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On localization of source by hidden Gaussian processes with small noise

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Abstract
We consider the problem of identification of the position of some source by observations of K detectors receiving signals from this source. The time of arriving of the signal to the k-th detector depends of the distance between this detector and the source. The signals are observed in the presence of small Gaussian noise. The properties of the MLE and Bayesian estimators are studied in the asymptotic of small noise.

Keywords Partially observed linear system · Parameter estimation · Hidden process · Small noise · MLE · BE

1 Introduction

Consider the problem of estimation of the position $\vartheta_0 = \left(x_0, y_0\right)^\top$ of the source $\mathbb{S}_0$ by the observations of the signals from this source received by K detectors $\mathbb{D}_1, \ldots, \mathbb{D}_K$ (see Fig. 1).

If we denote $\vartheta_k = \left(x_k, y_k\right)^\top \in \mathcal{R}^2$ the position of $\mathbb{D}_k$ and suppose that the source starts emission at the moment $t = 0$, then the signal arrives at this detector at the moment $\tau_k\left(\vartheta_0\right) = \nu^{-1}\left\|\vartheta_k - \vartheta_0\right\|$. Here $\nu > 0$ is the rate of propagation of the signals and $\|\cdot\|$ is Euclidean distance in $\mathcal{R}^2$. The set $\Theta \subset \mathcal{R}^2$ is supposed to be open, convex and bounded.

The k-th detector receives the signal $Y_k = \left(Y_k(t), 0 \leq t \leq T\right)$ from the source $\mathbb{S}$ and additive Gaussian noise according to equation

$$\mathrm{d}X_k(t) = a_k(t)\bar{\psi}\left(t - \tau_k\left(\vartheta_0\right)\right)Y_k(t)\mathrm{d}t + \varepsilon\sigma_k(t)\mathrm{d}W_k(t), \quad X_k(0) = 0. \qquad (1)$$

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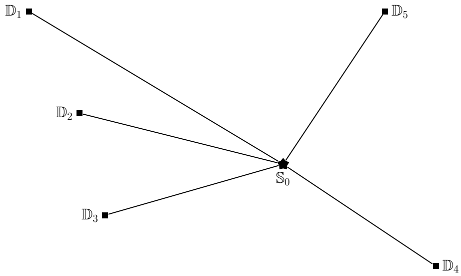

Fig. 1 Model of observations. $\mathbb{S}_0$ is position of the source and $\mathbb{D}_k, k = 1, \ldots, 5$ are positions of the sensors

Here $a_k(\cdot), \sigma_k(\cdot)$ and $\bar\psi(\cdot)$ are known functions and $W_k(\cdot), k = 1, \ldots, K$ are independent Wiener processes. The parameter $\varepsilon > 0$ controls the level of noise. In this work we study the properties of estimators of ϑ_0 in the asymptotic of *small noise*, i.e., as $\varepsilon \to 0$. This is equivalent to the situation with *large signal*, which is rather reasonable in many real situations. This problem is quite close to the inverse problem, where we have K sources $\mathbb{D}_1, \ldots, \mathbb{D}_K$ of signals with known positions and known moments of emission and one detector $\mathbb{S}_0$. The detector receives K signals X^K and has to estimate its own position. This is typical situation in the global positioning system (GPS/INS). The algorithms calculating the positions of different objects (cars, jets, ships et cet.) used in GPS/ISN are based essentially on the adaptive Kalman filtering theory, see, e.g., Almagbile et al. (2010), Gustaffson (2000), Hutchinson (1984), Luo (2013), Wang et al. (2006) and references therein. The same time it seems that the mathematical theory of statistical estimation of the position was not yet sufficiently well developed. This work is continuation of the study initiated in the papers Chernoyarov and Kutoyants (2020), Chernoyarov et al. (2020), Farinetto et al. (2020), where the observed processes are inhomogeneous Poisson.

The function $\bar\psi(t) = 0$ for $t < 0$ and reflects the form of the signal at the moment of its arriving. We consider three different cases: *smooth* $\psi_\delta(t) = t\delta^{-1}\mathbb{1}_{\{0\leq t\leq\delta\}} + \mathbb{1}_{\{t>\delta\}}$, *change-point* type $\psi(t) = \mathbb{1}_{\{t>0\}}$ and *cusp* type $\psi_{\delta,\kappa}(t) = \frac{1}{2}\left(1 + \operatorname{sgn}(2t-\delta)\left|\frac{2t}{\delta} - 1\right|^\kappa\right)\mathbb{1}_{\{0\leq t\leq\delta\}} + \mathbb{1}_{\{t>\delta\}}$, respectively. The examples of such functions are given in Fig. 2.

The parameter $\delta > 0$ is known and small. In the cusp case $\kappa \in (0, \frac{1}{2})$.

The signals $Y_k(\cdot), k = 1, \ldots, K$ satisfy the linear stochastic differential equations

$$\mathrm{d}Y_k(t) = -f_k(t)Y_k(t)\mathrm{d}t + \varepsilon b_k(t)\mathrm{d}V_k(t), \qquad Y_k(0) = y_{k,0} \neq 0. \tag{2}$$

The functions $f_k(\cdot)$ and $b_k(\cdot)$ are known and the Wiener processes $V_k(\cdot), k = 1, \ldots, K$ are independent. The Wiener processes $W_k(\cdot), k = 1, \ldots, K$ and $V_k(\cdot), k = 1, \ldots, K$ are supposed to be independent too.

In this work we consider the problem of estimation ϑ_0 by the observations $X^T = \left(X_1, \ldots, X_K\right)$, where $X_k = \left(X_k(t), 0 \leq t \leq T\right)$. The processes $Y_k(\cdot)$ are non-observable and can be called *hidden*. The processes (2) are Markov, therefore we have the problem of parameter estimation for continuous time hidden Markov processes. Note that similar problems for discrete time models were intensively studied by many authors, see, e.g., Bickel et al. (1998), Cappé et al. (2005), Elliott

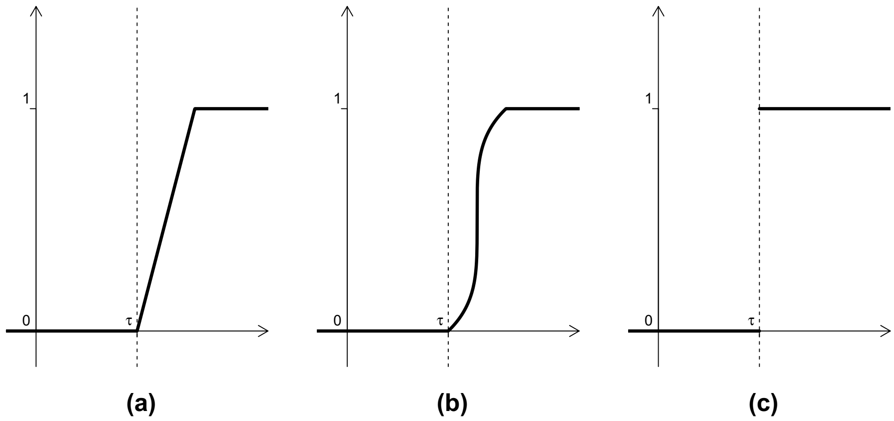

Fig. 2 Examples: **a** $\psi_\delta(\cdot)$, **b** $\psi_{\delta,\kappa}(\cdot)$ and **c**) $\psi(\cdot)$

et al. (1995), Ephraim and Mehrav (2002) and references therein. For continuous time models of partially observed linear stochastic differential equations see, e.g., Kutoyants (1984), Konecny (1990), Kallianpur and Selukar (1991), Kutoyants (1994). The problem of parameter estimation for hidden telegraph process observed in white Gaussian noise was studied in Chigansky (2009), Khasminskii and Kutoyants (2018).

We suppose that the following conditions are always satisfied in this work.

Conditions $\mathscr{R}$.

1. *The functions* $a_k(\cdot), \sigma_k(\cdot), k = 1, \dots, K$ *are bounded and separated from 0 and the functions* $,f_k(\cdot), b_k(\cdot), k = 1, \dots, K$ *are bounded.*
2. *The set* $\Theta \subset \mathcal{R}^2$ *is open, convex, bounded and such that all arrival times* $\tau_k(\vartheta), k = 1, \dots, K$ *belong to* $[0, T]$.
3. *There are at least three detectors which are not on the same line.*

Under the made assumptions the measures corresponding to the observations X^T for different values of $\vartheta \in \Theta$ are equivalent and the likelihood ratio function $L\left(\vartheta, X^K\right)$ is given by the formula (see Liptser and Shiryayev 2001)

$$\ln L\left(\vartheta, X^K\right) = \sum_{k=1}^{K} \int_0^{\mathrm{T}} \frac{G_k\left(\tau_k, t\right)}{\varepsilon^2 \sigma_k(t)} \, \mathrm{d}X_k(t) - \sum_{k=1}^{K} \int_0^{\mathrm{T}} \frac{G_k\left(\tau_k, t\right)^2}{2\varepsilon^2} \, \mathrm{d}t. \tag{3}$$

Here we denoted $\tau_k = \tau_k(\vartheta)$, $G_k\left(\tau_k, t\right) = a_k(t) M_k\left(\tau_k(\vartheta), t\right) \sigma_k(t)^{-1}$, $M_k\left(\tau_k, t\right) = \bar{\psi}\left(t - \tau_k\right) m_k\left(\tau_k, t\right)$, where $m_k\left(\tau_k, \cdot\right)$ are conditional expectations $m_k\left(\tau_k, t\right) = \mathbf{E}_\vartheta\left(Y_k(t) | X_k(s), 0 \le s \le t\right)$. The maximum likelihood estimator (MLE) $\hat{\vartheta}_\varepsilon$ and Bayesian estimator (BE) $\tilde{\vartheta}_\varepsilon$ for quadratic loss function are defined by the relations

$$L\left(\hat{\vartheta}_\varepsilon, X^K\right) = \sup_{\vartheta\in\Theta} L\left(\vartheta, X^K\right), \qquad \tilde{\vartheta}_\varepsilon = \frac{\int_\Theta \vartheta p(\vartheta) L\left(\vartheta, X^K\right) \mathrm{d}\vartheta}{\int_\Theta p(\vartheta) L\left(\vartheta, X^K\right) \mathrm{d}\vartheta}. \tag{4}$$

Here $p(\vartheta), \vartheta \in \Theta$ is prior density which is supposed to be known, continuous and positive function on Θ.

The goal of this work is to study the asymptotic ($\varepsilon \to 0$) behavior of the MLE $\hat{\vartheta}_\varepsilon$ and BE $\tilde{\vartheta}_\varepsilon$. It is shown that the mean squared error $\mathbf{E}_{\vartheta_0}\left\|\hat{\vartheta}_\varepsilon - \vartheta_0\right\|^2 \sim C\varepsilon^\gamma$ where $\gamma > 0$ depends on the type of regularity of the model of observations. We show that the rates are: *in smooth case* $\mathbf{E}_{\vartheta_0}\left\|\hat{\vartheta}_\varepsilon - \vartheta_0\right\|^2 \sim C\varepsilon^2$, in *in cusp-type case* $\mathbf{E}_{\vartheta_0}\left\|\hat{\vartheta}_\varepsilon - \vartheta_0\right\|^2 \sim C\varepsilon^{\frac{4}{2\kappa+1}}$, and in *in change-point case* $\mathbf{E}_{\vartheta_0}\left\|\hat{\vartheta}_\varepsilon - \vartheta_0\right\|^2 \sim C\varepsilon^4$.

The proofs in all three cases are based on two general results by Ibragimov and Khasminskii (1981). Note that the Theorems 1.10.1 and 1.10.3 in Ibragimov and Khasminskii (1981) describe the asymptotic behavior of the MLE and BE in quite general situation. The conditions of these theorems are given in terms of normalized likelihood ratio random fields. Therefore the proofs of our results consist in the verification of the properties of the corresponding likelihood ratio random fields.

2 Notation and auxiliary results

As $\tau_k\left(\vartheta_0\right) = \nu^{-1}\left\|\vartheta_k - \vartheta_0\right\|$ and $\vartheta_k = \left(x_k, y_k\right)^\top, \vartheta_0 = \left(x_0, y_0\right)^\top$ we can write,

$$\begin{aligned}
&\frac{\partial \tau_k\left(\vartheta_0\right)}{\partial x_0} = -\frac{x_k - x_0}{\nu\rho_k} = -\frac{\mu_{k,x}}{\nu}, \qquad \frac{\partial \tau_k\left(\vartheta_0\right)}{\partial y_0} = -\frac{y_k - y_0}{\nu\rho_k} = -\frac{\mu_{k,y}}{\nu},\\
&\rho_k = \left\|\vartheta_k - \vartheta_0\right\|, \qquad \mu_k = \left(\mu_{k,x}, \mu_{k,y}\right)^\top, \qquad \left\|\mu_k\right\| = 1,\\
&h_k(t) = \frac{a_k(t)}{\sigma_k(t)}, \quad \mathbb{B}_k^+ = \left(w: \quad \langle\mu_k, w\rangle > 0\right), \ \mathbb{B}_k^- = \left(w: \quad \langle\mu_k, w\rangle < 0\right).
\end{aligned} \tag{5}$$

Here $a^\top$ means transposition of the vector a and $\langle a, b\rangle$ is scalar product. We have

$$\tau_k\left(\vartheta_0 + \nu\varphi_\varepsilon w\right) = \tau_k\left(\vartheta_0\right) - \varphi_\varepsilon\langle\mu_k, w\rangle + O\left(\varphi_\varepsilon^2\right), \quad w \in \mathcal{R}^2, \quad \varphi_\varepsilon \to 0.$$

The random processes $m_k\left(\tau_k, \cdot\right)$, where $\tau_k = \tau_k(\vartheta)$, satisfy Kalman–Bucy filtration equations (see Kalman and Bucy 1961; Liptser and Shiryayev 2001; Arato 1983)

$$\begin{aligned}
m_k\left(\tau_k, t\right) &= y_k(t), \qquad\qquad 0 \le t \le \tau_k(\vartheta),\\
\mathrm{d}m_k\left(\tau_k, t\right) &= -\left[f_k(t) + \gamma_k\left(\tau_k, t\right) h_k(t)^2 \bar{\psi}\left(t - \tau_k\right)^2\right] m_k\left(\tau_k, t\right)\mathrm{d}t\\
&\quad + \frac{\gamma_k\left(\tau_k, t\right) h_k(t) \bar{\psi}\left(t - \tau_k\right)}{\sigma_k(t)}\,\mathrm{d}X_k(t), \quad \tau_k(\vartheta) \le t \le T.
\end{aligned} \tag{6}$$

Here we denoted $y_k(t)$ solution of Eq. (2) as $\varepsilon = 0$

$$y_k(t)=y_{k,0}\exp\left\{-\int_0^t f_k(s)\,\mathrm{d}s\right\},\quad 0\le t\le T.$$

We have the representation

$$\mathrm{d}X_t=a_k(t)\bar\psi\left(t-\tau_k\left(\vartheta_0\right)\right)m_k\left(\vartheta_0,t\right)\mathrm{d}t+\varepsilon\sigma_k(t)\mathrm{d}\bar W_k(t),\quad X_0=0, \tag{7}$$

where $\bar W_k(\cdot)$ is *innovation* Wiener process defined by Eq. (7) (see Theorem 7.12 in Liptser and Shiryayev (2001)). The function

$$\gamma_k\left(\tau_k,t\right)=\varepsilon^{-2}\mathbf{E}_\vartheta\left(m_k\left(\tau_k,t\right)-Y_k(t)\right)^2$$

is solution of the Ricatti equation

$$\begin{aligned}\gamma_k\left(\tau_k,t\right)&=\int_0^t\exp\left\{-2\int_s^t f_k(r)\mathrm{d}r\right\}b_k(s)^2\mathrm{d}s,\quad 0\le t\le\tau_k(\vartheta),\\ \frac{\partial\gamma_k\left(\tau_k,t\right)}{\partial t}&=-2f_k(t)\gamma_k\left(\tau_k,t\right)+b_k(t)^2\\ &\quad-\gamma_k\left(\tau_k,t\right)^2h_k(t)^2\bar\psi\left(t-\tau_k\right)^2,\quad \tau_k(\vartheta)\le t\le T.\end{aligned} \tag{8}$$

Note that the functions $\gamma_k\left(\tau_k,\cdot\right), k=1,\dots,K$ are bounded and do not depend on ε.

