Supplementary Material for
"Robust test for structural instability in dynamic factor models"*

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To verify the theorems in this paper, it is required to bound the derivatives of the objective function.

Lemma S1. *Suppose that assumption* **(A2)** *holds. Then, for* $i, j, m, s = 1, \ldots, r$ *and* $k, q = 1, \ldots, p$*, we have*

$$
\begin{aligned}
\sup_{\theta\in\Theta} |h_{\alpha,t}(\theta)| &\leq \frac{1}{(2\pi\lambda_L)^{r\alpha/2}}\left(2+\frac{1}{\alpha}\right), \\
\sup_{\theta\in\Theta}\left|\frac{\partial h_{\alpha,t}(\theta)}{\partial c_i}\right| &\leq \frac{1+\alpha}{(2\pi\lambda_L)^{r\alpha/2}}\sqrt{\frac{r}{\alpha e}}\frac{\sqrt{\lambda_U}}{\lambda_L}, \\
\sup_{\theta\in\Theta}\left|\frac{\partial h_{\alpha,t}(\theta)}{\partial A_{k,ij}}\right| &\leq \frac{1+\alpha}{(2\pi\lambda_L)^{r\alpha/2}}\sqrt{\frac{r}{\alpha e}}\frac{\sqrt{\lambda_U}}{\lambda_L}|Y_{t-k,j}|, \\
\sup_{\theta\in\Theta}\left|\frac{\partial h_{\alpha,t}(\theta)}{\partial \Sigma_{ij}}\right| &\leq \frac{1}{(2\pi\lambda_L)^{r\alpha/2}}\frac{1}{\lambda_L}\left\{2\alpha+1+\frac{2r(1+\alpha)}{\alpha e}\frac{\lambda_U}{\lambda_L}\right\}, \\
\sup_{\theta\in\Theta}\left|\frac{\partial^2 h_{\alpha,t}(\theta)}{\partial c_i\partial c_j}\right| &\leq \frac{1+\alpha}{(2\pi\lambda_L)^{r\alpha/2}}\frac{1}{\lambda_L}\left(\frac{2r}{e}\frac{\lambda_U}{\lambda_L}+1\right), \\
\sup_{\theta\in\Theta}\left|\frac{\partial^2 h_{\alpha,t}(\theta)}{\partial c_i\partial A_{k,ms}}\right| &\leq \frac{1+\alpha}{(2\pi\lambda_L)^{r\alpha/2}}\frac{1}{\lambda_L}\left(\frac{2r}{e}\frac{\lambda_U}{\lambda_L}+1\right)|Y_{t-k,s}|, \\
\sup_{\theta\in\Theta}\left|\frac{\partial^2 h_{\alpha,t}(\theta)}{\partial c_i\partial \Sigma_{ms}}\right| &\leq \frac{1+\alpha}{(2\pi\lambda_L)^{r\alpha/2}}\sqrt{\frac{r}{\alpha e}}\frac{\sqrt{\lambda_U}}{\lambda_L^2}\left(\alpha+2+\frac{4\sqrt{2}r}{e}\frac{\lambda_U}{\lambda_L}\right), \\
\sup_{\theta\in\Theta}\left|\frac{\partial^2 h_{\alpha,t}(\theta)}{\partial A_{k,ij}\partial A_{q,ms}}\right| &\leq \frac{1+\alpha}{(2\pi\lambda_L)^{r\alpha/2}}\frac{1}{\lambda_L}\left(\frac{2r}{e}\frac{\lambda_U}{\lambda_L}+1\right)|Y_{t-k,j}||Y_{t-q,s}|,
\end{aligned}
$$

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$$\sup_{\theta\in\Theta}\left|\frac{\partial^2 h_{\alpha,t}(\theta)}{\partial A_{k,ij}\partial\Sigma_{ms}}\right| \leq \frac{1+\alpha}{(2\pi\lambda_L)^{r\alpha/2}}\sqrt{\frac{r}{\alpha e}}\frac{\sqrt{\lambda_U}}{\lambda_L^2}\left(\alpha+2+\frac{4\sqrt{2}r}{e}\frac{\lambda_U}{\lambda_L}\right)|Y_{t-k,j}|,$$

$$\sup_{\theta\in\Theta}\left|\frac{\partial^2 h_{\alpha,t}(\theta)}{\partial\Sigma_{ij}\partial\Sigma_{ms}}\right| \leq \frac{1}{(2\pi\lambda_L)^{r\alpha/2}}\frac{1}{\lambda_L^2}\left\{(\alpha+2)(2\alpha+1)+\right.$$
$$\left.\frac{4r(1+\alpha)(\alpha+2)}{\alpha e}\frac{\lambda_U}{\lambda_L}+\frac{16r^2(1+\alpha)}{\alpha e^2}\frac{\lambda_U^2}{\lambda_L^2}\right\},$$

where c_i and $Y_{t,i}$ are the i-th element of c and Y_t, and $A_{k,ij}$ and Σ_{ij} are the (i,j)-th element of A_k and Σ, respectively.

