Supplementary material to
'Mellin-Meijer-kernel density estimation on $\mathbb{R}^+$'

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A Operational properties of the Mellin transform

The operational properties of the Mellin transform include:

$$\mathcal{M}\left(\sum_{q=1}^{Q} a_q f_q; z\right) = \sum_{q=1}^{Q} a_q \mathcal{M}(f_q; z), \qquad z \in \bigcap_{q=1}^{Q} \mathcal{S}_{f_q}, a_1, \ldots, a_Q \in \mathbb{R} \tag{A.1}$$

$$\mathcal{M}(f(ax); z) = a^{-z}\mathcal{M}(f; z), \qquad z \in \mathcal{S}_f, a \in \mathbb{R}^+ \tag{A.2}$$

$$\mathcal{M}(f(x^a); z) = \frac{1}{|a|}\mathcal{M}(f; \frac{z}{a}), \qquad \frac{z}{a} \in \mathcal{S}_f, a \in \mathbb{R}\backslash\{0\} \tag{A.3}$$

$$\mathcal{M}(x^y f(x); z) = \mathcal{M}(f; z+y), \qquad z+y \in \mathcal{S}_f, y \in \mathbb{C} \tag{A.4}$$

$$\mathcal{M}(\log^n(x) f(x); z) = \frac{d^n \mathcal{M}(f; z)}{dz^n}, \qquad z \in \mathcal{S}_f, n \in \mathbb{N} \tag{A.5}$$

$$\mathcal{M}\left(\frac{d^n f}{dx^n}(x); z\right) = (-1)^n (z-n)(z-n+1)\ldots(z-1)\mathcal{M}(f; z-n), \qquad z-n \in \mathcal{S}_f, n \in \mathbb{N} \tag{A.6}$$

$$\mathcal{M}\left(x^n \frac{d^n f}{dx^n}(x); z\right) = (-1)^n z(z+1)\ldots(z+n-1)\mathcal{M}(f; z), \qquad z \in \mathcal{S}_f, n \in \mathbb{N} \tag{A.7}$$

$$\mathcal{M}\left(\left(x\frac{d}{dx}\right)^n f(x); z\right) = (-1)^n z^n \mathcal{M}(f; z), \qquad z \in \mathcal{S}_f, n \in \mathbb{N}, \tag{A.8}$$

where $\left(x\frac{d}{dx}\right)$ is the differential operator ('Mellin derivative'). Proofs of these results are straightforward; see Sneddon (1974, Chapter 4).

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B Meijer parameterisation of the most common distributions on $\mathbb{R}^+$

Common name	Density	Parameters	ν	γ	ξ	θ
Chi	$f(x) = \frac{2^{1-k/2}x^{k-1}e^{-x^2/2}}{\Gamma\left(\frac{k}{2}\right)}$	$k > 0$	$\sqrt{k}$	$\frac{1}{\sqrt{2k}}$	$\frac{1}{2}$	0
Chi-squared	$f(x) = \frac{x^{\frac{k}{2}-1}e^{-x/2}}{2^{k/2}\Gamma\left(\frac{k}{2}\right)}$	$k > 0$	k	$\sqrt{\frac{2}{k}}$	1	0
Erlang	$f(x) = \frac{x^{k-1}e^{-x/\mu}}{\mu^k(k-1)!}$	$\mu > 0, k \in \mathbb{N}$	μk	$\sqrt{\frac{1}{k}}$	1	0
Gamma	$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$	$\alpha, \beta > 0$	$\frac{\alpha}{\beta}$	$\sqrt{\frac{1}{\alpha}}$	1	0
Maxwell	$f(x) = \frac{\sqrt{2}}{\sigma^3\sqrt{\pi}}\, x^2 e^{-x^2/(2\sigma^2)}$	$\sigma > 0$	$\sqrt{3}\,\sigma$	$\frac{1}{\sqrt{6}}$	$\frac{1}{2}$	0
Nakagami	$f(x) = \frac{2m^m}{\Gamma(m)\Omega^m}\, x^{2m-1}e^{-\frac{mx^2}{\Omega}}$	$m, \Omega > 0$	$\sqrt{\Omega}$	$\frac{1}{2\sqrt{m}}$	$\frac{1}{2}$	0
Rayleigh	$f(x) = \frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}$	$\sigma > 0$	$\sqrt{2}\,\sigma$	$\frac{1}{2}$	$\frac{1}{2}$	0
Stacy	$f(x) = \frac{1}{\Gamma\left(\frac{d}{p}\right)}\frac{p}{a^d}x^{d-1}e^{-(x/a)^p}$	$a, d, p > 0$	$a\left(\frac{d}{p}\right)^{1/p}$	$\frac{1}{\sqrt{dp}}$	$\frac{1}{p}$	0
Weibull	$f(x) = \frac{\eta}{\mu}\left(\frac{x}{\mu}\right)^{\eta-1} e^{-\left(\frac{x}{\mu}\right)^\eta}$	$\mu, \eta > 0$	μ	$\frac{1}{\eta}$	$\frac{1}{\eta}$	0
Beta prime	$f(x) = \frac{x^{\alpha-1}(1+x)^{-\alpha-\beta}}{\mathcal{B}(\alpha,\beta)}$	$\alpha, \beta > 0$	α/β	$\sqrt{\frac{1}{\alpha}+\frac{1}{\beta}}$	1	$\tan^{-1}\sqrt{\frac{\alpha}{\beta}}$
Burr	$f(x) = ck\,\frac{x^{c-1}}{(1+x^c)^{k+1}}$	$c, k > 0$	1	$\frac{1}{c}\sqrt{1+\frac{1}{k}}$	$\frac{1}{c}$	$\tan^{-1}\sqrt{\frac{1}{k}}$
Dagum	$f(x) = \frac{apx^{ap-1}}{b^{ap}\left(1+\left(\frac{x}{b}\right)^a\right)^{p+1}}$	$a, b, p > 0$	b	$\frac{1}{a}\sqrt{1+\frac{1}{p}}$	$\frac{1}{a}$	$\tan^{-1}\sqrt{\frac{1}{p}}$
Fisher-Snedecor	$f(x) = \frac{(d_1/d_2)^{d_1/2}}{\mathcal{B}\left(\frac{d_1}{2},\frac{d_2}{2}\right)}\frac{x^{\frac{d_1}{2}-1}}{\left(1+\frac{d_1}{d_2}x\right)^{\frac{d_1+d_2}{2}}}$	$d_1, d_2 > 0$	1	$\sqrt{\frac{2}{d_1}+\frac{2}{d_2}}$	1	$\tan^{-1}\sqrt{\frac{d_1}{d_2}}$
Generalised Pareto	$f(x) = \frac{1}{\sigma}\left(1+\frac{\zeta x}{\sigma}\right)^{-\frac{1}{\zeta}-1}$	$\sigma, \zeta > 0$	$\frac{\sigma}{\zeta}$	$\sqrt{\zeta+1}$	1	$\tan^{-1}\sqrt{\zeta}$
Log-logistic	$f(x) = \frac{\beta}{\alpha}\,\frac{\left(\frac{x}{\alpha}\right)^{\beta-1}}{\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^2}$	$\alpha, \beta > 0$	α	$\frac{\sqrt{2}}{\beta}$	$\frac{1}{\beta}$	$\frac{\pi}{4}$
Singh-Maddala	$f(x) = \frac{aq}{b}\,\frac{x^{a-1}}{\left(1+\left(\frac{x}{b}\right)^a\right)^{q+1}}$	$a, b, q > 0$	$\frac{b}{q^{1/a}}$	$\frac{1}{a}\sqrt{1+\frac{1}{q}}$	$\frac{1}{a}$	$\tan^{-1}\sqrt{\frac{1}{q}}$
Amoroso	$f(x) = \frac{\lvert p\rvert}{\Gamma(\alpha)\sigma}\left(\frac{x}{\sigma}\right)^{\alpha p-1}e^{-(x/\sigma)^p}$	$\alpha, \sigma > 0,\ p \neq 0$	$\sigma\alpha^{1/p}$	$\frac{1}{\lvert p\rvert\sqrt{\alpha}}$	$\frac{1}{\lvert p\rvert}$	$\frac{\pi}{4}\left(1-\frac{p}{\lvert p\rvert}\right)$
Fréchet	$f(x) = \frac{\alpha}{s}\left(\frac{x}{s}\right)^{-1-\alpha}e^{-\left(\frac{x}{s}\right)^{-\alpha}}$	$\alpha, s > 0$	s	$\frac{1}{\alpha}$	$\frac{1}{\alpha}$	$\frac{\pi}{2}$
Inverse Gamma	$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}x^{-\alpha-1}e^{-\beta/x}$	$\alpha, \beta > 0$	$\frac{\beta}{\alpha}$	$\sqrt{\frac{1}{\alpha}}$	1	$\frac{\pi}{2}$
Lévy	$f(x) = \sqrt{\frac{c}{2\pi}}\,\frac{e^{-c/(2x)}}{x^{3/2}}$	$c > 0$	c	$\sqrt{2}$	1	$\frac{\pi}{2}$