We denote $x_k\left(\cdot,\vartheta_0\right)$, the limit ($\varepsilon=0$) deterministic function of the random process (1). Then we obtain the following expression

$$x_k\left(t,\vartheta_0\right)=\mathbb{1}_{\left\{t>\tau_k\left(\vartheta_0\right)\right\}}\int_{\tau_k\left(\vartheta_0\right)}^t a_k(s)y_k(s)\bar\psi\left(s-\tau_k\left(\vartheta_0\right)\right)\mathrm{d}s.$$

Let us study the random process $m_k\left(\tau_k,t\right)$ as function of τ_k. Consider the difference $\Delta m_k(\tau,t)=m_k(\tau,t)-m_k\left(\tau_0,t\right)$, $\Delta_k\gamma(\tau,t)=\gamma_k(\tau,t)-\gamma_k\left(\tau_0,t\right)$ where $\tau=\tau_k(\vartheta),\tau_0=\tau_k\left(\vartheta_0\right)$. We take the *worst* (discontinuous) function $\bar\psi(t-\tau)=\psi(t-\tau)=\mathbb{1}_{\{t\ge\tau\}}$. We omit for instant the index k in other functions too.

Recall that for $t\in[0,T]$ we have the equality

$$\mathrm{d}m\left(\tau_0,t\right)=-f(t)m\left(\tau_0,t\right)\mathrm{d}t+\varepsilon\gamma\left(\tau_0,t\right)h(t)\mathbb{1}_{\left\{t\ge\tau_0\right\}}\mathrm{d}\bar W(t), \tag{9}$$

with initial value $y_0\neq 0$.

Lemma 1 *Let the conditions* $\mathscr{R}$ *be fulfilled and* $\tau<\tau_0$, *then*

$$\mathbf{E}_{\vartheta_0}|\Delta m(\tau,t)|^2\le C_1\left|\tau_0-\tau\right|\mathbb{1}_{\left\{\tau\le t\le\tau_0\right\}}+\varepsilon^2C_2\left|\tau-\tau_0\right|^2\mathbb{1}_{\left\{\tau_0\le t\le T\right\}}, \tag{10}$$

$$\left|\gamma(\tau,t)-\gamma\left(\tau_0\right)\right|\le C\left|\tau_0-\tau\right|\mathbb{1}_{\{t\ge\tau\}}, \tag{11}$$

where the constants $C_1>0$ and $C_2>0$ *do not depend on* ε.

Proof Suppose that $\tau < \tau_0$. The case $\tau > \tau_0$ can be considered by a similar way. From (6) and (9) we obtain

$$\Delta m(\tau,t) = \left[m(\tau,t) - y(t)\right]\mathbb{1}_{\{\tau\leq t\leq\tau_0\}}, \qquad 0 \leq t \leq \tau_0,$$
$$\mathrm{d}\Delta m(\tau,t) = -S(\tau,t)\Delta m(\tau,t)\mathrm{d}t + \varepsilon h(t)\Delta\gamma(\tau,t)\,\mathrm{d}\bar W(t), \quad \tau_0 < t \leq T,$$

where we put $S(\tau,t) = f(t) + \gamma(\tau,t)h(t)^2$.

Then we obtain the representation

$$\begin{aligned}\Delta m(\tau,t) &= \left[m(\tau,t) - y(t)\right]\mathbb{1}_{\{\tau\leq t\leq\tau_0\}} \\ &\quad + \mathbb{1}_{\{t>\tau_0\}}\left[m\left(\tau,\tau_0\right) - y\left(\tau_0\right)\right]e^{-\int_{\tau_0}^t S(\tau,v)\mathrm{d}v} \\ &\quad + \varepsilon\mathbb{1}_{\{t>\tau_0\}}\int_{\tau_0}^t e^{-\int_s^t S(\tau,v)\mathrm{d}v}h(s)\Delta\gamma(\tau,s)\,\mathrm{d}\bar W(s),\end{aligned} \tag{12}$$

where $\Delta\gamma(\tau,t) = \gamma(\tau,t) - \gamma\left(\tau_0,t\right)$. For the first term and $t \in \left[\tau,\tau_0\right]$ we have

$$\begin{aligned}m(\tau,t) - y(t) &= y(\tau)e^{-\int_\tau^t S(\tau,v)\mathrm{d}v} - y(t) \\ &\qquad + \varepsilon\int_\tau^t e^{-\int_\tau^s S(\tau,v)\mathrm{d}v}\gamma(\tau,s)h(s)\mathrm{d}\bar W(s) \\ &= \varepsilon\int_\tau^t e^{-\int_\tau^s S(\tau,v)\mathrm{d}v}\gamma(\tau,s)h(s)\mathrm{d}\bar W(s).\end{aligned}$$

Let us write the corresponding equation for $\Delta\gamma(\tau,t)$. Denote

$$g(t) = \int_0^t e^{-2\int_s^t f(v)\mathrm{d}v}b(s)^2\mathrm{d}s.$$

Then we have (see 8) $\gamma\left(\tau_0,t\right) = g(t), 0 \leq t \leq \tau_0$,

$$\frac{\partial\gamma\left(\tau_0,t\right)}{\partial t} = -2f(t)\gamma\left(\tau_0,t\right) + b(t)^2 - h(t)^2\gamma\left(\tau_0,t\right)^2, \qquad \tau_0 \leq t \leq T$$

and $\gamma(\tau,t) = g(t), 0 \leq t \leq \tau$,

$$\frac{\partial\gamma(\tau,t)}{\partial t} = -2f(t)\gamma(\tau,t) + b(t)^2 - h(t)^2\gamma(\tau,t)^2, \qquad \tau \leq t \leq T.$$

As $\tau_0 > \tau$ we obtain: $\Delta\gamma(\tau,t) = 0, 0 \leq t \leq \tau, \Delta\gamma(\tau,t) = \gamma(\tau,t) - g(t), \tau \leq t \leq \tau_0$,

$$\frac{\partial\Delta(\tau,t)}{\partial t} = -2f(t)\Delta(\tau,t) - h(t)^2\left[\gamma(\tau,t) + \gamma\left(\tau_0,t\right)\right]\Delta\gamma(\tau,t),\ \ \tau_0 \leq t \leq T$$

with initial value $\Delta\gamma\left(\tau,\tau_0\right) = \gamma\left(\tau,\tau_0\right) - g\left(\tau_0\right)$. Hence for $t \in \left[\tau_0,T\right]$ we have

$$\Delta\gamma(\tau,t)=\Delta\gamma\left(\tau,\tau_0\right)\exp\left\{-\int_{\tau_0}^{t}\left[2f(s)+h(s)^2\left[\gamma(\tau,t)+\gamma\left(\tau_0,t\right)\right]\right]\mathrm{d}s\right\}$$

and $|\Delta\gamma(\tau,t)|\leq\left|\gamma\left(\tau,\tau_0\right)-g\left(\tau_0\right)\right|$.

Recall that the function $\gamma(\tau,t)$ is bounded. On the interval $\left[\tau,\tau_0\right]$ we have

$$\frac{\partial\Delta\gamma(\tau,t)}{\partial t}=-2f(t)\Delta\gamma(\tau,t)-\gamma(\tau,t)^2h(t)^2,\qquad \Delta\gamma(\tau,\tau)=0$$

and

$$\Delta\gamma\left(\tau,\tau_0\right)=-\int_{\tau}^{\tau_0}e^{-2\int_s^t f(v)\mathrm{d}v}\gamma(\tau,s)^2h(s)^2\mathrm{d}s.$$

Therefore $\left|\Delta\gamma\left(\tau,\tau_0\right)\right|\leq C\left|\tau-\tau_0\right|,\qquad |\Delta\gamma(\tau,t)|\leq C\left|\tau-\tau_0\right|$.

This estimate and (12) allows us to write (10). □

3 Smooth case

Consider the model of observations (1), where $\bar{\psi}\left(t-\tau_k\right)=\psi_\delta\left(t-\tau_k\right)=\left(t-\tau_k\right)\delta^{-1}\mathbb{1}_{\left\{0\leq t-\tau_k\leq\delta\right\}}+\mathbb{1}_{\left\{t\geq\delta+\tau_k\right\}}$ and the processes $Y_k(\cdot),k=1,\dots,K$ satisfy Eq. (2). Recall that we have to estimate $\vartheta_0=\left(x_0,y_0\right)^\top$ by observations $X^K=\left(X_1,\dots,X_K\right)$, where $X_k=\left(X_k(t),0\leq t\leq T\right)$.

We write $\tau_k=\tau_k(\vartheta)$, $\theta=(x,y)^\top$ and for the derivative $\dot{m}_k\left(\tau_k,t\right)=\partial m_k\left(\tau_k,t\right)/\partial\tau_k$ we obtain equations: $\dot{m}_k\left(\tau_k,t\right)=0,0\leq t\leq\tau_k(\vartheta)$,

$$\begin{aligned}\mathrm{d}\dot{m}_k\left(\tau_k,t\right)=&-\left[f_k(t)+\Gamma_k\left(\tau_k,t\right)h_k(t)^2\psi_\delta\left(t-\tau_k\right)\right]\dot{m}_k\left(\tau_k,t\right)\mathrm{d}t\\&-\left[\dot{\Gamma}_k\left(\tau_k,t\right)\psi_\delta\left(t-\tau_k\right)-\Gamma_k\left(\tau_k,t\right)\delta^{-1}\right]h_k(t)^2m_k\left(\tau_k,t\right)\mathrm{d}t\\&+\frac{\dot{\Gamma}_k\left(\tau_k,t\right)h_k(t)}{\sigma_k(t)}\mathrm{d}X_t,\quad \tau_k(\vartheta)<t\leq\tau_k(\vartheta)+\delta,\\ \mathrm{d}\dot{m}_k\left(\tau_k,t\right)=&-\left[f_k(t)+\gamma_k\left(\tau_k,t\right)h_k(t)^2\right]\dot{m}_k\left(\tau_k,t\right)\mathrm{d}t+\frac{\dot{\gamma}_k\left(\tau_k,t\right)h_k(t)}{\sigma_k(t)}\mathrm{d}X_t,\\&-\dot{\gamma}_k\left(\tau_k,t\right)h_k(t)^2m_k\left(\tau_k,t\right)\mathrm{d}t,\quad \tau_k(\vartheta)+\delta<t\leq T,\end{aligned}\tag{13}$$

where we denoted $\Gamma_k\left(\tau_k,t\right)=\gamma_k\left(\tau_k,t\right)\psi_\delta\left(t-\tau_k\right)$ and used the equality $\partial\psi_\delta\left(t-\tau_k\right)/\partial\tau_k=-\delta^{-1}\mathbb{1}_{\left\{0\leq t-\tau_k\leq\delta\right\}}$.

The solution of Eq. (13) at point $\varepsilon=0,\vartheta=\vartheta_0$ we denote $z_k\left(t,\vartheta_0\right),0\leq t\leq T$. The function $z_k\left(t,\vartheta_0\right)$ satisfies equations

$$\frac{\partial z_k\left(t,\vartheta_0\right)}{\partial t}=0,\quad 0\leq t\leq \tau_k\left(\vartheta_0\right),$$
$$\frac{\partial z_k\left(t,\vartheta_0\right)}{\partial t}=-\left[f_k(t)+\Gamma_k\left(\tau_k\left(\vartheta_0\right),t\right)h_k(t)^2\frac{\left(t-\tau_k\left(\vartheta_0\right)\right)}{\delta}\right]z_k\left(t,\vartheta_0\right)$$
$$+\Gamma_k\left(\tau_k\left(\vartheta_0\right),t\right)h_k(t)^2\delta^{-1}y_k(t),\qquad \tau_k\left(\vartheta_0\right)<t\leq\tau_k\left(\vartheta_0\right)+\delta,$$
$$\frac{\partial z_k\left(t,\vartheta_0\right)}{\partial t}=-\left[f_k(t)+\gamma_k\left(\tau_k\left(\vartheta_0\right),t\right)h_k(t)^2\right]z_k\left(t,\vartheta_0\right),\ \tau_k\left(\vartheta_0\right)+\delta<t\leq T.$$

Of course the solutions of this linear equations can be written explicitly.

