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Supplement to "Hypothesis tests for high-dimensional covariance structures"

Aki Ishii · Kazuyoshi Yata · Makoto Aoshima

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Abstract In this supplement, we give details of the asymptotic normality in Lemma 5, an additional test procedure, an R-code to calculate $\boldsymbol{y}_{ij(l)}$s, proofs of the theoretical results in the main work in Ishii et al. (2020) together with additional propositions and proofs of the propositions. The equation numbers and the mathematical symbols used in the supplement are the same as those which are made reference to in the main document.

Appendix A: Details of the asymptotic normality in Lemma 5

In Lemma 5, we established the asymptotic normality under the SSE model. However, in general, high-dimensional statistics do not hold the asymptotic normality under the SSE model. See Aoshima and Yata (2018, 2019) and Ishii et al. (2019) for the details. We emphasize that the asymptotic normality given in Lemma 5 is a rare case for the SSE model. In this section, we explain the details of the asymptotic normality.

We assume $r_1 = 1$ in (7). Then, we write that

$$\Delta = 2\|\boldsymbol{\Omega}_1\|_F^2 + \text{tr}(\boldsymbol{\Omega}_2^2) - \text{tr}(\boldsymbol{\Sigma}_{*(1)}^2),$$

Aki Ishii
Department of Information Sciences, Tokyo University of Science, Chiba 278-8510, Japan
E-mail: a.ishii@rs.tus.ac.jp

Kazuyoshi Yata
Institute of Mathematics, University of Tsukuba, Ibaraki 305-8571, Japan
E-mail: yata@math.tsukuba.ac.jp

Makoto Aoshima
Institute of Mathematics, University of Tsukuba, Ibaraki 305-8571, Japan
E-mail: aoshima@math.tsukuba.ac.jp

where $\boldsymbol{\Sigma}_{*(1)}=\sum_{s=2}^{q}\kappa_s \boldsymbol{A}_s$. From (11) and (21), it follows that

$$\begin{aligned}\widetilde{\Delta}_n+B_n(2)=&2\sum_{i<j}^{n}\sum_{s=2}^{q}\frac{\mathrm{tr}(\boldsymbol{Y}_{ij(1),1}\boldsymbol{Y}_{ij(2),s})+\mathrm{tr}(\boldsymbol{Y}_{ij(1),s}\boldsymbol{Y}_{ij(2),1})}{n(n-1)}\\&+2\sum_{i<j}^{n}\sum_{s,s'=2}^{q}\frac{\mathrm{tr}(\boldsymbol{Y}_{ij(1),s}\boldsymbol{Y}_{ij(2),s'})}{n(n-1)}\quad(=V_1+V_2,\ \text{say}).\qquad\text{(A.1)}\end{aligned}$$

Thus, one can ignore the following term:

$$2\sum_{i<j}^{n}\frac{\mathrm{tr}(\boldsymbol{Y}_{ij(1),1}\boldsymbol{Y}_{ij(2),1})}{n(n-1)}.$$

This is the key to prove the asymptotic normality of $\widetilde{\Delta}_n$ since its variance is huge under (C-v) because $\mathrm{Var}\{2\sum_{i<j}^{n}\mathrm{tr}(\boldsymbol{Y}_{ij(1),1}\boldsymbol{Y}_{ij(2),1})/n^2\}=O(\kappa_1^4/n^2)$ under (A-i). For V_1, by noting that $\boldsymbol{A}_1\boldsymbol{A}_s=\boldsymbol{O}$ for $s>1$, we have that

$$\begin{aligned}V_1=&2\sum_{i<j}^{n}\sum_{s=2}^{q}\frac{\mathrm{tr}(\boldsymbol{Y}_{ij(1),1}\boldsymbol{Y}_{ij(2),s})+\mathrm{tr}(\boldsymbol{Y}_{ij(1),s}\boldsymbol{Y}_{ij(2),1})}{n(n-1)}\\=&4\sum_{i<j}^{n}\frac{\boldsymbol{y}_{ij(1)}^T\boldsymbol{A}_1\boldsymbol{y}_{ij(2)}\boldsymbol{y}_{ij(1)}^T\boldsymbol{A}_{(1)}\boldsymbol{y}_{ij(2)}}{n(n-1)}.\end{aligned}$$