Table B.1: Meijer parameterisation (ν, γ, ξ, θ in $L_{\nu,\gamma,\xi,\theta}$, Section 3) for the most common $\mathbb{R}^+$-supported densities, ordered according to the values of θ: $\theta = 0$, $\theta \in (0, \pi/2)$ and $\theta = \pi/2$. The Amoroso distribution includes the cases $\theta = 0$ an $\theta = \pi/2$, depending on the sign of its parameter p.

C Proofs

First we state a technical lemma that will be used repeatedly in the proofs below. Tricomi and Erdélyi (1951) gave the following asymptotic expansion for the ratio of two Gamma functions. Let $t, \alpha, \beta \in \mathbb{C}$. Then, as $|t| \to \infty$,

$$\frac{\Gamma(t+\alpha)}{\Gamma(t+\beta)} = t^{\alpha-\beta}\left(\sum_{k=0}^{M-1} \frac{1}{k!}\frac{\Gamma(1+\alpha-\beta)}{\Gamma(1+\alpha-\beta-k)} B_k^{(1+\alpha-\beta)}(\alpha)\frac{1}{t^k} + R_M(t)\right), \tag{C.1}$$

where $|R_M(t)| = O(|t|^{-M})$, provided that $|\alpha|, |\beta|$ are bounded and $|\arg(t+\alpha)| < \pi$. Here $B_k^{(a)}(x)$ are the generalised Bernoulli polynomials, which are polynomials in a and x of degree k, see Temme (1996, Section 1.1). The first such polynomials are $B_0^{(a)}(x) = 1$, $B_1^{(a)}(x) = x - a/2$ and $B_2^{(a)}(x) = (3a^2 + 12x^2 - a(1 + 12x))/12$. The following result, proved in Fields (1970), essentially gives a uniform version of (C.1) which allows $|\alpha|$ and $|\beta|$ to grow 'large' as well, if more slowly than $|t|$.

Lemma C.1. *Let $t, \alpha, \beta \in \mathbb{C}$. Then, for all $M = 1, 2, \ldots$, one has, as $|t| \to \infty$,*

$$\frac{\Gamma(t+\alpha)}{\Gamma(t+\beta)} = t^{\alpha-\beta}\left(\sum_{k=0}^{M-1} \frac{1}{k!}\frac{\Gamma(1+\alpha-\beta)}{\Gamma(1+\alpha-\beta-k)} B_k^{(1+\alpha-\beta)}(\alpha)\frac{1}{t^k} + R_M(t,\alpha,\beta)\right),$$

where $|R_M(t,\alpha,\beta)| = O\left(|t|^{-M}(1+|\alpha-\beta|)^M(1+|\alpha|+|\alpha-\beta|)^M)\right)$, provided that $|\arg(t+\alpha)| < \pi$ and $(1+|\alpha-\beta|)(1+|\alpha|+|\alpha-\beta|) = o(|t|)$.

Now we prove the main theoretical results.

Proof of Proposition 1

Let $X_j \sim \text{Gamma}(\alpha_j, \beta_j)$, $j = 1, 2$, be independent. Then $\frac{\alpha_2\beta_1}{\alpha_1\beta_2}\frac{X_1}{X_2}$ follows the Fisher-Snedecor distribution $F(2\alpha_1, 2\alpha_2)$ (Johnson et al, 1994, Section 27.8). Hence the F-distribution is the distribution of the product of a Gamma(α_1, β_1) r.v. and an Inverse Gamma(α_2, β_2) r.v., rescaled by the constant $\alpha_2\beta_1/\alpha_1\beta_2$. Denoting f_F the $F(2\alpha_1, 2\alpha_2)$-density, (9)–(10)–(11) yield

$$\mathcal{M}(f_{\mathrm{F}}; z) = \left(\frac{\alpha_2}{\alpha_1}\right)^{z-1}\frac{\Gamma(\alpha_1+z-1)\Gamma(\alpha_2+1-z)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}, \quad 1-\alpha_1 < \Re(z) < 1+\alpha_2. \tag{C.2}$$

Identifying (12) and (C.2), $L_{1,\gamma,1,\theta}$ is seen to be the $F\left(\frac{2}{\gamma^2\cos^2\theta}, \frac{2}{\gamma^2\sin^2\theta}\right)$-density. From Lemma 1, it follows that $L_{\nu,\gamma,\xi,\theta}$ is the density of the positive random variable $Y = \nu X^\xi$, where $X \sim F\left(\frac{2\xi^2}{\gamma^2\cos^2\theta}, \frac{2\xi^2}{\gamma^2\sin^2\theta}\right)$.