Proof. From **(A2)**, we have

$$\lambda_L^r \leq \lambda_{min}(\Sigma)^r \leq \det(\Sigma) \leq \lambda_{max}(\Sigma)^r \leq \lambda_U^r. \tag{S.1}$$

Since Σ^{-1} is positive definite, it holds that

$$\begin{aligned}|h_{\alpha,t}(\theta)| &= \frac{1}{(2\pi)^{r\alpha/2}\det(\Sigma)^{\alpha/2}}\left|\frac{1}{(1+\alpha)^{r/2}}-\left(1+\frac{1}{\alpha}\right)\exp\left(-\frac{\alpha}{2}\epsilon_t^T\Sigma^{-1}\epsilon_t\right)\right| \\ &\leq \frac{1}{(2\pi\lambda_L)^{r\alpha/2}}\left(2+\frac{1}{\alpha}\right),\end{aligned}$$

for all $\theta\in\Theta$. Hence, the first part of the lemma is established.

Next, we handle the first derivatives of $h_{\alpha,t}(\theta)$. For $i,j=1,\ldots,r$, we denote Σ_{ij}^{-1} and $\epsilon_{t,i}$ by the (i,j)-th element of Σ^{-1} and i-th element of ϵ_t, respectively. Note that Σ^{-1} can be decomposed by UDU^T, where U is an orthogonal matrix and D is a diagonal matrix of which diagonal elements are inverse of the eigenvalues of Σ. Thus, by **(A2)**,

$$\frac{1}{\lambda_U}\epsilon_t^T\epsilon_t \leq \epsilon_t^T\Sigma^{-1}\epsilon_t = \epsilon_t^T UDU^T\epsilon_t = (U^T\epsilon_t)^T D(U^T\epsilon_t) \leq \frac{1}{\lambda_L}\epsilon_t^T\epsilon_t,$$

so that

$$\exp\left(-\frac{\alpha}{2}\frac{1}{\lambda_L}\epsilon_t^T\epsilon_t\right) \leq \exp\left(-\frac{\alpha}{2}\epsilon_t^T\Sigma^{-1}\epsilon_t\right) \leq \exp\left(-\frac{\alpha}{2}\frac{1}{\lambda_U}\epsilon_t^T\epsilon_t\right). \tag{S.2}$$

By Jensen's inequality and (S.2), it can be shown that

$$\exp\left(-\frac{\alpha}{2}\epsilon_t^T\Sigma^{-1}\epsilon_t\right)\sum_{l=1}^{r}|\epsilon_{t,l}| \leq \sqrt{r}\exp\left(-\frac{\alpha}{2}\frac{1}{\lambda_U}\epsilon_t^T\epsilon_t\right)\sqrt{\epsilon_t^T\epsilon_t} \leq \sqrt{\frac{r\lambda_U}{\alpha e}}. \tag{S.3}$$

Combining (S.1), (S.3), and the fact that for $i,j=1,\ldots,r$,

$$|\Sigma_{ij}^{-1}| \leq \|\Sigma^{-1}\| = \sqrt{\lambda_{max}((\Sigma^{-1})^T\Sigma^{-1})} = \frac{1}{\lambda_{min}(\Sigma)} \leq \frac{1}{\lambda_L}, \tag{S.4}$$

