6 Supplementary material to "Estimation for high-frequency data under parametric market microstructure noise", Simon Clinet and Yoann Potiron: Proofs

6.1 Preliminaries

Due to our assumptions of local boundedness on b_t, $\widetilde{b}_t$, c_t and $\widetilde{c}_t$, (12) and (18), it is sufficient (see, e.g., Lemma 4.4.9 along with Proposition 2.2.1 in Jacod and Protter (2011)) to assume throughout the proofs the following stronger assumption.

(H) We have that b_t, $\widetilde{b}_t$, c_t and $\widetilde{c}_t$ are bounded. Moreover, there exists $K > 0$ such that $\|\widehat{\theta} - \theta_0\| \leq K/n$, and $\max_{i,j,k} \left| Q_{t_i^{(k)}}^{(k,j)} \right| \leq K$.

Since the last two properties on $\widehat{\theta}$ and Q are not directly implied by Proposition 2.2.1 from Jacod and Protter (2011), we now detail a general localization procedure in the next proposition, which we apply to the above particular cases in Corollary 17. In the next lemma, if A is a random event, $\overline{A}$ stands for $\Omega - A$.

Proposition 16. *(Localization) Let $(A_n^K)_{n \in \mathbb{N}, K \in \mathbb{R}_+}$ be a doubly-indexed family of events such that $\lim_{K \to +\infty} \sup_{n \in \mathbb{N}} \mathbb{P}[\overline{A_n^K}] = 0$. Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of $\mathbb{R}^d$-valued random variables for some $d \geq 1$, and X another $\mathbb{R}^d$-valued random variable, and assume that either of the following properties hold.*

1. *(Local convergence in probability) For any $K \geq 0$, $(X_n - X)\mathbb{1}_{A_n^K} \to^{\mathbb{P}} 0$.*
2. *(Local convergence in distribution) For any $K \geq 0$, for any f continuous and bounded, $\mathbb{E}[f(X_n)\mathbb{1}_{A_n^K}] \to \mathbb{E}[f(X)]$.*

Then we have respectively

1. *$X_n \to^{\mathbb{P}} X$.*
2. *$X_n \to^d X$.*

Proof. We prove the convergence in probability first. Fix $\epsilon > 0$ and $\eta > 0$, and note that

$$\begin{aligned}\mathbb{P}[|X_n - X| \geq \eta] &\leq \mathbb{P}\left[|X_n - X|\mathbb{1}_{A_n^K} \geq \frac{\eta}{2}\right] + \mathbb{P}\left[|X_n - X|\mathbb{1}_{\overline{A_n^K}} \geq \frac{\eta}{2}\right] \\ &\leq \mathbb{P}\left[|X_n - X|\mathbb{1}_{A_n^K} \geq \frac{\eta}{2}\right] + \mathbb{P}\left[\overline{A_n^K}\right].\end{aligned}$$

By taking K large enough, we can assume that the second term in the right-hand side is dominated by ϵ. Next, by taking n large enough, we may assume the first term to be smaller than ϵ as well by the local convergence in probability. This proves $X_n \to^{\mathbb{P}} X$. Next we prove the convergence in distribution. We have

$$\begin{aligned}|\mathbb{E}[f(X_n)] - \mathbb{E}[f(X)]| &= |\mathbb{E}[f(X_n)\mathbb{1}_{A_n^K}] - \mathbb{E}[f(X)] + \mathbb{E}[f(X_n)\mathbb{1}_{\overline{A_n^K}}]| \\ &\leq |\mathbb{E}[f(X_n)\mathbb{1}_{A_n^K}] - \mathbb{E}[f(X)]| + C\mathbb{P}\left[\overline{A_n^K}\right]\end{aligned}$$

for some constant C using the boundedness of f. Again, taking K large enough makes the third term arbitrary small, and then taking $n \to +\infty$ makes the difference between the first two terms tend to 0, wich proves $X_n \to^d X$. ☐

Corollary 17. *When proving the consistency of the estimator $\widehat{\Xi}$ toward Ξ or the asymptotic normality $n^\kappa(\widehat{\Xi} - \Xi) \to \mathcal{MN}(AB, AVAR)$, we may assume that there exists $K > 0$ (which may be arbitrary large) such that $\|\widehat{\theta} - \theta_0\| \leq K/n$, and $\max_{i,j,k} \left|Q^{(k,j)}_{t_i^{(k)}}\right| \leq K$.*

Proof. We show the case $\|\widehat{\theta} - \theta_0\| \leq K/n$, the case $\max_{i,j,k} |Q^{(k,j)}_{t_i^{(k)}}| \leq K$ being the same. For the consistency, we apply the previous proposition with $X_n = \widehat{\Xi}$, $X = \Xi$, and $A_n^K = \left\{\|\widehat{\theta} - \theta_0\| \leq K/n\right\}$. By hypothesis (6), $n(\widehat{\theta} - \theta_0)$ is stochastically bounded which exactly means that $\lim_{K\to+\infty} \sup_{n\in\mathbb{N}} \mathbb{P}[n\|\widehat{\theta} - \theta_0\| \geq K] = 0$ (recall that $\widehat{\theta}$ depends on n). For the central limit theory, apply the local convergence distribution with $X_n = n^\kappa(\widehat{\Xi} - \Xi)$, $X \sim \mathcal{MN}(AB, AVAR)$, and again $A_n^K = \left\{\|\widehat{\theta} - \theta_0\| \leq K/n\right\}$. ☐

All along the proofs, C is a constant that may vary from one line to the next. We further provide some notation related to the decomposition (13) of the efficient price, i.e. that

$$\begin{aligned} X_t \;&=\; X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + \int_0^t \int_{\mathbb{R}} \delta(s,z)\mathbf{1}_{\{||\delta(s,z)||\leq 1\}}(\mu - \nu)(ds,dz) \\ &\quad + \int_0^t \int_{\mathbb{R}} \delta(s,z)\mathbf{1}_{\{||\delta(s,z)||>1\}}\mu(ds,dz), \\ &:= X_0 + B_t + M_t^c + M_t^d + J_t^b. \end{aligned} \tag{43}$$

Note that in this decomposition M_t^c (resp. M_t^d) is a continuous (resp. purely discontinuous) local martingale (see the discussion in Section 2.1.2 in Jacod and Protter (2011)). Finally, we introduce $\Delta_i X(\theta) := \Delta_i X + \psi_i(\theta)$ where $\psi_i(\theta) := \mu_i(\theta_0) - \mu_i(\theta)$. In particular, note that $\Delta_i \widehat{X} = \Delta_i X(\widehat{\theta})$. Similarly we define $\Delta_i X'(\theta) := \Delta_i X' + \psi_i(\theta)$ and $\Delta_i \widehat{X}' = \Delta_i X'(\widehat{\theta})$, corresponding to the estimated increments when the jump part J has been removed. Moreover, $\mathbb{E}_s$ is defined as the conditional expectation given $\mathcal{F}_s$.

6.2 Proof of Theorem 2

For this proof, due to our assumptions in Theorem 2 and using the same argument as for Assumption **(H)** we further make the following assumption.

(H') We have that $n\Delta_i t$ and $\widetilde{v}_t$ are bounded and bounded away from 0.

Note that (25) is a particular case of (26) when $\phi = 0$. In what follows, we directly prove the general case (26). First of all, as $N/n \to^{\mathbb{P}} F$, it is sufficient to show the stable convergence in law

$$n^{1/2}(\widetilde{\Xi} - \Xi) \to \frac{2}{3}\int_0^T \widetilde{v}_s \sigma_s dX'_s + \int_0^T \sqrt{\frac{2}{3}\widetilde{u}_s - \frac{4}{9}\widetilde{v}_s^2 \sigma_s^2} dB_s. \tag{44}$$

Second, note that if we can prove that

$$n^{1/2} \sum_{i=1}^N \left(\Delta_i \widehat{X}\right)^2 \mathbf{1}_{\{|\Delta_i \widehat{X}| \leq w_i\}} = n^{1/2} \sum_{i=1}^N \left(\Delta_i X'\right)^2 + o_{\mathbb{P}}(1), \tag{45}$$

then (25) holds in view of Theorem 1 (p. 585) in Li et al (2014) together with the assumptions of Theorem 2. Accordingly, we show (45) in what follows. On the account of the decomposition (15), we have

$$\begin{aligned} n^{1/2}\sum_{i=1}^{N}\left(\Delta_i\widehat{X}\right)^2\mathbf{1}_{\{|\Delta_i\widehat{X}|\le w_i\}} &= n^{1/2}\sum_{i=1}^{N}\left(\Delta_i\widehat{X}'\right)^2\mathbf{1}_{\{|\Delta_i\widehat{X}|\le w_i\}} + 2n^{1/2}\sum_{i=1}^{N}\Delta_i\widehat{X}'\Delta_i J\mathbf{1}_{\{|\Delta_i\widehat{X}|\le w_i\}} \\ &\quad + n^{1/2}\sum_{i=1}^{N}\Delta_i J^2\mathbf{1}_{\{|\Delta_i\widehat{X}|\le w_i\}}, \\ &:= I + II + III. \end{aligned}$$

We will show in what follows that $I = n^{1/2}\sum_{i=1}^{N}\left(\Delta_i X'\right)^2 + o_{\mathbb{P}}(1)$, $II = o_{\mathbb{P}}(1)$, and $III = o_{\mathbb{P}}(1)$.

We start with I. By definition, we have

$$I = n^{1/2}\sum_{i=1}^{N}\left(\Delta_i\widehat{X}'\right)^2 - n^{1/2}\sum_{i=1}^{N}\left(\Delta_i\widehat{X}'\right)^2\mathbf{1}_{\{|\Delta_i\widehat{X}|> w_i\}}.$$

We show now that $n^{1/2}\sum_{i=1}^{N}\left(\Delta_i\widehat{X}'\right)^2\mathbf{1}_{\{|\Delta_i\widehat{X}|> w_i\}} = o_{\mathbb{P}}(1)$. We have that

$$\begin{aligned} n^{1/2}\sum_{i=1}^{N}\left(\Delta_i\widehat{X}'\right)^2\mathbf{1}_{\{|\Delta_i\widehat{X}|> w_i\}} &\le n^{1/2}\sum_{i=1}^{N}\left(\Delta_i\widehat{X}'\right)^2\mathbf{1}_{\{|\Delta_i\widehat{X}'|> w_i/2\}} + n^{1/2}\sum_{i=1}^{N}\left(\Delta_i\widehat{X}'\right)^2\mathbf{1}_{\{|\Delta_i J|> w_i/2\}} \\ &:= A + B. \end{aligned}$$

We first deal with A. By the domination $\mathbf{1}_{\{|\Delta_i\widehat{X}'|> w_i/2\}} \le 2^k \mid \Delta_i\widehat{X}' \mid^k w_i^{-k}$, we have for any $k>0$:

$$|A| \le Cn^{1/2}\sum_{i=1}^{N} w_i^{-k}|\Delta_i\widehat{X}'|^{2+k}. \tag{46}$$

Now, note that by Assumption **(H)** along with the fact that ψ_i is C^3 in θ and that Θ is a compact set, we easily obtain that for any $k\ge 1$, $|\psi_i(\widehat{\theta})|^k \le Cn^{-k}$. From here, by Assumption **(H')** we deduce by Burkhölder-Davis-Gundy inequality that

$$\mathbb{E}|\Delta_i\widehat{X}'|^k \le C(n^{-k/2}+n^{-k}) \le Cn^{-k/2}, \tag{47}$$

and so we can conclude that taking k large enough, $A = o_{\mathbb{P}}(1)$ as a result of the boundedness of $n\Delta_i t$, and $N/n \to F$.