Proof of Proposition 2

If $\xi > 2\gamma^2\sin^2\theta$, then $\{z \in \mathbb{C} : 2 \leq \Re(z) \leq 3\} \subset \mathcal{S}_{L_{\nu,\gamma,\xi,\theta}}$. Then, the mean μ of $L_{\nu,\gamma,\xi,\theta}$ is $\mu = \int_0^\infty x L_{\nu,\gamma,\xi,\theta}(x)\,dx = \mathcal{M}(L_{\nu,\gamma,\xi,\theta};2)$, that is,

$$\mu = \nu\left(\frac{1}{\tan^2\theta}\right)^{\xi} \frac{\Gamma\left(\frac{\xi^2}{\gamma^2\cos^2\theta}+\xi\right)\Gamma\left(\frac{\xi^2}{\gamma^2\sin^2\theta}-\xi\right)}{\Gamma\left(\frac{\xi^2}{\gamma^2\cos^2\theta}\right)\Gamma\left(\frac{\xi^2}{\gamma^2\sin^2\theta}\right)}.$$

Also, as $\int_0^\infty x^2 L_{\nu,\gamma,\xi,\theta}(x)\,dx = \mathcal{M}(L_{\nu,\gamma,\xi,\theta};3)$, the standard deviation of $L_{\nu,\gamma,\xi,\theta}$ is

$$\sigma = \sqrt{\mathcal{M}(L_{\nu,\gamma,\xi,\theta};3) - \mathcal{M}^2(L_{\nu,\gamma,\xi,\theta};2)},$$

which is

$$\sigma = \nu\left(\frac{1}{\tan^2\theta}\right)^{\xi} \frac{\Gamma\left(\frac{\xi^2}{\gamma^2\cos^2\theta}+\xi\right)\Gamma\left(\frac{\xi^2}{\gamma^2\sin^2\theta}-\xi\right)}{\Gamma\left(\frac{\xi^2}{\gamma^2\cos^2\theta}\right)\Gamma\left(\frac{\xi^2}{\gamma^2\sin^2\theta}\right)} \times \sqrt{\frac{\Gamma\left(\frac{\xi^2}{\gamma^2\cos^2\theta}\right)\Gamma\left(\frac{\xi^2}{\gamma^2\sin^2\theta}\right)\Gamma\left(\frac{\xi^2}{\gamma^2\cos^2\theta}+2\xi\right)\Gamma\left(\frac{\xi^2}{\gamma^2\sin^2\theta}-2\xi\right)}{\Gamma^2\left(\frac{\xi^2}{\gamma^2\cos^2\theta}+\xi\right)\Gamma^2\left(\frac{\xi^2}{\gamma^2\sin^2\theta}-\xi\right)} - 1}.$$

The coefficient of variation of $L_{\nu,\gamma,\xi,\theta}$ is thus

$$\chi = \frac{\sigma}{\mu} = \sqrt{\frac{\Gamma\left(\frac{\xi^2}{\gamma^2\cos^2\theta}\right)\Gamma\left(\frac{\xi^2}{\gamma^2\sin^2\theta}\right)\Gamma\left(\frac{\xi^2}{\gamma^2\cos^2\theta}+2\xi\right)\Gamma\left(\frac{\xi^2}{\gamma^2\sin^2\theta}-2\xi\right)}{\Gamma^2\left(\frac{\xi^2}{\gamma^2\cos^2\theta}+\xi\right)\Gamma^2\left(\frac{\xi^2}{\gamma^2\sin^2\theta}-\xi\right)} - 1}. \tag{C.3}$$

Now assume that $\theta \notin \{0, \pi/2\}$ (the proof is simpler if $\theta \in \{0,\pi/2\}$, and is omitted). By (C.1), we have, as $\gamma \to 0$,

$$\begin{aligned}\frac{\Gamma\left(\frac{\xi^2}{\gamma^2\cos^2\theta}\right)}{\Gamma\left(\frac{\xi^2}{\gamma^2\cos^2\theta}+\xi\right)} \times \frac{\Gamma\left(\frac{\xi^2}{\gamma^2\cos^2\theta}+2\xi\right)}{\Gamma\left(\frac{\xi^2}{\gamma^2\cos^2\theta}+\xi\right)} &= \left(\frac{\xi^2}{\gamma^2\cos^2\theta}\right)^{-\xi}\left(1+\frac{1-\xi}{2\xi}\gamma^2\cos^2\theta + O(\gamma^4)\right) \\ &\quad \times \left(\frac{\xi^2}{\gamma^2\cos^2\theta}\right)^{\xi}\left(1+\frac{3\xi-1}{2\xi}\gamma^2\cos^2\theta + O(\gamma^4)\right) \\ &= 1+\gamma^2\cos^2\theta + O(\gamma^4).\end{aligned}$$

Similarly,

$$\frac{\Gamma\left(\frac{\xi^2}{\gamma^2\sin^2\theta}\right)}{\Gamma\left(\frac{\xi^2}{\gamma^2\sin^2\theta}-\xi\right)} \times \frac{\Gamma\left(\frac{\xi^2}{\gamma^2\sin^2\theta}-2\xi\right)}{\Gamma\left(\frac{\xi^2}{\gamma^2\sin^2\theta}-\xi\right)} = 1+\gamma^2\sin^2\theta + O(\gamma^4).$$

Hence

$$\frac{\Gamma\left(\frac{\xi^2}{\gamma^2\cos^2\theta}\right)\Gamma\left(\frac{\xi^2}{\gamma^2\cos^2\theta}+2\xi\right)}{\Gamma^2\left(\frac{\xi^2}{\gamma^2\cos^2\theta}+\xi\right)} \frac{\Gamma\left(\frac{\xi^2}{\gamma^2\sin^2\theta}\right)\Gamma\left(\frac{\xi^2}{\gamma^2\sin^2\theta}-2\xi\right)}{\Gamma^2\left(\frac{\xi^2}{\gamma^2\sin^2\theta}-\xi\right)} = 1+\gamma^2+O(\gamma^4),$$

and the announced result follows from (C.3).

Proof of Proposition 3

The proof is given for the case $\theta \notin \{0, \pi/2\}$ only (the proof is simpler if $\theta \in \{0, \pi/2\}$, and is omitted).

(*i*) As $\gamma \to 0$, Lemma C.1 ascertains that

$$\left(\frac{\xi^2}{\gamma^2 \cos^2\theta}\right)^{-\xi(z-1)} \frac{\Gamma\left(\frac{\xi^2}{\gamma^2\cos^2\theta} + \xi(z-1)\right)}{\Gamma\left(\frac{\xi^2}{\gamma^2\cos^2\theta}\right)} = 1 + \frac{\gamma^2\cos^2\theta}{2\xi^2}\xi(z-1)(\xi(z-1)-1) + \rho_1(\gamma, z)$$
$$= 1 + \frac{\gamma^2\cos^2\theta}{2}(z-1)(z-1-\frac{1}{\xi}) + \rho_1(\gamma, z),$$

where $|\rho_1(\gamma, z)| = O(\gamma^4(1+|z-1|)^2)$, provided $|z-1| = o(\gamma^{-2})$. Similarly,

$$\left(\frac{\xi^2}{\gamma^2 \sin^2\theta}\right)^{-\xi(1-z)} \frac{\Gamma\left(\frac{\xi^2}{\gamma^2\sin^2\theta} + \xi(1-z)\right)}{\Gamma\left(\frac{\xi^2}{\gamma^2\sin^2\theta}\right)} = 1 + \frac{\gamma^2\sin^2\theta}{2\xi^2}\xi(1-z)(\xi(1-z)-1) + \rho_2(\gamma, z)$$
$$= 1 + \frac{\gamma^2\sin^2\theta}{2}(z-1)(z-1+\frac{1}{\xi}) + \rho_2(\gamma, z),$$

where $|\rho_2(\gamma, z)| = O(\gamma^4(1+|z-1|)^2)$, provided $|z-1| = o(\gamma^{-2})$. Also, for any $\Delta \in \mathbb{R}$, the binomial series expands as

$$(1+\Delta\gamma^2)^{z-1} = 1 + \Delta\gamma^2(z-1) + \rho_3(\gamma, z),$$

where $|\rho_3(\gamma, z)| = O(\gamma^4|(z-1)(z-2)|)$, provided $|z-1| = o(\gamma^{-2})$ as $\gamma \to 0$. Multiplying these factors yields