we have

$$\begin{aligned}
\left|\frac{\partial h_{\alpha,t}(\theta)}{\partial c_i}\right| &= \left|-\frac{1+\alpha}{(2\pi)^{r\alpha/2}\det(\Sigma)^{\alpha/2}}\exp\left(-\frac{\alpha}{2}\epsilon_t^T\Sigma^{-1}\epsilon_t\right)\sum_{l=1}^{r}\Sigma_{il}^{-1}\epsilon_{t,l}\right| \\
&\le \frac{1+\alpha}{(2\pi)^{r\alpha/2}\det(\Sigma)^{\alpha/2}}\exp\left(-\frac{\alpha}{2}\epsilon_t^T\Sigma^{-1}\epsilon_t\right)\sum_{l=1}^{r}|\Sigma_{il}^{-1}||\epsilon_{t,l}| \\
&\le \frac{1+\alpha}{(2\pi\lambda_L)^{r\alpha/2}}\frac{1}{\lambda_L}\sqrt{\frac{r\lambda_U}{\alpha e}},
\end{aligned}$$

and

$$\begin{aligned}
\left|\frac{\partial h_{\alpha,t}(\theta)}{\partial A_{k,ij}}\right| &= \left|-\frac{1+\alpha}{(2\pi)^{r\alpha/2}\det(\Sigma)^{\alpha/2}}\exp\left(-\frac{\alpha}{2}\epsilon_t^T\Sigma^{-1}\epsilon_t\right)Y_{t-k,j}\sum_{l=1}^{r}\Sigma_{il}^{-1}\epsilon_{t,l}\right| \\
&= \left|\frac{\partial h_{\alpha,t}(\theta)}{\partial c_i}Y_{t-k,j}\right| \\
&\le \frac{1+\alpha}{(2\pi\lambda_L)^{r\alpha/2}}\frac{1}{\lambda_L}\sqrt{\frac{r\lambda_U}{\alpha e}}|Y_{t-k,j}|.
\end{aligned}$$

Therefore, the second and third parts of the lemma are validated. Due to (S.2), (S.4), and the fact that $|x^TAx| \le r\|A\|x^Tx$ for $x \in \mathbb{R}^r$ and $A \in \mathbb{R}^{r\times r}$, we have

$$\begin{aligned}
&\exp\left(-\frac{\alpha}{2}\epsilon_t^T\Sigma^{-1}\epsilon_t\right)\left|\epsilon_t^T\Sigma^{-1}((1-\delta_{ij})J_{ij}+J_{ji})\Sigma^{-1}\epsilon_t\right| \\
\le\ & \exp\left(-\frac{\alpha}{2}\frac{1}{\lambda_U}\epsilon_t^T\epsilon_t\right)r\left\|\Sigma^{-1}((1-\delta_{ij})J_{ij}+J_{ji})\Sigma^{-1}\right\|\epsilon_t^T\epsilon_t \\
\le\ & \frac{2r}{\lambda_L^2}\exp\left(-\frac{\alpha}{2}\frac{1}{\lambda_U}\epsilon_t^T\epsilon_t\right)\epsilon_t^T\epsilon_t \\
\le\ & \frac{2r}{\lambda_L^2}\frac{2\lambda_U}{\alpha e} = \frac{4r}{\alpha e}\frac{\lambda_U}{\lambda_L^2},
\end{aligned} \tag{S.5}$$

where J_{ij} is a $r\times r$ matrix of which (i,j)-th element is equal to 1 and all other elements are 0, and $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. Note that since Σ is symmetric, $\partial\det(\Sigma)/\partial\Sigma_{ij} = \det(\Sigma)(2-\delta_{ij})\Sigma_{ij}^{-1}$ and $\partial\Sigma^{-1}/\partial\Sigma_{ij} = -\Sigma^{-1}((1-\delta_{ij})J_{ij}+J_{ji})\Sigma^{-1}$. From these facts together with (S.1), (S.4), and (S.5), it holds that