Now, we deal with B. Remark that by **(H')** and Hölder's inequality we have

$$\begin{aligned} |B| &\le 2n^{1/2}\sum_{i=1}^{N}\left(\Delta_i\widehat{X}'\right)^2|\Delta_i J||w_i|^{-1} \\ &\le Cn^{1/2+\bar{\omega}}\sum_{i=1}^{N}\left(\Delta_i\widehat{X}'\right)^2|\Delta_i J| \\ &\le Cn^{1/2+\bar{\omega}}\left(\sum_{i=1}^{N}\left(\Delta_i\widehat{X}'\right)^{2p}\right)^{1/p}\left(\sum_{i=1}^{N}|\Delta_i J|^q\right)^{1/q} \end{aligned}$$

where $1/p + 1/q = 1$ and $p, q > 1$. By (46) we get $\left(\sum_{i=1}^N \left(\Delta_i \widehat{X}'\right)^{2p}\right)^{1/p} = O_{\mathbb{P}}(n^{1/p-1})$ and since $q > 1$, we also have $\sum_{i=1}^N |\Delta_i J|^q = O_{\mathbb{P}}(1)$ because the jumps are summable. Indeed, note first that by application of Theorem 3.3.1, Case A, p.70 from Jacod and Protter (2011) under assumption (A-c), with $f(x) = |x|^q = o(x)$ for $x \to 0$ since $q > 1$, we have the convergence $\sum_{i=1}^N |\Delta_i J|^q \to^{\mathbb{P}} \sum_{0<s\leq T} |\Delta_s J|^q$. The stochastic boundedness of the left-hand side will therefore be proved if we show that the limit is finite almost surely. We can write

$$\sum_{0<s\leq T} |\Delta_s J|^q = \sum_{0<s\leq T} |\Delta_s J|^q \mathbf{1}_{\{|\Delta_s J|\geq 1\}} + \sum_{0<s\leq T} |\Delta_s J|^q \mathbf{1}_{\{|\Delta_s J|<1\}}.$$

The first term of the right-hand side is clearly finite since there is only a finite number of jumps larger than 1 on the interval $[0, T]$. Moreover, for the second term, using that $|x|^q < |x|$ for $x \in [0, 1)$ when $q > 1$, and using that the jumps are summable yields

$$\sum_{0<s\leq T} |\Delta_s J|^q \mathbf{1}_{\{|\Delta_s J|<1\}} \leq \sum_{0<s\leq T} |\Delta_s J| < +\infty \text{ a.s.}$$

Overall this yields $B = O_{\mathbb{P}}(n^{1/p+\bar{\omega}-1/2})$, which tends to 0 as soon as p is taken larger than $(1/2-\bar{\omega})^{-1}$, which is possible since $\bar{\omega} < 1/2$. Now we conclude for I by showing that we have

$$n^{1/2} \sum_{i=1}^N \left(\Delta_i \widehat{X}'\right)^2 = n^{1/2} \sum_{i=1}^N \left(\Delta_i X'\right)^2 + o_{\mathbb{P}}(1). \tag{48}$$

Note that

$$n^{1/2} \sum_{i=1}^N \left(\left(\Delta_i \widehat{X}'\right)^2 - \left(\Delta_i X'\right)^2\right) = 2n^{1/2} \sum_{i=1}^N \Delta_i X' \psi_i(\widehat{\theta}) + n^{1/2} \sum_{i=1}^N \psi_i(\widehat{\theta})^2,$$

and the second term in the right-hand side of the equation is negligible as a direct consequence of the domination $|\psi_i(\widehat{\theta})| \leq C/n$. We show now that the first term is also negligible. By the mean value theorem, we also have for some $\overline{\theta} \in [\theta_0, \widehat{\theta}]$ that

$$n^{1/2} \sum_{i=1}^N \Delta_i X' \psi_i(\widehat{\theta}) = n^{1/2}(\widehat{\theta} - \theta_0)^T \sum_{i=1}^N \Delta_i X' \partial_\theta \psi_i(\theta_0) + \frac{n^{1/2}(\widehat{\theta} - \theta_0)^T}{2} \sum_{i=1}^N \Delta_i X' \partial_\theta^2 \psi_i(\overline{\theta})(\widehat{\theta} - \theta_0) \tag{49}$$

Using that $\widehat{\theta} - \theta_0 = O_{\mathbb{P}}(1/n)$, and the fact that $\|\partial_\theta^2 \psi(\overline{\theta})\| \leq C$ we deduce that the second term is negligible. Finally, note that $\sum_{i=1}^N \Delta_i X' \partial_\theta \psi_i(\theta_0)$ can be decomposed as the sum of $\sum_{i=1}^N \Delta_i \breve{B} \partial_\theta \psi_i(\theta_0)$, where $\breve{B}_t = \int_0^t b'_s ds$, and which is easily proved to be negligible given the local boundedness of b and δ, and $\sum_{i=1}^N \Delta_i M^c \partial_\theta \psi_i(\theta_0)$, which is a sum of martingale increments with respect to the filtration $\mathcal{H}_t = \mathcal{F}_t \vee \sigma\{Q_{t_i}, 1 \leq i \leq N\}$. Thus, by (2.2.35) in Jacod and Protter (2011), proving that this term tends to 0 boils down to showing that

$$n^{-1} \sum_{i=1}^N \mathbb{E}\left[(\Delta_i M^c)^2 \|\partial_\theta \psi_i(\theta_0)\|^2\right] \to 0,$$

which is immediate since $\|\partial_\theta \psi_i(\theta_0)^2\| \leq C$, $N/n \to^{\mathbb{P}} F$ and $\mathbb{E}(\Delta_i M^c)^2 \leq C/n$ by Assumption **(H')**.

We now turn to II. As by (47) along with Assumption **(H')** , we have for any $k > 0$ the inequality $\mathbb{P}\left[|\Delta_i \widehat{X}'| > w_i/2\right] \leq C n^{k(\bar{\omega}-1/2)}$, we can assume without loss of generality, by taking k sufficiently large, that we can add the indicator $\mathbf{1}_{\left\{|\Delta_i \widehat{X}'| \leq w_i/2\right\}}$ in II, i.e. that

$$II = 2n^{1/2} \sum_{i=1}^{N} \Delta_i \widehat{X}' \Delta_i J \mathbf{1}_{\{|\Delta_i \widehat{X}| \leq w_i\}} \mathbf{1}_{\left\{|\Delta_i \widehat{X}'| \leq w_i/2\right\}},$$

$$\leq 2n^{1/2} \sum_{i=1}^{N} \Delta_i \widehat{X}' \Delta_i J \mathbf{1}_{\{|\Delta_i J| \leq 3w_i/2\}} \mathbf{1}_{\left\{|\Delta_i \widehat{X}'| \leq w_i/2\right\}},$$

so that

$$|II| \leq 2n^{1/2} \sum_{i=1}^{N} |\Delta_i \widehat{X}'| |\Delta_i J|^{1-r} |\Delta_i J|^r \mathbf{1}_{\{|\Delta_i J| \leq 3w_i/2\}} \mathbf{1}_{\left\{|\Delta_i \widehat{X}'| \leq w_i/2\right\}},$$

$$\leq C n^{1/2-\bar{\omega}(2-r)} \underbrace{\sum_{i=1}^{N} |\Delta_i J|^r}_{O_{\mathbb{P}}(1)},$$

where we recall that $r > 0$ is the jump index of J. Given that $\bar{\omega} \in (1/(2(2-r)), 1/2)$, we immediately deduce that $II = o_{\mathbb{P}}(1)$. Finally, we can show that $III = o_{\mathbb{P}}(1)$ with the same line of reasoning as for II.

6.3 Proof of Corollary 4

We show (30), as (29) is a particular case where $\phi = 0$. This amounts to proving that $\widehat{AB}$ and $\widehat{AVAR}$ are consistent.

We show first that $\widehat{AB}$ is consistent. As in the previous proofs (in this case this is actually quite easier as we only show the consistency), we can remove the truncation and the parametric noise part and replace $\Delta_i \widehat{X}$ by $\Delta_i X'$. We obtain that

$$\widehat{AB} = \sum_{i=1}^{B} \frac{2}{3} \overline{\upsilon\sigma}_i (X'_{t_{ih}} - X'_{t_{(i-1)h}}) + o_{\mathbb{P}}(1),$$

where

$$\overline{\upsilon\sigma}_i = \frac{N^{1/2} \sum_{j=h(i-1)+1}^{hi} (\Delta_j X')^3}{\sum_{j=h(i-1)+1}^{hi} (\Delta_j X')^2}.$$

A Taylor expansion on the function $f(x,y) = x/y$ along with a local version of the convergence (24), the fact that $\sum_{i=1}^{N} \left(\Delta_i X'\right)^2 \to^{\mathbb{P}} \Xi$, that σ_t and υ_t are bounded and bounded away from 0 and that $N/n \to^{\mathbb{P}} F$ yields

$$\widehat{AB} = \sum_{i=1}^{B} \frac{2}{3} \upsilon_{t_{i-1}} \sigma_{t_{i-1}} (X'_{t_{ih}} - X'_{t_{(i-1)h}}) + o_{\mathbb{P}}(1).$$

Applying Theorem I.4.31 (iii) on p. 47 in Jacod and Shiryaev (2003) together with the fact that σ_t and v_t are bounded and bounded away from 0, we conclude that $\widehat{AB} \to^{\mathbb{P}} AB$.

We show now that $\widehat{AVAR}$ is consistent. In this case we can again by similar arguments remove the truncation and substitute $\Delta_i \widehat{X}$ by $\Delta_i X'$, i.e. it holds that

$$\widehat{AVAR} = \frac{2N}{3}\sum_{i=1}^{N}(\Delta_i X')^4 - \frac{4}{9}\sum_{i=1}^{B}(\overline{v\sigma}_i)^2(X'_{t_{ih}} - X'_{t_{(i-1)h}})^2 + o_{\mathbb{P}}(1).$$

By (23) together with the fact that $N/n \to^{\mathbb{P}} F$, we deduce that

$$\frac{2N}{3}\sum_{i=1}^{N}(\Delta_i X')^4 \to^{\mathbb{P}} \frac{2}{3}\int_0^T u_s \sigma_s^4 ds.$$

Furthermore, using similar techniques as for $\widetilde{AB}$, we obtain that

$$\frac{4}{9}\sum_{i=1}^{B}(\overline{v\sigma}_i)^2(X'_{t_{ih}} - X'_{t_{(i-1)h}})^2 \to^{\mathbb{P}} \frac{4}{9}\int_0^T v_s^2 \sigma_s^4 ds.$$

We have thus shown that $\widehat{AVAR} \to^{\mathbb{P}} AVAR$.