$$(1+\Delta\gamma^2)^{z-1} \left(\frac{1}{\tan^2\theta}\right)^{\xi(z-1)} \frac{\Gamma\left(\frac{\xi^2}{\gamma^2\cos^2\theta} + \xi(z-1)\right)}{\Gamma\left(\frac{\xi^2}{\gamma^2\cos^2\theta}\right)} \frac{\Gamma\left(\frac{\xi^2}{\gamma^2\sin^2\theta} + \xi(1-z)\right)}{\Gamma\left(\frac{\xi^2}{\gamma^2\sin^2\theta}\right)}$$
$$= 1 + \frac{\gamma^2}{2}(z-1)\left(\cos^2\theta\,(z-1-\frac{1}{\xi}) + \sin^2\theta\,(z-1+\frac{1}{\xi}) + 2\Delta\right) + \rho(\gamma, z)$$
$$= 1 + \frac{\gamma^2}{2}(z-1)\left(z-1-\frac{\cos 2\theta}{\xi} + 2\Delta\right) + \rho(\gamma, z)$$

where $|\rho(\gamma, z)| = O(\gamma^4(1+|z-1|)^2)$, provided $|z-1| = o(\gamma^{-2})$. Taking $\Delta = \frac{1}{2}(1 + \frac{\cos 2\theta}{\xi})$ yields the announced result.

(*ii*) It can be checked from (12) and Lemma 1 that

$$L_{\nu,\gamma,\xi,\theta}(x) = \frac{1}{\nu\xi}\frac{1}{\Gamma\left(\frac{\xi^2}{\gamma^2\cos^2\theta}\right)\Gamma\left(\frac{\xi^2}{\gamma^2\sin^2\theta}\right)}(\tan^2\theta)^{\xi}\, G_{1,1}^{1,1}\left(\tan^2\theta\left(\frac{x}{\nu}\right)^{1/\xi}\,\middle|\, \begin{matrix} -\frac{\xi^2}{\gamma^2 Als\sin^2\theta}+1-\xi \\ \frac{\xi^2}{\gamma^2\cos^2\theta}-\xi \end{matrix}\right).$$

From Bateman (1954, item (15), p. 349), we have

$$\left\{G_{1,1}^{1,1}\left(\cdot\,\middle|\,{}^{1-b}_{a}\right)\right\}^2 = \frac{\Gamma^2(a+b)}{\Gamma(2a+2b)}G_{1,1}^{1,1}\left(\cdot\,\middle|\,{}^{1-2b}_{2a}\right),$$

which yields

$$\mathcal{M}\left(\left\{G_{1,1}^{1,1}\left(\cdot\,\middle|\,{}^{1-b}_{a}\right)\right\}^2; z\right) = \frac{\Gamma^2(a+b)}{\Gamma(2a+2b)}\Gamma(z+2a)\Gamma(2b-z),$$

by (3). Then we obtain, after some algebraic work,

$$\mathcal{M}(L^2_{\nu,\gamma,\xi,\theta};z) = \frac{1}{\xi}\,\nu^{z-2}\left(\frac{1}{\tan^2\theta}\right)^{\xi(z-2)} \frac{\mathcal{B}\left(\frac{2\xi^2}{\gamma^2\cos^2\theta},\frac{2\xi^2}{\gamma^2\sin^2\theta}\right)}{\mathcal{B}^2\left(\frac{\xi^2}{\gamma^2\cos^2\theta},\frac{\xi^2}{\gamma^2\sin^2\theta}\right)} \times \frac{\Gamma\left(\frac{2\xi^2}{\gamma^2\cos^2\theta}+\xi(z-2)\right)}{\Gamma\left(\frac{2\xi^2}{\gamma^2\cos^2\theta}\right)}\,\frac{\Gamma\left(\frac{2\xi^2}{\gamma^2\sin^2\theta}+\xi(2-z)\right)}{\Gamma\left(\frac{2\xi^2}{\gamma^2\sin^2\theta}\right)}, \quad \text{(C.4)}$$

where $\mathcal{B}(\cdot,\cdot)$ is the Beta function, on the strip of holomorphy

$$\mathcal{S}_{L^2_{\nu,\gamma,\xi,\theta}} = \left\{z\in\mathbb{C} : 2-\frac{2\xi}{\gamma^2\cos^2\theta} < \Re(z) < 2+\frac{2\xi}{\gamma^2\sin^2\theta}\right\}. \quad \text{(C.5)}$$

Now, resorting to Lemma C.1, one obtains, as $\gamma\to 0$,

$$\nu^{z-2}\left(\frac{1}{\tan^2\theta}\right)^{\xi(z-2)}\frac{\Gamma\left(\frac{2\xi^2}{\gamma^2\cos^2\theta}+\xi(z-2)\right)}{\Gamma\left(\frac{2\xi^2}{\gamma^2\cos^2\theta}\right)}\,\frac{\Gamma\left(\frac{2\xi^2}{\gamma^2\sin^2\theta}+\xi(2-z)\right)}{\Gamma\left(\frac{2\xi^2}{\gamma^2\sin^2\theta}\right)} = 1+\omega(\gamma,z),$$

where $|\omega(\gamma,z)| = O(\gamma^2(1+|z-2|))$ for $|z-2| = o(\gamma^{-2})$. On the other hand, for any $a,b>0$,

$$\begin{aligned}\frac{\mathcal{B}(2a,2b)}{\mathcal{B}^2(a,b)} &= \frac{\Gamma(2a)}{\Gamma(a)}\frac{\Gamma(2b)}{\Gamma(b)}\frac{\Gamma(a+b)}{\Gamma(2(a+b))}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \\ &= \frac{1}{2\sqrt{\pi}}\frac{\Gamma(a+1/2)}{\Gamma(a)}\frac{\Gamma(b+1/2)}{\Gamma(b)}\frac{\Gamma(a+b)}{\Gamma(a+b+1/2)} \qquad \text{(duplication formula).}\end{aligned}$$

Now, as $a,b\to\infty$, use (C.1) and see

$$\begin{aligned}\frac{\mathcal{B}(2a,2b)}{\mathcal{B}^2(a,b)} &= \frac{1}{2\sqrt{\pi}}\,a^{1/2}(1+O(a^{-1}))\,b^{1/2}(1+O(b^{-1}))\,(a+b)^{-1/2}(1+O(a+b)^{-1})) \\ &= \frac{1}{2\sqrt{\pi}}\frac{1}{\left(\frac{1}{a}+\frac{1}{b}\right)^{1/2}}\,(1+O(a^{-1})+O(b^{-1})+O((a+b)^{-1})).\end{aligned}$$

With $a=\frac{\xi^2}{\gamma^2\cos^2\theta}$ and $b=\frac{\xi^2}{\gamma^2\sin^2\theta}$, see that $1/a+1/b=\gamma^2/\xi^2$, hence

$$\frac{\mathcal{B}\left(\frac{2\xi^2}{\gamma^2\cos^2\theta},\frac{2\xi^2}{\gamma^2\sin^2\theta}\right)}{\mathcal{B}^2\left(\frac{\xi^2}{\gamma^2\cos^2\theta},\frac{\xi^2}{\gamma^2\sin^2\theta}\right)} = \frac{1}{2\sqrt{\pi}}\,\frac{\xi}{\gamma}(1+O(\gamma^2)).$$

It follows

$$\mathcal{M}(L^2_{\nu,\gamma,\xi,\theta};z) = \frac{1}{2\sqrt{\pi}}\,\frac{1}{\gamma}\,(1+\omega(\gamma,z)),$$

where $|\omega(\gamma,z)| = O(\gamma^2(1+|z-2|))$ for $|z-2| = o(\gamma^{-2})$.

