Supplement to “Hypothesis tests for high-dimensional covariance structures”

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Abstract In this supplement, we give details of the asymptotic normality in Lemma 5, an additional test procedure, an R-code to calculate $y_{ij(l)}$, proofs of the theoretical results in the main work in Ishii et al. (2020) together with additional propositions and proofs of the propositions. The equation numbers and the mathematical symbols used in the supplement are the same as those which are made reference to in the main document.

Appendix A: Details of the asymptotic normality in Lemma 5

In Lemma 5, we established the asymptotic normality under the SSE model. However, in general, high-dimensional statistics do not hold the asymptotic normality under the SSE model. See Aoshima and Yata (2018, 2019) and Ishii et al. (2019) for the details. We emphasize that the asymptotic normality given in Lemma 5 is a rare case for the SSE model. In this section, we explain the details of the asymptotic normality.

We assume $r_1 = 1$ in (7). Then, we write that

$$\Delta = 2\|\Omega_1\|_F^2 + \text{tr}(\Omega_2^2) - \text{tr}(\Sigma_{x_{1(1)}}^2),$$

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where $\Sigma_{s(1)} = \sum_{s=2}^{q} \kappa_s A_s$. From (11) and (21), it follows that

$$
\Delta_n + B_n(2) = 2 \sum_{i<j} \sum_{s=2}^{q} \frac{\text{tr}(Y_{ij(1),1} Y_{ij(2),s}) + \text{tr}(Y_{ij(1),s} Y_{ij(2),1})}{n(n-1)}
+ 2 \sum_{i<j} \sum_{s,s'=2}^{q} \frac{\text{tr}(Y_{ij(1),s} Y_{ij(2),s'})}{n(n-1)} (= V_1 + V_2, \text{ say}). \quad (A.1)
$$

Thus, one can ignore the following term:

$$
2 \sum_{i<j}^{n} \frac{\text{tr}(Y_{ij(1),1} Y_{ij(2),1})}{n(n-1)}.
$$

This is the key to prove the asymptotic normality of $\Delta_n$ since its variance is huge under (C-v) because $\text{Var}\{2 \sum_{i<j} \text{tr}(Y_{ij(1),1} Y_{ij(2),1})/n^2\} = O(n^3/n^2)$ under (A-i). For $V_1$, by noting that $A_1 A_s = O$ for $s > 1$, we have that

$$
V_1 = 2 \sum_{i<j}^{n} \sum_{s=2}^{q} \frac{\text{tr}(Y_{ij(1),1} Y_{ij(2),s}) + \text{tr}(Y_{ij(1),s} Y_{ij(2),1})}{n(n-1)}
= 4 \sum_{i<j}^{n} \frac{y_{ij(1)}^T A_1 y_{ij(2)} y_{ij(1)}^T A_1 y_{ij(2)}}{n(n-1)}.
$$

Note that $E(V_1) = 2 \| \Omega \|_F^2$. Now, we give asymptotic properties of $V_1$. Let $y_{ij(1),1} = a_i^T y_{ij(l)}$ and $y_{ij(2),2} = (a_2, \ldots, a_p)^T y_{ij(l)}$ for all $i, j, l$, where $A_1 = a_1 a_1^T$ and $a_2, \ldots, a_p$ are eigenvectors of $A_{(1)}$ such that $A_{(1)} = \sum_{j=2}^{p} a_j a_j^T$. Then, it follows that

$$
V_1 = 4 \sum_{i<j}^{n} \frac{y_{ij(1),1} y_{ij(2),1} y_{ij(1),2} y_{ij(2),2}}{n(n-1)}.
$$

Note that

$$
E(y_{ij(l),1} y_{ij(l),2}) = \text{Cov}(a_i^T x_j, (a_2, \ldots, a_p)^T x_j) = (a_2, \ldots, a_p)^T \Sigma a_i
$$

for $l = 1, 2; i < j$, and $\| (a_2, \ldots, a_p)^T \Sigma a_i \| = \| \Omega_i \|_F$. Thus, $V$ is an unbiased estimator of the squared norm of the covariance vector between $a_i^T x_j$ and $(a_2, \ldots, a_p)^T x_j$. Yata and Aoshima (2013, 2016) gave asymptotic properties of the ECDM estimator for such a covariance vector. Let

$$
L_V = 8 \kappa_1^2 \text{tr}(\Omega_2^2)/n^2.
$$

From Theorem 3.2 in Yata and Aoshima (2016), we have the following result.