6.4 Proof of Theorem 7

It is immediate to see that (32) holds as a consequence of (31) along with Theorem 3.3 in Vetter (2010). Accordingly, we show that (31) holds in what follows, i.e. that

$$n^{1/2}\widehat{\Xi} = n^{1/2}\widetilde{\Xi} + o_{\mathbb{P}}(1).$$

First, we show that we can assume without loss of generality that the price process X is continuous, i.e. $J = 0$. To do so, we introduce $\widehat{\Xi}'$ as the estimator applied to X' in lieu of X. We show that

$$n^{1/2}\left(\widehat{\Xi} - \widehat{\Xi}'\right) \to^{\mathbb{P}} 0. \tag{50}$$

From (15), we can easily obtain the key decomposition

$$\Delta_i \widehat{X} = \Delta_i X(\widehat{\theta}) = \underbrace{\Delta_i \breve{B} + \psi_i(\widehat{\theta})}_{\Delta_i B'} + \Delta_i M^c + \Delta_i J, \tag{51}$$

and by assumption **(H)**, also recall that we have $|\psi_i(\widehat{\theta})| \leq |\sup_{\theta\in\Theta} \partial_\theta \psi_i(\theta)|\,|\widehat{\theta} - \theta_0| \leq C/n$. Thus, remark that all usual conditional moment estimates for $\Delta_i \breve{B}$ are also true for $\Delta_i B'$. More precisely, replacing $\Delta_i \breve{B}$ by $\Delta_i B'$ and $\mathcal{F}_i$ by $\mathcal{G}_i = \mathcal{F}_i \vee \sigma\{Q_{t_i}, 0 \leq i \leq n\}$ in the proof of Lemma 13.2.6 (p. 384) in Jacod and Protter (2011), all the conditional estimates are preserved and thus the lemma holds true in the presence of the error term $\psi_i(\widehat{\theta})$. Indeed, the three key ingredients for the original proof of Lemma 13.2.6 are the following (with our own notations): defining

$$U_i = \frac{|\Delta_i X'|}{\Delta_n^{1/2}}, V_i = \left(\frac{\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}} \gamma(z)^{1/r}\mu(ds,dz)}{\Delta_n^{\bar{\varpi}}}\right) \wedge 1 \text{ and } W_i = \left(\frac{\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}} \gamma(z)^{1/r}\mu(ds,dz)}{\Delta_n^{1/2}}\right) \wedge 1,$$

we have (see (13.2.22)-(13.2.23) in Jacod and Protter (2011), pp.384-385), for any $m > 0$,

$$\mathbb{E}[(U_i)^m|\mathcal{G}_{i-1}] \leq C_m, \tag{52}$$

$$\mathbb{E}[(V_i)^m|\mathcal{G}_{i-1}] \leq \Delta_n^{(1-r\bar{\omega})(1\wedge(m/r))}\phi_n, \tag{53}$$

$$\mathbb{E}[(W_i)^m|\mathcal{G}_{i-1}] \leq \Delta_n^{(1-r/2)(1\wedge(m/r))}\phi_n, \tag{54}$$

where $C_m > 0$ is a constant possibly depending on m, and ϕ_n is a suitable deterministic sequence tending to 0 as $n \to +\infty$. Note that in the presence of the term $\psi_i(\widehat{\theta})$, that is, if V_i and W_i are unchanged but U_i is changed to $\widehat{U}_i = \frac{|\Delta_i \widehat{X}'|}{\Delta_n^{1/2}}$, the conditional deviations (52)-(54) remain unchanged since

$$\Big|\mathbb{E}[(U_i)^m|\mathcal{G}_{i-1}] - \mathbb{E}[(\widehat{U}_i)^m|\mathcal{G}_{i-1}]\Big| = O_{\mathbb{P}}(n^{-1}\Delta_n^{-1/2}) \to 0$$

using that $\psi_i(\widehat{\theta}) \leq C/n$. Therefore, Lemma 13.2.6 from Jacod and Protter (2011) still holds when X and X' are respectively replaced by $\widehat{X}$ and $\widehat{X}'$. Applied with $F(x_1, x_2) = |x_1||x_2|$, $k = 2$, $p' = s' = 2$, $s = 1$ and $\theta = 0$, this directly yields that for all $q \geq 1$ and for some deterministic sequence a_n going to 0,

$$\mathbb{E}\left||\Delta_i\widehat{X}||\Delta_{i-1}\widehat{X}|\mathbf{1}_{\{|\Delta_i\widehat{X}|\leq w\}}\mathbf{1}_{\{|\Delta_{i-1}\widehat{X}|\leq w\}} - |\Delta_i\widehat{X}'||\Delta_{i-1}\widehat{X}'|\mathbf{1}_{\{|\Delta_i\widehat{X}'|\leq w\}}\mathbf{1}_{\{|\Delta_{i-1}\widehat{X}'|\leq w\}}\right|^q \leq Ca_n\Delta_n^{(2q-r)\bar{\omega}+1},$$

where we have used that $q/r > 1$ and $\bar{\omega} < 1/2$, and where we recall that $\Delta_i\widehat{X}' = \Delta_i X'(\widehat{\theta})$. Given the definitions of $\widehat{\Xi}$ and $\widehat{\Xi}'$, applying the above domination with $q = 1$, we directly deduce the estimate

$$n^{1/2}\mathbb{E}|\widehat{\Xi} - \widehat{\Xi}'| \leq a_n n^{1/2-(2-r)\bar{\omega}} \to 0,$$

since $\bar{\omega} \in (1/(2(2-r)), 1/2)$. From now on, by (50), we are left to show $n^{1/2}(\widehat{\Xi}' - \widetilde{\Xi}) \to^{\mathbb{P}} 0$. By definition, we have that

$$\begin{aligned} n^{1/2}\widehat{\Xi}' &= \frac{\pi n^{1/2}}{2}\sum_{i=2}^n \big|\Delta_i\widehat{X}'\big|\mathbf{1}_{\{|\Delta_i\widehat{X}'|\leq w\}}\big|\Delta_{i-1}\widehat{X}'\big|\mathbf{1}_{\{|\Delta_{i-1}\widehat{X}'|\leq w\}}, \\ &= \frac{\pi n^{1/2}}{2}\sum_{i=2}^n \big|(\Delta_i X' + \psi_i(\widehat{\theta}))(\Delta_{i-1}X' + \psi_{i-1}(\widehat{\theta}))\big|\mathbf{1}_{\{|\Delta_i\widehat{X}'|\leq w\}}\mathbf{1}_{\{|\Delta_{i-1}\widehat{X}'|\leq w\}}. \end{aligned}$$

If we introduce $\breve{\Xi} = \frac{\pi}{2}\sum_{i=2}^n \big|\Delta_i X'\big|\mathbf{1}_{\{|\Delta_i\widehat{X}'|\leq w\}}\big|\Delta_{i-1}X'\big|\mathbf{1}_{\{|\Delta_{i-1}\widehat{X}'|\leq w\}}$, we have

$$\begin{aligned} n^{1/2}\big|\widehat{\Xi}' - \breve{\Xi}\big| &= \frac{\pi n^{1/2}}{2}\sum_{i=2}^n \big|\Delta_i X'\big|\mathbf{1}_{\{|\Delta_i\widehat{X}'|\leq w\}}\Big(\big|\Delta_{i-1}\widehat{X}'\big| - \big|\Delta_{i-1}X'\big|\Big)\mathbf{1}_{\{|\Delta_{i-1}\widehat{X}'|\leq w\}} \\ &\quad + \frac{\pi n^{1/2}}{2}\sum_{i=2}^n \Big(\big|\Delta_i\widehat{X}'\big| - \big|\Delta_i X'\big|\Big)\mathbf{1}_{\{|\Delta_i\widehat{X}'|\leq w\}}\big|\Delta_{i-1}\widehat{X}'\big|\mathbf{1}_{\{|\Delta_{i-1}\widehat{X}'|\leq w\}} \\ &= I + II. \end{aligned}$$

We prove (31) in two steps in what follows. First, we show that $n^{1/2}\big|\widetilde{\Xi} - \breve{\Xi}\big| = o_{\mathbb{P}}(1)$. Second, we prove that $I = o_{\mathbb{P}}(1)$ and $II = o_{\mathbb{P}}(1)$. We have

$$n^{1/2}\big|\widetilde{\Xi} - \breve{\Xi}\big| \leq \frac{\pi n^{1/2}}{2} \sum_{i=2}^{n} |\Delta_i X'||\Delta_{i-1} X'| \big|\mathbf{1}_{\{|\Delta_i \widehat{X}'| \leq w\}} \mathbf{1}_{\{|\Delta_{i-1} \widehat{X}'| \leq w\}} - \mathbf{1}_{\{|\Delta_i X'| \leq w\}} \mathbf{1}_{\{|\Delta_{i-1} X'| \leq w\}}\big|,$$

so that by standard inequalities we can deduce $n^{1/2}\big|\widetilde{\Xi} - \breve{\Xi}\big| \to^{\mathbb{P}} 0$ if

$$\mathbb{E}\left|\mathbf{1}_{\{|\Delta_i \widehat{X}'| \leq w\}} - \mathbf{1}_{\{|\Delta_i X'| \leq w\}}\right| \leq C n^{-\beta} \tag{55}$$

for some $\beta > 0$ large enough (where C possibly depends on β). Let us thus show now (55). Introducing $\breve{\Delta}$ as the symmetric difference operator, we have

$$\begin{aligned}\left|\mathbf{1}_{\{|\Delta_i \widehat{X}'| \leq w\}} - \mathbf{1}_{\{|\Delta_i X'| \leq w\}}\right| &= \mathbf{1}_{\{|\Delta_i \widehat{X}'| \leq w\} \breve{\Delta} \{|\Delta_i X'| \leq w\}} \\ &\leq \mathbf{1}_{\{|\Delta_i X' - w| \leq |\psi_i(\widehat{\theta})|\}} \\ &\leq \mathbf{1}_{\{|\Delta_i X' - w| \leq C/n\}}.\end{aligned}$$

Now, letting $\gamma \in (\bar{\omega}, 1/2)$ and $q > 0$, since $\{|\Delta_i X' - w| \leq C/n\} \cap \{|\Delta_i X'| \leq n^{-\gamma}\} = \emptyset$ for n large enough, we automatically have

$$\mathbf{1}_{\{|\Delta_i X' - w| \leq C/n\}} \leq \mathbf{1}_{\{|\Delta_i X'| > n^{-\gamma}\}} \leq n^{\gamma q} |\Delta_i X'|^q,$$

hence

$$\begin{aligned}\mathbb{E}\left|\mathbf{1}_{\{|\Delta_i \widehat{X}'| \leq w\}} - \mathbf{1}_{\{|\Delta_i X'| \leq w\}}\right| &\leq n^{\gamma q} \mathbb{E}|\Delta_i X'|^q \\ &\leq C n^{q(\gamma - 1/2)},\end{aligned}$$

and taking q large enough we get (55). Finally, we prove that $I = o_{\mathbb{P}}(1)$. The proof for II is similar. First note that since X' is continuous and $\psi_i(\widehat{\theta}) < K/n$, we can get rid of the indicator functions in I following the same line of reasoning as for (46). Moreover, following arguments similar to that of (55), I is asymptotically unaffected if $\mathbf{1}_{\{|\psi_{i-1}(\widehat{\theta})| < |\Delta_{i-1} X'|\}}$ is present in the sum. Without loss of generality, we can therefore assume that

$$I = \frac{\pi n^{1/2}}{2} \sum_{i=2}^{n} \big|\Delta_i X'\big| \mathbf{1}_{\{|\psi_{i-1}(\widehat{\theta})| < |\Delta_{i-1} X'|\}} \left(\big|\Delta_{i-1} \widehat{X}'\big| - \big|\Delta_{i-1} X'\big|\right) + o_{\mathbb{P}}(1).$$