$$\begin{aligned}
\left|\frac{\partial h_{\alpha,t}(\theta)}{\partial\Sigma ij}\right| &= \Bigg|-\frac{1}{(2\pi)^{r\alpha/2}\det(\Sigma)^{\alpha/2}}\Bigg[\frac{\alpha}{2}(2-\delta_{ij})\Sigma_{ij}^{-1}\left\{\frac{1}{(1+\alpha)^{r/2}}-\left(1+\frac{1}{\alpha}\right)\exp\left(-\frac{\alpha}{2}\epsilon_t^T\Sigma^{-1}\epsilon_t\right)\right\} \\
&\qquad +\frac{1+\alpha}{2}\exp\left(-\frac{\alpha}{2}\epsilon_t^T\Sigma^{-1}\epsilon_t\right)\epsilon_t^T\Sigma^{-1}((1-\delta_{ij})J_{ij}+J_{ji})\Sigma^{-1}\epsilon_t\Bigg]\Bigg| \\
&\le \frac{1}{(2\pi)^{r\alpha/2}\det(\Sigma)^{\alpha/2}}\Bigg\{\frac{\alpha}{2}(2-\delta_{ij})|\Sigma_{ij}^{-1}|\left|\frac{1}{(1+\alpha)^{r/2}}-\left(1+\frac{1}{\alpha}\right)\exp\left(-\frac{\alpha}{2}\epsilon_t^T\Sigma^{-1}\epsilon_t\right)\right| \\
&\qquad +\frac{1+\alpha}{2}\exp\left(-\frac{\alpha}{2}\epsilon_t^T\Sigma^{-1}\epsilon_t\right)\left|\epsilon_t^T\Sigma^{-1}((1-\delta_{ij})J_{ij}+J_{ji})\Sigma^{-1}\epsilon_t\right|\Bigg\} \\
&\le \frac{1}{(2\pi\lambda_L)^{r\alpha/2}}\left\{\frac{\alpha}{\lambda_L}\left(2+\frac{1}{\alpha}\right)+\frac{1+\alpha}{2}\frac{4r}{\alpha e}\frac{\lambda_U}{\lambda_L^2}\right\}.
\end{aligned}$$

Thus, the fourth part of the lemma is verified.

Now, we consider the second derivatives of $h_{\alpha,t}(\theta)$. By (S.2), (S.4), and Jensen's inequality, it holds that

$$\begin{aligned}
\exp\left(-\frac{\alpha}{2}\epsilon_t^T\Sigma^{-1}\epsilon_t\right)\left(\sum_{l=1}^r|\Sigma_{jl}^{-1}||\epsilon_{t,l}|\right)\left(\sum_{l=1}^r|\Sigma_{il}^{-1}||\epsilon_{t,l}|\right) &\leq \frac{1}{\lambda_L^2}\exp\left(-\frac{\alpha}{2}\frac{1}{\lambda_U}\epsilon_t^T\epsilon_t\right)\left(\sum_{l=1}^r|\epsilon_{t,l}|\right)^2 \\
&\leq \frac{r}{\lambda_L^2}\exp\left(-\frac{\alpha}{2}\frac{1}{\lambda_U}\epsilon_t^T\epsilon_t\right)\epsilon_t^T\epsilon_t \\
&\leq \frac{r}{\lambda_L^2}\frac{2\lambda_U}{\alpha e} = \frac{2r}{\alpha e}\frac{\lambda_U}{\lambda_L^2}. \qquad \text{(S.6)}
\end{aligned}$$

Hence, from (S.1), (S.4), and (S.6), we have

$$\begin{aligned}
\left|\frac{\partial^2 h_{\alpha,t}(\theta)}{\partial c_i\partial c_j}\right| &= \left|-\frac{1+\alpha}{(2\pi)^{r\alpha/2}\det(\Sigma)^{\alpha/2}}\left\{\alpha\exp\left(-\frac{\alpha}{2}\epsilon_t^T\Sigma^{-1}\epsilon_t\right)\left(\sum_{l=1}^r\Sigma_{jl}^{-1}\epsilon_{t,l}\right)\left(\sum_{l=1}^r\Sigma_{il}^{-1}\epsilon_{t,l}\right)\right.\right. \\
&\qquad \left.\left.-\Sigma_{ij}^{-1}\exp\left(-\frac{\alpha}{2}\epsilon_t^T\Sigma^{-1}\epsilon_t\right)\right\}\right| \\
&\leq \frac{1+\alpha}{(2\pi)^{r\alpha/2}\det(\Sigma)^{\alpha/2}}\left\{\alpha\exp\left(-\frac{\alpha}{2}\epsilon_t^T\Sigma^{-1}\epsilon_t\right)\left(\sum_{l=1}^r|\Sigma_{jl}^{-1}||\epsilon_{t,l}|\right)\left(\sum_{l=1}^r|\Sigma_{il}^{-1}||\epsilon_{t,l}|\right)\right. \\
&\qquad \left.+|\Sigma_{ij}^{-1}|\exp\left(-\frac{\alpha}{2}\epsilon_t^T\Sigma^{-1}\epsilon_t\right)\right\} \\
&\leq \frac{1+\alpha}{(2\pi\lambda_L)^{r\alpha/2}}\left(\alpha\frac{2r}{\alpha e}\frac{\lambda_U}{\lambda_L^2}+\frac{1}{\lambda_L}\right),
\end{aligned}$$