**Proposition A1**  Assume (A-ii) and (C-i'). Assume also

$$(C-iii') \limsup_{m \to \infty} \frac{\| \Omega_m \|_F^2}{L_V} < \infty.$$
Then, it holds that as $m \to \infty$

$$\frac{V_1 - 2\|\Omega_1\|_F^2}{L_{V_1}^{1/2}} \Rightarrow N(0, 1).$$

Next, we consider the variance of $V_2$ in (A.1). Note that

$$E(V_2) = \text{tr}(\Omega_2^2) - \text{tr}(\Sigma_{*}^2) \leq \Delta.$$

We write that

$$\sum_{s,s'=2}^q \frac{\text{tr}(Y_{ij(s),s} Y_{ij(s'),s'})}{n(n-1)} = 2 \sum_{i<j} \frac{\text{tr}\{A(1) (y_{ij(1)} y_{ij(1)}^T - \Sigma_*) A(1) (y_{ij(2)} y_{ij(2)}^T - \Sigma_*)\}}{n(n-1)}.$$

Then, from Lemma 1, we have the following result.

**Proposition A2** Assume (A-i). It holds that as $m \to \infty$

$$\text{Var}(V_2) = O(\text{tr}(\Omega_2^2)^2/n^2) + O(\Delta \text{tr}(\Omega_2^2)/n).$$

From Proposition A2, under (A-i), (C-iii') and (C-v), we have that as $m \to \infty$

$$\text{Var}(V_2) = o(L) \quad \text{(A.2)}$$

because $\text{tr}(\Omega_2^2) = o(k_1^2)$ under (C-v). By noting that $\|\Omega_1\|_F^2 \leq \Delta$, under (C-iii') and (C-v), it follows that as $m \to \infty$

$$L_V/L = 1 + o(1).$$

Thus, the union of (C-iii') and (C-v) implies (C-iii''). Then, from (A.1) with Proposition A1 and (A.2), under (A-ii), (C-i'), (C-iii') and (C-v), it holds that as $m \to \infty$

$$\frac{\tilde{\Delta}_n + B_n(2) - \Delta}{L^{1/2}} = \frac{V_1 - 2\|\Omega_1\|_F^2}{L_{V_1}^{1/2}} + o_P(1) \Rightarrow N(0, 1). \quad \text{(A.3)}$$

Hence, we can establish the asymptotic normality even under the SSE model.
Appendix B: Test of the eigenvector

In this section, we apply the asymptotic normality given in Proposition A1 to testing whether a given vector is the eigenvector of \( \Sigma \) or not.

We assume \( \nu_1 > 0 \) and \( r_1 = 1 \) in (7). Then, we consider the following test:

\[
H_0: \Sigma a_1 = \nu_1 a_1 \quad \text{vs.} \quad H_1: \Sigma a_1 \neq \nu_1 a_1,
\]

(B.1)

where \( a_1 \) is a given vector. Note that \( \|\Omega_1\|_F = a_1^T \Sigma A_1 \Sigma a_1 = 0 \) under \( H_0 \) and \( \|\Omega_1\|_F > 0 \) under \( H_1 \) from the facts that \( A_1 = a_1 a_1^T \) and \( A_1 a_1 = 0 \). Thus, the asymptotic normality in Proposition A1 is available.

Let

\[
Q_1 = 2 \sum_{i<j} \frac{(y_{ij}(1) A_1 y_{ij}(2))^2}{n(n-1)} \quad \text{and} \quad Q_2 = 2 \sum_{i<j} \frac{(y_{ij}(1) A_1 y_{ij}(2))^2}{n(n-1)}.
\]

Note that \( E(Q_1) = \nu_1^2 \) and \( E(Q_2) = \text{tr}(\Omega_1^2) \). Let

\[
\tilde{T}_{n(V)} = \frac{nV_1}{\sqrt{8Q_1 Q_2}}.
\]

Then, we propose a test procedure for (B.1) by

\[
\text{rejecting } H_0 \iff \tilde{T}_{n(V)} > z_a.
\]

We have the following results.

**Proposition B1** Assume (A-ii) and (C-i'). For the test procedure (B.2), we have that

\[
\text{Size} = \alpha + o(1) \quad \text{and} \quad \text{Power} = \Phi \left( 2 \frac{\|\Omega_1\|_F^2}{L^{1/2} \frac{1}{L^{1/2}}} - z_a \frac{L^{1/2}}{L^{1/2}} \right) + o(1) \quad \text{as } m \to \infty,
\]

where \( L_{V*} = 8\nu_1^2 \text{tr}(\Sigma_1^2(1))/n^2 \).

**Proposition B2** Assume (A-i). Assume also \( L^{1/2} / \|\Omega_1\|_F \to 0 \) as \( m \to \infty \). For the test procedure (B.2), we have (18).