Introduce two deterministic functions

$$\dot M_{k,x}^o\left(\tau_k\left(\vartheta_0\right),t\right)=\nu\,\frac{\partial\left[m_k\left(\tau_k(\vartheta),t\right)\psi_\delta\left(t-\tau_k(\vartheta)\right)\right]}{\partial\tau_k}\Bigg|_{\vartheta=\vartheta_0,\varepsilon=0}\frac{\partial\tau_k\left(\vartheta_0\right)}{\partial x_0}$$
$$=\left[\delta^{-1}y_k(t)\mathbb{1}_{\left\{\tau_k(\vartheta_0)\leq t\leq\tau_k(\vartheta_0)+\delta\right\}}-z_k\left(t,\vartheta_0\right)\psi_\delta\left(t-\tau_k\left(\vartheta_0\right)\right)\right]\mu_{k,x},$$
$$\dot M_{k,y}^o\left(\tau_k\left(\vartheta_0\right),t\right)=\nu\,\frac{\partial\left[m_k\left(\tau_k(\vartheta),t\right)\psi_\delta\left(t-\tau_k(\vartheta)\right)\right]}{\partial\tau_k}\Bigg|_{\vartheta=\vartheta_0,\varepsilon=0}\frac{\partial\tau_k\left(\vartheta_0\right)}{\partial y_0}$$
$$=\left[\delta^{-1}y_k(t)\mathbb{1}_{\left\{\tau_k(\vartheta_0)\leq t\leq\tau_k(\vartheta_0)+\delta\right\}}-z_k\left(t,\vartheta_0\right)\psi_\delta\left(t-\tau_k\left(\vartheta_0\right)\right)\right]\mu_{k,y}.$$

and 2×2 Fisher information matrix $\mathrm{I}\left(\vartheta_0\right)=\left(\mathrm{I}\left(\vartheta_0\right)_{l,m}\right)$

$$\mathrm{I}_{1,1}\left(\vartheta_0\right)=\sum_{k=1}^K\int_{\tau_k(\vartheta_0)}^{\mathrm{T}}h_k(t)^2\dot M_{k,x}^o\left(\tau_k\left(\vartheta_0\right),t\right)^2\mathrm{d}t,$$
$$\mathrm{I}_{2,2}\left(\vartheta_0\right)=\sum_{k=1}^K\int_{\tau_k(\vartheta_0)}^{\mathrm{T}}h_k(t)^2\dot M_{k,y}^o\left(\tau_k\left(\vartheta_0\right),t\right)^2\mathrm{d}t,$$
$$\mathrm{I}_{1,2}\left(\vartheta_0\right)=\mathrm{I}_{2,1}\left(\vartheta_0\right)=\sum_{k=1}^K\int_{\tau_k(\vartheta_0)}^{\mathrm{T}}h_k(t)^2\dot M_{k,x}^o\left(\tau_k\left(\vartheta_0\right),t\right)\dot M_{k,y}^o\left(\tau_k\left(\vartheta_0\right),t\right)\mathrm{d}t.$$

The family of measures which corresponds to the solutions of Eq. (1) with different $\vartheta\in\Theta$ is locally asymptotically normal (LAN) (see Lemma 2 below). Therefore we have the Hajek–Le Cam's lower minimax bound on the risks of all estimators $\bar\vartheta_\varepsilon$:

$$\lim_{\delta\to 0}\ \varliminf_{\varepsilon\to 0}\ \sup_{\left\|\vartheta-\vartheta_0\right\|\leq\delta}\varepsilon^{-2}\mathbf{E}_\vartheta\left\|\bar\vartheta_\varepsilon-\vartheta\right\|^2\geq\mathbf{E}_{\vartheta_0}\|\zeta\|^2,\quad \zeta\sim\mathcal{N}\left(0,\mathrm{I}\left(\vartheta_0\right)^{-1}\right)\tag{14}$$

(see, e.g., Ibragimov and Khasminskii (1981)). As usual, we call the estimator ϑ_ε^* asymptotically efficient if for this estimator and all $\vartheta_0\in\Theta$ we have equality in (14).

The properties of the estimators are given in the following theorem.

Theorem 1 *Let the conditions $\mathscr{R}$ be fulfilled and the Fisher information matrix is uniformly non-degenerate*

$$\hat{\kappa} = \inf_{\vartheta_0 \in \Theta} \inf_{e \in \mathcal{R}^2, \|e\|=1} e^\top \mathrm{I}(\vartheta_0) e > 0.$$

Then the MLE $\hat{\vartheta}_\varepsilon$ and BE $\tilde{\vartheta}_\varepsilon$ are uniformly consistent on compacts $\mathbb{K} \in \Theta$, asymptotically normal

$$\nu^{-1}\varepsilon^{-1}\left(\hat{\vartheta}_\varepsilon - \vartheta_0\right) \Longrightarrow \mathcal{N}\left(0, \mathrm{I}\left(\vartheta_0\right)^{-1}\right), \quad \varepsilon^{-1}\left(\tilde{\vartheta}_\varepsilon - \vartheta_0\right) \Longrightarrow \mathcal{N}\left(0, \mathrm{I}\left(\vartheta_0\right)^{-1}\right),$$

we have the convergence of moments: for any $p > 0$

$$\varepsilon^{-p}\mathbf{E}_{\vartheta_0}\left\|\hat{\vartheta}_\varepsilon - \vartheta_0\right\|^p \longrightarrow \mathbf{E}_{\vartheta_0}\|\zeta\|^p, \qquad \varepsilon^{-p}\mathbf{E}_{\vartheta_0}\left\|\tilde{\vartheta}_\varepsilon - \vartheta_0\right\|^p \longrightarrow \mathbf{E}_{\vartheta_0}\|\zeta\|^p,$$

where $\zeta \sim \mathcal{N}\left(0, \mathrm{I}\left(\vartheta_0\right)^{-1}\right)$. The both estimators are asymptotically efficient.

Proof The proof of this theorem is based on the general results obtained for the MLE and BE by Ibragimov and Khasminskii (1981) (see Theorems 1.10.1 and 1.10.3). Below we verify the properties of the normalized likelihood ratio random field

$$Z_\varepsilon(w) = \frac{L\left(\vartheta_0 + \varepsilon w, X^K\right)}{L\left(\vartheta_0, X^K\right)}, \quad w = (u, v) \in \mathbb{W}_\varepsilon = \left\{w : \ \vartheta_0 + \varepsilon w \in \Theta\right\},$$

which are the conditions of the mentioned theorems. Introduce the random field

$$Z(w) = \exp\left\{w^\top \Delta\left(\vartheta_0\right) - \frac{1}{2} w^\top \mathrm{I}\left(\vartheta_0\right) w\right\}, \quad \Delta\left(\vartheta_0\right) \sim \mathcal{N}\left(0, \mathrm{I}\left(\vartheta_0\right)\right), \ w \in \mathcal{R}^2.$$

We prove three lemmas below.

Lemma 2 *The finite-dimensional distributions of the random fields $Z_\varepsilon(\cdot)$ converge to the finite-dimensional distributions of $Z(\cdot)$ and this convergence is uniform on compacts $\mathbb{K} \in \Theta$.*

Proof Denote: $D_{k,\varepsilon}\left(\vartheta_0, w, t\right) = \varepsilon^{-1}\left[M_k\left(\tau_k\left(\vartheta_w\right), t\right) - M_k\left(\tau_k\left(\vartheta_0\right), t\right)\right], \vartheta_w = \vartheta_0 + \varepsilon w$.
Then the log-likelihood ratio admits the representation (see 9)

$$\ln Z_\varepsilon(w) = \sum_{k=1}^{K} \int_{\tau_k(\vartheta_w)\wedge\tau_k(\vartheta_0)}^{\mathrm{T}} h_k(t) D_{k,\varepsilon}\left(\vartheta_0, w, t\right) \mathrm{d}\bar{W}_k(t)$$
$$- \frac{1}{2}\sum_{k=1}^{K} \int_{\tau_k(\vartheta_w)\wedge\tau_k(\vartheta_0)}^{\mathrm{T}} h_k(t)^2 D_{k,\varepsilon}\left(\vartheta_0, w, t\right)^2 \mathrm{d}t.$$

The random processes $m_k\left(\tau_k,t\right)$ are differentiable with probability 1 w.r.t. τ_k and the derivative $\dot m_k\left(\vartheta_0,t\right)$ satisfies the estimate

$$\mathbf{E}_{\vartheta_0}\left|\dot m_k\left(\tau_k\left(\vartheta_0\right),t\right)-z_k\left(\vartheta_0,t\right)\right|^2\leq C\varepsilon^2.$$

To prove this estimate we have to write the difference of the equations for $\dot m_k\left(\tau_k\left(\vartheta_0\right),t\right)$ and $z_k\left(\vartheta_0,t\right)$ and then after simple transformations to use the Gronwall-Bellman lemma. See similar estimates can be found in Kutoyants (1994).

By Taylor formula we obtain the relation

$$D_{k,\varepsilon}\left(\vartheta_0,w,t\right)=\langle \dot M_k^o\left(\tau_k\left(\vartheta_0\right),t\right),w\rangle+o(1)$$

where $\dot M_k^o\left(\tau_k\left(\vartheta_0\right),t\right)=\left(\dot M_{k,x}^o\left(\tau_k\left(\vartheta_0\right),t\right),\dot M_{k,y}^o\left(\tau_k\left(\vartheta_0\right),t\right)\right)^\top$. Introduce the random vector

$$\Delta_\varepsilon\left(\vartheta_0\right)=\sum_{k=1}^K\int_{\tau_k\left(\vartheta_0\right)}^{\mathrm{T}}h_k(t)\dot M_k^o\left(\tau_k\left(\vartheta_0\right),t\right)\mathrm{d}\bar W_k(t).$$

Recall that $\bar W_k(t)$ depends on ε. Then we have

$$\begin{aligned}&\int_{\tau_k\left(\vartheta_w\right)\wedge\tau_k\left(\vartheta_0\right)}^{\mathrm{T}}h_k(t)D_{k,\varepsilon}\left(\vartheta_0,w,t\right)\,\mathrm{d}\bar W_k(t)\\&\quad=\int_{\tau_k\left(\vartheta_0\right)}^{\mathrm{T}}h_k(t)\langle\dot M_k^o\left(\tau_k\left(\vartheta_0\right),t\right),w\rangle\,\mathrm{d}\bar W_k(t)+o(1)\end{aligned}$$

because $\tau_k\left(\vartheta_w\right)=\tau_k\left(\vartheta_0\right)+O(\varepsilon)$ and

$$\begin{aligned}&\int_{\tau_k\left(\vartheta_w\right)\wedge\tau_k\left(\vartheta_0\right)}^{\mathrm{T}}h_k(t)^2D_{k,\varepsilon}\left(\vartheta_0,w,t\right)^2\mathrm{d}t\\&\quad=\int_{\tau_k\left(\vartheta_0\right)}^{\mathrm{T}}h_k(t)^2\langle\dot M_k^o\left(\tau_k\left(\vartheta_0\right),t\right),w\rangle^2\mathrm{d}t+o(1).\end{aligned}$$

Note that

$$\sum_{k=1}^K\int_{\tau_k\left(\vartheta_0\right)}^{\mathrm{T}}h_k(t)^2\langle\dot M_k^o\left(\tau_k\left(\vartheta_0\right),t\right),w\rangle^2\mathrm{d}t=w^\top\mathrm{I}\left(\vartheta_0\right)w.$$

Hence for the normalized likelihood ratio function we obtain the representation called LAN (local asymptotic normality)

$$Z_\varepsilon(w)=\exp\left\{\langle w,\Delta_\varepsilon\left(\vartheta_0\right)\rangle-\frac{1}{2}w^\top\mathrm{I}\left(\vartheta_0\right)w+o(1)\right\},$$

where

$$\Delta_\varepsilon\left(\vartheta_0\right)\sim\mathcal{N}\left(0,\mathrm{I}\left(\vartheta_0\right)\right).$$

This representation of $Z_\varepsilon(\cdot)$ provides the convergence of finite dimensional distributions. Moreover, it can be shown that all convergences are uniform on compacts $\mathbb{K}\subset\Theta$. □

Lemma 3 *There exists constant $C>0$ such that for any $R>0$ and $\left\|w_1\right\|+\left\|w_2\right\|<R$ we have*

$$\sup_{\vartheta_0\in\mathbb{K}}\mathbf{E}_{\vartheta_0}\left|Z_\varepsilon^{1/4}\left(w_2\right)-Z_\varepsilon^{1/4}\left(w_1\right)\right|^4\leq C\left(1+R^4\right)\left\|w_2-w_1\right\|^4. \tag{15}$$

Proof Following the proof of Lemma 5.1 in Kutoyants (1994) we first write

$$\mathbf{E}_{\vartheta_0}\left|Z_\varepsilon^{1/4}\left(w_2\right)-Z_\varepsilon^{1/4}\left(w_1\right)\right|^4=\mathbf{E}_{\vartheta_0}Z_\varepsilon\left(w_1\right)\left|\left(\frac{Z_\varepsilon\left(w_2\right)}{Z_\varepsilon\left(w_1\right)}\right)^{1/4}-1\right|^4$$
$$=\mathbf{E}_{\vartheta_0+\varepsilon w_1}\left|V_T-1\right|^4,$$

where we changed the measure and denoted

$$V_T=\left(\frac{Z_\varepsilon\left(w_2\right)}{Z_\varepsilon\left(w_1\right)}\right)^{1/4}=\exp\left\{\frac{1}{4}\sum_{k=1}^K\int_0^{\mathrm{T}}h_k(t)D_{k,\varepsilon}\left(\vartheta_0,w_1,w_2,t\right)\mathrm{d}\bar W_k(t)\right.$$
$$\left.-\frac{1}{8}\sum_{k=1}^K\int_0^{\mathrm{T}}h_k(t)^2D_{k,\varepsilon}\left(\vartheta_0,w_1,w_2,t\right)^2\mathrm{d}t\right\}.$$

Here $D_{k,\varepsilon}\left(\vartheta_0,w_1,w_2,t\right)=D_{k,\varepsilon}\left(\vartheta_0,w_2,t\right)-D_{k,\varepsilon}\left(\vartheta_0,w_1,t\right)$. Then for the process $V_t, 0\leq t\leq T$ we write Itô formula

$$\begin{aligned}V_T=1&-\frac{3}{32}\sum_{k=1}^K\int_0^{\mathrm{T}}h_k(t)^2V_tD_{k,\varepsilon}\left(\vartheta_0,w_1,w_2,t\right)^2\mathrm{d}t\\&+\frac{1}{4}\sum_{k=1}^K\int_0^{\mathrm{T}}h_k(t)V_tD_{k,\varepsilon}\left(\vartheta_0,w_1,w_2,t\right)\mathrm{d}\bar W_k(t).\end{aligned}\tag{16}$$

Hence

$$
\begin{aligned}
\mathbf{E}_{\vartheta_0+\varepsilon w_1}\left|V_T-1\right|^4 &\leq C\mathbf{E}_{\vartheta_0+\varepsilon w_1}\left|\sum_{k=1}^K\int_0^{\rm T} h_k(t)^2V_tD_{k,\varepsilon}\left(\vartheta_0,w_1,w_2,t\right)^2{\rm d}t\right|^4\\
&\quad +C\mathbf{E}_{\vartheta_0+\varepsilon w_1}\left|\sum_{k=1}^K\int_0^{\rm T} h_k(t)V_tD_{k,\varepsilon}\left(\vartheta_0,w_1,w_2,t\right){\rm d}\bar W_k(t)\right|^4\\
&\leq CT^3\sum_{k=1}^K\int_0^{\rm T}\mathbf{E}_{\vartheta_0+\varepsilon w_1}V_t^4h_k(t)^8D_{k,\varepsilon}\left(\vartheta_0,w_1,w_2,t\right)^8{\rm d}t\\
&\quad +CT\sum_{k=1}^K\int_0^{\rm T}\mathbf{E}_{\vartheta_0+\varepsilon w_1}V_t^4h_k(t)^4D_{k,\varepsilon}\left(\vartheta_0,w_1,w_2,t\right)^4{\rm d}t\\
&\leq CT^3\sum_{k=1}^K\int_0^{\rm T}\mathbf{E}_{\vartheta_0+\varepsilon w_2}h_k(t)^8D_{k,\varepsilon}\left(\vartheta_0,w_1,w_2,t\right)^8{\rm d}t\\
&\quad +CT\sum_{k=1}^K\int_0^{\rm T}\mathbf{E}_{\vartheta_0+\varepsilon w_2}h_k(t)^4D_{k,\varepsilon}\left(\vartheta_0,w_1,w_2,t\right)^4{\rm d}t.
\end{aligned}
$$