Note that $E(V_1)=2\|\boldsymbol{\Omega}_1\|_F^2$. Now, we give asymptotic properties of V_1. Let $y_{ij(l),1}=\boldsymbol{a}_1^T\boldsymbol{y}_{ij(l)}$ and $\boldsymbol{y}_{ij(l),2}=(\boldsymbol{a}_2,\dots,\boldsymbol{a}_p)^T\boldsymbol{y}_{ij(l)}$ for all i,j,l, where $\boldsymbol{A}_1=\boldsymbol{a}_1\boldsymbol{a}_1^T$ and $\boldsymbol{a}_2,\dots,\boldsymbol{a}_p$ are eigenvectors of $\boldsymbol{A}_{(1)}$ such that $\boldsymbol{A}_{(1)}=\sum_{j=2}^{p}\boldsymbol{a}_j\boldsymbol{a}_j^T$. Then, it follows that

$$V_1=4\sum_{i<j}^{n}\frac{y_{ij(1),1}y_{ij(2),1}\boldsymbol{y}_{ij(1),2}^T\boldsymbol{y}_{ij(2),2}}{n(n-1)}.$$

Note that

$$E(y_{ij(l),1}\boldsymbol{y}_{ij(l),2})=\mathrm{Cov}\{\boldsymbol{a}_1^T\boldsymbol{x}_j,\ (\boldsymbol{a}_2,\dots,\boldsymbol{a}_p)^T\boldsymbol{x}_j\}=(\boldsymbol{a}_2,\dots,\boldsymbol{a}_p)^T\boldsymbol{\Sigma}\boldsymbol{a}_1$$

for $l=1,2;\ i<j$, and $\|(\boldsymbol{a}_2,\dots,\boldsymbol{a}_p)^T\boldsymbol{\Sigma}\boldsymbol{a}_1\|^2=\|\boldsymbol{\Omega}_1\|_F^2$. Thus, V is an unbiased estimator of the squared norm of the covariance vector between $\boldsymbol{a}_1^T\boldsymbol{x}_j$ and $(\boldsymbol{a}_2,\dots,\boldsymbol{a}_p)^T\boldsymbol{x}_j$. Yata and Aoshima (2013, 2016) gave asymptotic properties of the ECDM estimator for such a covariance vector. Let

$$L_V=8\kappa_1^2\mathrm{tr}(\boldsymbol{\Omega}_2^2)/n^2.$$

From Theorem 3.2 in Yata and Aoshima (2016), we have the following result.

Proposition A1 *Assume (A-ii) and (C-i'). Assume also*

(C-iii") $\displaystyle\limsup_{m\to\infty}\frac{\|\boldsymbol{\Omega}_1\|_F^2}{L_V^{1/2}}<\infty.$

Then, it holds that as $m \to \infty$

$$\frac{V_1 - 2\|\boldsymbol{\Omega}_1\|_F^2}{L_V^{1/2}} \Rightarrow N(0,1).$$

Next, we consider the variance of V_2 in (A.1). Note that

$$E(V_2) = \mathrm{tr}(\boldsymbol{\Omega}_2^2) - \mathrm{tr}(\boldsymbol{\Sigma}_{*(1)}^2) \le \Delta.$$

We write that

$$\begin{aligned}&\sum_{s,s'=2}^{q} \frac{\mathrm{tr}(\boldsymbol{Y}_{ij(1),s}\boldsymbol{Y}_{ij(2),s'})}{n(n-1)}\\ &= 2\sum_{i<j}^{n} \frac{\mathrm{tr}\big\{\boldsymbol{A}_{(1)}(\boldsymbol{y}_{ij(1)}\boldsymbol{y}_{ij(1)}^T - \boldsymbol{\Sigma}_*)\boldsymbol{A}_{(1)}(\boldsymbol{y}_{ij(2)}\boldsymbol{y}_{ij(2)}^T - \boldsymbol{\Sigma}_*)\big\}}{n(n-1)}.\end{aligned}$$

Then, from Lemma 1, we have the following result.