Proof of Theorem 2

Apply Parseval's identity (8) to $\hat{f}-f$ to get

$$\int_0^\infty x^{2c-1}\left(\hat{f}(x)-f(x)\right)^2 dx = \frac{1}{2\pi}\int_{\Re(z)=c}|\mathcal{M}(\hat{f}-f;z)|^2\,dz, \quad \text{(C.6)}$$

for any $c\in\mathcal{S}_{\hat{f}-f}$. Then we resort to the following lemma.

Lemma C.2. *Under Assumptions 1-4, the strip of holomorphy* $\mathcal{S}_{\hat{f}-f}$ *of* $\hat{f}-f$ *is such that*

$$\mathcal{S}_{\hat{f}-f} \supseteq \left\{z \in \mathbb{C} : 1-\min(\alpha, \xi/\cos^2\theta) \leq \Re(z) \leq 1+\min(\beta, \xi/\sin^2\theta)\right\}.$$

Proof. From (19) and (A.1),

$$\mathcal{M}(\hat{f};z) = \frac{1}{n}\sum_{k=1}^{n} \mathcal{M}(L_\eta^{(k)};z) X_k^{z-1},$$

with $\mathcal{S}_{\hat{f}} = \bigcap_{k=1}^n \mathcal{S}_{L_\eta^{(k)}}$. From (13) and (21), we see that

$$\begin{aligned}\mathcal{S}_{L_\eta^{(k)}} &= \left\{z \in \mathbb{C} : 1 - \frac{\xi(\eta^2+X_k)}{\eta^2\cos^2\theta} < \Re(z) < 1 + \frac{\xi(\eta^2+X_k)}{\eta^2\sin^2\theta}\right\} \\ &\supseteq \left\{z \in \mathbb{C} : 1 - \frac{\xi}{\cos^2\theta} \leq \Re(z) \leq 1 + \frac{\xi}{\sin^2\theta}\right\} \quad \text{for all } k,\end{aligned}$$

whence

$$\mathcal{S}_{\hat{f}} \supseteq \left\{z \in \mathbb{C} : 1 - \frac{\xi}{\cos^2\theta} \leq \Re(z) \leq 1 + \frac{\xi}{\sin^2\theta}\right\}.$$

Assumption 2 implies that

$$\{z \in \mathbb{C} : 1-\alpha \leq \Re(z) \leq 1+\beta\} \subseteq S_f. \tag{C.7}$$

The result follows as $\mathcal{S}_{\hat{f}-f} = \mathcal{S}_{\hat{f}} \cap \mathcal{S}_f$, from (A.1). □

Lemma C.2 ascertains that (C.6) is valid for any $c \in [1-\min(\alpha,\xi/\cos^2\theta),\ 1+\min(\beta,\xi/\sin^2\theta)]$. In particular, it is true for c satisfying (31), as $1-\min(\alpha,\xi/\cos^2\theta) \leq \max(2-\alpha, 1-\xi/\cos^2\theta)$ and $1+\min(\beta,\xi/\sin^2\theta) \geq \min((3+2\beta)/4, 1+\xi/\sin^2\theta)$.

Now, because $\mathcal{M}(\hat{f}-f;z)$ is holomorphic on $\mathcal{S}_{\hat{f}-f}$ and $\hat{f}-f$ is real-valued, $\mathcal{M}^*(\hat{f}-f;z) = \mathcal{M}(\hat{f}-f;z^*)$, where $\cdot^*$ denotes complex conjugation. Hence, $|\mathcal{M}(\hat{f}-f;z)|^2 = \mathcal{M}(\hat{f}-f;z) \times \mathcal{M}(\hat{f}-f;z^*)$. By (A.1), $\mathcal{M}(\hat{f}-f;z) = \mathcal{M}(\hat{f};z) - \mathcal{M}(f;z)$. Hence (C.6) is

$$\begin{aligned}\int_0^\infty x^{2c-1}\left(\hat{f}(x)-f(x)\right)^2 dx &= \frac{1}{2\pi}\int_{\Re(z)=c} \mathcal{M}(\hat{f};z)\mathcal{M}(\hat{f};z^*)\,dz \\ &\quad - \frac{1}{2\pi}\int_{\Re(z)=c} \mathcal{M}(f;z)\mathcal{M}(\hat{f};z^*)\,dz \\ &\quad - \frac{1}{2\pi}\int_{\Re(z)=c} \mathcal{M}(\hat{f};z)\mathcal{M}(f;z^*)\,dz \\ &\quad + \frac{1}{2\pi}\int_{\Re(z)=c} \mathcal{M}(f;z)\mathcal{M}(f;z^*)\,dz \\ &\doteq Ⓐ + Ⓑ + Ⓒ + Ⓓ,\end{aligned}$$

and

$$\mathbb{E}\left(\int_0^\infty x^{2c-1}\left(\hat{f}(x)-f(x)\right)^2 dx\right) = \mathbb{E}\left(Ⓐ\right) + \mathbb{E}\left(Ⓑ\right) + \mathbb{E}\left(Ⓒ\right) + Ⓓ. \tag{C.8}$$

From (20), we have

$$\mathcal{M}(\hat{f};z)\mathcal{M}(\hat{f};z^*) = \frac{1}{n^2}\sum_{k=1}^n \mathcal{M}(L_\eta^{(k)};z)\mathcal{M}(L_\eta^{(k)};z^*)X_k^{2\Re(z)-2} + \frac{1}{n^2}\sum_{k=1}^n\sum_{k'\neq k} \mathcal{M}(L_\eta^{(k)};z)\mathcal{M}(L_\eta^{(k')};z^*)X_k^{z-1}X_{k'}^{z^*-1},$$

whence

$$Ⓐ = \frac{1}{n^2}\sum_{k=1}^n X_k^{2c-2}\frac{1}{2\pi}\int_{\Re(z)=c} |\mathcal{M}(L_\eta^{(k)};z)|^2\,dz + \frac{1}{n^2}\sum_{k=1}^n\sum_{k'\neq k}\frac{1}{2\pi}\int_{\Re(z)=c}\mathcal{M}(L_\eta^{(k)};z)\mathcal{M}(L_\eta^{(k')};z^*)X_k^{z-1}X_{k'}^{z^*-1}\,dz. \quad \text{(C.9)}$$

Given that $c \in \bigcap_{k=1}^n \mathcal{S}_{L_\eta^{(k)}}$, it holds for all k

$$\frac{1}{2\pi}\int_{\Re(z)=c} |\mathcal{M}(L_\eta^{(k)};z)|^2\,dz = \int_0^\infty x^{2c-1}{L_\eta^{(k)}}^2(x)\,dx = \mathcal{M}({L_\eta^{(k)}}^2;2c),$$

from (8) back and forth. Hence the first term in (C.9), say Ⓐ-1, is

$$Ⓐ\text{-1} = \frac{1}{n^2}\sum_{k=1}^n X_k^{2c-2}\mathcal{M}({L_\eta^{(k)}}^2;2c). \quad \text{(C.10)}$$

Note that $c \in \bigcap_{k=1}^n \mathcal{S}_{L_\eta^{(k)}} \iff 2c \in \bigcap_{k=1}^n \mathcal{S}_{{L_\eta^{(k)}}^2}$, as seen from (13) and (C.5).