We checked the performance of the test procedure (B.2) by using a microarray data set in Section 6.

Appendix C: R-code to calculate \( y_{ij(l)} \) s

We give the following R-code to calculate \( y_{ij(l)} \) s given by (9):

**Input** Y(X); a p by n (\( \geq 4 \)) matrix X as \( X = (x_1, ..., x_n) \).

**Output** The \((i, j, l)\) element is \( y_{ij(l)} \) for all \( i < j; l = 1, 2 \).
Y <- function(X){
p <- dim(X)[1]
n <- dim(X)[2]
n1 <- as.integer(ceiling(n/2))
n2 <- n-n1
u1 <- n1/(n1-1)
u2 <- n2/(n2-1)
S <- c(3:(2*n-1))
L <- length(S)
X.var <- array(0, dim=c(2, L, p))
for (l in 1:L){
  V <- list()
  dv <- as.integer(floor(S[1]/2))
  if (dv >= n1){
    V <- append(V, list(c((dv-n1+1): dv)))
  } else{
    V <- append(V, list(c(1: dv), c((dv+n2+1): n))))
  }
  if (dv <= n1){
    V <- append(V, list(c((dv+1): (dv+n2))))
  } else{
    V <- append(V, list(c(c(1: (dv-n1)), c((dv+1): n))))
  }
  for (i in 1:2){
    X.var[i, 1, ] <- apply(X[, V[[i]]], 1, mean)
  }
}
y <- array(0, dim=c(n, n, 2, p))
for (j in 1:n){
  for (i in 1:j){
    if (i != j){
      y[i, j, 1, ] <- sqrt(u1)*(X[, i]-X.var[1, (i+j-2), ])
      y[i, j, 2, ] <- sqrt(u2)*(X[, j]-X.var[2, (i+j-2), ])
    }
  }
}
return(y)
Appendix D: Proofs

Throughout this section, we assume $\mu = 0$ without loss of generality. Let

$$\Delta_n = 2 \sum_{i<j} \text{tr}((x_i x_i^T - \Sigma_s)(x_j x_j^T - \Sigma_s))/\{n(n-1)\} \quad \text{and}$$

$$\Delta_n(t) = 2 \sum_{i<j} \sum_{s=1}^t \text{tr}((x_i x_i^T A_s - \kappa_s A_s)(x_j x_j^T A_s - \kappa_s A_s))/\{n(n-1)\}$$

for $t = 1, \ldots, q$. Let $\Sigma_0 = \Sigma - \Sigma_*$.

**D.1 Proofs of Lemma 1, Theorems 1 and 2**

From Lemma 5.1 in Yata and Aoshima (2016), we have that

$$\text{Var}(\Delta_n) = \left\{ K + 8 \text{tr}((\Sigma^2 - \Sigma_0^2) + 4 \sum_{j=1}^d (M_j - 2 \gamma^T_0 \Sigma_0 \gamma_j)^2) \right\} \{1 + o(1)\}$$

$$+ O\left( \frac{\text{tr}(\Sigma^4)}{n^2} \right) \quad (D.1)$$

as $m \to \infty$ under (A-i). We note that

$$\sum_{j=1}^d (\gamma_0^T \Sigma_0 \gamma_j)^2 \leq \text{tr}((\Sigma^2 - \Sigma_0^2) \leq \lambda_1^2 \Delta \leq \text{tr}(\Sigma^4)^{1/2} \Delta. \quad (D.2)$$

By noting that $\text{tr}(\Sigma^4)^{1/2} \Delta/n = o(K)$ under (C-i) and (C-iii), from (D.1) and (D.2), we can conclude the results of Lemma 1. By noting that $\text{tr}(\Sigma^4) \leq \text{tr}(\Sigma^2)^2$, from the first result of Lemma 1 and (D.2), we have that $\text{Var}(\Delta_n)/\Delta = o(1)$ under (A-i) and (C-ii). We can conclude the result of Theorem 1. From Corollary 5.2 in Yata and Aoshima (2016), we can conclude the result of Theorem 2.