Proposition A2 *Assume (A-i). It holds that as* $m \to \infty$

$$Var(V_2) = O(tr(\boldsymbol{\Omega}_2^2)^2/n^2) + O(\Delta tr(\boldsymbol{\Omega}_2^2)/n).$$

From Proposition A2, under (A-i), (C-iii') and (C-v), we have that as $m \to \infty$

$$\mathrm{Var}(V_2) = o(L) \tag{A.2}$$

because $\mathrm{tr}(\boldsymbol{\Omega}_2^2) = o(\kappa_1^2)$ under (C-v). By noting that $\|\boldsymbol{\Omega}_1\|_F^2 \le \Delta$, under (C-iii') and (C-v), it follows that as $m \to \infty$

$$L_V/L = 1 + o(1).$$

Thus, the union of (C-iii') and (C-v) implies (C-iii"). Then, from (A.1) with Proposition A1 and (A.2), under (A-ii), (C-i'), (C-iii') and (C-v), it holds that as $m \to \infty$

$$\frac{\widetilde{\Delta}_n + B_n(2) - \Delta}{L^{1/2}} = \frac{V_1 - 2\|\boldsymbol{\Omega}_1\|_F^2}{L_V^{1/2}} + o_P(1) \Rightarrow N(0,1). \tag{A.3}$$

Hence, we can establish the asymptotic normality even under the SSE model.

Appendix B: Test of the eigenvector

In this section, we apply the asymptotic normality given in Proposition A1 to testing whether a given vector is the eigenvector of $\boldsymbol{\Sigma}$ or not.

We assume $\kappa_1 > 0$ and $r_1 = 1$ in (7). Then, we consider the following test:

$$H_0 : \boldsymbol{\Sigma}\boldsymbol{a}_1 = \kappa_1\boldsymbol{a}_1 \quad \text{vs.} \quad H_1 : \boldsymbol{\Sigma}\boldsymbol{a}_1 \neq \kappa_1\boldsymbol{a}_1, \tag{B.1}$$

where $\boldsymbol{a}_1$ is a given vector. Note that $\|\boldsymbol{\Omega}_1\|_F^2 = \boldsymbol{a}_1^T\boldsymbol{\Sigma}\boldsymbol{A}_{(1)}\boldsymbol{\Sigma}\boldsymbol{a}_1 = 0$ under H_0 and $\|\boldsymbol{\Omega}_1\|_F^2 > 0$ under H_1 from the facts that $\boldsymbol{A}_1 = \boldsymbol{a}_1\boldsymbol{a}_1^T$ and $\boldsymbol{A}_{(1)}\boldsymbol{a}_1 = \boldsymbol{0}$. Thus, the asymptotic normality in Proposition A1 is available.

Let

$$Q_1 = 2\sum_{i<j}^{n} \frac{(\boldsymbol{y}_{ij(1)}^T\boldsymbol{A}_1\boldsymbol{y}_{ij(2)})^2}{n(n-1)} \quad \text{and} \quad Q_2 = 2\sum_{i<j}^{n} \frac{(\boldsymbol{y}_{ij(1)}^T\boldsymbol{A}_{(1)}\boldsymbol{y}_{ij(2)})^2}{n(n-1)}.$$

Note that $E(Q_1) = \kappa_1^2$ and $E(Q_2) = \text{tr}(\boldsymbol{\Omega}_2^2)$. Let

$$\widetilde{T}_{n(V)} = \frac{nV_1}{\sqrt{8Q_1Q_2}}.$$

Then, we propose a test procedure for (B.1) by

$$\text{rejecting } H_0 \Longleftrightarrow \widetilde{T}_{n(V)} > z_\alpha. \tag{B.2}$$

We have the following results.

Proposition B1 *Assume (A-ii) and (C-i'). For the test procedure (B.2), we have that*

$$\textit{Size} = \alpha + o(1) \quad \textit{and} \quad \textit{Power} = \Phi\left(\frac{2\|\boldsymbol{\Omega}_1\|_F^2}{L_V^{1/2}} - z_\alpha\frac{L_{V*}^{1/2}}{L_V^{1/2}}\right) + o(1) \quad \textit{as } m \to \infty,$$

where $L_{V*} = 8\kappa_1^2 tr(\boldsymbol{\Sigma}_{*(1)}^2)/n^2$.