Next, we decompose I as follows, using the identity for $|y| \leq |x|$, $|x + y| - |x| = y\operatorname{sgn}(x)$ with sgn the usual sign function:

$$I = \frac{\pi n^{1/2}}{2} \sum_{i=2}^{n} \psi_{i-1}(\widehat{\theta}) \big|\Delta_i X'\big| \operatorname{sgn}(\Delta_{i-1} X') \mathbf{1}_{\{|\psi_{i-1}(\widehat{\theta})| \leq |\Delta_i X'|\}} + o_{\mathbb{P}}(1).$$

Again, the indicator function can be removed since its complement event is negligible (it can be majorated by e.g $C|\Delta_i X'|^p/n^p$ for any p where C possibly depends on p), which yields the approximation

$$\begin{aligned} I &= \frac{\pi n^{1/2}}{2} \sum_{i=2}^{n} \psi_{i-1}(\widehat{\theta}) \big|\Delta_i X'\big| \mathrm{sgn}(\Delta_{i-1} X') + o_{\mathbb{P}}(1) \\ &= \frac{\pi n^{1/2}(\widehat{\theta} - \theta_0)^T}{2} \sum_{i=2}^{n} \partial_\theta \psi_{i-1}(\theta_0) \big|\Delta_i X'\big| \mathrm{sgn}(\Delta_{i-1} X') + o_{\mathbb{P}}(1) \end{aligned}$$

where the second step is another application of the mean value theorem (as in the proof of Theorem 2). Now note that standard arguments yield

$$\mathbb{P}[\mathrm{sgn}(\Delta_{i-1} X') \neq \mathrm{sgn}(\Delta_{i-1} W)] = o_{\mathbb{P}}(n^{-p})$$

and

$$\begin{aligned} \mathbb{E}\big||\Delta_i X'| - |\sigma_{t_{i-2}} \Delta_i W|\big|^p &\leq \mathbb{E}|\Delta_i X' - \sigma_{t_{i-2}} \Delta_i W|^p \\ &\leq C n^{-p} \end{aligned}$$

for any $p > 0$ (where the constant C may depend on p) and where we have used (14), so that using $\widehat{\theta} - \theta_0 = O_{\mathbb{P}}(n^{-1})$ gives

$$I = \frac{\pi n^{1/2}(\widehat{\theta} - \theta_0)^T}{2} \sum_{i=2}^{n} \sigma_{t_{i-2}} \partial_\theta \psi_{i-1}(\theta_0) \big|\Delta_i W\big| \mathrm{sgn}(\Delta_{i-1} W) + o_{\mathbb{P}}(1)$$

which are conditionally centered and uncorrelated increments, with Var $[|\Delta_i W| \mathrm{sgn}(\Delta_{i-1} W)|\, \mathcal{F}_{i-2}] = O(n^{-1})$, so that $\sum_{i=2}^{n} \sigma_{t_{i-2}} \partial_\theta \psi_{i-1}(\theta_0) \big|\Delta_i W\big| \mathrm{sgn}(\Delta_{i-1} W) = O_{\mathbb{P}}(1)$. Therefore, using again that $\widehat{\theta} - \theta_0 = O_{\mathbb{P}}(n^{-1})$, we have $I \to^{\mathbb{P}} 0$.

6.5 Proof of Corollary 8

By the stable convergence of Theorem 7, the proof amounts to showing that $\widehat{AVAR}$ is consistent, which is actually a corollary to Theorem 11 in the special case $g(x) = x^2$.

6.6 Proof of Theorem 9

Following the discussion at the beginning of Appendix A.2 (p. 30) in Potiron and Mykland (2017) and Proposition 1 from Mykland and Zhang (2009), p. 1408, we can assume without loss of generality that the drift b_t is null as the price process X is continuous.

First, note that (37) is a straightforward consequence of (36) together with Theorem 1 (p. 25) in Potiron and Mykland (2017). Consequently, we only need to show (36). We now provide the proof of (36), i.e. that

$$\alpha^{-1} \widehat{\Xi} = \alpha^{-1} \widetilde{\Xi} + o_{\mathbb{P}}(1).$$

First, note that as a result of Remark 5 (p. 25) in Potiron and Mykland (2017), $n^{1/2}$ and α^{-1} are of the same order, and thus it is sufficient to show that

$$n^{1/2}\widehat{\Xi} = n^{1/2}\widetilde{\Xi} + o_{\mathbb{P}}(1).$$

Second, we have to reexpress the Hayashi-Yoshida estimator (35). To do so, we follow the beginning of Section 4.3 in Potiron and Mykland (2017) and introduce some (common) definition in the Hayashi-Yoshida literature. For any positive integer i, we consider the ith sampling time of the first asset $t_i^{(1)}$. We define two related random times, t_i^- and t_i^+, which correspond respectively to the closest sampling time of the second asset that is strictly smaller than $t_i^{(1)}$, and the closest sampling time of the second asset that is (not necessarily strictly) bigger than $t_i^{(1)}$. Formally, they are defined as

$$t_0^- = 0, \tag{56}$$
$$t_i^- = \max\{t_j^{(2)} : t_j^{(2)} < t_i^{(1)}\} \text{ for } i \geq 1, \tag{57}$$
$$t_i^+ = \min\{t_j^{(2)} : t_j^{(2)} \geq t_i^{(1)}\}. \tag{58}$$

Rearranging the terms in (35) gives us

$$\widetilde{\Xi} = \sum_{t_i^+ < t} \Delta_i X^{(1)} (X_{t_i^+}^{(2)} - X_{t_{i-1}^-}^{(2)}) + o_{\mathbb{P}}(n^{-1/2}). \tag{59}$$

We deduce that

$$\begin{aligned}
n^{1/2}\widehat{\Xi} &= n^{1/2} \sum_{t_i^+ < t} \Delta_i \widehat{X}^{(1)} (\widehat{X}_{t_i^+}^{(2)} - \widehat{X}_{t_{i-1}^-}^{(2)}) + o_{\mathbb{P}}(1), \\
&= n^{1/2}\widetilde{\Xi} + n^{1/2} \sum_{t_i^+ < t} \psi_i^{(1)}(\widehat{\theta}^{(1)}) \big((\phi(Q_{t_i^+}^{(2)}, \theta_0^{(2)}) - \phi(Q_{t_{i-1}^-}^{(2)}, \theta_0^{(2)})) - (\phi(Q_{t_i^+}^{(2)}, \widehat{\theta}^{(2)}) - \phi(Q_{t_{i-1}^-}^{(2)}, \widehat{\theta}^{(2)})) \big) \\
&\quad + n^{1/2} \sum_{t_i^+ < t} \Delta_i X^{(1)} \big((\phi(Q_{t_i^+}^{(2)}, \theta_0^{(2)}) - \phi(Q_{t_{i-1}^-}^{(2)}, \theta_0^{(2)})) - (\phi(Q_{t_i^+}^{(2)}, \widehat{\theta}^{(2)}) - \phi(Q_{t_{i-1}^-}^{(2)}, \widehat{\theta}^{(2)})) \big) \\
&\quad + n^{1/2} \sum_{t_i^+ < t} \psi_i^{(1)}(\widehat{\theta}^{(1)}) (X_{t_i^+}^{(2)} - X_{t_{i-1}^-}^{(2)}) + o_{\mathbb{P}}(1), \\
&:= n^{1/2}\widetilde{\Xi} + I + II + III + o_{\mathbb{P}}(1).
\end{aligned}$$

Our aim is to show that $I = o_{\mathbb{P}}(1)$, $II = o_{\mathbb{P}}(1)$ and $III = o_{\mathbb{P}}(1)$. We start with I. On the account that ϕ is C^3 in θ, and because $\max_i \|Q_{t_i}\|$ is bounded,

$$I \leq C n^{1/2} N |\widehat{\theta} - \theta_0|^2,$$

and this is $o_{\mathbb{P}}(1)$ by (18), Remark 5 (p. 25) and Lemma 8 (p. 31) in Potiron and Mykland (2017).

As for II, the proof of Theorem 2 (p. 46) in Li et al (2016) in the volatility case goes through with one change. To prove (69) in the cited paper, since

$$\big((\phi(Q_{t_i^+}^{(2)}, \theta_0^{(2)}) - \phi(Q_{t_{i-1}^-}^{(2)}, \theta_0^{(2)})) - (\phi(Q_{t_i^+}^{(2)}, \widehat{\theta}^{(2)}) - \phi(Q_{t_{i-1}^-}^{(2)}, \widehat{\theta}^{(2)})) \big)$$

is not $\mathcal{F}_{t_i}$-measurable, we need to use a Taylor expansion around θ_0. More specifically, let us prove (69) and in line with the notation of the cited paper, we define:

$$F_N(\theta) = \sum_{i=1}^{N^{(1)}} \underbrace{\Big((\phi(Q^{(2)}_{t_i^+},\theta) - \phi(Q^{(2)}_{t_{i-1}^-},\theta)) - (\phi(Q^{(2)}_{t_i^+},\theta_0^{(2)}) - \phi(Q^{(2)}_{t_{i-1}^-},\theta_0^{(2)}))\Big)}_{\chi_i(\theta)} \underbrace{\int_{t_{i-1}^{(1)}}^{t_i^{(1)}} \sigma_t^{(1)} dW_t^{(1)}}_{\Delta M_i^{c,(1)}}.$$

Note now that by the same Taylor expansion as in (49) and the same line of reasoning, we directly get that for $\theta \in \Theta$ such that $|\theta - \theta_0| \leq K/N$, for some $\overline{\theta} \in [\theta_0, \theta]$,

$$N^l |F_N(\theta) - F_N(\theta_0)|^{2l} \leq C_l N^l |\theta - \theta_0|^{2l} \left(\left| \sum_{i=1}^{N^{(1)}} \partial_\theta \chi_i(\theta_0) \Delta M_i^{c,(1)} \right|^{2l} + \left| \frac{1}{2} \sum_{i=1}^{N^{(1)}} \partial_\theta^2 \chi_i(\overline{\theta}) \Delta M_i^{c,(1)} \right|^{2l} |\theta - \theta_0|^{2l} \right).$$

Now, using that the first term is a sum of $\mathcal{H}_t$-martingale increments and Burkholder-Davis-Gundy inequality yields

$$\begin{aligned} \mathbb{E} \left| \sum_{i=1}^{N^{(1)}} \partial_\theta \chi_i(\theta_0) \Delta M_i^{c,(1)} \right|^{2l} &\leq C \mathbb{E} \left| \sum_{i=1}^{N^{(1)}} |\partial_\theta \chi_i(\theta_0)|^2 \Delta_i t^{(1)} \right|^l \\ &\leq C. \end{aligned}$$

Similarly, Jensen inequality applied to the measure $(N^{(1)})^{-1} \sum_{i=1}^{N^{(1)}}$, the boundedness of $|\partial_\theta^2 \chi_i(\overline{\theta})|$, and direct calculation of moments for $\Delta M_i^{c,(1)}$ yield

$$\begin{aligned} \mathbb{E} \left| \frac{1}{2} \sum_{i=1}^{N^{(1)}} \partial_\theta^2 \chi_i(\overline{\theta}) \Delta M_i^{c,(1)} \right|^{2l} &\leq C N^{2l-1} \mathbb{E} \sum_{i=1}^{N^{(1)}} \left| \Delta M_i^{c,(1)} \right|^{2l} \\ &\leq C N^l. \end{aligned}$$

Combined with $|\theta - \theta_0| \leq K/N$, this gives

$$N^l \mathbb{E} \sup_{\theta \in \Theta | |\theta - \theta_0| \leq K/N} |F_N(\theta) - F_N(\theta_0)|^{2l} \to 0$$

which is (69) from Li et al (2016). Then, one can proceed as in the proof of Theorem 2 (p. 46) in Li et al (2016).