Further, recall that $\vartheta_w=\vartheta_0+\varepsilon w$ and denote $\vartheta(s)=\vartheta_0+\varepsilon w_1+\varepsilon s\left(w_2-w_1\right)$, then we can write

$$
\begin{aligned}
&D_{k,\varepsilon}\left(\vartheta_0,w_1,w_2,t\right)\\
&=\varepsilon^{-1}\left[\psi_\delta\left(t-\tau_k\left(\vartheta_{w_2}\right)\right)m_k\left(\tau_k\left(\vartheta_{w_2}\right),t\right)-\psi_\delta\left(t-\tau_k\left(\vartheta_{w_1}\right)\right)m_k\left(\tau_k\left(\vartheta_{w_1}\right),t\right)\right]\\
&=\nu^{-1}\int_0^1\left[\dot\psi_\delta\left(t-\tau_k(\vartheta(s))\right)m_k\left(\tau_k(\vartheta(s),t)\right)\right.\\
&\quad\left.+\psi_\delta\left(t-\tau_k(\vartheta(s))\right)\dot m_k\left(\tau_k(\vartheta(s)),t\right)\right]\langle\mu_k,w_2-w_1\rangle{\rm d}s.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\sum_{k=1}^K\int_0^{\rm T}\mathbf{E}_{\vartheta_0+\varepsilon w_2}h_k(t)^8D_{k,\varepsilon}\left(\vartheta_0,w_1,w_2,t\right)^8{\rm d}t\leq C\left\|w_2-w_1\right\|^8,\\
&\sum_{k=1}^K\int_0^{\rm T}\mathbf{E}_{\vartheta_0+\varepsilon w_2}h_k(t)^4D_{k,\varepsilon}\left(\vartheta_0,w_1,w_2,t\right)^4{\rm d}t\leq C\left\|w_2-w_1\right\|^4
\end{aligned}
$$

Finally we obtain (15). □

Lemma 4 *There exists constant* $c_*>0$ *such that*

$$
\sup_{\vartheta_0\in\mathbb{K}}\mathbf{E}_{\vartheta_0}Z_\varepsilon(w)^{1/2}\leq e^{-c_*\|w\|^2}. \tag{17}
$$

Proof Consider the integral

$$\int_0^T h_k(t)^2 D^2_{k,\varepsilon(\vartheta_0,w,t)}\mathrm{d}t = \int_0^T \frac{h_k(t)^2}{\varepsilon^2}\left[M\left(\tau_k\left(\vartheta_w\right),t\right) - M\left(\tau_k\left(\vartheta_0\right),t\right)\right]^2\mathrm{d}t.$$

Denote

$$m_{k,w}(t) = m_k\left(\tau_k\left(\vartheta_w\right),t\right), \quad \Gamma_{k,w}(t) = \Gamma_k\left(\tau_k\left(\vartheta_w\right),t\right),$$
$$\psi_{\delta,k,w}(t) = \psi_\delta\left(t - \tau_k\left(\vartheta_w\right)\right).$$

Random functions $m_{k,0}(t)$ and $m_{k,w}(t)$ satisfy equations

$$\begin{aligned}
m_{k,0}(t) &= y_k(t), \quad 0 \le t \le \tau_k\left(\vartheta_0\right),\\
\mathrm{d}m_{k,0}(t) &= -f_k(t)m_{k,0}(t)\mathrm{d}t + \varepsilon\Gamma_{k,0}(t)h_k(t)\mathrm{d}\bar{W}_t, \quad \tau_k\left(\vartheta_0\right) < t \le T,\\
m_{k,w}(t) &= y_k(t), \quad 0 \le t \le \tau_k\left(\vartheta_w\right),\\
\mathrm{d}m_{k,w}(t) &= -\left[f_k(t) + \Gamma_{k,w}(t)h_k(t)^2\psi_{\delta,k,w}(t)\right]m_{k,w}(t)\mathrm{d}t\\
&\quad + \Gamma_{k,w}(t)h_k(t)^2\psi_{\delta,k,0}(t)m_{k,0}(t)\mathrm{d}t\\
&\quad + \varepsilon\Gamma_{k,w}(t)h_k(t)\mathrm{d}\bar{W}_t, \qquad \tau_k\left(\vartheta_w\right) < t \le T.
\end{aligned}$$

The solution of the first equation for $\varepsilon = 0$ is denoted as $y_k(t) = y_k\left(\vartheta_0, t\right)$ and given in (8). The solution $m_{k,w}(t)$ for $\varepsilon = 0$ and $t \in \left[\tau_k\left(\vartheta_w\right), T\right]$ (we denote it as $y_k\left(\vartheta_w, t\right) = y_k\left(\vartheta_w, \vartheta_0, t\right)$) satisfies equation

$$\begin{aligned}
\frac{\partial y_k\left(\vartheta_w,t\right)}{\partial t} &= -\left[f_k(t) + \Gamma_{k,w}(t)h_k(t)^2\psi_{\delta,k,w}(t)\right]y_k\left(\vartheta_w,t\right)\\
&\quad + \Gamma_{k,w}(t)h_k(t)^2\psi_{\delta,k,0}(t)y_k(t),\ y_k\left(\vartheta_w,\tau_k\left(\vartheta_w\right)\right) = y_k\left(\tau_k\left(\vartheta_w\right)\right).
\end{aligned}$$

If we put $q_k\left(\vartheta_0,t\right) = m_k\left(\tau_k\left(\vartheta_0\right),t\right) - y_k(t)$ and $q_k\left(\vartheta_w,t\right) = m_k\left(\tau_k\left(\vartheta_w\right),t\right) - y_k\left(\vartheta_w,t\right)$, then for these differences it is possible to verify the relations

$$\begin{aligned}
q_k\left(\vartheta_0,t\right) &= \varepsilon\int_0^t e^{-\int_s^t f_k(r)\mathrm{d}r}\Gamma_{k,0}(s)h_k(s)\mathrm{d}\bar{W}_s,\\
q_k\left(\vartheta_w,t\right) &= \varepsilon\int_0^t e^{-\int_s^t Q_k(\vartheta_w,r)\mathrm{d}r}\Gamma_{k,w}(s)h_k(s)\mathrm{d}\bar{W}_s\\
&\quad + \int_0^t e^{-\int_s^t Q_k(\vartheta_w,r)\mathrm{d}r}\Gamma_{k,w}\left(\vartheta_w,s\right)h_k(s)^2\psi_{\delta,k}(0,s)q_k(s)\mathrm{d}s,
\end{aligned}$$

where $Q_k\left(\vartheta_w,t\right) = f_k(t) + \Gamma_k\left(\vartheta_w,t\right)h_k(t)^2\psi_\delta\left(t - \tau_k\left(\vartheta_w\right)\right)$.

Thus we have

$$m_k\left(\vartheta_0,t\right) - y_k(t) = \varepsilon\xi_1(t), \qquad m_k\left(\vartheta_w,t\right) - y_k\left(\vartheta_w,t\right) = \varepsilon\xi_2(t),$$

where $\xi_1(t)$ and $\xi_2(t)$ are Gaussian processes with bounded variances.

Therefore, as it follows from Lemma 2.4 in Kutoyants (1994), it is sufficient to consider the deterministic integral

$$F\left(\vartheta_w,\vartheta_0\right)=\sum_{k=1}^{K}\int_0^{\mathrm{T}}\frac{h_k(t)^2}{\varepsilon^2}\left[\psi_{\delta,k,w}(t)y_k\left(\vartheta_w,t\right)-\psi_{\delta,k,0}(t)y_k(t)\right]^2\mathrm{d}t$$

and to show that there exists a constant $\kappa_*>0$ such that

$$F\left(\vartheta_w,\vartheta_0\right)\geq\kappa_*\|w\|^2. \tag{18}$$

Remind that for $\|w\|<R$

$$F\left(\vartheta_w,\vartheta_0\right)=w^{\top}\mathrm{I}\left(\vartheta_0\right)w(1+o(1)).$$

Hence for some $\varepsilon_1>0$ and $\varepsilon_1\|w\|\leq\varepsilon_1 R=d$ and all $\varepsilon\leq\varepsilon_1$ we can write

$$F\left(\vartheta_w,\vartheta_0\right)\geq\frac{1}{2}w^{\top}\mathrm{I}\left(\vartheta_0\right)w\geq\frac{\|w\|^2}{2}\inf_{\|e\|=1}e^{\top}\mathrm{I}\left(\vartheta_0\right)e\geq\frac{\hat{\kappa}}{2}\|w\|^2.$$

Here we denoted $\hat{\kappa}>0$ the constant from the condition of the Theorem 1.

Introduce the function

$$G\left(\vartheta_*,\vartheta_0\right)=\sum_{k=1}^{K}\int_0^{\mathrm{T}}h_k(t)^2\left[\psi_\delta\left(t-\tau_k\left(\vartheta_*\right)\right)y_k\left(\vartheta_*,t\right)\right.$$
$$\left.-\psi_\delta\left(t-\tau_k\left(\vartheta_0\right)\right)y_k\left(\vartheta_0,t\right)\right]^2\mathrm{d}t,$$

where $\vartheta_*=\left(x_*,y_*\right)\in\Theta$ and $\left\|\vartheta_*-\vartheta_0\right\|\geq d$. We have to show that

$$\tilde{\kappa}=\inf_{\vartheta_*:\left\|\vartheta_*-\vartheta_0\right\|\geq d}G\left(\vartheta_*,\vartheta_0\right)>0.$$

Suppose that there exists ϑ_* such that $G\left(\vartheta_*,\vartheta_0\right)=0$ and $\left\|\vartheta_*-\vartheta_0\right\|\geq d$. Then we have $\tau_k\left(\vartheta_*\right)=\tau_k\left(\vartheta_0\right)$ and $y_k\left(\vartheta_*,t\right)=y_k\left(\vartheta_0,t\right)$ for all k. This means that for two different positions of the source ϑ_* and ϑ_0 we have k equalities

$$\left\|\vartheta_k-\vartheta_*\right\|=\left\|\vartheta_k-\vartheta_0\right\|,\qquad k=1,\ldots,K$$

but as we have at least three detectors not on the same line such equalities for all k are impossible to have. This is the condition of identifiability.

Therefore for $\varepsilon\|w\|>d$ there exists a constant $c_*>0$ such that

$$F\left(\vartheta_w,\vartheta_0\right)\geq\varepsilon^{-2}G\left(\vartheta_0+\varepsilon w,\vartheta_0\right)\geq\frac{\tilde{\kappa}}{\varepsilon^2}\geq c_*^2\tilde{\kappa}\|w\|^2$$

because

$$\|w\|\leq\varepsilon^{-1}\sup_{\vartheta,\vartheta'\in\Theta}\left\|\vartheta-\vartheta'\right\|\qquad\text{and}\qquad\varepsilon^{-1}\geq\frac{\|w\|}{\sup_{\vartheta,\vartheta'\in\Theta}\|\vartheta-\vartheta'\|}=c_*\|w\|.$$

Therefore we obtained (18). Now the proof of the estimate (17) follows from the proof of Lemma 2.4 in Kutoyants (1994). □

The properties of the normalized likelihood ratio function $Z_\varepsilon(w), w \in \mathbb{W}_n$ established in Lemmas 2–4 are sufficient conditions for the Theorems 3.1.1, 3.1.3 and 3.2.1 in Ibragimov and Khasminskii (1981) and therefore the MLE and BE mentioned in the Theorem 1 of this work follow from the mentioned theorems in Ibragimov and Khasminskii (1981). □

4 Cusp-type case

The cusp-type models are proposed as alternative to change-point models with the following motivation. The real signals in technical devices can not have discontinuous characteristics. Say, electrical current can not have pure jump at the moment of signal arriving. Usually we have continuous curves but with strong increasing at the moment of signal arriving. We presented different types of increasing functions: smooth, cusp-type and change-point type. The cusp-type model is intermediate between smooth model case (rate of mean square error is ε^2) and change-point type models (rate of mean square error is ε^4). We suppose that the cusp-type model with κ close to 0 is better approximation of real "change-point" situation, because it proposes a continuous curve close in $L_2[0, T]$ to discontinuous curve.

We return to the considered above model of K detectors with observations

$$\mathrm{d}X_k(t) = a_k(t)\psi_{\delta,\kappa}\big(t - \tau_k(\vartheta)\big)\,Y_k(t)\mathrm{d}t + \varepsilon\sigma_k(t)\mathrm{d}W_t, \quad 0 \le t \le T \tag{19}$$

where $X_k(0) = 0$, $k = 1, \ldots, K$ and the function

$$\psi_{\delta,\kappa}(t) = \frac{1}{2}\left(1 + \operatorname{sgn}(2t - \delta)\left|\frac{2t}{\delta} - 1\right|^{\kappa}\right)\mathbb{1}_{\{0\le t\le\delta\}} + \mathbb{1}_{\{\delta\le t\le T\}}.$$

The *hidden* Gaussian processes $Y_k(\cdot), k = 1, \ldots, K$, as before, satisfy the same linear equations

$$\mathrm{d}Y_k(t) = -f_k(t)Y_k(t)\mathrm{d}t + \varepsilon b_k(t)\mathrm{d}V_k(t), \quad Y_k(0) = y_{k,0} > 0, \quad 0 \le t \le T. \tag{20}$$

We suppose that the conditions $\mathscr{R}$ are fulfilled and as before we study the properties of estimators as $\varepsilon \to 0$, i.e., *small noise* asymptotic.

The delay $\tau_k(\vartheta) = \rho_k/\nu = \nu^{-1}\left\|\vartheta_k - \vartheta_0\right\|$ and we have to estimate the position $\vartheta_0 = \left(x_0, y_0\right)$ by observations $X^K = \left(X_1, \ldots, X_K\right)$. Here the process $X_k = \left(X_k(t), 0 \le t \le T\right)$.

The likelihood ratio function and estimators $\hat\vartheta_\varepsilon$, $\tilde\vartheta_\varepsilon$ are defined by the same relations (3)–(6).