**D.2 Proofs of Theorem 3 and Corollary 1**

We note that

$$K^{1/2}/\Delta = o(1) \quad \text{as } m \to \infty \quad (D.3)$$

under (C-ii) because $\text{tr}(\Sigma^2) = \Delta(1 + o(1))$ under $\text{tr}(\Sigma^2)/\text{tr}(\Sigma^2_0) = o(1)$. Thus, from Theorem 1, we have that

$$P(T_n > z_\alpha) = P(\Delta_n/\Delta > z_\alpha K^{1/2}/\Delta) = P\{1 + oP(1) > o(1)\} = 1 + o(1)$$

under (A-i) and (C-ii). It concludes the result of Corollary 1.
Next, we consider the proof of Theorem 3. By using Theorem 2, we can conclude the result of the size in Theorem 3. From Theorem 2, under (A-ii), (C-i) and (C-iii), it holds that

\[ P(T_n > z_0) = P\left( (\hat{\Delta}_n - \Delta)/K^{1/2} > z_0 K^{1/2}/K^{1/2} - \Delta/K^{1/2} \right) = \Phi\left( \Delta/K^{1/2} - z_0 K^{1/2}/K^{1/2} \right) + o(1). \]  

(D.4)

Thus, we can conclude the results of the power when (C-iii) is met in Theorem 3. From (D.3) we note that

\[ \Phi\left( \Delta/K^{1/2} - z_0 K^{1/2}/K^{1/2} \right) = 1 + o(1) \]  

(D.5)

under (C-ii), so that from Corollary 1 we obtain the result of the power when (C-ii) is met. Hence, by considering the convergent subsequence of \( \Delta=K^{1/2} \), we can conclude the result of the power in Theorem 3. The proofs are completed.

D.3 Proofs of Proposition 1 and Corollary 2

Note that \( \Delta = \sum_{j=1}^p (\lambda_j - 1)^2 \) (= \( \Delta_1 \), say) when \( \Sigma = I_p \). If \( \text{tr}(\Sigma)/\text{tr}(\Sigma^2) = o(1) \) as \( p \to \infty \), it holds that \( \Delta_1/\text{tr}(\Sigma^2) = 1 + o(1) \), so that (C-ii) with \( \Sigma = I_p \) holds. Thus, under (C-iii) with \( \Sigma = I_p \), it follows that \( \Delta_1/p^{1/2} = o(1) \), so that \( \text{tr}(\Sigma^4)/\text{tr}(\Sigma^2)^2 \leq \lambda_1^2 \text{tr}(\Sigma^2)/\text{tr}(\Sigma^2)^2 = o(1) \) under (C-ii). It concludes the result of Proposition 1. From Theorem 3 in view of (D.4), (D.5) and Corollary 1, we can conclude the results of Corollary 2.

D.4 Proofs of Proposition 2 and Lemma 2

Assume (A-i), (7) and \( 2 \leq q_* < q \). We first consider the proof of Proposition 2. Let \( A_+ = \sum_{s=1}^{q_*} A_s \) and \( A_{(*)} = \sum_{s=q_*+1}^q A_s \). Then, we write that

\[
\hat{\Delta}_n - \hat{\Delta}_n(q_*) - \Delta = 2 \sum_{i<j} \left( \frac{\text{tr}\{(x_i x_j^T - \Sigma) A_{(*)} (x_i x_j^T - \Sigma) A_{(*)}\}}{n(n-1)} + \frac{\text{tr}\{(x_i x_j^T - \Sigma) A_{(*)} (x_j x_j^T - \Sigma) A_{(*)}\}}{n(n-1)} + \frac{\sum_{s \neq s'} \text{tr}\{(x_i x_j^T - \Sigma) A_{(*)} (x_j x_j^T - \Sigma) A_{(*)}\}}{n(n-1)} + \frac{\sum_{j=1}^n \text{tr}\{(\Sigma - \Sigma)(x_i x_j^T - \Sigma)\}}{n} \right).
\]  

(D.6)
Note that
\[
\sum_{t, t'=1}^{d} (\gamma_t^T A_t \gamma_{t'} \gamma_t^T A_{t'} \gamma_{t'})^2 \leq \sum_{t=1}^{d} (\gamma_t^T A_t \Sigma A_t \gamma_{t}) (\gamma_t^T A_t \Sigma A_t \gamma_{t})
\]
\[
\leq \left\{ \sum_{t=1}^{d} (\gamma_t^T A_t \Sigma A_t \gamma_{t})^2 \sum_{t'=1}^{d} (\gamma_{t'}^T A_{t'} \Sigma A_{t'} \gamma_{t'})^2 \right\}^{1/2}
\]
\[
\leq \text{tr}\{(\Sigma A_t \Sigma)^4\}^{1/2} \text{tr}\{(\Sigma A_t \Sigma)^4\}^{1/2}.
\]

(D.7)

Also, note that when \( \Sigma = \Sigma_* \),
\[
\sum_{s \neq s', t, t'=1}^{q_*} (\gamma_t^T A_t \gamma_{t'} A_{t'} \gamma_{t'})^2 \leq \sum_{s, s', t=1}^{d} (\gamma_t^T \Sigma_* A_t \gamma_{t}) (\gamma_t^T \Sigma_* A_t \gamma_{t})
\]
\[
= \sum_{t=1}^{d} (\gamma_t^T A_t \Sigma_* \gamma_{t})^2 \leq \text{tr}\{(\Sigma_* A_t \Sigma_* \Sigma_* A_t \Sigma_*)\}.
\]