We turn to III, which is slightly more complicated to deal with. We decompose the increment of the second asset in three parts and rewrite III as

$$\begin{aligned} III &= n^{1/2} \Big(\sum_{t_i^+ < t} \psi_i^{(1)}(\widehat{\theta}^{(1)})(X^{(2)}_{t_i^+} - X^{(2)}_{t_i^{(1)}}) + \sum_{t_i^+ < t} \psi_i^{(1)}(\widehat{\theta}^{(1)})(X^{(2)}_{t_i^{(1)}} - X^{(2)}_{t_{i-1}^{(1)}}) + \sum_{t_i^+ < t} \psi_i^{(1)}(\widehat{\theta}^{(1)})(X^{(2)}_{t_{i-1}^{(1)}} - X^{(2)}_{t_{i-1}^-}) \Big) \\ &:= n^{1/2}(III_A + III_B + III_C). \end{aligned}$$

The problem with III_A is that it is not adapted to a simple filtration. To circumvent this difficulty, we need to rearrange the terms of the sum again. We follow Potiron and Mykland (2017) (Section 4.3) and we define the new sampling times t_i^{1C} as $t_0^{1C} := t_0^{(1)}$, and recursively for i any nonnegative integer

$$t_{i+1}^{1C} := \min \big\{ t_u^{(1)} : \text{ there exists } j \in \mathbb{N} \text{ such that } t_i^{1C} \leq t_j^{(2)} < t_u^{(1)} \big\}. \quad (60)$$

In analogy with (56), (57) and (58), we introduce the following times

$$t_0^{1C,-} := 0, \tag{61}$$

$$t_{i-1}^{1C,-} := \max\{t_j^{(2)} : t_j^{(2)} < t_{i-1}^{1C}\} \text{ for } i \geq 2 \tag{62}$$

$$t_{i-1}^{1C,+} := \min\{t_j^{(2)} : t_j^{(2)} \geq t_{i-1}^{1C}\} \text{ for } i \geq 1. \tag{63}$$

In light of this definition, we can rewrite III_A as

$$III_A = \sum_{t_i^{1C,+} < t} \underbrace{\big((\phi(Q^{(1)}_{t_i^{1C}}, \widehat{\theta}^{(1)}) - \phi(Q^{(1)}_{t_{i-1}^{1C}}, \widehat{\theta}^{(1)})) - (\phi(Q^{(1)}_{t_i^{1C}}, \theta_0^{(1)}) - \phi(Q^{(1)}_{t_{i-1}^{1C}}, \theta_0^{(1)}))\big)(X^{(2)}_{t_i^{1C,+}} - X^{(2)}_{t_i^{1C}})}_{M_i(\widehat{\theta}^{(1)})},$$

where $M_i(\theta)$ is $\mathcal{F}_{t_{i+1}^{1C}}$-measurable. By the mean value theorem, we also have for some $\overline{\theta} \in [\theta_0^{(1)}, \widehat{\theta}^{(1)}]$ that

$$n^{1/2} \sum_{i=1}^{N^{(1)}} M_i(\widehat{\theta}^{(1)}) = n^{1/2}(\widehat{\theta}^{(1)} - \theta_0^{(1)})^T \sum_{i=1}^{N^{(1)}} \partial_\theta M_i(\theta_0^{(1)}) + \frac{n^{1/2}(\widehat{\theta}^{(1)} - \theta_0^{(1)})^T}{2} \sum_{i=1}^{N^{(1)}} \partial_\theta^2 M_i(\overline{\theta})(\widehat{\theta}^{(1)} - \theta_0^{(1)}).$$

Following the same line of reasoning as for the proof of (49) in the volatility case, we can show that the two terms go to 0 in probability, so that we have shown that $n^{1/2} III_A = o_{\mathbb{P}}(1)$. The other two terms III_B and III_C do not require rearranging the terms. Specifically, $n^{1/2} III_B$ can be shown $o_{\mathbb{P}}(1)$ following exactly the proof of Theorem 2 (p. 46) in Li et al (2016). Regarding the third term $n^{1/2} III_C$, we can show that it is $o_{\mathbb{P}}(1)$ using a Taylor expansion similarly as for III_A.

6.7 Proof of Corollary 10

Although the quantities introduced are quite involved to formally define $\widetilde{AB}$ and $\widetilde{AVAR}$, the proof works the same way as for the proof of (30) in Corollary 4, along with techniques and estimates from Potiron and Mykland (2017).

6.8 Proof of Theorem 11

All along this proof, we use the notations k_n, Δ_n, w_n in lieu of respectively k, Δ and w in order to emphasize their dependence on n. We have to show that $n^{1/2}\left(\widehat{\Xi} - \widetilde{\Xi}'\right) = o_{\mathbb{P}}(1)$ where

$$\widehat{\Xi} = \Delta_n \sum_{i=1}^{[T/\Delta_n]-k_n+1} \left\{ g(\widehat{c}_i) - \frac{1}{2k_n} \sum_{j,k,l,m=1}^{d} \partial^2_{jk,lm} g(\widehat{c}_i) \left(\widehat{c}_i^{jl} \widehat{c}_i^{km} + \widehat{c}_i^{jm} \widehat{c}_i^{kl} \right) \right\}, \tag{64}$$

with

$$\widehat{c}_i^{lm} = \frac{1}{k_n \Delta_n} \sum_{j=0}^{k_n-1} \Delta_{i+j} \widehat{X}^l \Delta_{i+j} \widehat{X}^m \mathbf{1}_{\{\|\Delta_{i+j}\widehat{X}\| \leq w_n\}}. \tag{65}$$

We start by showing that we can assume without loss of generality that X is continuous, i.e replace X by X' in all the expressions. To do so, consider $\widehat{\Xi}'$ and $\widehat{c}_i'$ the estimators applied to the continuous part X' in lieu of X. Without loss of generality, we assume in what follows that X, $\widehat{\theta}$ and θ_0 are 1-dimensional quantities. The multi-dimensional case can be derived by a straightforward adaptation.

Lemma 18. *We have*

$$n^{1/2}\left(\widehat{\Xi}-\widehat{\Xi}'\right) \to^{\mathbb{P}} 0.$$

Proof. Recall that we have the key decomposition

$$\Delta_i \widehat{X} = \Delta_i X(\widehat{\theta}) = \underbrace{\Delta_i \breve{B} + \psi_i(\widehat{\theta})}_{\Delta_i B'} + \Delta_i M^c + \Delta_i J, \tag{66}$$

where we recall that $\breve{B}_t = \int_0^t b'_s ds$. Now, we apply exactly the same line of reasoning as for the proof of Theorem 7. We replace again $\Delta_i \breve{B}$ by $\Delta_i B'$ and $\mathcal{F}_i$ by $\mathcal{G}_i = \mathcal{F}_i \vee \sigma\{Q_{t_i}, 0 \leq i \leq n\}$ in the proof of Lemma 13.2.6 (p. 384) in Jacod and Protter (2011), all the conditional estimates are preserved and thus the lemma remains valid in the presence of the term $\psi_i(\widehat{\theta})$. Applied with $F(x) = x^2$, $k = 1$, $p' = s' = 2$, $s = 1$ and $\theta = 0$, this directly yields that for all $q \geq 1$ and for some deterministic sequence a_n shrinking to 0, we have that

$$\mathbb{E}\left| |\Delta_i \widehat{X}|^2 \mathbf{1}_{\left\{|\Delta_i \widehat{X}| \leq w_n\right\}} - |\Delta_i \widehat{X}'|^2 \mathbf{1}_{\left\{|\Delta_i \widehat{X}'| \leq w_n\right\}} \right|^q \leq C a_n \Delta_n^{(2q-r)\bar{\omega}+1}. \tag{67}$$

As a by-product, we also deduce

$$\mathbb{E}\left|\widehat{c}_i - \widetilde{c}'_i\right|^q \leq C a_n \Delta_n^{(2q-r)\bar{\omega}+1-q}. \tag{68}$$

Moreover, replacing again $\mathcal{F}_i$ by $\mathcal{G}_i$ and $\Delta_i \breve{B}$ by $\Delta_i B'$ in the calculation we can also see that the second inequality of (4.10) in Jacod and Rosenbaum (2013) remains true in the presence of $\psi_i(\widehat{\theta})$, that is, introducing $\alpha_i = |\Delta_i \widehat{X}'|^2 - \sigma_{t_i}^2 \Delta_n$, we have

$$|\mathbb{E}[\alpha_i | \mathcal{G}_i]| \leq C \Delta_n^{3/2}. \tag{69}$$

Now, remark that by the proof of Lemma 4.4 (p. 1479, case $v = 1$) in Jacod and Rosenbaum (2013), $n^{1/2}\big(\widehat{\Xi} - \widehat{\Xi}'\big) \to^{\mathbb{P}} 0$ is an immediate consequence of our estimates (68) and (69), along with the polynomial condition (40) on g. $\square$

From now on, by virtue of Lemma 18, we only have to prove $n^{1/2}(\widehat{\Xi}' - \widetilde{\Xi}') \to^{\mathbb{P}} 0$. We now want to show that in the definition of $\widehat{\Xi}'$, we can substitute $\widetilde{c}'_i$ by $\overline{c}'_i$, where

$$\overline{c}'^{lm}_i = \frac{1}{k_n \Delta_n} \sum_{j=0}^{k_n - 1} \Delta_{i+j} \widehat{X}'^l \Delta_{i+j} \widehat{X}'^m \mathbf{1}_{\{|\Delta_{i+j} X'| \leq w_n\}}, \tag{70}$$

that is when the indicator function is applied to X' itself instead of $\widehat{X}'$. We first state a technical lemma.