The second term in (C.9), say Ⓐ-2, has expectation

$$\mathbb{E}\left(Ⓐ\text{-2}\right) = \left(1-\frac{1}{n}\right)\frac{1}{2\pi}\int_{\Re(z)=c}\mathbb{E}\left(\mathcal{M}(L_\eta^{(k)};z)X_k^{z-1}\right)\mathbb{E}\left(\mathcal{M}(L_\eta^{(k)};z^*)X_k^{z^*-1}\right)\,dz$$
$$\doteq \left(1-\frac{1}{n}\right)\mathbb{E}\left(Ⓐ\text{-2-a}\right),$$

for a generic $k \in \{1,\ldots,n\}$. Interchanging expectation and integral is justified as c belongs to both $\mathcal{S}_f$ and $\mathcal{S}_{L_\eta^{(k)}}$ (for all k), making the corresponding integrals both absolutely convergent. Likewise,

$$\mathbb{E}\left(Ⓑ\right) = -\frac{1}{2\pi}\int_{\Re(z)=c}\mathcal{M}(f;z)\mathbb{E}\left(\mathcal{M}(L_\eta^{(k)};z^*)X_k^{z^*-1}\right)\,dz$$
$$\text{and } \mathbb{E}\left(Ⓒ\right) = -\frac{1}{2\pi}\int_{\Re(z)=c}\mathcal{M}(f;z^*)\mathbb{E}\left(\mathcal{M}(L_\eta^{(k)};z)X_k^{z-1}\right)\,dz.$$

It is easily seen that

$$\mathbb{E}\left(Ⓐ\text{-2-a}\right)+\mathbb{E}\left(Ⓑ\right)+\mathbb{E}\left(Ⓒ\right)+Ⓓ = \frac{1}{2\pi}\int_{\Re(z)=c}\left|\mathbb{E}\left(\mathcal{M}(L_\eta^{(k)};z)X_k^{z-1}\right)-\mathcal{M}(f;z)\right|^2\,dz, \quad \text{(C.11)}$$

which is clearly the integrated squared bias term, say IB_c^2, in the Weighted Mean Integrated Square Error expression (C.8). The remaining $\mathbb{E}\left(Ⓐ\text{-1}\right)-\frac{1}{n}\mathbb{E}\left(Ⓐ\text{-2-a}\right)$ thus forms the integrated variance, say IV_c. Below, we show that $\text{IB}_c^2 = O(\eta^4)$ and $\text{IV}_c = O((n\eta)^{-1})$ as $n\to\infty$, under our assumptions.

Integrated squared bias term: Under condition (31), $c > 2-\alpha$, hence $0 < \frac{c+\alpha-2}{c+\alpha-1} < 1$. Let $\epsilon \doteq \epsilon_n \to 0$ as

$n \to \infty$, such that $\epsilon \sim \eta^b$ for

$$0 < b < \frac{c+\alpha-2}{c+\alpha-1}. \tag{C.12}$$

Note thas this implies $\eta/\epsilon \to 0$ as $n \to \infty$. Write

$$\mathcal{M}(L_\eta^{(k)};z)X_k^{z-1} = \mathcal{M}(L_\eta^{(k)};z)X_k^{z-1}\mathbb{1}_{\{X_k \geq \eta^2\left(\frac{1}{\epsilon^2}-1\right)\}} + \mathcal{M}(L_\eta^{(k)};z)X_k^{z-1}\mathbb{1}_{\{X_k < \eta^2\left(\frac{1}{\epsilon^2}-1\right)\}}, \tag{C.13}$$

where $\mathbb{1}_{\{\cdot\}}$ is the indicator function, equal to 1 if the condition $\{\cdot\}$ is satisfied and 0 otherwise. See that $X_k \geq \eta^2\left(\frac{1}{\epsilon^2}-1\right) \iff \frac{\eta}{\sqrt{\eta^2+X_k}} \leq \epsilon \to 0$, hence one can make use of the asymptotic expansion (16) with (21)-(22) to write, as $n \to \infty$,

$$\mathcal{M}(L_\eta^{(k)};z)X_k^{z-1}\mathbb{1}_{\{X_k \geq \eta^2\left(\frac{1}{\epsilon^2}-1\right)\}} = \left(1 + \frac{1}{2}\frac{\eta^2}{\eta^2+X_k}z(z-1) + R_k(\eta,z)\right)X_k^{z-1}\mathbb{1}_{\{X_k \geq \eta^2\left(\frac{1}{\epsilon^2}-1\right)\}}$$

where $|R_k(\eta,z)| \leq C\frac{\eta^4}{(\eta^2+X_k)^2}(1+|z-1|)^2$ for some constant C. From this and (C.13) we have

$$\begin{aligned}
\mathbb{E}\left(\mathcal{M}(L_\eta^{(k)};z)X_k^{z-1}\right) - \mathcal{M}(f;z) = {} & \mathbb{E}\left(X_k^{z-1}\mathbb{1}_{\{X_k \geq \eta^2\left(\frac{1}{\epsilon^2}-1\right)\}}\right) - \mathbb{E}\left(X_k^{z-1}\right) \\
& + \frac{1}{2}\eta^2 z(z-1)\mathbb{E}\left(\frac{1}{\eta^2+X_k}X_k^{z-1}\mathbb{1}_{\{X_k \geq \eta^2\left(\frac{1}{\epsilon^2}-1\right)\}}\right) \\
& + \mathbb{E}\left(R_k(\eta,z)X_k^{z-1}\mathbb{1}_{\{X_k \geq \eta^2\left(\frac{1}{\epsilon^2}-1\right)\}}\right) \\
& + \mathbb{E}\left(\mathcal{M}(L_\eta^{(k)};z)X_k^{z-1}\mathbb{1}_{\{X_k < \eta^2\left(\frac{1}{\epsilon^2}-1\right)\}}\right),
\end{aligned}$$

that is,

$$\begin{aligned}
\mathbb{E}\left(\mathcal{M}(L_\eta^{(k)};z)X_k^{z-1}\right) - \mathcal{M}(f;z) = {} & \frac{1}{2}\eta^2 z(z-1)\mathbb{E}\left(\frac{1}{\eta^2+X_k}X_k^{z-1}\mathbb{1}_{\{X_k \geq \eta^2\left(\frac{1}{\epsilon^2}-1\right)\}}\right) \\
& + \mathbb{E}\left(\left(\mathcal{M}(L_\eta^{(k)};z)-1\right)X_k^{z-1}\mathbb{1}_{\{X_k < \eta^2\left(\frac{1}{\epsilon^2}-1\right)\}}\right) \\
& + \mathbb{E}\left(R_k(\eta,z)X_k^{z-1}\mathbb{1}_{\{X_k \geq \eta^2\left(\frac{1}{\epsilon^2}-1\right)\}}\right).
\end{aligned}$$