To describe the asymptotic behavior of these estimators we need notations (5) and the following ones:

$$
\begin{aligned}
&\pi_k(\vartheta_0)=h_k\left(\tau_k\left(\vartheta_0+\delta/2\right)\right)y_k\left(\tau_k(\vartheta_0)+\delta/2\right),\\
&\pi_k=\pi_k(\vartheta_0),\quad Q(\kappa)^2=\int_{\mathcal{R}}\left[\operatorname{sgn}(s-1)|s-1|^\kappa-\operatorname{sgn}(s)|s|^\kappa\right]^2\mathrm{d}s,\\
&\varphi_\varepsilon=\varepsilon^{\frac{1}{H}}\;Q(\kappa)^{-\frac{1}{H}}\;2^{\frac{1-\kappa}{H}}\;\delta^{\frac{\kappa}{H}},\qquad H=\kappa+\frac{1}{2}.
\end{aligned}
$$

Introduce the limit likelihood ratio random field

$$
\begin{aligned}
Z(w)=\exp\Biggl\{&\sum_{k=1}^{K}\left[\pi_k W_k^{H,+}\left(\langle\mu_k,w\rangle\right)-\frac{\pi_k^2}{2}\left|\langle\mu_k,w\rangle\right|^{2H}\right]\mathbb{1}_{\{\mathbb{B}_k^+\}}\\
&+\sum_{k=1}^{K}\left[\pi_k W_k^{H,-}\left(-\langle\mu_k,w\rangle\right)-\frac{\pi_k^2}{2}\left|\langle\mu_k,w\rangle\right|^{2H}\right]\mathbb{1}_{\{\mathbb{B}_k^-\}}\Biggr\},\quad w\in\mathcal{R}^2.
\end{aligned}
$$

Here $W_k^{H,+}(s), s\geq 0, k=1,\ldots,K$ and $W_k^{H,-}(s), s\geq 0, k=1,\ldots,K$ are independent fractional Brownian motions with Hurst parameter H, i.e., independent Gaussian processes with properties: $\mathbf{E}W_k^{H,+}(s)=0$ and

$$
\mathbf{E}W_k^{H,+}(s_1)W_k^{H,+}(s_2)=\frac{1}{2}\left(\left|s_1\right|^{2H}+\left|s_2\right|^{2H}-\left|s_1-s_2\right|^{2H}\right).
$$

The limit distributions of the estimators are given by the random vectors $\hat{w}$ and $\tilde{w}$ defined by the following relations

$$
Z(\hat{w})=\sup_{w\in\mathcal{R}^2}Z(w),\qquad \tilde{w}=\frac{\int_{\mathcal{R}^2}wZ(w)\mathrm{d}w}{\int_{\mathcal{R}^2}Z(w)\mathrm{d}w}.
$$

We have the following minimax lower bound for mean square errors of any estimator

$$
\lim_{\delta_*\to 0}\varliminf_{\varepsilon\to 0}\sup_{\left\|\vartheta-\vartheta_0\right\|\leq\delta_*}\varphi_\varepsilon^{-2}\mathbf{E}_\vartheta\left\|\bar{\vartheta}_\varepsilon-\vartheta\right\|^2\geq\mathbf{E}_{\vartheta_0}\|\tilde{w}\|^p. \tag{21}
$$

This lower bound can be considered as a particular case of the lower bound given in the Theorem 1.9.1 in Ibragimov and Khasminskii (1981). We call the estimator ϑ_ε^* asymptotically efficient if for all $\vartheta_0\in\Theta$ for this estimator we have equality in (21).

Theorem 2 *Suppose that the conditions are fulfilled, then the MLE $\hat{\vartheta}_\varepsilon$ and BE $\tilde{\vartheta}_\varepsilon$ are uniformly consistent, have different limit distributions*

$$
\varphi_\varepsilon^{-1}\left(\hat{\vartheta}_\varepsilon-\vartheta_0\right)\Longrightarrow\hat{w},\qquad \varphi_\varepsilon^{-1}\left(\tilde{\vartheta}_\varepsilon-\vartheta_0\right)\Longrightarrow\tilde{w}, \tag{22}
$$

we have uniform convergence of moments: for any $p>0$

$$
\varphi_\varepsilon^{-p}\mathbf{E}_{\vartheta_0}\left\|\hat{\vartheta}_\varepsilon-\vartheta_0\right\|^p\longrightarrow\mathbf{E}_{\vartheta_0}\|\hat{w}\|^p,\qquad \varphi_\varepsilon^{-p}\mathbf{E}_{\vartheta_0}\left\|\tilde{\vartheta}_\varepsilon-\vartheta_0\right\|^p\longrightarrow\mathbf{E}_{\vartheta_0}\|\tilde{w}\|^p
$$

and BE are asymptotically efficient.

Proof We verify the conditions of the general Theorems 1.10.1 and 1.10.2 in Ibragimov and Khasminskii (1981). These conditions are given in terms of the properties of the normalized likelihood ratio random field

$$Z_\varepsilon(w)=\frac{L\left(\vartheta_0+\nu\varphi_\varepsilon w,X^K\right)}{L\left(\vartheta_0,X^K\right)},\qquad w\in\mathbb{W}_\varepsilon=\left(w:\ \vartheta_0+\nu\varphi_\varepsilon w\in\Theta\right).$$

We do it with the help of three lemmas.

Lemma 5 *The finite-dimensional distributions of the random fields $Z_\varepsilon(\cdot)$ converge to the finite-dimensional distributions of $Z(\cdot)$ and this convergence is uniform on compacts $\mathbb{K}\in\Theta$.*

Proof The normalized log-likelihood ratio random field is

$$\begin{aligned}\ln Z(w)=&\sum_{k=1}^{K}\int_0^{\rm T}\frac{h_k(t)}{\varepsilon}\left[M_k\left(\tau_k\left(\vartheta_w\right),t\right)-M_k\left(\tau_k\left(\vartheta_0\right),t\right)\right]{\rm d}\bar W_k(t)\\&-\sum_{k=1}^{K}\int_0^{\rm T}\frac{h_k(t)^2}{2\varepsilon^2}\left[M_k\left(\tau_k\left(\vartheta_w\right),t\right)-M_k\left(\tau_k\left(\vartheta_0\right),t\right)\right]^2{\rm d}t,\end{aligned}$$

where we used notation $\vartheta_w=\vartheta_0+\nu\varphi_\varepsilon w$,

$$\begin{aligned}&\psi_{\delta,\kappa,k}(w,t)=\psi_{\delta,\kappa}\left(t-\tau_k\left(\vartheta_w\right)\right),\qquad m_{k,w}(t)=m\left(\tau_k\left(\vartheta_w\right),t\right),\\&M_k\left(\tau_k\left(\vartheta_w\right),t\right)=\psi_{\delta,\kappa,k}(w,t)m_{k,w}(t)-\psi_{\delta,\kappa,k}(0,t)m_{k,0}(t).\end{aligned}$$

We have

$$\begin{aligned}M_k\left(\tau_k\left(\vartheta_w\right),t\right)-M_k\left(\tau_k\left(\vartheta_0\right),t\right)=&\left[\psi_{\delta,\kappa,k}(w,t)-\psi_{\delta,\kappa,k}(0,t)\right]m_{k,w}(t)\\&+\psi_{\delta,\kappa,k}(0,t)\left[m_{k,w}(t)-m_{k,0}(t)\right].\end{aligned}$$

It can be shown that

$$\begin{aligned}\varepsilon^{-2}\mathbf{E}_{\vartheta_0}\left|m_{k,w}(t)-m_{k,0}(t)\right|^2&\leq C\,\varepsilon^{-2}\varphi_\varepsilon^2=C\,\varepsilon^{\frac{1-2\kappa}{\kappa+\frac12}}\rightarrow 0,\\\mathbf{E}_{\vartheta_0}\left|m_{k,0}(t)-y_k(t)\right|^2&\leq C\,\varepsilon^2.\end{aligned}$$

For example, to verify the first estimate we have to write the equations for $m_{k,w}(t)$ and $m_{k,0}(t)$, take the difference $\left|m_{k,w}(t)-m_{k,0}(t)\right|$ and then to use Gronwall-Bellman lemma and Cauchy-Schwartz inequality. Hence to study

$$\varepsilon^{-2}\int_0^{\mathrm{T}} h_k(t)^2\left[M_k\left(\tau_k\left(\vartheta_w\right),t\right)-M_k\left(\tau_k\left(\vartheta_0\right),t\right)\right]^2\mathrm{d}t$$

it is sufficient to find the limit of the integral

$$\begin{aligned}H_{k,\varepsilon}(w)&=\varepsilon^{-2}\int_0^{\mathrm{T}} h_k(t)^2\left[\psi_{\delta,\kappa,k}(w,t)-\psi_{\delta,\kappa,k}(0,t)\right]^2 m_{k,w}(t)^2\mathrm{d}t\\&=\varepsilon^{-2}\int_0^{\mathrm{T}} h_k(t)^2\left[\psi_{\delta,\kappa,k}(w,t)-\psi_{\delta,\kappa,k}(0,t)\right]^2 m_{k,0}(t)^2\mathrm{d}t(1+o(1))\\&=\varepsilon^{-2}\int_0^{\mathrm{T}} h_k(t)^2\left[\psi_{\delta,\kappa,k}(w,t)-\psi_{\delta,\kappa,k}(0,t)\right]^2 y_k(t)^2\mathrm{d}t(1+o(1)).\end{aligned}$$

Suppose that $w\in\mathbb{B}_k^-$, i.e., for small ε we have

$$\tau_k\left(\vartheta_0+\nu\varphi_\varepsilon w\right)=\tau_k\left(\vartheta_0\right)-\varphi_\varepsilon\langle\mu_k,w\rangle+O\left(\varphi_\varepsilon^2\right)>\tau_k\left(\vartheta_0\right).$$

Therefore $H_{k,\varepsilon}(w)=\hat H_{k,\varepsilon}(w)(1+o(1))$, where

$$\hat H_{k,\varepsilon}(w)=\varepsilon^{-2}\int_{\tau_k\left(\vartheta_0\right)-\varphi_\varepsilon\langle\mu_k,w\rangle}^{\tau_k\left(\vartheta_0\right)+\delta} h_k(t)^2\left[\psi_{\delta,\kappa,k}(w,t)-\psi_{\delta,\kappa,k}(0,t)\right]^2 y_k(t)^2\mathrm{d}t.$$

In this integral we change the variables several times:

$$t=s+\tau_k\left(\vartheta_0\right),\qquad s=\frac{\delta}{2}(r+1),\qquad r=-\frac{2\varphi_\varepsilon\langle\mu_k,w\rangle}{\delta}\,q,$$

obtain $h_k(t)=h_k\left(\tau_k\left(\vartheta_0+\delta/2\right)\right)+o(1)$, and $y_k(t)=y_k\left(\tau_k\left(\vartheta_0\right)+\delta/2\right)+o(1)$ and

$$\begin{aligned}\hat H_{k,\varepsilon}(w)&=\varepsilon^{-2}\int_{\tau_k\left(\vartheta_0\right)-\varphi_\varepsilon\langle\mu_k,w\rangle}^{\tau_k\left(\vartheta_0\right)+\delta} h_k(t)^2\left[\psi_{\delta,\kappa,k}(w,t)-\psi_{\delta,\kappa,k}(0,t)\right]^2 y_k(t)^2\mathrm{d}t\\&=\frac{\pi_k^2}{Q(\kappa)^2}\left|\langle\mu_k,w\rangle\right|^{2\kappa+1}\int_{\mathcal{R}}\left[\operatorname{sgn}(q-1)|q-1|^\kappa-\operatorname{sgn}(q)|q|^\kappa\right]^2\mathrm{d}q(1+o(1))\\&=\pi_k^2\left|\langle\mu_k,w\rangle\right|^{2\kappa+1}(1+o(1)).\end{aligned}$$

Recall here that

$$\begin{aligned}\pi_k^2&=h_k\left(\tau_k\left(\vartheta_0+\delta/2\right)\right)^2 y_k\left(\tau_k\left(\vartheta_0\right)+\delta/2\right)^2,\\Q(\kappa)^2&=\int_{\mathcal{R}}\left[\operatorname{sgn}(q-1)|q-1|^\kappa-\operatorname{sgn}(q)|q|^\kappa\right]^2\mathrm{d}q.\end{aligned}$$

Therefore we obtained the relation

$$\begin{aligned}&\varepsilon^{-2}\int_0^{\mathrm{T}} h_k(t)^2\left[M_k\left(\tau_k\left(\vartheta_w\right),t\right)-M_k\left(\tau_k\left(\vartheta_0\right),t\right)\right]^2{\rm d}t\\&\quad=\pi_k\left(\vartheta_0\right)^2Q\left(\kappa,\vartheta_0\right)^2\left|\langle\mu_k,w\rangle\right|^{2\kappa+1}(1+o(1)).\end{aligned}$$

For stochastic integral we have similar relations

$$\begin{aligned}&\int_0^{\mathrm{T}}\frac{h_k(t)}{\varepsilon}\left[M_k\left(\tau_k\left(\vartheta_w\right),t\right)-M_k\left(\tau_k\left(\vartheta_0\right),t\right)\right]{\rm d}\bar W_k(t)\\&\quad=\frac{\pi_k}{Q(\kappa)}\int_{-\varphi_\varepsilon^{-1}}^{\varphi_\varepsilon^{-1}}\left[\operatorname{sgn}\left(q-\langle\mu_k,w\rangle\right)\left|q-\langle\mu_k,w\rangle\right|^{\kappa}\right.\\&\quad\left.-\operatorname{sgn}(q)|q|^{\kappa}\right]{\rm d}w_{k,\varepsilon}(q)(1+o(1))\\&\quad=\pi_k\left(\vartheta_0\right)W_{k,\varepsilon}\left(\langle\mu_k,w\rangle\right)(1+o(1)).\end{aligned}$$

Here $w_{k,\varepsilon}(\cdot)$ are independent two-sided Wiener processes obtained from the $\bar W_k(\cdot)$ after the mentioned change of variables and the corresponding normalization as follows

$$\begin{aligned}\bar W_k(t)&\longrightarrow\bar W_k\left(s+\tau_k\left(\vartheta_0\right)\right)-\bar W_k\left(\tau_k\left(\vartheta_0\right)\right)\\&\longrightarrow\tilde W_k(r)=\sqrt{\frac{2}{\delta}}\left[\bar W_k\left(\frac{\delta}{2}(r+1)\right)-\bar W_k\left(\frac{\delta}{2}\right)\right]\\&\longrightarrow w_{k,\varepsilon}(\cdot)=\sqrt{\frac{\delta}{2\varphi_\varepsilon}}\tilde W_k\left(\frac{2\varphi_\varepsilon}{\delta}q\right).\end{aligned}$$

The Gaussian random function

$$\begin{aligned}&W_{k,\varepsilon}\left(\langle\mu_k,w\rangle\right)\\&\quad=Q(\kappa)^{-1}\int_{-\varphi_\varepsilon^{-1}}^{\varphi_\varepsilon^{-1}}\left[\operatorname{sgn}\left(q-\langle\mu_k,w\rangle\right)\left|q-\langle\mu_k,w\rangle\right|^{\kappa}-\operatorname{sgn}(q)|q|^{\kappa}\right]{\rm d}w_{k,\varepsilon}(q)\end{aligned}$$

has the properties

$$\begin{aligned}&\mathbf{E}_{\vartheta_0}W_{k,\varepsilon}\left(\langle\mu_k,w\rangle\right)=0,\qquad\mathbf{E}_{\vartheta_0}W_{k,\varepsilon}\left(\langle\mu_k,w\rangle\right)^2=\left|\langle\mu_k,w\rangle\right|^{2H}(1+o(1)),\\&\mathbf{E}_{\vartheta_0}W_{k,\varepsilon}\left(\langle\mu_k,w_1\rangle\right)W_{k,\varepsilon}\left(\langle\mu_k,w_2\rangle\right)\\&\qquad=\frac{1}{2}\left(\left|\langle\mu_k,w_1\rangle\right|^{2H}+\left|\langle\mu_k,w_2\rangle\right|^{2H}-\left|\langle\mu_k,w_2-w_1\rangle\right|^{2H}\right)(1+o(1)).\end{aligned}$$