and since $\partial h_{\alpha,t}(\theta)/\partial A_{k,ms} = (\partial h_{\alpha,t}(\theta)/\partial c_m)Y_{t-k,s}$,

$$\left|\frac{\partial^2 h_{\alpha,t}(\theta)}{\partial c_i\partial A_{k,ms}}\right| = \left|\frac{\partial^2 h_{\alpha,t}(\theta)}{\partial c_i\partial c_m}Y_{t-k,s}\right| \leq \frac{1+\alpha}{(2\pi\lambda_L)^{r\alpha/2}}\left(\alpha\frac{2r}{\alpha e}\frac{\lambda_U}{\lambda_L^2}+\frac{1}{\lambda_L}\right)|Y_{t-k,s}|. \qquad \text{(S.7)}$$

Hence, fifth and sixth parts of the lemma are established. Due to (S.1), (S.3), (S.4), and (S.5), it can be shown that

$$\begin{aligned}
\left|\frac{\partial^2 h_{\alpha,t}(\theta)}{\partial c_i\partial\Sigma_{ms}}\right| &= \left|-\frac{1+\alpha}{(2\pi)^{r\alpha/2}\det(\Sigma)^{\alpha/2}}\left[-\frac{\alpha}{2}(2-\delta_{ms})\Sigma_{ms}^{-1}\exp\left(-\frac{\alpha}{2}\epsilon_t^T\Sigma^{-1}\epsilon_t\right)\sum_{l=1}^r\Sigma_{il}^{-1}\epsilon_{t,l}\right.\right. \\
&\qquad +\frac{\alpha}{2}\exp\left(-\frac{\alpha}{2}\epsilon_t^T\Sigma^{-1}\epsilon_t\right)\left(\sum_{l=1}^r\Sigma_{il}^{-1}\epsilon_{t,l}\right)\epsilon_t^T\Sigma^{-1}((1-\delta_{ms})J_{ms}+J_{sm})\Sigma^{-1}\epsilon_t \\
&\qquad \left.\left.-\exp\left(-\frac{\alpha}{2}\epsilon_t^T\Sigma^{-1}\epsilon_t\right)\left\{(1-\delta_{ms})\Sigma_{im}^{-1}\sum_{l=1}^r\Sigma_{sl}^{-1}\epsilon_{t,l}+\Sigma_{is}^{-1}\sum_{l=1}^r\Sigma_{ml}^{-1}\epsilon_{t,l}\right\}\right]\right| \\
&\leq \frac{1+\alpha}{(2\pi)^{r\alpha/2}\det(\Sigma)^{\alpha/2}}\left[\frac{\alpha}{2}(2-\delta_{ms})|\Sigma_{ms}^{-1}|\exp\left(-\frac{\alpha}{2}\epsilon_t^T\Sigma^{-1}\epsilon_t\right)\sum_{l=1}^r|\Sigma_{il}^{-1}||\epsilon_{t,l}|\right. \\
&\qquad +\frac{\alpha}{2}\left\{\exp\left(-\frac{\alpha}{4}\epsilon^T\Sigma^{-1}\epsilon_t\right)\sum_{l=1}^r|\Sigma_{il}^{-1}||\epsilon_{t,l}|\right\}
\end{aligned}$$

$$
\begin{aligned}
&\times \left\{ \exp\left(-\frac{\alpha}{4}\epsilon^T \Sigma^{-1}\epsilon_t\right) \left|\epsilon_t^T \Sigma^{-1}((1-\delta_{ms})J_{ms} + J_{sm})\Sigma^{-1}\epsilon_t\right| \right\} \\
&+(1-\delta_{ms})|\Sigma^{-1}_{im}| \exp\left(-\frac{\alpha}{2}\epsilon_t^T \Sigma^{-1}\epsilon_t\right) \sum_{l=1}^{r} |\Sigma^{-1}_{sl}||\epsilon_{t,l}| \\
&+|\Sigma^{-1}_{is}| \exp\left(-\frac{\alpha}{2}\epsilon_t^T \Sigma^{-1}\epsilon_t\right) \sum_{l=1}^{r} |\Sigma^{-1}_{ml}||\epsilon_{t,l}| \Bigg] \\
\leq\ & \frac{1+\alpha}{(2\pi\lambda_L)^{r\alpha/2}} \left(\frac{\alpha}{\lambda_L}\frac{1}{\lambda_L}\sqrt{\frac{r\lambda_U}{\alpha e}} + \frac{\alpha}{2}\frac{1}{\lambda_L}\sqrt{\frac{2r\lambda_U}{\alpha e}}\frac{8r}{\alpha e}\frac{\lambda_U}{\lambda_L^2} + \frac{1}{\lambda_L}\frac{1}{\lambda_L}\sqrt{\frac{r\lambda_U}{\alpha e}} + \frac{1}{\lambda_L}\frac{1}{\lambda_L}\sqrt{\frac{r\lambda_U}{\alpha e}} \right) \\
=\ & \frac{1+\alpha}{(2\pi\lambda_L)^{r\alpha/2}} \sqrt{\frac{r}{\alpha e}} \frac{\sqrt{\lambda_U}}{\lambda_L^2} \left(\alpha + 2 + \frac{4\sqrt{2}r}{e}\frac{\lambda_U}{\lambda_L} \right), \qquad \text{(S.8)}
\end{aligned}
$$

