., Diggle (1985); Berman and Diggle (1989)). Moreover, Theorem 3 applies to the random variables

$$\frac{\boldsymbol{\Phi}(rW) - \mathbb{E}[\boldsymbol{\Phi}(rW)]}{a_r\sqrt{r^d}}$$

with rate function $\widetilde{\mathcal{J}}^*(y) = \frac{1}{2\alpha|W|}y^2$, which is in accordance with (Hwang (2000), Theorem 3.1).

Our large deviation results specialize for a class of Poisson processes, called cluster Poisson point processes. We remember that $\boldsymbol{\Phi}$ is a *Poisson cluster point process* in $\mathbb{R}^d$ if

$$\boldsymbol{\Phi} = \bigcup_{x_i \in \boldsymbol{\Phi}_p} (x_i + N_i),$$

where $\boldsymbol{\Phi}_p$ is a Poisson point process $\mathbb{R}^d$ with intensity $\lambda_p(x) \equiv \alpha > 0$, and $\{N_i\}_{i \geq 1}$ is a sequence of i.i.d. point processes in $\mathbb{R}^d$ as well as N_0. If the *typical cluster* N_0 is such that $\mathbb{E}[N_0(\mathbb{R}^d)] < \infty$, then $\boldsymbol{\Phi}$ is a point process with constant intensity $\alpha\mathbb{E}[N_0(\mathbb{R}^d)]$. Equivalently, $\boldsymbol{\Phi}$ may be viewed as a Boolean model Θ_0 as in (2), with Hausdorff dimension $n = 0$, where in this case the mark space $\mathbf{K}$ is the space of counting measures (point processes) in $\mathbb{R}^d$, $\boldsymbol{\Psi}(\cdot \times \mathbf{K}) = \boldsymbol{\Phi}_p$, and typical grain $Z_0 \equiv N_0$. Note that, by interpreting N_0 as sequences of points in $\mathbb{R}^d$, we may equivalently write $\mathbb{E}[N_0(\mathbb{R}^d)] = \mathbb{E}[\mathcal{H}^0(N_0)]$. It follows that Theorem 2 applies, and we obtain here as particular case the same results proved in (Burton and Dehling (1990), Theorem 3.2):

(a) $\lim_{r\to\infty} \frac{1}{r^d} \log \mathbb{E}[e^{t\Phi(rW)}] = \alpha|W|\Big(\mathbb{E}[e^{tN_0(\mathbb{R}^d)}] - 1\Big)$
(b) $\{\Phi(rW)/r^d\}_{r>0}$ satisfies the LDP with rate function

$$J^*(y) = \sup_{t\in\mathbb{R}} \left\{ ty - \alpha|W|(\mathbb{E}[e^{tN_0(\mathbb{R}^d)}] - 1) \right\}.$$

Such a result is also in accordance with Theorem 5.1.1 in Bordenave and Torrisi (2007), which provides a large deviation principle for Poisson cluster point processes and Hawkes point processes; here the authors do not study moderate deviations; however, they present a sample-path versions of their results.

With regards to moderate deviations for the Poisson cluster process Φ, Theorem 3 in this paper allows to recover the results proved in (Jiang et al. (1992), Theorem 3.1) and in (Hwang(2000), Theorem 3.3), i.e., for every family of positive numbers $\{a_r\}_{r>0}$ such that (18) holds, the family of random variables

$$\left\{ \frac{\Phi(rW) - \mathbb{E}[\Phi(rW)]}{a_r\sqrt{r^d}} \right\}_{r>0}$$

satisfies the LDP with velocity a_r^2, and rate function $\widetilde{J}^*(y) = \frac{y^2}{2\alpha|W|\mathbb{E}[(N_0(\mathbb{R}^d))^2]}$. Actually (Hwang (2000), Theorem 3.3) is proved under a stronger condition, i.e., $\mathbb{E}[e^{t\mathcal{H}^n(Z_0)}] < \infty$ for all $t \in \mathbb{R}$.

As already mentioned in the Introduction, Corollary 2 on asymptotic normality and the related result on the estimator $\widehat{\lambda}^{(r)}_{\Theta_n}$ derived in Remark 3 are in accordance with known results in the literature, retrieved here as a by-product of moderate deviation results. Among the papers cited in the Introduction, it is worth mentioning Pawlas (2003), where a central limit theorem for random measures generated by stationary processes of compact sets is proved. More precisely, the more general quantity (9) with $\mathcal{H}^n$ replaced by an arbitrary translation invariant Borel measure on $\mathbb{R}^d$, and Φ stationary marked point process, non-necessarily Poisson, is considered; the normal convergence is obtained through the study of the characteristic function, under the assumption that a similar central limit theorem holds for Ψ, together with second order conditions ensuring the existence of appropriate variances. Our Corollary 2 may be obtained also as a special case in Pawlas(2003).

We already observed that the assumption n strictly less than d is crucial for the validity of

$$\mathcal{H}^n(\Theta_n \cap W) \overset{\text{a.s.}}{=} \sum_{(x_i,s_i)\in\Psi} \mathcal{H}^n((x_i + Z(s_i)) \cap W). \tag{30}$$

As the window W increases to the whole space, the border effects vanish and $\mathcal{H}^n(\Theta_n \cap W)$ behaves like a compound Poisson process, as pointed out in Remarks 1–2. Moreover, (30) reminds the definition of a multidimensional shot-noise Poisson processes, which is a random field $\{v(y)\}_{y\in\mathbb{R}^d}$ of the type

$$v(y) = \sum_{(x_i,s_i)\in\Psi} g(y - x_i, s_i),$$

where Ψ is a stationary marked Poisson point process and $g : \mathbb{R}^d \times \mathbf{K} \to \mathbb{R}$ is a measurable function, as defined in Heinrich and Schmidt (1985). If we replace W with $B_r(y)$ in (30), one can easily see that

$$\mathcal{H}^n(\Theta_n \cap W) \overset{\text{a.s.}}{=} \sum_{(x_i,s_i)\in\Psi} \mathcal{H}^n(B_r(y - x_i) \cap Z(s_i))$$

and one could wish to relate the results in Heinrich and Schmidt (1985) on shot-noise Poisson processes to our setting. For instance, one could find similarities between the asymptotic results discussed in Sect. 4 concerning the non-stationary case, and the asymptotic normality results for suitable high density Poisson shot-noise process in Heinrich and Schmidt (1985).

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