Therefore we have the convergence

$$W_{k,\varepsilon}\left(\left|\langle\mu_k,w\rangle\right|\right)\Longrightarrow W_k^{H,-}\left(\left|\langle\mu_k,w\rangle\right|\right),\qquad w\in\mathbb{B}_k^-.$$

For $w \in \mathbb{B}_k^+$ we have the similar limits. Therefore the one-dimensional distributions of $Z_\varepsilon(w)$ converge to the one-dimensional distributions of $Z(w)$. As it follows from the given proof this convergence is uniform on compacts $\mathbb{K} \subset \Theta$. To prove the convergence of finite-dimensional distributions

$$\left(Z_\varepsilon\left(w_1\right), \ldots, Z_\varepsilon\left(w_N\right)\right) \Longrightarrow \left(Z\left(w_1\right), \ldots, Z\left(w_N\right)\right)$$

we can use the same steps as in the given above proof. □

Lemma 6 *There exists constant* $C > 0$ *such that for any* $R > 0$ *and* $\|w_1\| + \|w_2\| < R$ *we have*

$$\sup_{\vartheta_0 \in \mathbb{K}} \mathbf{E}_{\vartheta_0} \left| Z_\varepsilon^{1/4}\left(w_2\right) - Z_\varepsilon^{1/4}\left(w_1\right)\right|^4 \leq C\left(1 + R^4\right) \left\|w_2 - w_1\right\|^{4\kappa+2}. \tag{23}$$

Proof Using (16) we can write

$$\begin{aligned}
\mathbf{E}_{\vartheta_0} & \left| Z_\varepsilon^{1/4}\left(w_2\right) - Z_\varepsilon^{1/4}\left(w_1\right)\right|^4 = \mathbf{E}_{\vartheta_0+\varphi_\varepsilon w_1} \left|V_T - 1\right|^4 \\
& \leq C\mathbf{E}_{\vartheta_0+\varphi_\varepsilon w_1} \left(\sum_{k=1}^K \int_0^{\mathrm{T}} V_t h_t^2 D_{k,\varepsilon}\left(\vartheta_0, w_1, w_2, t\right)^2 \mathrm{d}t \right)^4 \\
& \quad + C\mathbf{E}_{\vartheta_0+\varphi_\varepsilon w_1} \left(\sum_{k=1}^K \int_0^{\mathrm{T}} V_t h_t D_{k,\varepsilon}\left(\vartheta_0, w_1, w_2, t\right) \mathrm{d}\bar{W}_k(t) \right)^4 \\
& \leq C \sum_{k=1}^K \mathbf{E}_{\vartheta_0+\varphi_\varepsilon w_1} \left(\int_0^{\mathrm{T}} V_t D_{k,\varepsilon}\left(\vartheta_0, w_1, w_2, t\right)^2 \mathrm{d}t \right)^4 \\
& \quad + C \sum_{k=1}^K \mathbf{E}_{\vartheta_0+\varphi_\varepsilon w_1} \left(\int_0^{\mathrm{T}} V_t^2 D_{k,\varepsilon}\left(\vartheta_0, w_1, w_2, t\right)^2 \mathrm{d}t \right)^2.
\end{aligned}$$

Introduce notation $\vartheta_1 = \vartheta_0 + \nu\varphi_\varepsilon w_1, \vartheta_2 = \vartheta_0 + \nu\varphi_\varepsilon w_2$

$$\begin{aligned}
\Delta\psi_k(t) &= \psi_{\delta,\kappa}\left(t - \tau_k\left(\vartheta_1\right)\right) - \psi_{\delta,\kappa}\left(t - \tau_k\left(\vartheta_2\right)\right), \\
\Delta m_k(t) &= m\left(\tau_k\left(\vartheta_1\right), t\right) - m\left(\tau_k\left(\vartheta_2\right), t\right),
\end{aligned}$$

and write

$$\varepsilon D_{k,\varepsilon}\left(\vartheta_0, w_1, w_2, t\right) = m\left(\tau_k\left(\vartheta_1\right), t\right) \Delta\psi_k(t) + \psi_{\delta,\kappa}\left(t - \tau_k\left(\vartheta_2\right)\right) \Delta m(t).$$

We have

$$\begin{aligned}
\mathbf{E}_{\vartheta_1} & \left(\int_0^{\mathrm{T}} V_t^2 D_{k,\varepsilon}\left(\vartheta_0, w_1, w_2, t\right)^2 \mathrm{d}t \right)^2 \\
& \leq \frac{C}{\varepsilon^2} \mathbf{E}_{\vartheta_1} \left(\int_0^{\mathrm{T}} \left[\Delta\psi_k(t)^2 V_t^2 m\left(\tau_k\left(\vartheta_1\right), t\right)^2 + V_t^2 \Delta m_k(t)^2 \right] \mathrm{d}t \right)^2.
\end{aligned}$$

Recall that $m\left(\tau_k\left(\vartheta_1\right),t\right), 0\leq t\leq T$ is Gaussian process with bounded variance. Hence ($2ab\leq a^2+b^2$)

$$\begin{aligned}
2\mathbf{E}_{\vartheta_1}V_s^2V_t^2 m\left(\tau_k\left(\vartheta_1\right),t\right)^2 m\left(\tau_k\left(\vartheta_1\right),s\right)^2
&\leq \mathbf{E}_{\vartheta_1}V_t^4 m\left(\tau_k\left(\vartheta_1\right),t\right)^4+\mathbf{E}_{\vartheta_1}V_s^4 m\left(\tau_k\left(\vartheta_1\right),s\right)^4\\
&=\mathbf{E}_{\vartheta_2} m\left(\tau_k\left(\vartheta_1\right),t\right)^4+\mathbf{E}_{\vartheta_2} m\left(\tau_k\left(\vartheta_1\right),s\right)^4\leq C,
\end{aligned}$$

and

$$\begin{aligned}
2\mathbf{E}_{\vartheta_1}V_s^2V_t^2\Delta m_k(t)^2\Delta m_k(s)^2 &\leq \mathbf{E}_{\vartheta_1}V_s^4\Delta m_k(t)^4+\mathbf{E}_{\vartheta_1}V_s^4\Delta m_k(s)^4\\
&\leq \mathbf{E}_{\vartheta_2}\Delta m_k(t)^4+\mathbf{E}_{\vartheta_2}\Delta m_k(s)^4.
\end{aligned}$$

Note that the Gaussian process $m_k(\tau,t)$ is mean square differentiable w.r.t. τ and we have

$$\begin{aligned}
\mathbf{E}_{\vartheta}\Delta m_k(s)^4&=\mathbf{E}_{\vartheta}\left|m\left(\tau_k\left(\vartheta_1\right),t\right)-m\left(\tau_k\left(\vartheta_2\right),t\right)\right|^4\\
&\leq C\left|\tau_k\left(\vartheta_1\right)-\tau_k\left(\vartheta_2\right)\right|^4\leq C\varphi_\varepsilon^4\left\|w_2-w_1\right\|^4.
\end{aligned}$$

These estimates allow us to write

$$\begin{aligned}
&\mathbf{E}_{\vartheta_1}\left(\int_0^{\mathrm{T}} V_t^2 D_{k,\varepsilon}\left(\vartheta_0,w_1,w_2,t\right)^2\mathrm{d}t\right)^2\\
&=\varepsilon^{-4}\mathbf{E}_{\vartheta_1}\int_0^{\mathrm{T}}\int_0^{\mathrm{T}} V_t^2V_s^2 D_{k,\varepsilon}\left(\vartheta_0,w_1,w_2\right)^2 D_{k,\varepsilon}\left(\vartheta_0,w_1,w_2,s\right)^2\mathrm{d}t\,\mathrm{d}s\\
&\leq C\varepsilon^{-4}\left(\int_0^{\mathrm{T}}\Delta\psi_k(t)^2\mathrm{d}t\right)^2+\varepsilon^{-4}\varphi_\varepsilon^4\left\|w_2-w_1\right\|^4\\
&\leq C\frac{\varphi_\varepsilon^{4\kappa+2}}{\varepsilon^4}\left\|w_2-w_1\right\|^{4\kappa+2}\left(\int_{\mathcal{R}}\left[\operatorname{sgn}(s-1)|s-1|^{\kappa}-\operatorname{sgn}(s)|s|^{\kappa}\right]^2\mathrm{d}s\right)^2\\
&\quad +C+\varepsilon^{-4}\varphi_\varepsilon^4\left\|w_2-w_1\right\|^4\\
&\leq C\left\|w_2-w_1\right\|^{4\kappa+2}+C\,\varepsilon^{\frac{2-4\kappa}{\kappa+\frac{1}{2}}}\left\|w_2-w_1\right\|^4\\
&\leq C\left(1+R^{2-4\kappa}\right)\left\|w_2-w_1\right\|^{4\kappa+2}
\end{aligned}$$

for $\left\|w_1\right\|+\left\|w_2\right\|<R$.

For the term

$$\mathbf{E}_{\vartheta_1}\left(\int_0^{\mathrm{T}} V_t D_{k,\varepsilon}\left(\vartheta_0,w_1,w_2,t\right)^2\mathrm{d}t\right)^4$$

we use the similar relations like

$$\begin{aligned}&\mathbf{E}_{\vartheta_1} V_{t_1} V_{t_2} V_{t_3} V_{t_4} m\left(\tau_k\left(\vartheta_1\right), t_1\right) m\left(\tau_k\left(\vartheta_1\right), t_2\right) m\left(\tau_k\left(\vartheta_1\right), t_3\right) m\left(\tau_k\left(\vartheta_1\right), t_4\right)\\&\quad\leq C \sum_{i=1}^{4} \mathbf{E}_{\vartheta_1} V_{t_i}^4 m\left(\tau_k\left(\vartheta_1\right), t_i\right)^4 = C \sum_{i=1}^{4} \mathbf{E}_{\vartheta_i} m\left(\tau_k\left(\vartheta_1\right), t_i\right)^4 \leq C,\end{aligned}$$

and obtain the relations

$$\begin{aligned}&\sum_{k=1}^{K} \mathbf{E}_{\vartheta_1}\left(\int_0^{\mathrm{T}} V_t D_{k,\varepsilon}\left(\vartheta_0, w_1, w_2, t\right)^2 \mathrm{d}t\right)^4\\&\quad\leq C \frac{\varphi_\varepsilon^{8\kappa+4}}{\varepsilon^8}\left\|w_2 - w_1\right\|^{8\kappa+4} + C \frac{\varphi_\varepsilon^8}{\varepsilon^8}\left\|w_2 - w_1\right\|^8\\&\quad\leq C\left\|w_2 - w_1\right\|^{8\kappa+4} + C\varepsilon^{\frac{4-8\kappa}{\kappa+\frac{1}{2}}}\left\|w_2 - w_1\right\|^8\\&\quad\leq C\left(1 + R^4\right)\left\|w_2 - w_1\right\|^{4\kappa+2}.\end{aligned}$$

Therefore we obtained (23). □

Lemma 7 *There exists constant* $c_* > 0$ *such that*

$$\sup_{\vartheta_0 \in \mathbb{K}} \mathbf{E}_{\vartheta_0} Z_\varepsilon(w)^{1/2} \leq e^{-c_* \|w\|^{2H}}. \tag{24}$$

Proof We follow the same steps as in the proof of Lemma 4. The local relation is (below $\vartheta_w = \vartheta_0 + \nu\varphi_\varepsilon w$)

$$\begin{aligned}F\left(\vartheta_w, \vartheta_0\right) &= \sum_{k=1}^{K} \int_0^{\mathrm{T}} \frac{h_k(t)^2}{\varepsilon^2}\left[\psi_{\delta,\kappa}\left(t - \tau_k\left(\vartheta_w\right)\right) y_k\left(\vartheta_w, t\right)\right.\\&\quad\left.-\psi_{\delta,\kappa}\left(t - \tau_k\left(\vartheta_0\right)\right) y_k\left(\vartheta_0, t\right)\right]^2 \mathrm{d}t\\&= \sum_{k=1}^{K} \int_0^{\mathrm{T}} \frac{h_k(t)^2}{\varepsilon^2}\left[\psi_{\delta,\kappa}\left(t - \tau_k\left(\vartheta_w\right)\right)\right.\\&\quad\left.-\psi_{\delta,\kappa}\left(t - \tau_k\left(\vartheta_0\right)\right)\right]^2 y_k\left(\vartheta_0, t\right)^2 \mathrm{d}t(1 + o(1))\\&= \sum_{k=1}^{K} \pi_k^2\left|\langle \mu_k, w\rangle\right|^{2H}(1 + o(1)).\end{aligned}$$

Therefore there exists $\varepsilon_1 > 0$ and $d > 0$ such that $\varepsilon_1\|w\| < d$ and all $\varepsilon < \varepsilon_1$ we have

$$F\left(\vartheta_w, \vartheta_0\right) \geq \sum_{k=1}^{K} \pi_k^2\left|\langle \mu_k, w\rangle\right|^{2H}(1 + o(1)) \geq \frac{1}{2} \sum_{k=1}^{K} \pi_k^2\left|\langle \mu_k, w\rangle\right|^{2H} \geq \hat{c}\|w\|^{2H}.$$

To verify the last inequality with some $\hat c>0$ we write

$$\sum_{k=1}^K \pi_k^2\left|\langle\mu_k,w\rangle\right|^{2H}=\sum_{k=1}^K \pi_k^2\left|\langle\mu_k,e\rangle\right|^{2H}\|w\|^{2H},$$

where we denoted $e=w/\|w\|$. We have

$$\hat c=\inf_{e\in\mathcal{R}^2,\|e\|=1}\sum_{k=1}^K \pi_k^2\left|\langle\mu_k,e\rangle\right|^{2H}>0.$$

If $\hat c=0$, then there exists a vector e_* such that all scalar products $\langle\mu_k,e_*\rangle=0, k=1,\ldots,K$. As $K\geq 3$ and there exists at least three detectors not on the same line, the vector e_*, which is orthogonal to all vectors $\mu_k, k=1,\ldots,K$ does not exists.

For $\left\|\vartheta-\vartheta_0\right\|\geq d$ we have

$$\inf_{\vartheta_0\in\Theta}\inf_{\left\|\vartheta-\vartheta_0\right\|\geq d}\varepsilon^2F\left(\vartheta,\vartheta_0\right)>0.$$

The proof of this inequality is the same as the proof of similar relation in the Lemma 4.

Hence there exists $\tilde c>0$ such that

$$F\left(\vartheta,\vartheta_0\right)\geq\tilde c\|w\|^{2H}.$$

The end of the proof is the same as in Lemma 4. □

The properties of the normalized likelihood ratio random field $Z_\varepsilon(\cdot)$ established in the Lemmas 5–7 are sufficient to cite the Theorems 1.10.1 and 1.10.3 in Ibragimov and Khasminskii (1981), where the mentioned properties of estimators were established under such conditions. □

5 Change-point case

Let us consider once more the problem of one source detection with K detectors (see Fig. 1). The observations $X^K=\left(X_1,\ldots,X_K\right)$, where $X_k=\left(X_k(t),0\leq t\leq T\right)$ and

$$\mathrm{d}X_k(t)=a_k(t)\psi\left(t-\tau_k(\vartheta)\right)Y_k(t)\mathrm{d}t+\varepsilon\sigma_k(t)\mathrm{d}W_t,\quad 0\leq t\leq T,\tag{25}$$

where $X_k(0)=0$, $k=1,\ldots,K$ and the function

$$\psi(t)=\mathbb{1}_{\{t\geq 0\}}.$$

The *hidden* Gaussian processes $Y_k(\cdot), k=1,\ldots,K$, satisfy the same equations

$$\mathrm{d}Y_k(t) = -f_k(t)Y_k(t)\mathrm{d}t + \varepsilon b_k(t)\mathrm{d}V_k(t), \quad Y_k(0) = y_{k,0}, \quad 0 \le t \le T.$$

As before we suppose that the conditions $\mathscr{R}$ are fulfilled. The parameter $\varepsilon \in (0,1)$ is small and the asymptotic is the same: $\varepsilon \to 0$.