Lemma 19. *We have, for any $i \in \{1, \cdots, n\}$, any $j \in \{1, \cdots, 3\}$, and any $q \geq 1$,*

$$\mathbb{E}|\partial^j g(\widetilde{c}'_i)|^q \leq C \text{ and } \mathbb{E}|\partial^j g(\overline{c}'_i)|^q \leq C.$$

Proof. In view of (40), it is sufficient to prove that for any $q \geq 1$,

$$\mathbb{E}|\widehat{c}_i'|^q \leq C \text{ and } \mathbb{E}|\overline{c}_i'|^q \leq C.$$

Moreover, since $|\widehat{c}_i'|^q \leq C(|\widehat{c}_i' - \overline{c}_i'|^q + |\overline{c}_i' - \widetilde{c}_i|^q + |\widetilde{c}_i|^q)$, and as $\mathbb{E}|\widetilde{c}_i|^q \leq C$ as an easy consequence of (4.11) in Jacod and Rosenbaum (2013) (p. 1476) and the boundedness of c in Assumption **(H)**, it suffices to show the $\mathbb{L}_q$ boundedness of

$$\widehat{c}_i' - \overline{c}_i' = \frac{1}{k_n \Delta_n} \sum_{j=0}^{k_n - 1} |\Delta_{i+j} \widehat{X}'|^2 \left(\mathbf{1}_{\{|\Delta_{i+j} \widehat{X}'| \leq w_n\}} - \mathbf{1}_{\{|\Delta_{i+j} X'| \leq w_n\}} \right) \tag{71}$$

and

$$\begin{aligned} \overline{c}_i' - \widetilde{c}_i \leq\ & \frac{2}{k_n \Delta_n} \sum_{j=0}^{k_n-1} \Delta_{i+j} X' \psi_{i+j}(\widehat{\theta}) \mathbf{1}_{\{|\Delta_{i+j} X'| \leq w_n\}} + \frac{1}{k_n \Delta_n} \sum_{j=0}^{k_n-1} \psi_{i+j}(\widehat{\theta})^2, \\ :=\ & I + II. \end{aligned} \tag{72}$$

We first show the $\mathbb{L}_q$ boundedness of (71). First recall that in (55) we proved that

$$\mathbb{E} \left| \mathbf{1}_{\{|\Delta_i \widehat{X}'| \leq w_n\}} - \mathbf{1}_{\{|\Delta_i X'| \leq w_n\}} \right| \leq n^{-\beta}$$

for any $\beta > 0$. Thus, by Cauchy-Schwarz inequality and Jensen's inequality we easily get that $\mathbb{E}|\widehat{c}_i' - \overline{c}_i'|^q \leq C$ considering β large enough.

We prove now the $\mathbb{L}_q$ boundedness of (72). By Jensen's inequality applied to

$$|k_n^{-1} \sum_{j=0}^{k_n-1} \Delta_{i+j} X' \psi_{i+j}(\widehat{\theta})|^q,$$

we have

$$\begin{aligned} \mathbb{E}|I|^q &\leq \frac{Cn^q}{k_n} \sum_{j=0}^{k_n-1} \mathbb{E}|\Delta_{i+j} X'|^q \underbrace{|\psi_{i+j}(\widehat{\theta})|^q}_{C/n^q} \\ &\leq Cn^{-q/2}. \end{aligned}$$

For II we have

$$\begin{aligned} \mathbb{E}|II|^q &\leq \frac{Cn^q}{k_n} \mathbb{E} \sum_{j=0}^{k_n-1} |\psi_{i+j}(\widehat{\theta})|^{2q} \\ &\leq Cn^{-q}, \end{aligned}$$

and thus this yields the $\mathbb{L}_q$ boundedness of $\overline{c}_i' - \widetilde{c}_i$, which concludes the proof. □

Lemma 20. *Let $\overline{\Xi}'$ be defined as $\widehat{\Xi}'$ where $\widehat{c}_i'$ is replaced by $\overline{c}_i'$. Then*

$$n^{1/2} \left(\widehat{\Xi}' - \overline{\Xi}' \right) \to^{\mathbb{P}} 0.$$

Proof. We have

$$n^{1/2}\left(\widehat{\Xi}' - \overline{\Xi}'\right) = n^{1/2}\Delta_n \sum_{i=1}^{[T/\Delta_n]-k_n+1} \left\{g(\widehat{c}_i') - g(\overline{c}_i')\right\} + \frac{n^{1/2}\Delta_n}{2k_n}\sum_{i=1}^{[T/\Delta_n]-k_n+1}\left\{h(\overline{c}_i') - h(\widehat{c}_i')\right\}, \tag{73}$$

with $h(x) = 2\partial^2 g(x)x^2$, so that proving our claim boils down to showing that both terms in the right-hand side of (73) are negligible. For the first one, we have

$$\sum_{i=1}^{[T/\Delta_n]-k_n+1} \left|g(\widehat{c}_i') - g(\overline{c}_i')\right| \le \frac{1}{k_n\Delta_n}\sum_{i=1}^{[T/\Delta_n]-k_n+1}\sum_{j=0}^{k_n-1} |\partial g(a_{i,j})|\,|\Delta_{i+j}\widehat{X}'|^2 \left|\mathbf{1}_{\{|\Delta_{i+j}\widehat{X}'|\le w_n\}} - \mathbf{1}_{\{|\Delta_{i+j}X'|\le w_n\}}\right|$$

for some (random) $a_{i,j}$ such that $|a_{i,j}| \le |\widehat{c}_i'| + |\overline{c}_i'|$ by the mean value theorem. Now, by Lemma 19 and the fact that g is of polynomial growth we get $\mathbb{E}|\partial g(a_{i,j})|^q \le C$ for any $q \ge 1$, and thus by Cauchy-Schwarz inequality we will have

$$n^{1/2}\Delta_n\sum_{i=1}^{[T/\Delta_n]-k_n+1}\left\{g(\widehat{c}_i') - g(\overline{c}_i')\right\} \to^{\mathbb{P}} 0$$

if we can prove that

$$\sum_{i=1}^{[T/\Delta_n]-k_n+1}\sum_{j=0}^{k_n-1}\left(\mathbb{E}\left[|\Delta_{i+j}\widehat{X}'|^4\left|\mathbf{1}_{\{|\Delta_{i+j}\widehat{X}|\le w_n\}} - \mathbf{1}_{\{|\Delta_{i+j}X'|\le w_n\}}\right|\right]\right)^{1/2} = o(k_n n^{-1/2}),$$

i.e. that

$$\sum_{i=1}^{[T/\Delta_n]-k_n+1}\left(\mathbb{E}\left[|\Delta_i\widehat{X}'|^4\left|\mathbf{1}_{\{|\Delta_i\widehat{X}'|\le w_n\}} - \mathbf{1}_{\{|\Delta_iX'|\le w_n\}}\right|\right]\right)^{1/2} = o(n^{-1/2}).$$

Recalling $|\Delta_i\widehat{X}'|^4 \le C(|\Delta_iX'|^4 + |\psi_i(\widehat{\theta})|^4)$, we have that

$$\sum_{i=1}^{[T/\Delta_n]-k_n+1}\left(\mathbb{E}\left[|\Delta_iX'|^4\left|\mathbf{1}_{\{|\Delta_i\widehat{X}'|\le w_n\}} - \mathbf{1}_{\{|\Delta_iX'|\le w_n\}}\right|\right]\right)^{1/2} = O(n^{-\beta/4}) = o(n^{-1/2})$$

since β can be taken arbitrary big, using again Cauchy-Schwarz inequality along with the fact that $\mathbb{E}|\Delta_iX'|^q \le Cn^{-q/2}$, and (55). Finally, it is immediate to prove

$$\sum_{i=1}^{[T/\Delta_n]-k_n+1}\left(\mathbb{E}\left[|\psi_i(\widehat{\theta})|^4\left|\mathbf{1}_{\{|\Delta_i\widehat{X}'|\le w_n\}} - \mathbf{1}_{\{|\Delta_iX'|\le w_n\}}\right|\right]\right)^{1/2} = o\left(n^{-1/2}\right),$$

given that $|\psi_i(\widehat{\theta})|^4 \le K/n^4$. The second term on the right-hand side of (73) is proved in the same way. $\square$

In the 1-dimensional setting, we now introduce the following notation for $\theta \in \Theta$:

$$c_i'(\theta) = \frac{1}{k_n \Delta_n} \sum_{j=0}^{k_n-1} |\Delta_{i+j} X'(\theta)|^2 1_{\{|\Delta_{i+j} X'| \le w_n\}},$$

where we recall that for any $i \in \{1, \cdots, n\}$, $\Delta_i X'(\theta) = \Delta_i X' + \psi_i(\theta)$. Note that $\overline{c}_i' = c_i'(\widehat{\theta})$, and $\widetilde{c}_i = c_i'(\theta_0)$. We define

$$E_n := n^{1/2} \Delta_n \sum_{i=1}^{[T/\Delta_n]-k_n+1} \left\{ g(\overline{c}_i') - g(\widetilde{c}_i) \right\}.$$

By the mean value theorem along with the chain rule we have for some $\overline{\theta} \in [\theta_0, \widehat{\theta}]$,

$$\begin{aligned} E_n &= \frac{2n^{1/2}}{k_n} (\widehat{\theta} - \theta_0) \sum_{i=1}^{[T/\Delta_n]-k_n+1} \partial g(\widetilde{c}_i) \sum_{j=0}^{k_n-1} \Delta_{i+j} X' \partial_\theta \psi_{i+j}(\theta_0) 1_{\{|\Delta_{i+j} X'| \le w_n\}} \\ &+ \frac{n^{1/2}}{k_n} (\widehat{\theta} - \theta_0)^2 \sum_{i=1}^{[T/\Delta_n]-k_n+1} \partial g(c_i'(\overline{\theta})) \sum_{j=0}^{k_n-1} \Delta_{i+j} X'(\overline{\theta}) \partial_\theta^2 \psi_{i+j}(\overline{\theta}) 1_{\{|\Delta_{i+j} X'| \le w_n\}} \\ &+ \frac{n^{1/2}}{k_n} (\widehat{\theta} - \theta_0)^2 \sum_{i=1}^{[T/\Delta_n]-k_n+1} \partial g(c_i'(\overline{\theta})) \sum_{j=0}^{k_n-1} \partial_\theta \psi_{i+j}(\overline{\theta})^2 1_{\{|\Delta_{i+j} X'| \le w_n\}} \\ &+ \frac{2n^{1/2}}{k_n^2 \Delta_n} (\widehat{\theta} - \theta_0)^2 \sum_{i=1}^{[T/\Delta_n]-k_n+1} \partial^2 g(c_i'(\overline{\theta})) \left\{ \sum_{j=0}^{k_n-1} \Delta_{i+j} X'(\overline{\theta}) \partial_\theta \psi_{i+j}(\overline{\theta}) 1_{\{|\Delta_{i+j} X'| \le w_n\}} \right\}^2, \\ &:= I + II + III + IV. \end{aligned}$$

We now show that each term is $o_{\mathbb{P}}(1)$.