Then, from (D.6), when \( \Sigma = \Sigma_* \), we have that as \( m \to \infty \)
\[
\text{Var}\{\tilde{\Delta}_n - \Delta_n(q_*)\} = (4\Psi/n^2)\{1 + o(1)\} + O(\text{tr}(\Sigma_*^4)/n^2).
\]

(D.8)

Let \( u_n(l) = n(l)/(n(l) - 1) \) for \( l = 1, 2 \). We note that
\[
y_{ij}(1) = \frac{x_{i}}{u_n(1)} - \sum_{k \in \mathcal{V}_{n(1)(i+j)} \setminus \{i\}} u_n^{1/2} \frac{x_{k}}{n(1)}
\]
and
\[
y_{ij}(2) = \frac{x_{j}}{u_n(2)} - \sum_{k \in \mathcal{V}_{n(2)(i+j)} \setminus \{j\}} u_n^{1/2} \frac{x_{k}}{n(2)}
\]
for all \( i < j \). Similar to (A.4) in Yata and Aoshima (2016), when \( \Sigma = \Sigma_* \), it holds that
\[
\text{Var}\{\tilde{\Delta}_n + B_n(q_* + 1) - \{\tilde{\Delta}_n - \Delta_n(q_*)\}\} = o(\Psi/n^2).
\]

Hence, when \( \Sigma = \Sigma_* \), from (D.8) we have that
\[
\text{Var}\{\tilde{\Delta}_n + B_n(q_* + 1)\}
\]
\[
= \text{Var}\{\tilde{\Delta}_n - \Delta_n(q_*)\} + \text{Var}\{\tilde{\Delta}_n + B_n(q_* + 1) - \{\tilde{\Delta}_n - \Delta_n(q_*)\}\}
\]
\[
+ O\left[ \text{Var}\{\tilde{\Delta}_n - \Delta_n(q_*)\}^{1/2} \text{Var}\{\tilde{\Delta}_n + B_n(q_* + 1) - \{\tilde{\Delta}_n - \Delta_n(q_*)\}\}^{1/2}\right]
\]
\[
= (4\Psi/n^2)\{1 + o(1)\} + O(\text{tr}(\Sigma_*^4)/n^2).
\]

(D.10)

As for \( q_* < 2 \) or \( q_* = q \), we obtain the result similarly. The proof of Proposition 2 is completed.

As for Lemma 2, by noting that
\[
\text{tr}(\Sigma A_j \Sigma A_{j'}) \sum_{s=1}^{d} \gamma_s^T A_j \gamma_s \gamma_s^T A_{j'} \gamma_{s} \leq \text{tr}(\Sigma A_j \Sigma A_{j'})^2 + \left( \sum_{s=1}^{d} \gamma_s^T A_j \gamma_s \gamma_s^T A_{j'} \gamma_{s} \right)^2
\]
and (D.9), we can conclude the result.
D.5 Proofs of Lemma 3, Theorem 4 and Corollary 3

Assume (A-i) and (7). We first consider the proof of Lemma 3. From (20) we write that

$$U_n - \text{tr}(\Sigma^2) = B_n(1) + 2 \sum_{i<j}^{n} \frac{\sum_l^2 \text{tr}\{ (y_{ij(l)}y_{ij(l)}^T - \Sigma)\Sigma_s \}}{n(n-1)}. \quad (D.11)$$

We note that as $m \to \infty$

$$\sum_{j,j'=1}^{q} \frac{(\Sigma_s A_s \Sigma_s A_s)_{ij}^2}{r_j r_{j'} n^2} \leq \sum_{j,j'=1}^{q} \frac{\text{tr}\{(\Sigma A_j)\Sigma_j\} \text{tr}\{(\Sigma A_{j'})\Sigma_{j'}\}}{r_j r_{j'} n^2} \leq \text{tr}(\Sigma_s^2)/n^2 = o(\text{tr}(\Sigma^2)^2); \quad (D.12)$$

because it holds that

$$\text{tr}(\Sigma A_j \Sigma A_{j'}) \leq \text{tr}\{(\Sigma A_j)^2\} \text{tr}\{(\Sigma A_{j'})^2\} \leq \text{tr}(\Sigma A_j)^2 \text{tr}(\Sigma A_{j'})^2 = r_j^2 r_{j'}^2 \kappa_j^2 \kappa_{j'},$$