The delay $\tau_k(\vartheta) = \rho_k/\nu = \nu^{-1}\left\|\vartheta_k - \vartheta_0\right\|$ and we have to estimate the position $\vartheta_0 = \left(x_0, y_0\right)^\top$ by observations X^K.

The likelihood ratio function is

$$L\left(\vartheta, X^K\right) = \exp\left\{\sum_{k=1}^{K}\int_{\tau_k(\vartheta)}^{\mathrm{T}} \frac{a_k(t)m_k\left(\tau_k(\vartheta), t\right)}{\varepsilon^2\sigma_k(t)^2}\mathrm{d}X_k(t) - \sum_{k=1}^{K}\int_{\tau_k(\vartheta)}^{\mathrm{T}} \frac{a_k(t)^2 m_k\left(\tau_k(\vartheta), t\right)^2}{2\varepsilon^2\sigma_k(t)^2}\mathrm{d}t\right\}, \quad \vartheta \in \Theta.$$

The estimators $\hat\vartheta_\varepsilon$, $\tilde\vartheta_\varepsilon$ are defined by the relations (3)–(6).

We need the same notations (5) and

$$\pi_k\left(\vartheta_0\right) = h_k\left(\tau_k\left(\vartheta_0\right)\right)y_k\left(\tau_k\left(\vartheta_0\right)\right), \qquad \pi_k = \pi_k\left(\vartheta_0\right).$$

The limit likelihood ratio random field is

$$Z(w) = \exp\left\{\sum_{k=1}^{K}\left[\pi_k W_k^+\left(\langle\mu_k, w\rangle\right) - \frac{\pi_k^2}{2}\left|\langle\mu_k, w\rangle\right|\right]\mathbb{1}_{\{\mathbb{B}_k^+\}} + \sum_{k=1}^{K}\left[\pi_k W_k^-\left(-\langle\mu_k, w\rangle\right) - \frac{\pi_k^2}{2}\left|\langle\mu_k, w\rangle\right|\right]\mathbb{1}_{\{\mathbb{B}_k^-\}}\right\}, \quad w \in \mathcal{R}^2.$$

Here $W_k^+(s), s \ge 0, k = 1, \dots, K$ and $W_k^-(s), s \ge 0, k = 1, \dots, K$ are independent two-sided Brownian motions.

The limit random vectors $\hat w$ and $\tilde w$ are defined by the same relations

$$Z(\hat w) = \sup_{w\in\mathcal{R}^2} Z(w), \quad \tilde w = \frac{\int_{\mathcal{R}^2} wZ(w)\mathrm{d}w}{\int_{\mathcal{R}^2} Z(w)\mathrm{d}w}.$$

The asymptotically efficient estimators in this problem are defined with the help of the following lower bound

$$\lim_{\nu\to 0}\varliminf_{\varepsilon\to 0}\sup_{\left\|\vartheta-\vartheta_0\right\|\le\nu} \varepsilon^{-4}\mathbf{E}_\vartheta\|\bar\vartheta_\varepsilon - \vartheta\|^2 \ge \mathbf{E}_{\vartheta_0}\|\tilde w\|^2.$$

We call the estimator ϑ_ε^* asymptotically efficient if for this estimator we have

$$\lim_{\nu\to 0}\lim_{\varepsilon\to 0}\sup_{\left\|\vartheta-\vartheta_0\right\|\le\nu} \varepsilon^{-4}\mathbf{E}_\vartheta\|\vartheta_\varepsilon^* - \vartheta\|^2 = \mathbf{E}_{\vartheta_0}\|\tilde w\|^2$$

for all $\vartheta_0 \in \Theta$.

Theorem 3 *Suppose that the conditions are fulfilled, then the MLE* $\hat{\vartheta}_\varepsilon$ and BE $\tilde{\vartheta}_\varepsilon$ *are uniformly consistent, have different limit distributions*

$$\varepsilon^{-2}\left(\hat{\vartheta}_\varepsilon - \vartheta_0\right) \Longrightarrow \hat{w}, \quad \varepsilon^{-2}\left(\tilde{\vartheta}_\varepsilon - \vartheta_0\right) \Longrightarrow \tilde{w}, \tag{26}$$

the moments converge: for any $p > 0$

$$\varepsilon^{-2p}\mathbf{E}_{\vartheta_0}\left\|\hat{\vartheta}_\varepsilon - \vartheta_0\right\|^p \longrightarrow \mathbf{E}_{\vartheta_0}\|\hat{w}\|^p, \quad \varepsilon^{-2p}\mathbf{E}_{\vartheta_0}\left\|\tilde{\vartheta}_\varepsilon - \vartheta_0\right\|^p \longrightarrow \mathbf{E}_{\vartheta_0}\|\tilde{w}\|^p$$

and BE are asymptotically efficient.

Proof Introduce the normalized likelihood ratio

$$Z_\varepsilon(w) = \frac{L\left(\vartheta_0 + \nu\varepsilon^2 w, X^K\right)}{L\left(\vartheta_0, X^K\right)}, \qquad w \in \mathbb{W}_\varepsilon = \left(w : \vartheta_0 + \nu\varepsilon^2 w \in \Theta\right).$$

Once more we have to prove three lemmas and then to cite the Theorems 1.10.1 and 1.10.3 in Ibragimov and Khasminskii (1981).

Lemma 8 *The finite-dimensional distributions of the random fields* $Z_\varepsilon(\cdot)$ *converge to the finite-dimensional distributions of* $Z(\cdot)$ *and this convergence is uniform on compacts* $\mathbb{K} \in \Theta$.

Proof Suppose that $w \in \mathbb{B}_k^-$. Then

$$\tau_k\left(\vartheta_0 + \varphi_\varepsilon w\right) = \tau_k\left(\vartheta_0\right) - \varphi_\varepsilon\langle\mu_k, w\rangle + O\left(\varphi_\varepsilon^2\right) > \tau_k\left(\vartheta_0\right)$$

and

$$\begin{aligned}\ln Z_\varepsilon(w) &\\ = \sum_{k=1}^K \int_{\tau_k(\vartheta_0)}^{\mathrm{T}} &\frac{h_k(t)\left[\mathbb{1}_{\{t>\tau_k(\vartheta_w)\}} m_k\left(\tau_k\left(\vartheta_w\right), t\right) - m_k\left(\tau_k\left(\vartheta_w\right), t\right)\right]}{\varepsilon\sigma_k(t)}\mathrm{d}\bar{W}_k(t)\\ &- \sum_{k=1}^K \int_{\tau_k(\vartheta_0)}^{\mathrm{T}} \frac{h_k(t)^2\left[\mathbb{1}_{\{t>\tau_k(\vartheta_w)\}} m_k\left(\tau_k\left(\vartheta_w\right), t\right) - m_k\left(\tau_k\left(\vartheta_w\right), t\right)\right]^2}{2\varepsilon^2\sigma_k(t)^2}\mathrm{d}t.\end{aligned}$$

For ordinary integral we can write

$$
\begin{aligned}
&\int_{\tau_k(\vartheta_0)}^{\mathrm{T}} \frac{h_k(t)^2\left[\mathbb{1}_{\{t>\tau_k(\vartheta_w)\}} m_k\left(\tau_k\left(\vartheta_w\right),t\right)-m_k\left(\tau_k\left(\vartheta_w\right),t\right)\right]^2}{\varepsilon^2\sigma_k(t)^2}\mathrm{d}t\\
&=\int_{\tau_k(\vartheta_0)}^{\tau_k(\vartheta_0+\varphi_\varepsilon w)}\frac{h_k(t)^2 m_k\left(\tau_k\left(\vartheta_w\right),t\right)^2}{\varepsilon^2\sigma_k(t)^2}\mathrm{d}t(1+o(1))\\
&=\frac{h_k\left(\tau_k\left(\vartheta_0\right)\right)^2}{\varepsilon^2\sigma_k\left(\tau_k\left(\vartheta_0\right)\right)^2}\int_{\tau_k(\vartheta_0)}^{\tau_k(\vartheta_0+\varphi_\varepsilon w)} m_k\left(\tau_k\left(\vartheta_w\right),t\right)^2\mathrm{d}t(1+o(1))\\
&=\frac{h_k\left(\tau_k\left(\vartheta_0\right)\right)^2}{\varepsilon^2\sigma_k\left(\tau_k\left(\vartheta_0\right)\right)^2}\int_{\tau_k(\vartheta_0)}^{\tau_k(\vartheta_0+\varphi_\varepsilon w)} m_k\left(\tau_k\left(\vartheta_0\right),t\right)^2\mathrm{d}t(1+o(1))\\
&=\frac{h_k\left(\tau_k\left(\vartheta_0\right)\right)^2}{\varepsilon^2\sigma_k\left(\tau_k\left(\vartheta_0\right)\right)^2}\int_{\tau_k(\vartheta_0)}^{\tau_k(\vartheta_0)-\varphi_\varepsilon\langle\mu_k,w\rangle} y_k(t)^2\mathrm{d}t(1+o(1))\\
&=\frac{h_k\left(\tau_k\left(\vartheta_0\right)\right)^2 y_k\left(\tau_k\left(\vartheta_0\right)\right)^2}{\sigma_k\left(\tau_k\left(\vartheta_0\right)\right)^2}\frac{\varphi_\varepsilon}{\varepsilon^2}\ \left|\langle\mu_k,w\rangle\right|(1+o(1))\\
&\longrightarrow \pi_k\left(\vartheta_0\right)^2\ \left|\langle\mu_k,w\rangle\right|.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\sum_{k=1}^K\int_{\tau_k(\vartheta_0)}^{\mathrm{T}} \frac{h_k(t)^2\left[\mathbb{1}_{\{t>\tau_k(\vartheta_w)\}} m_k\left(\tau_k\left(\vartheta_w\right),t\right)-m_k\left(\tau_k\left(\vartheta_w\right),t\right)\right]^2}{2\varepsilon^2\sigma_k(t)^2}\mathrm{d}t\\
&\qquad\longrightarrow \sum_{k=1}^K \pi_k\left(\vartheta_0\right)^2\ \left|\langle\mu_k,w\rangle\right|.
\end{aligned}
$$

The similar calculations for stochastic integral leads to the following relations

$$
\begin{aligned}
&\int_{\tau_k(\vartheta_0)}^{\mathrm{T}} \frac{h_k(t)\left[\mathbb{1}_{\{t>\tau_k(\vartheta_w)\}} m_k\left(\tau_k\left(\vartheta_w\right),t\right)-m_k\left(\tau_k\left(\vartheta_w\right),t\right)\right]}{\varepsilon\sigma_k(t)}\mathrm{d}\bar W_k(t)\\
&=\int_{\tau_k(\vartheta_0)}^{\tau_k(\vartheta_0+\varphi_\varepsilon w)}\frac{h_k(t) m_k\left(\tau_k\left(\vartheta_w\right),t\right)}{\varepsilon\sigma_k(t)}\mathrm{d}\bar W_k(t)t(1+o(1))\\
&=\frac{h_k\left(\tau_k\left(\vartheta_0\right)\right)}{\varepsilon\sigma_k\left(\tau_k\left(\vartheta_0\right)\right)}\int_{\tau_k(\vartheta_0)}^{\tau_k(\vartheta_0+\varphi_\varepsilon w)} m_k\left(\tau_k\left(\vartheta_w\right),t\right)\mathrm{d}\bar W_k(t)(1+o(1))\\
&=\frac{h_k\left(\tau_k\left(\vartheta_0\right)\right)}{\varepsilon\sigma_k\left(\tau_k\left(\vartheta_0\right)\right)}\int_{\tau_k(\vartheta_0)}^{\tau_k(\vartheta_0+\varphi_\varepsilon w)} m_k\left(\tau_k\left(\vartheta_0\right),t\right)\mathrm{d}\bar W_k(t)(1+o(1))\\
&=\frac{h_k\left(\tau_k\left(\vartheta_0\right)\right)}{\varepsilon\sigma_k\left(\tau_k\left(\vartheta_0\right)\right)}\int_{\tau_k(\vartheta_0)}^{\tau_k(\vartheta_0)-\varphi_\varepsilon\langle\mu_k,w\rangle} y_k(t)\mathrm{d}\bar W_k(t)(1+o(1))\\
&=\pi_k\left(\vartheta_0\right)\frac{\bar W_k\left(\tau_k\left(\vartheta_0\right)-\varphi_\varepsilon\langle\mu_k,w\rangle\right)-\bar W_k\left(\tau_k\left(\vartheta_0\right)\right)}{\varepsilon}(1+o(1))\\
&=\pi_k\left(\vartheta_0\right)\tilde W_{k,\varepsilon}\left(\left|\langle\mu_k,w\rangle\right|\right)(1+o(1))\Longrightarrow \pi_k\left(\vartheta_0\right)W_k^-\left(\left|\langle\mu_k,w\rangle\right|\right)
\end{aligned}
$$

and

$$\sum_{k=1}^{K}\int_{\tau_k(\vartheta_0)}^{\mathrm{T}}\frac{h_k(t)\left[\mathbb{1}_{\{t>\tau_k(\vartheta_w)\}}m_k\left(\tau_k\left(\vartheta_w\right),t\right)-m_k\left(\tau_k\left(\vartheta_w\right),t\right)\right]}{\varepsilon\sigma_k(t)}\mathrm{d}\bar{W}_k(t)$$
$$\Longrightarrow\sum_{k=1}^{K}\pi_k\left(\vartheta_0\right)W_k^-\left(\left|\langle\mu_k,w\rangle\right|\right).$$

Therefore we obtained convergence of one-dimensional distributions. The proof of the convergence of multidimensional distributions follows the same steps but is cumbersome. □

Lemma 9 *There exists constant $C>0$ such that for any $R>0$ and $\|w_1\|+\|w_2\|<R$ we have*

$$\sup_{\vartheta_0\in\mathbb{K}}\mathbf{E}_{\vartheta_0}\left|Z_\varepsilon^{1/8}\left(w_2\right)-Z_\varepsilon^{1/8}\left(w_1\right)\right|^8\leq C\left(1+R^8\right)\left\|w_2-w_1\right\|^4. \tag{27}$$