Proposition B2 *Assume (A-i). Assume also* $L_V^{1/2}/\|\boldsymbol{\Omega}_1\|_F^2 \to 0$ *as* $m \to \infty$. *For the test procedure (B.2), we have (18).*

We checked the performance of the test procedure (B.2) by using a microarray data set in Section 6.

Appendix C: R-code to calculate $\boldsymbol{y}_{ij(l)}$s

We give the following R-code to calculate $\boldsymbol{y}_{ij(l)}$s given by (9):

Input Y(X); a p by n (≥ 4) matrix X as $X = (\boldsymbol{x}_1, ..., \boldsymbol{x}_n)$.
Output The (i, j, l) element is $\boldsymbol{y}_{ij(l)}$ for all $i < j; l = 1, 2$.

```
Y <- function(X){
 p <- dim(X)[1]
 n <- dim(X)[2]
 n1 <- as.integer(ceiling(n/2))
 n2 <- n-n1
 u1 <- n1/(n1-1)
 u2 <- n2/(n2-1)
 S <- c(3:(2*n-1))
 L <- length(S)
 X_var <- array(0, dim=c(2, L, p))
 for (l in 1:L){
  V <- list()
  dv <- as.integer(floor(S[l]/2))
  if (dv >= n1){
    V <- append(V, list(c((dv-n1+1): dv)))
  } else{
    V <- append(V, list(c(c(1: dv), c((dv+n2+1): n))))
  }
  if (dv <= n1){
    V <- append(V, list(c((dv+1): (dv+n2))))
  } else{
    V <- append(V, list(c(c(1: (dv-n1)), c((dv+1): n))))
  }
  for (i in 1:2){
    X_var[i, l, ] <- apply(X[, V[[i]]], 1, mean)
  }
 }
 y <- array(0, dim=c(n, n, 2, p))
 for (j in 1:n){
  for (i in 1:j){
    if (i != j){
    y[i, j, 1, ] <- sqrt(u1)*(X[, i]-X_var[1, (i+j-2), ])
    y[i, j, 2, ] <- sqrt(u2)*(X[, j]-X_var[2, (i+j-2), ])
   }
  }
 }
 return(y)
}
```

Appendix D: Proofs

Throughout this section, we assume $\boldsymbol{\mu} = \mathbf{0}$ without loss of generality. Let

$$\acute{\Delta}_n = 2\sum_{i<j}^{n} \mathrm{tr}\{(\boldsymbol{x}_i\boldsymbol{x}_i^T - \boldsymbol{\Sigma}_*)(\boldsymbol{x}_j\boldsymbol{x}_j^T - \boldsymbol{\Sigma}_*)\}/\{n(n-1)\} \quad \text{and}$$

$$\acute{\Delta}_n(t) = 2\sum_{i<j}^{n}\sum_{s=1}^{t} \mathrm{tr}\{(\boldsymbol{x}_i\boldsymbol{x}_i^T\boldsymbol{A}_s - \kappa_s\boldsymbol{A}_s)(\boldsymbol{x}_j\boldsymbol{x}_j^T\boldsymbol{A}_s - \kappa_s\boldsymbol{A}_s)\}/\{n(n-1)\}$$

for $t = 1, \dots, q$. Let $\boldsymbol{\Sigma}_0 = \boldsymbol{\Sigma} - \boldsymbol{\Sigma}_*$.

D.1 Proofs of Lemma 1, Theorems 1 and 2

From Lemma 5.1 in Yata and Aoshima (2016), we have that

$$\begin{aligned}\mathrm{Var}(\widehat{\Delta}_n) =& \Big\{K + \frac{8\mathrm{tr}\{(\boldsymbol{\Sigma}\boldsymbol{\Sigma}_0)^2\} + 4\sum_{j=1}^{d}(M_j-2)(\boldsymbol{\gamma}_j^T\boldsymbol{\Sigma}_0\boldsymbol{\gamma}_j)^2}{n}\Big\}\{1+o(1)\} \\ &+ O\Big(\frac{\mathrm{tr}(\boldsymbol{\Sigma}^4)}{n^2}\Big) \end{aligned} \tag{D.1}$$

as $m \to \infty$ under (A-i). We note that

$$\sum_{j=1}^{d}(\boldsymbol{\gamma}_j^T\boldsymbol{\Sigma}_0\boldsymbol{\gamma}_j)^2 \le \mathrm{tr}\{(\boldsymbol{\Sigma}\boldsymbol{\Sigma}_0)^2\} \le \lambda_1\mathrm{tr}(\boldsymbol{\Sigma}\boldsymbol{\Sigma}_0^2) \le \lambda_1^2\Delta \le \mathrm{tr}(\boldsymbol{\Sigma}^4)^{1/2}\Delta. \tag{D.2}$$