for all $j, j'$. From Lemma 2 it follows that

$$B_n(1) = o_P(\text{tr}(\Sigma^2)). \quad (D.13)$$

For the second term in (D.11), we have that for $l = 1, 2$

$$\text{Var}\left(\sum_{i<j}^{n} \frac{\text{tr}\{ (y_{ij(l)}y_{ij(l)}^T - \Sigma)\Sigma_s \}}{n(n-1)}\right) = O(\text{tr}(\Sigma_s^2)/n) = o(\text{tr}(\Sigma^2)^2)$$

from the fact that $\text{tr}(\Sigma^2) \leq \text{tr}(\Sigma_s^2)^2 = \text{tr}(\Sigma_s^2)^2$. Then, from (D.11) and (D.13), it holds that

$$U_n/\text{tr}(\Sigma^2) = 1 + o_P(1). \quad (D.14)$$

We have that under (C-iv)

$$\sum_{s=1}^{q} E\left\{ \left( \sum_{i<j}^{n} \frac{y_{ij(1)}^T A_s y_{ij(1)} y_{ij(2)}^T A_s y_{ij(2)}}{n(n-1)} \right)^2 \right\} = O\left( \sum_{s=1}^{q} \text{tr}\{(\Sigma A_s)^2\}^2 \right) = o(\text{tr}(\Sigma_s^2)^2), \quad (D.15)$$

so that $\tilde{\psi}_n = U_n^2 + o_P(\text{tr}(\Sigma_s^2)^2)$ from Markov’s inequality. Then, from (D.14), we can conclude the result of Lemma 3.

Next, we consider the proof of Corollary 3. From Lemma 2, (D.12) and $\text{tr}(\Sigma_s^2) \leq \text{tr}(\Sigma^2)$, it holds that $B_n(1)/\Delta = o_P(1)$ under (C-ii). Thus, from Theorem 1 and (21), it holds that $\Delta_n/\Delta = 1 + o_P(1)$ under (C-ii). Similar to the proof of Corollary 1, we can conclude the result of Corollary 3.

For Theorem 4, from (24) and Theorem 3 in view of (D.4) and (D.5), we can conclude the result.
D.6 Proofs of Lemmas 4, 5 and Theorem 5

Assume (A-i), (7) and \( r_1 = 1 \). We first consider the proof of Lemma 4. Similar to (D.2), it holds that \( \text{tr}\{(\Omega_2 \Sigma_0)^2\} \leq \text{tr}(\Omega_2)^{1/2} \Delta \). Note that

\[
\text{tr}\{(\Sigma A_i \Sigma_0 A_i(1))^2\} \leq \|\Sigma_1/2 A_i \Sigma_0 A_i(1)\|_F^2 = \|\Sigma_1/2 A_i \Omega_1 \Omega_2^{1/2}\|_F^2 \\
\leq \lambda_{\text{max}}(\Omega_2) \|\Omega_1\|_F^2 \leq \Delta \text{tr}(\Omega_2)^{1/4} \kappa_1
\]

because \( A_i \Sigma_0 A_i(1) = A_i \Sigma A_i(1) \) and \( \Delta = 2\|\Omega_1\|_F^2 + \|A_i(1) \Sigma_0 A_i(1)\|_F^2 \), where \( \lambda_{\text{max}}(\Omega_2) \) denotes the largest eigenvalue of \( \Omega_2 \). By noting that \( A_i \Sigma_0 A_i = O \) and \( \|\Sigma A_i(1) \Sigma_0 A_i(1)\|^2 = \|\Omega_2 \Sigma_0\|^2 \), from (D.2) we have that as \( m \to \infty \)

\[
\begin{align*}
\text{Var}\left(\sum_{j=1}^n \frac{\text{tr}\{(\Sigma - \Sigma_*) (x_i x_i^T - \Sigma)\}}{n}\right) &= O\left(\frac{\text{tr}(\Omega_2)^{1/4} \{\text{tr}(\Omega_2)^{1/4} + \kappa_1\}}{n}\right).
\end{align*}
\]

(D.16)

Similar to (D.6) and (D.7), from (D.16) we have that

\[
\begin{align*}
\text{Var}\{\hat{\Delta}_n - \hat{\Delta}_n(1)\} &= L\{1 + o(1)\} + O\left(\frac{\text{tr}(\Omega_2)^{1/4} \{\text{tr}(\Omega_2)^{1/4} + \kappa_1\}}{n}\right) \\
&+ O\left(\frac{\text{tr}(\Omega_2)^{1/2} \{\kappa_1^2 + \text{tr}(\Omega_2)^{1/2}\}}{n^2}\right).
\end{align*}
\]