and thus, the seventh part of the lemma is validated. Since $\partial h_{\alpha,t}(\theta)/\partial A_{k,ij} = (\partial h_{\alpha,t}(\theta)/\partial c_i)Y_{t-k,j}$, from (S.7) and (S.8), we have

$$
\left|\frac{\partial^2 h_{\alpha,t}(\theta)}{\partial A_{k,ij}\partial A_{q,ms}}\right| = \left|\frac{\partial^2 h_{\alpha,t}(\theta)}{\partial c_i \partial A_{q,ms}} Y_{t-k,j}\right| \leq \frac{1+\alpha}{(2\pi\lambda_L)^{r\alpha/2}} \left(\frac{2r}{e}\frac{\lambda_U}{\lambda_L^2} + \frac{1}{\lambda_L} \right) |Y_{t-q,s}||Y_{t-k,j}|,
$$

and

$$
\left|\frac{\partial^2 h_{\alpha,t}(\theta)}{\partial A_{k,ij}\partial \Sigma_{ms}}\right| = \left|\frac{\partial^2 h_{\alpha,t}(\theta)}{\partial c_i \partial \Sigma_{ms}} Y_{t-k,j}\right| \leq \frac{1+\alpha}{(2\pi\lambda_L)^{r\alpha/2}} \sqrt{\frac{r}{\alpha e}} \frac{\sqrt{\lambda_U}}{\lambda_L^2} \left(\alpha + 2 + \frac{4\sqrt{2}r}{e}\frac{\lambda_U}{\lambda_L} \right) |Y_{t-k,j}|.
$$

Therefore, the eighth and ninth parts of the lemma are established. In a similar way to (S.5), we obtain

$$
\begin{aligned}
& \exp\left(-\frac{\alpha}{2}\epsilon_t^T \Sigma^{-1}\epsilon_t\right) \left|\epsilon_t^T \Sigma^{-1}((1-\delta_{ms})J_{ms} + J_{sm})\Sigma^{-1}\epsilon_t\right| \left|\epsilon_t^T \Sigma^{-1}((1-\delta_{ij})J_{ij} + J_{ji})\Sigma^{-1}\epsilon_t\right| \\
\leq\ & r^2 \exp\left(-\frac{\alpha}{2}\frac{1}{\lambda_U}\epsilon_t^T\epsilon_t\right) (\epsilon_t^T\epsilon_t)^2 \left\|\Sigma^{-1}((1-\delta_{ms})J_{ms} + J_{sm})\Sigma^{-1}\right\| \left\|\Sigma^{-1}((1-\delta_{ij})J_{ij} + J_{ji})\Sigma^{-1}\right\| \\
\leq\ & \frac{4r^2}{\lambda_L^4} \exp\left(-\frac{\alpha}{2}\frac{1}{\lambda_U}\epsilon_t^T\epsilon_t\right) (\epsilon_t^T\epsilon_t)^2 \\
\leq\ & \frac{4r^2}{\lambda_L^4}\frac{16\lambda_U^2}{\alpha^2 e^2} = \frac{64r^2}{\alpha^2 e^2}\frac{\lambda_U^2}{\lambda_L^4}, \qquad \text{(S.9)}
\end{aligned}
$$