Lemma 21. *We have*

$$I = \frac{2n^{1/2}}{k_n} (\widehat{\theta} - \theta_0) \sum_{i=1}^{[T/\Delta_n]-k_n+1} \partial g(\widetilde{c}_i) \sum_{j=0}^{k_n-1} \Delta_{i+j} X' \partial_\theta \psi_{i+j}(\theta_0) 1_{\{|\Delta_{i+j} X'| \le w_n\}} \to^{\mathbb{P}} 0.$$

Proof. Since Assumption **(H)** yields $\frac{2n^{1/2}}{k_n}(\widehat{\theta} - \theta_0) = O_{\mathbb{P}}(k_n^{-1} n^{-1/2})$, it suffices to prove that

$$\sum_{i=1}^{[T/\Delta_n]-k_n+1} \partial g(\widetilde{c}_i) \sum_{j=0}^{k_n-1} \Delta_{i+j} X' \partial_\theta \psi_{i+j}(\theta_0) 1_{\{|\Delta_{i+j} X| \le w_n\}} = o_{\mathbb{P}}(k_n n^{1/2}). \tag{74}$$

Recalling the decomposition $\Delta_{i+j} X' = \Delta_{i+j} \breve{B} + \Delta_{i+j} M^c$, we first show that the above term is negligible when $\Delta_{i+j} X'$ is replaced by $\Delta_{i+j} M^c$. In that case, by virtue of the domination $1_{\{|\Delta_{i+j} M^c| \ge w_n\}} \le w_n^{-1} |\Delta_{i+j} M^c|$, Burkhölder-Davis-Gundy inequality, Hölder's inequality, along with the fact that $|\partial g(\widetilde{c}_i)|$ is $\mathbb{L}_q$ bounded by Lemma 19, the indicator function can be removed without loss of generality. Thus, introducing

$$A_n = \sum_{i=1}^{[T/\Delta_n]-k_n+1} \partial g(\widetilde{c}_i) \sum_{j=0}^{k_n-1} \Delta_{i+j} M^c \partial_\theta \psi_{i+j}(\theta_0),$$

and

$$B_n = \sum_{i=1}^{[T/\Delta_n]-k_n+1} \partial g(c_{t_i}) \sum_{j=0}^{k_n-1} \Delta_{i+j} M^c \partial_\theta \psi_{i+j}(\theta_0),$$

we show that $A_n - B_n = o_{\mathbb{P}}(k_n n^{1/2})$ and $B_n = o_{\mathbb{P}}(k_n n^{1/2})$ separately. We have for some $\xi_i \in [\widetilde{c}_i, c_{t_i}]$,

$$|A_n - B_n| \leq \sum_{i=1}^{[T/\Delta_n]-k_n+1} \left|\partial^2 g(\xi_i)\right| |\widetilde{c}_i - c_{t_i}| \sum_{j=0}^{k_n-1} |\Delta_{i+j} M^c||\partial_\theta \psi_{i+j}(\theta_0)|.$$

Moreover, by (4.11) in Jacod and Rosenbaum (2013) (p. 1476), we have the estimate

$$\mathbb{E}\left[|\widetilde{c}_i - c_{t_i}|^2\right] \leq C\left(k_n^{-1} + k_n \Delta_n\right). \tag{75}$$

Thus, by application of Hölder's inequality, the fact that $\partial^2 g(\xi_i)$ is $\mathbb{L}_q$ bounded by Lemma 19, and that for any $q \geq 1$:

$$\begin{aligned}
\mathbb{E}\left[|\Delta_{i+j} M^c|^q |\partial_\theta \psi_{i+j}(\theta_0)|^q\right] &\leq C\mathbb{E}[|\Delta_{i+j} M^c|^q] \\
&\leq C n^{-q/2},
\end{aligned}$$

we deduce that

$$\mathbb{E}|A_n - B_n| \leq C k_n n^{1/2} \left(k_n^{-1} + k_n \Delta_n\right)^{1/2} = o_{\mathbb{P}}(k_n n^{1/2}).$$

As for B_n, we note that it can be expressed as a sum of martingale increments with respect to the filtration $\mathcal{H}_t = \mathcal{F}_t \vee \sigma\{Q_{t_i}, i = 0, \cdots, n\}$, and we have $B_n = \sum_{i=1}^{[T/\Delta_n]} \chi_i$ with

$$\chi_i = \sum_{l=(i-k_n+1)\wedge 1}^{i} \partial g(\sigma_{t_l}^2) \partial_\theta \psi_i(\theta_0) \Delta_i M^c.$$

Thus, by property (2.2.35) p. 56 in Jacod and Protter (2011), proving that $B_n = o_{\mathbb{P}}(k_n n^{1/2})$ boils down to showing that

$$\widetilde{B}_n := n^{-1} k_n^{-2} \sum_{i=1}^{[T/\Delta_n]} \mathbb{E}\chi_i^2 \to 0. \tag{76}$$

Now, using the boundedness of c, we have

$$\begin{aligned}
\mathbb{E}\chi_i^2 &\leq C k_n^2 \mathbb{E} \partial_\theta \psi_i(\theta_0)^2 \left(\Delta_i M^c\right)^2 \\
&\leq C k_n^2 n^{-1}.
\end{aligned}$$

Therefore $\widetilde{B}_n = O_{\mathbb{P}}(n^{-1})$ which proves (76) and thus (74) when replacing $\Delta_{i+j} X'$ by $\Delta_{i+j} M^c$. Finally, the case where we consider the drift term $\Delta_{i+j} \breve{B}$ in lieu of $\Delta_{i+j} X'$ follows immediately from the fact that $\mathbb{E}|\Delta_{i+j} \breve{B}|^k \leq C n^{-k}$ for any $k \geq 1$. □

Lemma 22. *We have that $II = o_{\mathbb{P}}(1)$, $III = o_{\mathbb{P}}(1)$, $IV = o_{\mathbb{P}}(1)$.*

Proof. Proving the first claim is equivalent to showing that

$$\widetilde{II} := \sum_{i=1}^{[T/\Delta_n]-k_n+1} \partial g(c_i'(\overline{\theta})) \sum_{j=0}^{k_n-1} \Delta_{i+j} X'(\overline{\theta}) \partial_\theta^2 \psi_{i+j}(\overline{\theta}) 1_{\{|\Delta_{i+j}X'| \leq w_n\}} = o_{\mathbb{P}}(k_n n^{3/2}).$$

Note, again, that by Assumption **(H)** and the fact that $\overline{\theta}$ belongs to a compact set, we have $|\partial_\theta^2 \psi_{i+j}(\overline{\theta})| \leq C$. Thus

$$\begin{aligned}
\mathbb{E}\left|\widetilde{II}\right| &\leq C \sum_{i=1}^{[T/\Delta_n]-k_n+1} \mathbb{E}\left[|\partial g(c_i'(\overline{\theta}))| \sum_{j=0}^{k_n-1} |\Delta_{i+j} X'(\overline{\theta})| \right] \\
&\leq C \sum_{i=1}^{[T/\Delta_n]-k_n+1} \sum_{j=0}^{k_n-1} \left(\mathbb{E}\partial g(c_i'(\overline{\theta}))^2\right)^{1/2} \left(\mathbb{E}|\Delta_{i+j} X'(\overline{\theta})|^2\right)^{1/2} \\
&\leq C k_n n^{1/2} = o_{\mathbb{P}}(k_n n^{3/2}),
\end{aligned}$$

where we have used Lemma 19, and the fact that for any $q \geq 1$,

$$\mathbb{E}|\Delta_{i+j} X'(\overline{\theta})|^q \leq C \left(\mathbb{E}|\Delta_{i+j} X'|^q + \mathbb{E}\left[\underbrace{(\overline{\theta} - \theta_0)^q}_{\leq K/n^q} \underbrace{\sup_{\theta \in \Theta} |\partial_\theta \psi_i(\theta)|^q}_{\leq K} \right] \right) \leq C \left(n^{-q/2} + n^{-q} \right). \tag{77}$$

For the second claim, we have (bounding the indicator function from above by 1) the estimate

$$\begin{aligned}
\widetilde{III} &\leq \sum_{i=1}^{[T/\Delta_n]-k_n+1} \partial g(c_i'(\overline{\theta})) \sum_{j=0}^{k_n-1} \partial_\theta \psi_{i+j}(\overline{\theta})^2 \\
&\leq C k_n \underbrace{\sum_{i=1}^{[T/\Delta_n]-k_n+1} |\partial g(c_i'(\overline{\theta}))|}_{O_{\mathbb{P}}(n)} \\
&= O_{\mathbb{P}}(k_n n) = o_{\mathbb{P}}(k_n n^{3/2}),
\end{aligned}$$

so that $III = o_{\mathbb{P}}(1)$. Finally we show that $IV = o_{\mathbb{P}}(1)$, that is

$$\widetilde{IV} := \sum_{i=1}^{[T/\Delta_n]-k_n+1} \partial^2 g(c_i'(\overline{\theta})) \left\{ \sum_{j=0}^{k_n-1} \Delta_{i+j} X'(\overline{\theta}) \partial_\theta \psi_{i+j}(\overline{\theta}) 1_{\{|\Delta_{i+j}X'| \leq w_n\}} \right\}^2 = o_{\mathbb{P}}(k_n^2 n^{1/2}). \tag{78}$$

By Cauchy-Schwarz inequality and the fact that $|\partial_\theta \psi_{i+j}(\overline{\theta})|^2 \leq C$, we get the domination

$$\begin{aligned}
\mathbb{E}|\widetilde{IV}| &\leq C k_n \mathbb{E}\left[\sum_{i=1}^{[T/\Delta_n]-k_n+1} |\partial^2 g(c_i'(\overline{\theta}))| \sum_{j=0}^{k_n-1} |\Delta_{i+j} X'(\overline{\theta})|^2 \right] \\
&\leq C k_n \sum_{i=1}^{[T/\Delta_n]-k_n+1} \sum_{j=0}^{k_n-1} \underbrace{\left(\mathbb{E}\partial^2 g(c_i'(\overline{\theta}))^2\right)^{1/2}}_{\leq C} \underbrace{\left(\mathbb{E}|\Delta_{i+j} X'(\overline{\theta})|^4\right)^{1/2}}_{O(n^{-1})} \\
&\leq C k_n^2 = o(k_n^2 n^{1/2}),
\end{aligned}$$

where we have used (77) with $q = 4$, and we are done. □

Similarly we have by the mean value theorem that

$$\frac{n^{1/2}\Delta_n}{k_n}\sum_{i=1}^{[T/\Delta_n]-k_n+1}\left\{h(\overline{c}_i')-h(\widetilde{c}_i)\right\}$$

is equal to

$$\frac{2n^{1/2}}{k_n^2}(\widehat{\theta}-\theta_0)\sum_{i=1}^{[T/\Delta_n]-k_n+1}\partial h(c_i'(\overline{\theta}))\sum_{j=0}^{k_n-1}\Delta_{i+j}X'(\overline{\theta})\partial_\theta\psi_{i+j}(\overline{\theta})1_{\{|\Delta_{i+j}X'|\leq w_n\}}.$$

Lemma 23. *We have*

$$\frac{n^{1/2}\Delta_n}{k_n}\sum_{i=1}^{[T/\Delta_n]-k_n+1}\left\{h(\overline{c}_i')-h(\widetilde{c}_i)\right\}\to^{\mathbb{P}}0.$$

Proof. By Assumption **(H)** we have

$$\mathbb{E}\left|\frac{n^{1/2}\Delta_n}{k_n}\sum_{i=1}^{[T/\Delta_n]-k_n+1}\left\{h(\overline{c}_i')-h(\widetilde{c}_i)\right\}\right|\leq\frac{C}{n^{1/2}k_n^2}\sum_{i=1}^{[T/\Delta_n]-k_n+1}\sum_{j=0}^{k_n-1}\mathbb{E}\left[|\partial h(c_i'(\overline{\theta}))||\Delta_{i+j}X'(\overline{\theta})|\right].$$

Since ∂h is also of polynomial growth, we deduce as for Lemma 19 that for any $q\geq 1$, $\mathbb{E}|\partial h(c_i'(\overline{\theta}))|^q\leq C$, and so an application of Cauchy-Schwarz inequality yields

$$\mathbb{E}\left|\frac{n^{1/2}\Delta_n}{k_n}\sum_{i=1}^{[T/\Delta_n]-k_n+1}\left\{h(\overline{c}_i)-h(\widetilde{c}_i)\right\}\right|\leq C/k_n\to 0.$$

□

We prove now the theorem.