Then, similar to (D.8) and (D.10), we have that

\[
\text{Var}\{\hat{\Delta}_n + B_n(2)\} = \text{Var}\{\hat{\Delta}_n - \hat{\Delta}_n(1)\}\{1 + o(1)\}.
\]

It concludes the first result of Lemma 4. Note that

\[
\frac{\text{tr}(\Omega_2)^{1/4} \{\text{tr}(\Omega_2)^{1/4} + \kappa_1\}}{n} = o(L) \quad \text{and} \quad \frac{\text{tr}(\Omega_2)^{1/2} \{\kappa_1^2 + \text{tr}(\Omega_2)^{1/2}\}}{n^2} = o(L)
\]

under (C-i') and (C-iii'). Thus, from the first result of Lemma 4, we can conclude the second result of Lemma 4.

For Lemma 5 and Theorem 5, from (A.3), it concludes the result of Lemma 5. From (26) and Lemma 5, we can conclude the result of Theorem 5.

D.7 Proofs of Lemma 6, Theorem 6 and Corollary 4

Assume (A-i), (7) and \( r_1 = 1 \). We first consider the proof of Lemma 6. Let

\[
\xi_s = 2 \sum_{i<j}^n y_{ij}^T A_i y_{ij(1)} y_{ij(2)}^T A_i y_{ij(2)} / \{r_s n (n - 1)\}
\]

for \( s = 1, \ldots, q \). We write that

\[
U_n^2 - \xi_1^2 = 2 \xi_1 \sum_{s=2}^q \xi_s + \left( \sum_{s=2}^q \xi_s \right)^2.
\]

(D.17)
Similar to (D.11) to (D.14), we have that as $m \to \infty$
\[
\sum_{s=2}^{q} \xi_s = \left( \sum_{s=2}^{q} r_s \kappa_s^2 \right) \{1 + o_P(1)\}.
\]

Note that $\text{Var}(\xi_1) = O(\kappa_1^2/n)$, so that $\xi_1 = \kappa_1^2 \{1 + o_P(1)\}$. Thus, from (D.17) it holds that
\[
U_n^2 - \xi_1^2 = \Psi_1 \{1 + o_P(1)\}.
\]

We note that when $q_s \geq 2$,
\[
\sum_{s=2}^{q_s} \text{tr}(\{\Sigma A_s\}^4) = \sum_{s=2}^{q_s} \kappa_s^4 = o(\Psi_1)
\]
under (C-v) because $\text{tr}(\Sigma^2) \geq (\sum_{s=2}^{q_s} r_s \kappa_s^2)^2 + \kappa_1^4$ and $\sum_{s=2}^{q_s} \kappa_s^4 \leq (\sum_{s=2}^{q_s} r_s \kappa_s^2)^2$. Similar to (D.15), from (D.18) and (D.19), we can conclude the result of Lemma 6.

Next, we consider the proof of Corollary 4. From (26), it holds that $B_n(2)/\Delta = o_P(1)$ under (C-ii'). From Lemma 4 we have that $\text{Var}\{\hat{\Delta}_n + B_n(2)/\Delta\} = o(1)$ under (C-ii'). Thus, it follows that $\hat{\Delta}_n/\Delta = 1 + o_P(1)$ under (C-ii'). Note that $\text{tr}(\Sigma_n^2) - \kappa_1^2 + 2\|\Omega_1\|_F^2 \leq \text{tr}(\Sigma_n^2) - \kappa_1^2 + \Delta = \text{tr}(\Sigma^2) - \kappa_1^2 = \text{tr}(\Omega_1^2) + 2\|\Omega_1\|_F^2$. It holds that $\sum_{s=2}^{q_s} r_s \kappa_s^2 \leq \text{tr}(\Omega_1^2)$, so that $L_s = O(L)$. Then, similar to the proof of Corollary 1, from Lemma 6 we can conclude the result of Corollary 4.

For Theorem 6, similar to the proof of Theorem 3, from Theorem 5 and Lemma 6 we can conclude the result.

D.8 Proofs of Proposition 3 and Corollary 5

From (A.4) and (A.5) in Yata et al. (2018), we can conclude the result of Proposition 3. From Theorem 4 in view of (D.4)-(D.5), (28), Proposition 3 and Corollary 3, we can conclude the results of Corollary 5.