By noting that $\mathrm{tr}(\boldsymbol{\Sigma}^4)^{1/2}\Delta/n = o(K)$ under (C-i) and (C-iii), from (D.1) and (D.2), we can conclude the results of Lemma 1. By noting that $\mathrm{tr}(\boldsymbol{\Sigma}^4) \le \mathrm{tr}(\boldsymbol{\Sigma}^2)^2$, from the first result of Lemma 1 and (D.2), we have that $\mathrm{Var}(\widehat{\Delta}_n)/\Delta = o(1)$ under (A-i) and (C-ii). We can conclude the result of Theorem 1. From Corollary 5.2 in Yata and Aoshima (2016), we can conclude the result of Theorem 2.

D.2 Proofs of Theorem 3 and Corollary 1

We note that

$$K_*^{1/2}/\Delta = o(1) \quad \text{as } m \to \infty \tag{D.3}$$

under (C-ii) because $\mathrm{tr}(\boldsymbol{\Sigma}_*^2) = \Delta\{1+o(1)\}$ under $\mathrm{tr}(\boldsymbol{\Sigma}^2)/\mathrm{tr}(\boldsymbol{\Sigma}_*^2) = o(1)$. Thus, from Theorem 1, we have that

$$P(T_n > z_\alpha) = P(\widehat{\Delta}_n/\Delta > z_\alpha K_*^{1/2}/\Delta) = P\{1 + o_P(1) > o(1)\} = 1 + o(1)$$

under (A-i) and (C-ii). It concludes the result of Corollary 1.

Next, we consider the proof of Theorem 3. By using Theorem 2, we can conclude the result of the size in Theorem 3. From Theorem 2, under (A-ii), (C-i) and (C-iii), it holds that

$$\begin{aligned} P(T_n > z_\alpha) &= P\big\{(\widehat{\Delta}_n - \Delta)/K^{1/2} > z_\alpha K_*^{1/2}/K^{1/2} - \Delta/K^{1/2}\big\} \\ &= \Phi\big(\Delta/K^{1/2} - z_\alpha K_*^{1/2}/K^{1/2}\big) + o(1). \end{aligned} \tag{D.4}$$

Thus, we can conclude the results of the power when (C-iii) is met in Theorem 3. From (D.3) we note that

$$\Phi(\Delta/K^{1/2} - z_\alpha K_*^{1/2}/K^{1/2}) = 1 + o(1) \tag{D.5}$$

under (C-ii), so that from Corollary 1 we obtain the result of the power when (C-ii) is met. Hence, by considering the convergent subsequence of $\Delta/K^{1/2}$, we can conclude the result of the power in Theorem 3. The proofs are completed.

D.3 Proofs of Proposition 1 and Corollary 2

Note that $\Delta = \sum_{j=1}^{p}(\lambda_j - 1)^2$ $(= \Delta_{\mathrm{I}},$ say) when $\boldsymbol{\Sigma}_* = \boldsymbol{I}_p$. If $\mathrm{tr}(\boldsymbol{\Sigma})/\mathrm{tr}(\boldsymbol{\Sigma}^2) = o(1)$ as $p \to \infty$, it holds that $\Delta_{\mathrm{I}}/\mathrm{tr}(\boldsymbol{\Sigma}^2) = 1+o(1)$, so that (C-ii) with $\boldsymbol{\Sigma}_* = \boldsymbol{I}_p$ holds. Thus, under (C-iii) with $\boldsymbol{\Sigma}_* = \boldsymbol{I}_p$, it follows that $\mathrm{tr}(\boldsymbol{\Sigma}^2)/p \in (0,\infty)$ as $p \to \infty$. If $\liminf_{p\to\infty} \lambda_1/p^{1/2} > 0$ and $\mathrm{tr}(\boldsymbol{\Sigma}^2)/p \in (0,\infty)$ as $p \to \infty$, (C-ii) with $\boldsymbol{\Sigma}_* = \boldsymbol{I}_p$ holds. Thus, under (C-iii) with $\boldsymbol{\Sigma}_* = \boldsymbol{I}_p$ it follows that $\lambda_1/p^{1/2} = o(1)$, so that