Hence the integrated squared bias (C.11) is such that

$$\begin{aligned}
\mathrm{IB}_c^2 \leq {} & 2\left(\frac{1}{4}\eta^4\frac{1}{2\pi}\int_{\Re(z)=c}\left|z(z-1)\mathbb{E}\left(\frac{1}{\eta^2+X_k}X_k^{z-1}\mathbb{1}_{\{X_k \geq \eta^2\left(\frac{1}{\epsilon^2}-1\right)\}}\right)\right|^2 dz\right. \\
& + \frac{1}{2\pi}\int_{\Re(z)=c}\left|\mathbb{E}\left(\left(\mathcal{M}(L_\eta^{(k)};z)-1\right)X_k^{z-1}\mathbb{1}_{\{X_k < \eta^2\left(\frac{1}{\epsilon^2}-1\right)\}}\right)\right|^2 dz \\
& \left. + \frac{1}{2\pi}\int_{\Re(z)=c}\left|\mathbb{E}\left(R_k(\eta,z)X_k^{z-1}\mathbb{1}_{\{X_k \geq \eta^2\left(\frac{1}{\epsilon^2}-1\right)\}}\right)\right|^2 dz\right) \\
\doteq {} & 2 \times (Ⓔ + Ⓕ + Ⓖ).
\end{aligned} \tag{C.14}$$

As $\frac{1}{(\eta^2+X_k)} \leq \frac{1}{X_k}$, Ⓔ $\leq \frac{1}{4}\eta^4\frac{1}{2\pi}\int_{\Re(z)=c}\left|z(z-1)\mathbb{E}\left(X_k^{z-2}\right)\right|^2 dz$. By combining (A.4) and (A.7), it is seen that $z(z-1)\mathbb{E}\left(X_k^{z-2}\right) = z(z-1)\mathcal{M}(f;z-1) = \mathcal{M}(xf''(x);z)$ if $z-1 \in \mathcal{S}_f$, which is the case here by (C.7) and because $\Re(z) = c > 2 - \alpha$ by (31). With (8), Ⓔ $\leq \frac{1}{4}\eta^4 \int_0^\infty x^{2c+1} f''^2(x)\, dx$, hence

$$\text{Ⓔ} = O(\eta^4). \tag{C.15}$$

Given that $c \in \bigcap_{k=1}^n \mathcal{S}_{L_\eta^{(k)}}$, $\sup_{z\in\mathbb{C}:\Re(z)=c} \max_{k=1,\ldots,n} |\mathcal{M}(L_\eta^{(k)}; z)| \leq C$ for some constant C and

$$\text{Ⓕ} \leq (1+C)^2 \frac{1}{2\pi} \int_{\Re(z)=c} \left| \mathbb{E}\left(X_k^{z-1} \mathbb{1}_{\{X_k < \eta^2\left(\frac{1}{\epsilon^2}-1\right)\}} \right) \right|^2 dz.$$

Now,

$$\mathbb{E}\left(X_k^{z-1} \mathbb{1}_{\{X_k < \eta^2\left(\frac{1}{\epsilon^2}-1\right)\}} \right) = \int_0^{\eta^2\left(\frac{1}{\epsilon^2}-1\right)} x^{z-1} f(x)\, dx = \mathcal{M}\left(f(x) \mathbb{1}_{\{x < \eta^2\left(\frac{1}{\epsilon^2}-1\right)\}}; z \right).$$

Clearly the strip of holomorphy of f is contained in that of any of its restriction on $\mathbb{R}^+$, so by (8) again,

$$\text{Ⓕ} \leq (1+C)^2 \int_0^\infty x^{2c-1} f^2(x) \mathbb{1}_{\{x<\eta^2\left(\frac{1}{\epsilon^2}-1\right)\}}\, dx = (1+C)^2 \int_0^{\eta^2\left(\frac{1}{\epsilon^2}-1\right)} x^{2c-1} f^2(x)\, dx.$$

By Assumpion 2, $\mathbb{E}(X^{-\alpha}) < \infty$, which implies $f(x) = o(x^{\alpha-1})$ as $x \to 0$. Hence

$$\text{Ⓕ} = o\left(\left(\frac{\eta^2}{\epsilon^2} \right)^{2c+2\alpha-2} \right),$$

following Example 4 in (Paris and Kaminski, 2001, Section 1.1.1). With $\epsilon \sim \eta^b$ and condition (C.12), it can be checked that this is

$$\text{Ⓕ} = o(\eta^4). \tag{C.16}$$

Finally,

$$\begin{aligned}
|R_k(\eta, z)| \mathbb{1}_{\{X_k \geq \eta^2\left(\frac{1}{\epsilon^2}-1\right)\}} &\leq C \frac{\eta^4}{(\eta^2 + X_k)^2} (1 + |z-1|)^2 \mathbb{1}_{\{X_k \geq \eta^2\left(\frac{1}{\epsilon^2}-1\right)\}} \\
&= C \frac{\eta^2}{\eta^2 + X_k} (1+|z-1|)^2 \frac{\eta^2}{\eta^2 + X_k} \mathbb{1}_{\{X_k \geq \eta^2\left(\frac{1}{\epsilon^2}-1\right)\}} \\
&\leq C \frac{\eta^2}{X_k} (1+|z-1|)^2 \epsilon^2,
\end{aligned}$$

and it follows

$$\text{Ⓖ} \leq C \frac{1}{2\pi} \eta^4 \epsilon^4 \int_{\Re(z)=c} (1+|z-1|)^2 \left| \mathbb{E}\left(X_k^{z-2} \right) \right|^2 dz.$$

The integral may be seen to be bounded by (A.8), as $z - 1 \in S_f$ for $\Re(z) = c > 2 - \alpha$ under condition (31), hence

$$\text{Ⓖ} = O(\eta^4 \epsilon^4) = o(\eta^4). \tag{C.17}$$

It follows from (C.14), (C.15), (C.16) and (C.17) that

$$\text{IB}_c^2 = O(\eta^4).$$

Integrated variance term: Consider again $\epsilon \doteq \epsilon_n \to 0$ with $\eta/\epsilon \to 0$ as $n \to \infty$. Then write (C.10) as

$$\text{Ⓐ-1} = \frac{1}{n^2}\sum_{k=1}^n X_k^{2c-2}\mathcal{M}(L_\eta^{(k)2};2c)\,\mathbb{1}_{\left\{X_k \geq \eta^2\left(\frac{1}{\epsilon^2}-1\right)\right\}} + \frac{1}{n^2}\sum_{k=1}^n X_k^{2c-2}\mathcal{M}(L_\eta^{(k)2};2c)\,\mathbb{1}_{\left\{X_k < \eta^2\left(\frac{1}{\epsilon^2}-1\right)\right\}}$$
$$\doteq \text{Ⓐ-1-a} + \text{Ⓐ-1-b}.$$