Proof of Theorem 11. Recall that by Lemma 18 we only need to prove that $n^{1/2}(\widehat{\Xi}'-\widetilde{\Xi}')\to^{\mathbb{P}}0$. We have

$$n^{1/2}\left(\widehat{\Xi}'-\widetilde{\Xi}'\right)=n^{1/2}\left(\widehat{\Xi}'-\overline{\Xi}'\right)+n^{1/2}\left(\overline{\Xi}'-\widetilde{\Xi}'\right).$$

The first term above is negligible by virtue of Lemma 20. Moreover, since

$$\begin{aligned}n^{1/2}\left(\overline{\Xi}'-\widetilde{\Xi}'\right)&=n^{1/2}\Delta_n\sum_{i=1}^{[T/\Delta_n]-k_n+1}\left\{g(\overline{c}_i')-g(\widetilde{c}_i)\right\}\\&\quad+\frac{n^{1/2}\Delta_n}{2k_n}\sum_{i=1}^{[T/\Delta_n]-k_n+1}\left\{h(\widetilde{c}_i)-h(\overline{c}_i')\right\},\end{aligned}$$

the assertion $n^{1/2}\left(\overline{\Xi}'-\widetilde{\Xi}'\right)\to^{\mathbb{P}}0$ is an immediate consequence of Lemma 21, Lemma 22 and Lemma 23. Combined with Theorem 3.2 (p. 1469, applied to X') in Jacod and Rosenbaum (2013), this yields the central limit theorem. □

6.9 Proof of Corollary 12

By Slutsky's Lemma, all we need to prove is that $\widehat{AVAR} \to^{\mathbb{P}} AVAR$. Given the form of $\widehat{AVAR}$, this can be shown using exactly the same line of reasoning as for the general theorem replacing g by $\overline{h}$ in all our estimates and combining the results with Corollary 3.7 in Jacod and Rosenbaum (2013) in lieu of Theorem 3.2, except that there is no scaling by $n^{1/2}$ in front of the estimates and no bias term. Since the C^3 property of g is only used once when handling the bias term in Lemma 23, the fact that $\overline{h}$ is only of class C^2 is not problematic.

6.10 Proof of Theorem 14 and Corollary 15

In Vetter (2015), the author introduces

$$A_i = \frac{2n}{k_n} \sum_{j=1}^{k_n} \int_{(i+j-1)T/n}^{(i+j)T/n} (X_s - X_{(i+j-1)T/n}) dX_s$$

and

$$B_i := \frac{n}{k_n} \int_{iT/n}^{(i+k_n)T/n} \sigma_s^2 ds.$$

Accordingly, we define

$$\widehat{A}_i := \frac{2n}{k_n} \sum_{j=1}^{k_n} \left\{ \int_{(i+j-1)T/n}^{(i+j)T/n} (X_s - X_{(i+j-1)T/n}) dX_s + \psi_{i+j}(\widehat{\theta}) \Delta_{i+j} X \right\},$$
$$\widehat{B}_i := \frac{n}{k_n} \left\{ \int_{iT/n}^{(i+k_n)T/n} \sigma_s^2 ds + \sum_{j=1}^{k_n} \psi_{i+j}(\widehat{\theta})^2 \right\},$$

along with the approximated increments for some arbitrary $p \geq 1$ and $1 \leq l \leq J(p) := [[nt/T - 2k_n]/((p+2)k_n)]$, where $[x]$ is defined as the floor function of x,

$$\widetilde{A}_{i+k_n} - \widetilde{A}_i := \frac{n}{k_n} \sigma_{a_l(p)T/n} \sum_{j=1}^{k_n} \left(\Delta_{i+k_n+j} W^2 - \Delta_{i+j} W^2 \right),$$

and

$$\widetilde{B}_{i+k_n} - \widetilde{B}_i := \frac{n}{k_n} \int_{iT/n}^{(i+k_n)T/n} \widetilde{\sigma}_{a_l(p)T/n} (W'_{(s+k_nT/n)} - W'_s) ds,$$

where $a_l(p) := (l-1)(p+2)k_n$. Note that $\widehat{c}_i = \widehat{A}_i + \widehat{B}_i$, and therefore $\widehat{\Xi}$ can be linked to the above quantities as follows:

$$\widehat{\Xi} = \sum_{i=0}^{[T/\Delta_n]-2k_n} \left\{ \frac{3}{2k_n} (\widehat{A}_{i+k_n} - \widehat{A}_i + \widehat{B}_{i+k_n} - \widehat{B}_i)^2 - \frac{6}{k_n^2} \widehat{q}_i \right\}. \tag{79}$$

Remark also that the approximated increments are independent of the information process and of $\widehat{\theta}$. Now note that the general proof in Vetter (2015) is conducted in the following two steps.

- Compute an estimate for the deviations $A_{i+k_n} - A_i - (\widetilde{A}_{i+k_n} - \widetilde{A}_i)$, $B_{i+k_n} - B_i - (\widetilde{B}_{i+k_n} - \widetilde{B}_i)$, and $\widetilde{q}_i - \int_{t_i}^{t_{i+1}} \sigma_s^4 ds$.
- Systematically use the previous estimate to replace A_i (resp. B_i, $\widetilde{q}_i$) by its counterpart $\widetilde{A}_i$ (resp. $\widetilde{B}_i$, $\int_{t_i}^{t_{i+1}} \sigma_s^4 ds$) in all the encountered expressions.

Since $\widetilde{A}_i$, $\widetilde{B}_i$ and $\int_{t_i}^{t_{i+1}} \sigma_s^4 ds$ are independent of the information process and $\widehat{\theta}$, the second step holds in our setting as well with no modification in the proofs of Vetter (2015). Thus, all we have to do in order to prove the theorem is to adapt the first step replacing A_i, B_i and $\widetilde{q}_i$ by $\widehat{A}_i$, $\widehat{B}_i$ and $\widehat{q}_i$. More precisely, we adapt Lemma A.1 and the second equation in the proof of (A.8) p. 2411 (corresponding to the approximation of $\widetilde{q}_i$ by $\int_{t_i}^{t_{i+1}} \sigma_s^4 ds$) in Vetter (2015) as follows (in the next lemma, recall that $\widetilde{A}_i$ and $\widetilde{B}_i$ depend on some parameter $p \geq 1$).

Lemma 24. *We have for any $r \geq 1$, $p \geq 1$, and any $i \in \{a_l(p), \cdots, a_l(p) + pk_n\}$*

$$\mathbb{E}\left[\left|\widehat{A}_{i+k_n} - \widehat{A}_i - (\widetilde{A}_{i+k_n} - \widetilde{A}_i)\right|^r\right] \leq C(pn^{-1})^{r/2},$$

$$\mathbb{E}\left[\left|\widehat{B}_{i+k_n} - \widehat{B}_i - (\widetilde{B}_{i+k_n} - \widetilde{B}_i)\right|^r\right] \leq C(pn^{-1})^{r/2},$$

$$\mathbb{E}\left[\left|\widehat{A}_{i+k_n} - \widehat{A}_i\right|^r\right] \leq Cn^{-r/2},$$

and

$$\mathbb{E}\left[\left|\widehat{B}_{i+k_n} - \widehat{B}_i\right|^r\right] \leq Cn^{-r/2}.$$

Moreover we have uniformly in $t \in [0, T]$

$$\sqrt{\frac{n}{k_n}}\mathbb{E}\left|\sum_{i=1}^{[t/\Delta_n]-2k_n} \frac{6}{k_n^2}\widehat{q}_i - \frac{6}{k_n^2}\int_0^t \sigma_s^4 ds\right| = o(1).$$

Proof. By Lemma A.1 in Vetter (2015), it suffices to prove that we have

$$\mathbb{E}\left[\left|\widehat{A}_{i+k_n} - \widehat{A}_i - (A_{i+k_n} - A_i)\right|^r\right] \leq C(pn^{-1})^{r/2},$$

and a similar statement for $\widehat{B}_i$. Since $|\psi_k(\widehat{\theta})| \leq K/n$ for all $1 \leq k \leq n$, we obtain

$$\begin{aligned}
\mathbb{E}\left[\left|\widehat{A}_{i+k_n} - \widehat{A}_i - (A_{i+k_n} - A_i)\right|^r\right] &\leq \frac{2^r n^r}{k_n^r}\mathbb{E}\left[\left|\sum_{j=1}^{k_n}\left\{\psi_{i+k_n+j}(\widehat{\theta})\Delta_{i+k_n+j}X - \psi_{i+j}(\widehat{\theta})\Delta_{i+j}X\right\}\right|^r\right], \\
&\leq \frac{Cn^r}{k_n}\sum_{j=1}^{k_n}\mathbb{E}\left[\left|\psi_{i+k_n+j}(\widehat{\theta})\Delta_{i+k_n+j}X\right|^r + \left|\psi_{i+j}(\widehat{\theta})\Delta_{i+j}X\right|^r\right], \\
&\leq \frac{C}{k_n}\sum_{j=1}^{k_n}\underbrace{\mathbb{E}\left[|\Delta_{i+k_n+j}X|^r + |\Delta_{i+j}X|^r\right]}_{\leq Cn^{-r/2}}, \\
&\leq Cp^{r/2}n^{-r/2},
\end{aligned}$$

since $p \geq 1$, where we used Jensen's inequality at the second step and the domination $|\psi_i(\widehat{\theta})|^r \leq C/n^r$ at the third step. Proving the other three inequalities and the approximation for $\widehat{q}_i$ can be done by similar calculation. $\square$

Now, to prove Theorem 14, it is sufficient to follow closely the proof of Theorem 2.6 in Vetter (2015) replacing all occurrences of A_i, B_i and $\sum_{i=1}^{[t/\Delta_n]-2k_n} \frac{6}{k_n^2} \widehat{q}_i$ by $\widehat{A}_i$, $\widehat{B}_i$ and $\frac{6}{k_n^2} \int_0^t \sigma_s^4 ds$, and accordingly all applications of Lemma A.1 and the approximation for $\widehat{q}_i$ by Lemma 24 above.

A similar line of reasoning yields Corollary 15.

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