$$\mathrm{tr}(\boldsymbol{\Sigma}^4)/\mathrm{tr}(\boldsymbol{\Sigma}^2)^2 \le \lambda_1^2 \mathrm{tr}(\boldsymbol{\Sigma}^2)/\mathrm{tr}(\boldsymbol{\Sigma}^2)^2 = o(1)$$

under (C-iii). It concludes the result of Proposition 1. From Theorem 3 in view of (D.4), (D.5) and Corollary 1, we can conclude the results of Corollary 2.

D.4 Proofs of Proposition 2 and Lemma 2

Assume (A-i), (7) and $2 \le q_\star < q$. We first consider the proof of Proposition 2. Let $\boldsymbol{A}_\star = \sum_{s=1}^{q_\star} \boldsymbol{A}_s$ and $\boldsymbol{A}_{(\star)} = \sum_{s=q_\star+1}^{q} \boldsymbol{A}_s$. Then, we write that

$$\begin{aligned} \acute{\Delta}_n - \acute{\Delta}_n(q_\star) - \Delta = 2\sum_{i<j}^{n} \Big(& \frac{\mathrm{tr}\{(\boldsymbol{x}_i\boldsymbol{x}_i^T - \boldsymbol{\Sigma})\boldsymbol{A}_{(\star)}(\boldsymbol{x}_j\boldsymbol{x}_j^T - \boldsymbol{\Sigma})\boldsymbol{A}_{(\star)}\}}{n(n-1)} \\ & + 2\frac{\mathrm{tr}\{(\boldsymbol{x}_i\boldsymbol{x}_i^T - \boldsymbol{\Sigma})\boldsymbol{A}_\star(\boldsymbol{x}_j\boldsymbol{x}_j^T - \boldsymbol{\Sigma})\boldsymbol{A}_{(\star)}\}}{n(n-1)}\Big) \\ & + \sum_{s\ne s'}^{q_\star} \frac{\mathrm{tr}\{(\boldsymbol{x}_i\boldsymbol{x}_i^T - \boldsymbol{\Sigma})\boldsymbol{A}_s(\boldsymbol{x}_j\boldsymbol{x}_j^T - \boldsymbol{\Sigma})\boldsymbol{A}_{s'}\}}{n(n-1)}\Big) \\ + 2\sum_{j=1}^{n} & \frac{\mathrm{tr}\{(\boldsymbol{\Sigma} - \boldsymbol{\Sigma}_*)(\boldsymbol{x}_i\boldsymbol{x}_i^T - \boldsymbol{\Sigma})\}}{n}. \end{aligned} \tag{D.6}$$

Note that

$$\sum_{t,t'=1}^{d}(\boldsymbol{\gamma}_t^T\boldsymbol{A}_\star\boldsymbol{\gamma}_{t'}\boldsymbol{\gamma}_{t'}^T\boldsymbol{A}_{(\star)}\boldsymbol{\gamma}_t)^2 \le \sum_{t=1}^{d}(\boldsymbol{\gamma}_t^T\boldsymbol{A}_{(\star)}\boldsymbol{\Sigma}\boldsymbol{A}_{(\star)}\boldsymbol{\gamma}_t)(\boldsymbol{\gamma}_t^T\boldsymbol{A}_\star\boldsymbol{\Sigma}\boldsymbol{A}_\star\boldsymbol{\gamma}_t)$$
$$\le \Big\{\sum_{t=1}^{d}(\boldsymbol{\gamma}_t^T\boldsymbol{A}_{(\star)}\boldsymbol{\Sigma}\boldsymbol{A}_{(\star)}\boldsymbol{\gamma}_t)^2\sum_{t'=1}^{d}(\boldsymbol{\gamma}_{t'}^T\boldsymbol{A}_\star\boldsymbol{\Sigma}\boldsymbol{A}_\star\boldsymbol{\gamma}_{t'})^2\Big\}^{1/2}$$
$$\le \mathrm{tr}\{(\boldsymbol{\Sigma}\boldsymbol{A}_{(\star)})^4\}^{1/2}\mathrm{tr}\{(\boldsymbol{\Sigma}\boldsymbol{A}_\star)^4\}^{1/2}. \tag{D.7}$$