Seeing again that $X_k \geq \eta^2\left(\frac{1}{\epsilon^2}-1\right) \iff \frac{\eta}{\sqrt{\eta^2+X_k}} \leq \epsilon \to 0$, one can write the expansion (17) for $\mathcal{M}(L_\eta^{(k)2};2c)$ in Ⓐ-1-a, that is, making use of (21)-(22),

$$\mathcal{M}(L_\eta^{(k)2};2c) = \frac{1}{2\sqrt{\pi}}\frac{\sqrt{\eta^2+X_k}}{\eta}(1+\Omega_k(\eta,2c)),$$

where $|\Omega_k(\eta,2c)| = O\left(\frac{\eta^2}{\eta^2+X_k}(1+|2c-2|)\right) = O(\epsilon^2)$. Also, $\sqrt{\eta^2+X_k}/\sqrt{X_k} = \sqrt{1+\eta^2/X_k} \leq 1/\sqrt{1-\epsilon^2} \leq 1+\epsilon^2$, for n large enough. This means that, as $n \to \infty$,

$$\mathcal{M}(L_\eta^{(k)2};2c) = \frac{1}{2\sqrt{\pi}}\frac{\sqrt{X_k}}{\eta}(1+\Omega'_k(\eta,2c)),$$

where $|\Omega'_k(\eta,2c))| \leq C\epsilon^2$ for some constant C, yielding

$$\text{Ⓐ-1-a} = \frac{1}{n^2\eta}\frac{1}{2\sqrt{\pi}}\sum_{k=1}^n X_k^{2c-3/2}\,\mathbb{1}_{\left\{X_k \geq \eta^2\left(\frac{1}{\epsilon^2}-1\right)\right\}}(1+O(\epsilon^2)).$$

Assumption 2 ensures that $f(x) = o(x^{\alpha-1})$ as $x \to 0$, whence

$$\mathbb{P}\left(X_k \geq \eta^2\left(\frac{1}{\epsilon^2}-1\right)\right) = 1 - \int_0^{\eta^2\left(\frac{1}{\epsilon^2}-1\right)} f(x)\,dx = 1 - O\left(\left(\frac{\eta^2}{\epsilon^2}\right)^\alpha\right) = 1 - o(1).$$

It follows

$$\mathbb{E}\left(\text{Ⓐ-1-a}\right) = \frac{1}{n\eta}\frac{1}{2\sqrt{\pi}}\mathcal{M}(f;2c-1/2)\,(1-o(1))\left(1+O(\epsilon^2)\right).$$

This is $O((n\eta)^{-1})$ if $2c - 1/2 \in \mathcal{S}_f$, which is the case under condition (31).

Now, because $2c \in \bigcap_{k=1}^n \mathcal{S}_{L_\eta^{(k)2}}$, each $|\mathcal{M}(L_\eta^{(k)2};2c)|$ is finite and $\max_{1\leq k\leq n}|\mathcal{M}(L_\eta^{(k)2};2c)| \leq C$, for C some constant. Hence

$$\text{Ⓐ-1-b} \leq \frac{C}{n^2}\sum_{k=1}^n X_k^{2c-2}\,\mathbb{1}_{\left\{X_k < \eta^2\left(\frac{1}{\epsilon^2}-1\right)\right\}}. \tag{C.18}$$

Similarly to above,

$$\mathbb{E}\left(X^{2c-2}\,\mathbb{1}_{\left\{X < \eta^2\left(\frac{1}{\epsilon^2}-1\right)\right\}}\right) = \int_0^{\eta^2\left(\frac{1}{\epsilon^2}-1\right)} x^{2c-2}f(x)\,dx = o\left(\left(\frac{\eta^2}{\epsilon^2}\right)^{2c-2+\alpha}\right)$$

as $n \to \infty$, making use again of $f(x) = o(x^{\alpha-1})$ as $x \to 0$. Taking expectations in (C.18) yields

$$\mathbb{E}\left(\text{Ⓐ-1-b}\right) = o\left(n^{-1}\left(\frac{\eta^2}{\epsilon^2}\right)^{2c-2+\alpha}\right).$$

It can be checked that, for $c \geq 3/4 - \alpha/2$, $\left(\frac{\eta^2}{\epsilon^2}\right)^{2c-2+\alpha} = O(\eta^{-1})$. Hence, $\mathbb{E}\left(\text{Ⓐ-1-b}\right) = o((n\eta)^{-1})$, leading to

$$\mathbb{E}(\text{Ⓐ-1}) = O((n\eta)^{-1}).$$

The dominant term in $\mathbb{E}\left(\text{Ⓐ-2-a}\right)$ can be understood to be Ⓓ. Yet,

$$\begin{aligned} \text{Ⓓ} &= \frac{1}{2\pi}\int_{\Re(z)=c} \mathcal{M}(f;z)\mathcal{M}(f;z^*)\,dz \\ &= \frac{1}{2\pi}\int_{\Re(z)=c} |\mathcal{M}(f;z)|^2\,dz \\ &= \int_0^\infty x^{2c-1}f^2(x)\,dx \end{aligned}$$

which is bounded for any $c \in \mathcal{S}_f$. Hence $\mathbb{E}\left(\text{Ⓐ-2-a}\right)/n = O(n^{-1}) = o((n\eta)^{-1})$, which shows

$$\text{IV}_c = O((n\eta)^{-1}).$$

□

Proposition C.1. *Under Assumptions 1-4, with $\alpha > 1/2$ in Assumption 2 and $\xi/\cos^2\theta > 1/2$ in Assumption 3, the Mellin-Meijer kernel density estimator (19) is such that $\mathbb{E}\left(\int_0^\infty \left(\hat{f}(x) - f(x)\right)^2 dx\right) = O(\eta^4) + O((n\eta)^{-1}) \to 0$ as $n \to \infty$, provided that $\int_0^\infty (xf''(x))^2\,dx < \infty$.*

Proof. We just show that (C.15) holds true if $\int_0^\infty (xf''(x))^2\,dx < \infty$. From (C.14),

$$\text{Ⓔ} \leq \frac{1}{4}\eta^4\frac{1}{2\pi}\int_{\Re(z)=c}\left|z(z-1)\mathbb{E}\left(X_k^{z-2}\mathbb{1}_{\{X_k \geq \eta^2\left(\frac{1}{\epsilon^2}-1\right)\}}\right)\right|^2 dz.$$

Now,

$$\mathbb{E}\left(X_k^{z-2}\mathbb{1}_{\{X_k\geq\eta^2\left(\frac{1}{\epsilon^2}-1\right)\}}\right) = \int_{\eta^2\left(\frac{1}{\epsilon^2}-1\right)}^\infty x^{z-2}f(x)\,dx = \mathcal{M}\left(f(x)\mathbb{1}_{\{x\geq\eta^2\left(\frac{1}{\epsilon^2}-1\right)\}}; z-1\right).$$

The strip of holomorphy of $f(x)\mathbb{1}_{\{x\geq\cdot\}}$ is $(-\infty, 1+\beta)$, as $f(x)\mathbb{1}_{\{x\geq\cdot\}} \equiv 0$ for $x \simeq 0$ ('flat' head). So for any $c \leq 1+\beta$,

$$\begin{aligned} \text{Ⓔ} &\leq \frac{1}{4}\eta^4\frac{1}{2\pi}\int_{\Re(z)=c}\left|z(z-1)\mathcal{M}\left(f(x)\mathbb{1}_{\{x\geq\eta^2\left(\frac{1}{\epsilon^2}-1\right)\}}; z-1\right)\right|^2 dz \\ &= \frac{1}{4}\eta^4\int_0^\infty x^{2c-1}(xf''(x))^2\mathbb{1}_{\{x\geq\eta^2\left(\frac{1}{\epsilon^2}-1\right)\}}\,dx \\ &\leq \frac{1}{4}\eta^4\int_0^\infty x^{2c-1}(xf''(x))^2\,dx, \end{aligned}$$

by (8). Taking $c = 1/2$ yields the result, as $\int_0^\infty (xf''(x))^2\,dx < \infty$. □

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