D.9 Proofs of Proposition 4 and Corollary 6

Assume $\Sigma_* = \Sigma_D$. Note that
\[
\text{tr}(\Sigma^2)/\text{tr}(\Sigma_D^2) \in (0, \infty) \quad \text{as} \quad p \to \infty
\]
under (C-iii). We write that $h_1 = (h_{11}, \ldots, h_{1p})^T$. Note that $h_1^T(\Sigma - \Sigma_D)h_1 = \lambda_1 - \sum_{j=1}^{p} h_{1j}^2 \sigma_{jj}$. Also, note that $\Delta_D = \{h_1^T(\Sigma - \Sigma_D)h_1\}^2$ and $\sum_{j=1}^{p} h_{1j}^2 \sigma_{jj} = O(1)$ from the fact that $\sigma_{jj} = O(1)$ for all $j$. Then, if $\lim inf_{p \to \infty} \lambda_1^2/\text{tr}(\Sigma_D^2) > 0$, it holds that $\lim inf_{p \to \infty} \Delta_D/\text{tr}(\Sigma_D^2) > 0$, so that (C-ii) holds. Then, from
(D.20), under (C-iii) it follows that \( \lambda_{j}^2/\text{tr}(\Sigma^2) \to 0 \) as \( p \to \infty \). Thus, (C-iii) implies (C-i). On the other hand, we have that

\[
\sum_{j,j'=1}^{q} \left( \sum_{s=1}^{d} \gamma_{s} \gamma_{s}^{T} A_{j} \gamma_{s}^{T} A_{j'} \gamma_{s} \right)^2 \leq \max_{j=1,\ldots,p} \sigma_{jj}^2 \sum_{j,j'=1}^{q} \sum_{s=1}^{d} \gamma_{s} \gamma_{s}^{T} A_{j} \gamma_{s}^{T} A_{j'} \gamma_{s} \leq \max_{j=1,\ldots,p} \sigma_{jj}^2 \text{tr}(\Sigma^2)
\]

and

\[
\sum_{j,j'=1}^{q} \frac{\text{tr}(\Sigma A_{j} \Sigma A_{j'})^2}{r_{jj'}} = \sum_{j,j'=1}^{p} \sigma_{jj'}^4 \leq \text{tr}(\Sigma^4)
\]

because \( \sum_{s=1}^{d} \gamma_{s} \gamma_{s}^{T} A_{j} \gamma_{s}^{T} A_{j'} \gamma_{s} \leq \text{tr}(\Sigma A_{j})\text{tr}(\Sigma A_{j'}) = \sigma_{ii} \sigma_{jj} \). From (D.20) we note that \( \max_{j=1,\ldots,p} \sigma_{jj}^2 \text{tr}(\Sigma^2) = o(\text{tr}(\Sigma^2)^2) \) and \( \text{tr}(\Sigma^4) = o(\text{tr}(\Sigma^2)^2) \) under (C-i) and (C-iii). Thus, (C-iii) implies (C-iv). It concludes the results of Proposition 4.

Next, we consider the proof of Corollary 6. From Theorem 1, Lemma 2, (21) and (D.21), we have that \( \Delta_{n}/\Delta = 1+o_P(1) \) under (A-i) and (C-ii). Then, from Theorem 4 in view of (D.4)-(D.5), Proposition 4 and Corollary 3, we can conclude the results of Corollary 6.

D.10 Proof of Corollary 7

Note that (C-v') implies (C-v). Then, from Theorem 6 and Corollary 4, we can conclude the results.

D.11 Proofs of Propositions A1 and A2

Let \( H_a = (a_1, \ldots, a_p) \), \( x_{j,H} = H_a^T x_j = (x_{j1,H}, \ldots, x_{jp,H})^T \) \( (j = 1, \ldots, n) \) and \( \Gamma_H = H_a^T \Gamma \). We write that \( x_{j,H} = \Gamma_H w_j \). Then, for \( x_{j1,H} \) and \( (x_{j2,H}, \ldots, x_{jp,H})^T \), from Theorem 3.2 in Yata and Aoshima (2016), we can conclude the result of Proposition A1.

For Proposition A2, let \( x_{j,A} = A_{(1)} x_j \) \( (j = 1, \ldots, n) \). Note that \( \text{Var}(x_{j,A}) = \Omega_2 \). Then, from Lemma 1 we can conclude the result of Proposition A2.

D.12 Proofs of Propositions B1 and B2

From Lemma 3.3 in Yata and Aoshima (2016), we have that as \( m \to \infty \)

\[ Q_1/\kappa_1^2 = 1 + o_P(1) \quad \text{and} \quad Q_2/\text{tr}(\Omega_2^2) = 1 + o_P(1) \]

under (A-i). Then, from Corollary 4.1 in Yata and Aoshima (2016), we can conclude the result of Proposition B2. From Theorem 4 in view of (D.4)-(D.5), Propositions A1 and B2, we can conclude the result of Proposition B1.
References