and

$$
\begin{aligned}
& \exp\left(-\frac{\alpha}{2}\epsilon_t^T \Sigma^{-1}\epsilon_t\right) \left|\epsilon_t^T \Sigma^{-1}((1-\delta_{ms})J_{ms} + J_{sm})\Sigma^{-1}((1-\delta_{ij})J_{ij} + J_{ji})\Sigma^{-1}\epsilon_t\right| \\
\leq\ & r \exp\left(-\frac{\alpha}{2}\frac{1}{\lambda_U}\epsilon_t^T\epsilon_t\right) \epsilon_t^T\epsilon_t \left\|\Sigma^{-1}((1-\delta_{ms})J_{ms} + J_{sm})\Sigma^{-1}((1-\delta_{ij})J_{ij} + J_{ji})\Sigma^{-1}\right\| \\
\leq\ & \frac{4r}{\lambda_L^3} \exp\left(-\frac{\alpha}{2}\frac{1}{\lambda_U}\epsilon_t^T\epsilon_t\right) \epsilon_t^T\epsilon_t \\
\leq\ & \frac{4r}{\lambda_L^3}\frac{2\lambda_U}{\alpha e} = \frac{8r}{\alpha e}\frac{\lambda_U}{\lambda_L^3}. \qquad \text{(S.10)}
\end{aligned}
$$

Note that since Σ is symmetric, $\partial\Sigma_{ij}^{-1}/\partial\Sigma_{ms} = -\Sigma_{is}^{-1}\Sigma_{mj}^{-1} - (1-\delta_{ms})\Sigma_{im}^{-1}\Sigma_{sj}^{-1}$. Using this fact together with (S.1), (S.4), (S.5), (S.9), and (S.10), we have