Proof Suppose that w_1 and w_2 are such that $\tau_k(\vartheta_{w_1})<\tau_k(\vartheta_{w_2})$ and denote

$$\Delta M_k\left(w_1,w_2,t\right)=\mathbb{1}_{\left\{t>\tau_k(\vartheta_{w_2})\right\}}m_k\left(\tau_k(\vartheta_{w_2}),t\right)-\mathbb{1}_{\left\{t>\tau_k(\vartheta_{w_1})\right\}}m_k\left(\tau_k(\vartheta_{w_1}),t\right).$$

Using once more (16) we write

$$\begin{aligned}\mathbf{E}_{\vartheta_0}\left|Z_\varepsilon^{1/8}\left(w_2\right)-Z_\varepsilon^{1/8}\left(w_1\right)\right|^8&=\mathbf{E}_{\vartheta_0+\varphi_\varepsilon w_1}\left|V_T-1\right|^8\\&\leq C\varepsilon^{-16}\mathbf{E}_{\vartheta_{w_1}}\left(\sum_{k=1}^{K}\int_0^{\mathrm{T}}V_th_t^2\Delta M_k\left(w_1,w_2,t\right)^2\mathrm{d}t\right)^8\\&\quad+C\varepsilon^{-8}\mathbf{E}_{\vartheta_{w_1}}\left(\sum_{k=1}^{K}\int_0^{\mathrm{T}}V_th_t\Delta M_k\left(w_1,w_2,t\right)\mathrm{d}\bar{W}_k(t)\right)^8\\&\leq C\varepsilon^{-16}\sum_{k=1}^{K}\mathbf{E}_{\vartheta_{w_1}}\left(\int_0^{\mathrm{T}}V_t\Delta M_k\left(w_1,w_2,t\right)^2\mathrm{d}t\right)^8\\&\quad+C\varepsilon^{-8}\sum_{k=1}^{K}\mathbf{E}_{\vartheta_{w_1}}\left(\int_0^{\mathrm{T}}V_t^2\Delta M_k\left(w_1,w_2,t\right)^2\mathrm{d}t\right)^4.\end{aligned}$$

Here

$$\begin{aligned}V_t=\exp\Bigg\{&\sum_{k=1}^{K}\int_0^t\frac{h_k(s)}{8\varepsilon}\Delta M_k\left(w_1,w_2,s\right)\mathrm{d}\bar{W}_k(s)\\&-\sum_{k=1}^{K}\int_0^t\frac{h_k(s)^2}{16\varepsilon^2}\Delta M_k\left(w_1,w_2,s\right)^2\mathrm{d}s\Bigg\}.\end{aligned}$$

We have

$$\int_0^{\mathrm{T}} V_t^2\Delta M_k\left(w_1,w_2,t\right)^2{\rm d}t=\int_{\tau_k(\vartheta_{w_1})}^{\tau_k(\vartheta_{w_2})} V_t^2 m_k\left(\tau_k\left(\vartheta_{w_1}\right),t\right)^2{\rm d}t$$

$$+\int_{\tau_k(\vartheta_{w_2})}^{\mathrm{T}} V_t^2\left[m_k\left(\tau_k\left(\vartheta_{w_2}\right),t\right)-m_k\left(\tau_k\left(\vartheta_{w_1}\right),t\right)\right]^2{\rm d}t.$$

Let us denote $m_k(t)=m_k\left(\tau_k\left(\vartheta_{w_1}\right),t\right)$, then

$$\mathbf{E}_{\vartheta_{w_1}} V_{t_1}^2 m_k\left(t_1\right)^2 V_{t_2}^2 m_k\left(t_2\right)^2 V_{t_3}^2 m_k\left(t_3\right)^2 V_{t_4}^2 m_k\left(t_4\right)^2$$

$$\leq C\sum_{i=1}^4 \mathbf{E}_{\vartheta_{w_1}} V_{t_i}^8 m_k\left(t_i\right)^8=C\sum_{i=1}^4 \mathbf{E}_{\vartheta_{w_2}} m_k\left(t_i\right)^8\leq C.$$

Further

$$\mathbf{E}_{\vartheta_{w_1}}\left(\int_{\tau_k(\vartheta_{w_1})}^{\tau_k(\vartheta_{w_2})} V_t^2 m_k\left(\tau_k\left(\vartheta_{w_1}\right),t\right)^2{\rm d}t\right)^4$$

$$=\int_{\tau_k(\vartheta_{w_1})}^{\tau_k(\vartheta_{w_2})}\int_{\tau_k(\vartheta_{w_1})}^{\tau_k(\vartheta_{w_2})}\int_{\tau_k(\vartheta_{w_1})}^{\tau_k(\vartheta_{w_2})}\int_{\tau_k(\vartheta_{w_1})}^{\tau_k(\vartheta_{w_2})}\prod_{i=1}^4 \mathbf{E}_{\vartheta_{w_1}} V_{t_i}^2 m_k\left(\tau_k\left(\vartheta_{w_1}\right),t_i\right)^2{\rm d}t_i$$

$$\leq C\left|\tau_k(\vartheta_{w_2}-\tau_k(\vartheta_{w_1}\right|^4\leq C\varepsilon^8\left\|w_2-w_1\right\|^4$$

and

$$\mathbf{E}_{\vartheta_{w_1}}\left(\int_{\tau_k(\vartheta_{w_2})}^{\mathrm{T}} V_t^2\left[m_k\left(\tau_k\left(\vartheta_{w_2}\right),t\right)-m_k\left(\tau_k\left(\vartheta_{w_1}\right),t\right)\right]^2{\rm d}t\right)^4$$

$$\leq\left(T-\tau_k(\vartheta_{w_2})\right)^3\int_{\tau_k(\vartheta_{w_2})}^{\mathrm{T}}\mathbf{E}_{\vartheta_{w_1}} V_t^8\left[m_k\left(\tau_k\left(\vartheta_{w_2}\right),t\right)-m_k\left(\tau_k\left(\vartheta_{w_1}\right),t\right)\right]^8{\rm d}t$$

$$=T^3\int_{\tau_k(\vartheta_{w_2})}^{\mathrm{T}}\mathbf{E}_{\vartheta_{w_2}}\left[m_k\left(\tau_k\left(\vartheta_{w_2}\right),t\right)-m_k\left(\tau_k\left(\vartheta_{w_1}\right),t\right)\right]^8{\rm d}t$$

$$\leq C\left\|\tau_k\left(\vartheta_{w_2}\right)-\tau_k\left(\vartheta_{w_1}\right)\right\|^8\leq C\varepsilon^{16}\left\|w_2-w_1\right\|^8.$$

Therefore for $\left\|w_1\right\|<R,\left\|w_2\right\|<R$ we obtained

$$\varepsilon^{-8}\sum_{k=1}^K\mathbf{E}_{\vartheta_{w_1}}\left(\int_0^{\mathrm{T}} V_t^2\Delta M_k\left(w_1,w_2,t\right)^2{\rm d}t\right)^4$$

$$\leq C\left\|w_2-w_1\right\|^4+C\varepsilon^8\left\|w_2-w_1\right\|^8\leq C\left(1+R^4\right)\left\|w_2-w_1\right\|^4.$$

Using the similar arguments it is possible to verify the estimate

$$\varepsilon^{-16}\sum_{k=1}^{K}\mathbf{E}_{\vartheta_{w_1}}\left(\int_0^{\rm T} V_t\Delta M_k\left(w_1,w_2,t\right)^2{\rm d}t\right)^8$$
$$\leq C\left\|w_2-w_1\right\|^8+C\varepsilon^{16}\left\|w_2-w_1\right\|^{16}\leq C\left(1+R^8\right)\left\|w_2-w_1\right\|^8.$$

□

Lemma 10 *There exists constant* $c_*>0$ *such that*

$$\sup_{\vartheta_0\in\mathbb{K}}\mathbf{E}_{\vartheta_0}Z_\varepsilon(w)^{1/2}\leq e^{-c_*\|w\|}. \tag{28}$$

Proof Once more we follow the proof of the Lemma 4. First we show that for some $d>0$ and $\left\|\vartheta_w-\vartheta_0\right\|\leq d$ we have the estimate

$$\sum_{k=1}^{K}\int_0^{\rm T}\frac{h_t^2}{\varepsilon^2}\left[\mathbb{1}_{\left\{t\geq\tau_k\left(\vartheta_w\right)\right\}}y_k\left(\vartheta_w,t\right)-\mathbb{1}_{\left\{t\geq\tau_k\left(\vartheta_0\right)\right\}}y_k(t)\right]^2{\rm d}t\geq\bar c\|w\| \tag{29}$$

with some constant $\bar c>0$. Then for $\vartheta_0\in\mathbb{K}$

$$\inf_{\left\|\vartheta-\vartheta_0\right\|>d}\sum_{k=1}^{K}\int_0^{\rm T}h_t^2\left[\mathbb{1}_{\left\{t\geq\tau_k(\vartheta)\right\}}y_k(\vartheta,t)-\mathbb{1}_{\left\{t\geq\tau_k\left(\vartheta_0\right)\right\}}y_k(t)\right]^2{\rm d}t>0. \tag{30}$$

Consider the values $\varepsilon\|w\|\leq d$ and $\tau_k\left(\vartheta_w\right)>\tau_k\left(\vartheta_0\right)$

$$F_k\left(\vartheta_w,\vartheta_0\right)=\varepsilon^{-2}\int_0^{\rm T}h_t^2\left[\mathbb{1}_{\left\{t\geq\tau_k\left(\vartheta_w\right)\right\}}y_k\left(\vartheta_w,t\right)-\mathbb{1}_{\left\{t\geq\tau_k\left(\vartheta_0\right)\right\}}y_k(t)\right]^2{\rm d}t$$
$$\geq C\varepsilon^{-2}\int_0^{\rm T}\left[\mathbb{1}_{\left\{t\geq\tau_k\left(\vartheta_w\right)\right\}}y_k\left(\vartheta_w,t\right)-\mathbb{1}_{\left\{t\geq\tau_k\left(\vartheta_0\right)\right\}}y_k(t)\right]^2{\rm d}t$$
$$\geq C\varepsilon^{-2}\int_{\tau_k\left(\vartheta_0\right)}^{\tau_k\left(\vartheta_w\right)}y_k(t)^2{\rm d}t+C\varepsilon^{-2}\int_{\tau_k\left(\vartheta_w\right)}^{\rm T}\left[y_k\left(\vartheta_w,t\right)-y_k(t)\right]^2{\rm d}t$$
$$\geq C\varepsilon^{-2}y_k\left(\tau_k\left(\vartheta_0\right)\right)^2\left[\tau_k\left(\vartheta_w\right)-\tau_k\left(\vartheta_0\right)\right](1+o(1))-C\varepsilon^{-2}\left\|\vartheta_w-\vartheta_0\right\|^2$$
$$\geq Cy_k\left(\tau_k\left(\vartheta_0\right)\right)^2\left|\langle\mu_k,w\rangle\right|(1+o(1))-C\varepsilon^2\|w\|^2.$$

Therefore

$$\sum_{k=1}^{K}F_k\left(\vartheta_w,\vartheta_0\right)\geq C\sum_{k=1}^{K}y_k\left(\tau_k\left(\vartheta_0\right)\right)^2\left|\langle\mu_k,w\rangle\right|(1+o(1))-Cd\varepsilon\|w\|$$
$$\geq C\sum_{k=1}^{K}\left|\langle\mu_k,w\rangle\right|(1+o(1))-Cd\varepsilon\|w\|.$$

Once more we have the estimate

$$\sum_{k=1}^{K}\left|\langle\mu_k,w\rangle\right|=\|w\|\sum_{k=1}^{K}\left|\langle\mu_k,e\rangle\right|\geq\check c\|w\|.$$

Hence

$$\sum_{k=1}^{K}F_k\left(\vartheta_w,\vartheta_0\right)\geq(C\check c-Cd\varepsilon)\|w\|\geq c_1\|w\|$$

and we obtain (30).

Suppose that we have equality in (30). Then there exists ϑ_* such that $\left\|\vartheta_*-\vartheta_0\right\|>0$ and

$$\sum_{k=1}^{K}\int_0^{\mathrm{T}}h_t^2\left[\mathbb{1}_{\left\{t\geq\tau_k\left(\vartheta_*\right)\right\}}y_k\left(\vartheta_*,t\right)-\mathbb{1}_{\left\{t\geq\tau_k\left(\vartheta_0\right)\right\}}y_k(t)\right]^2\mathrm{d}t=0.$$

This is possible if and only if $\tau_k\left(\vartheta_*\right)=\tau_k\left(\vartheta_0\right)$ for all $k=1,\ldots,K$. These equalities simultaneously are impossible if there is at least three detectors not on the same line. Therefore (30) is valid. □

Now the properties of the MLE and BE follows from the Theorems 1.10.1 and 1.10.3 in Ibragimov and Khasminskii (1981). □

6 Discussion

The studied here model of observations (1) is motivated by two different problems. One is to detect the position of the source $\mathbb{S}_0$, which emmits K independent Gaussian processes in different directions and the second is to detect a position of object $\mathbb{S}_0$ which receives K independent Gaussian signals from K reper sources with known positions. If the independence of the signals in the second statement of the problem seems to be natural to suppose that one source can emit K independent signals merits to be discussed. For example, suppose that we have a radioactive source and the signals are the Gaussian approximation of the flux of the particles emited in different directions. According to the physical law all elementary events (radioactive decays) are independent and therefore the corresponding Gaussian processes can be considered as independent. Note that the case where the same Gaussian process $Y=(Y(t),0\leq t\leq T)$ is detected by K different sensors

$$\mathrm{d}X_k(t)=a_k(t)\bar\psi\left(t-\tau_k\left(\vartheta_0\right)\right)Y(t)\mathrm{d}t+\varepsilon\sigma_k(t)\mathrm{d}W_k(t),\quad X_k(0)=0$$

with different delays can be treated using the developped here technics.

We considered the position estimation problem in three different cases corresponding to three different types of regularity. It is supposed that the moment of the beginning of emission is known and this corresponds well to the usual situations in GPS models. There is another class of problems like identification of explosion, where the position and the moment of explosion are supposed to be unknown. It is possible to treat these problems too. The first important question is the question of identifiability conditions. In the work Arakelyan and Kutoyants (2019) we gave the example of such identifiability condition, but the further study is needed.

Another problem is the computational complexity of the estimation algorithms. The calculation of the MLE and BE requires the calculation of the solutions of the Kalman–Bucy filtration equations (6)–(8) for all (many) values of $\vartheta \in \Theta$. Of course, the numerical realization can be difficult problem. There is a possibility to use two-step procedure developed recently for partially observed continuous time dynamic systems Kutoyants (2019a, b).

It is also possible to separate the problem of localization in two. The first problem is estimation of arrival times $\tau_1, \dots, \tau_K$ by K independent Gaussian processes (1) with hidden processes (2). Then having K estimators, say, $\hat\tau_{1,\varepsilon}, \dots, \hat\tau_{K,\varepsilon}$ we can consider the problem of estimation ϑ_0 using least squares approach. See details in Chernoyarov and Kutoyants (2020) and Arakelyan and Kutoyants (2019). The estimator of ϑ_0 obtained by this method can be used as preliminary estimator for two step procedure like the given in Kutoyants (2019a), Kutoyants and Zhou (2019).

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