$$\begin{aligned}
&\left|\frac{\partial^2 h_{\alpha,t}(\theta)}{\partial\Sigma_{ij}\partial\Sigma_{ms}}\right| \\
= &\left| -\frac{1}{(2\pi)^{r\alpha/2}\det(\Sigma)^{\alpha/2}}\left[-\frac{\alpha}{2}(2-\delta_{ms})\Sigma_{ms}^{-1}\left\{\frac{\alpha}{2}(2-\delta_{ij})\Sigma_{ij}^{-1}\left(\frac{1}{(1+\alpha)^{r/2}} - \left(1+\frac{1}{\alpha}\right)\exp\left(-\frac{\alpha}{2}\epsilon_t^T\Sigma^{-1}\epsilon_t\right)\right)\right.\right.\right. \\
&\left. + \frac{1+\alpha}{2}\exp\left(-\frac{\alpha}{2}\epsilon_t^T\Sigma^{-1}\epsilon_t\right)\epsilon_t^T\Sigma^{-1}((1-\delta_{ij})J_{ij}+J_{ji})\Sigma^{-1}\epsilon_t\right\} \\
&-\frac{\alpha}{2}(2-\delta_{ij})\left\{\Sigma_{is}^{-1}\Sigma_{mj}^{-1} + (1-\delta_{ms})\Sigma_{im}^{-1}\Sigma_{sj}^{-1}\right\}\left\{\frac{1}{(1+\alpha)^{r/2}} - \left(1+\frac{1}{\alpha}\right)\exp\left(-\frac{\alpha}{2}\epsilon_t^T\Sigma^{-1}\epsilon_t\right)\right\} \\
&-\frac{\alpha(1+\alpha)}{4}(2-\delta_{ij})\Sigma_{ij}^{-1}\exp\left(-\frac{\alpha}{2}\epsilon_t^T\Sigma^{-1}\epsilon_t\right)\epsilon_t^T\Sigma^{-1}((1-\delta_{ms})J_{ms}+J_{sm})\Sigma^{-1}\epsilon_t \\
&+\frac{\alpha(1+\alpha)}{4}\exp\left(-\frac{\alpha}{2}\epsilon_t^T\Sigma^{-1}\epsilon_t\right)\left\{\epsilon_t^T\Sigma^{-1}((1-\delta_{ms})J_{ms}+J_{sm})\Sigma^{-1}\epsilon_t\right\} \\
&\qquad\times\left\{\epsilon_t^T\Sigma^{-1}((1-\delta_{ij})J_{ij}+J_{ji})\Sigma^{-1}\epsilon_t\right\} \\
&\left.\left.-(1+\alpha)\exp\left(-\frac{\alpha}{2}\epsilon_t^T\Sigma^{-1}\epsilon_t\right)\epsilon_t^T\Sigma^{-1}((1-\delta_{ms})J_{ms}+J_{sm})\Sigma^{-1}((1-\delta_{ij})J_{ij}+J_{ji})\Sigma^{-1}\epsilon_t\right]\right| \\
\le &\frac{1}{(2\pi)^{r\alpha/2}\det(\Sigma)^{\alpha/2}}\left[\frac{\alpha}{2}(2-\delta_{ms})|\Sigma_{ms}^{-1}|\left\{\frac{\alpha}{2}(2-\delta_{ij})|\Sigma_{ij}^{-1}|\left|\frac{1}{(1+\alpha)^{r/2}} - \left(1+\frac{1}{\alpha}\right)\exp\left(-\frac{\alpha}{2}\epsilon_t^T\Sigma^{-1}\epsilon_t\right)\right|\right.\right. \\
&\left. + \frac{1+\alpha}{2}\exp\left(-\frac{\alpha}{2}\epsilon_t^T\Sigma^{-1}\epsilon_t\right)\left|\epsilon_t^T\Sigma^{-1}((1-\delta_{ij})J_{ij}+J_{ji})\Sigma^{-1}\epsilon_t\right|\right\} \\
&+\frac{\alpha}{2}(2-\delta_{ij})\left|\Sigma_{is}^{-1}\Sigma_{mj}^{-1} + (1-\delta_{ms})\Sigma_{im}^{-1}\Sigma_{sj}^{-1}\right|\left|\frac{1}{(1+\alpha)^{r/2}} - \left(1+\frac{1}{\alpha}\right)\exp\left(-\frac{\alpha}{2}\epsilon_t^T\Sigma^{-1}\epsilon_t\right)\right| \\
&+\frac{\alpha(1+\alpha)}{4}(2-\delta_{ij})|\Sigma_{ij}^{-1}|\exp\left(-\frac{\alpha}{2}\epsilon_t^T\Sigma^{-1}\epsilon_t\right)\left|\epsilon_t^T\Sigma^{-1}((1-\delta_{ms})J_{ms}+J_{sm})\Sigma^{-1}\epsilon_t\right| \\
&+\frac{\alpha(1+\alpha)}{4}\exp\left(-\frac{\alpha}{2}\epsilon_t^T\Sigma^{-1}\epsilon_t\right)\left|\epsilon_t^T\Sigma^{-1}((1-\delta_{ms})J_{ms}+J_{sm})\Sigma^{-1}\epsilon_t\right| \\
&\qquad\times\left|\epsilon_t^T\Sigma^{-1}((1-\delta_{ij})J_{ij}+J_{ji})\Sigma^{-1}\epsilon_t\right| \\
&\left.+(1+\alpha)\exp\left(-\frac{\alpha}{2}\epsilon_t^T\Sigma^{-1}\epsilon_t\right)\left|\epsilon_t^T\Sigma^{-1}((1-\delta_{ms})J_{ms}+J_{sm})\Sigma^{-1}((1-\delta_{ij})J_{ij}+J_{ji})\Sigma^{-1}\epsilon_t\right|\right] \\
\le &\frac{1}{(2\pi\lambda_L)^{r\alpha/2}}\left[\frac{\alpha}{\lambda_L}\left\{\frac{\alpha}{\lambda_L}\left(2+\frac{1}{\alpha}\right) + \frac{1+\alpha}{2}\frac{4r}{\alpha e}\frac{\lambda_U}{\lambda_L^2}\right\} + \alpha\frac{2}{\lambda_L^2}\left(2+\frac{1}{\alpha}\right) + \frac{\alpha(1+\alpha)}{2}\frac{1}{\lambda_L}\frac{4r}{\alpha e}\frac{\lambda_U}{\lambda_L^2}\right. \\
&\left. + \frac{\alpha(1+\alpha)}{4}\frac{64r^2}{\alpha^2 e^2}\frac{\lambda_U^2}{\lambda_L^4} + (1+\alpha)\frac{8r}{\alpha e}\frac{\lambda_U}{\lambda_L^3}\right] \\
= &\frac{1}{(2\pi\lambda_L)^{r\alpha/2}}\frac{1}{\lambda_L^2}\left\{(\alpha+2)(2\alpha+1) + \frac{4r(1+\alpha)(\alpha+2)}{\alpha e}\frac{\lambda_U}{\lambda_L} + \frac{16r^2(1+\alpha)}{\alpha e^2}\frac{\lambda_U^2}{\lambda_L^2}\right\}.
\end{aligned}$$

Therefore, the tenth part of the lemma is verified. □