Also, note that when $\boldsymbol{\Sigma}=\boldsymbol{\Sigma}_*$,

$$\sum_{s\neq s'}^{q_\star}\sum_{t,t'=1}^{d}(\boldsymbol{\gamma}_t^T\boldsymbol{A}_s\boldsymbol{\gamma}_{t'}\boldsymbol{\gamma}_{t'}^T\boldsymbol{A}_{s'}\boldsymbol{\gamma}_t)^2 \le \sum_{s,s'}^{q_\star}\sum_{t=1}^{d}(\boldsymbol{\gamma}_t^T\boldsymbol{\Sigma}_*\boldsymbol{A}_s\boldsymbol{\gamma}_t)(\boldsymbol{\gamma}_t^T\boldsymbol{\Sigma}_*\boldsymbol{A}_{s'}\boldsymbol{\gamma}_t)$$
$$=\sum_{t=1}^{d}(\boldsymbol{\gamma}_t^T\boldsymbol{A}_\star\boldsymbol{\Sigma}_*\boldsymbol{\gamma}_t)^2 \le \mathrm{tr}\{(\boldsymbol{\Sigma}_*\boldsymbol{A}_\star)^4\}.$$

Then, from (D.6), when $\boldsymbol{\Sigma}=\boldsymbol{\Sigma}_*$, we have that as $m\to\infty$

$$\mathrm{Var}\{\acute{\Delta}_n-\acute{\Delta}_n(q_\star)\}=(4\Psi/n^2)\{1+o(1)\}+O\{\mathrm{tr}(\boldsymbol{\Sigma}_*^4)/n^2\}. \tag{D.8}$$

Let $u_{n(l)}=n_{(l)}/(n_{(l)}-1)$ for $l=1,2$. We note that

$$\boldsymbol{y}_{ij(1)}=\frac{\boldsymbol{x}_i}{u_{n(1)}^{1/2}}-\sum_{k\in\boldsymbol{V}_{n(1)(i+j)}\setminus\{i\}}u_{n(1)}^{1/2}\frac{\boldsymbol{x}_k}{n_{(1)}} \quad \text{and}$$
$$\boldsymbol{y}_{ij(2)}=\frac{\boldsymbol{x}_j}{u_{n(2)}^{1/2}}-\sum_{k\in\boldsymbol{V}_{n(2)(i+j)}\setminus\{j\}}u_{n(2)}^{1/2}\frac{\boldsymbol{x}_k}{n_{(2)}} \tag{D.9}$$

for all $i<j$. Similar to (A.4) in Yata and Aoshima (2016), when $\boldsymbol{\Sigma}=\boldsymbol{\Sigma}_*$, it holds that

$$\mathrm{Var}[\widetilde{\Delta}_n+B_n(q_\star+1)-\{\acute{\Delta}_n-\acute{\Delta}_n(q_\star)\}]=o(\Psi/n^2).$$

Hence, when $\boldsymbol{\Sigma}=\boldsymbol{\Sigma}_*$, from (D.8) we have that

$$\begin{aligned}&\mathrm{Var}\{\widetilde{\Delta}_n+B_n(q_\star+1)\}\\ &=\mathrm{Var}\{\acute{\Delta}_n-\acute{\Delta}_n(q_\star)\}+\mathrm{Var}[\widetilde{\Delta}_n+B_n(q_\star+1)-\{\acute{\Delta}_n-\acute{\Delta}_n(q_\star)\}]\\ &\quad+O\Big[\mathrm{Var}\{\acute{\Delta}_n-\acute{\Delta}_n(q_\star)\}^{1/2}\mathrm{Var}\big\{\widetilde{\Delta}_n+B_n(q_\star+1)-\big(\acute{\Delta}_n-\acute{\Delta}_n(q_\star)\big)\big\}^{1/2}\Big]\\ &=(4\Psi/n^2)\{1+o(1)\}+O\{\mathrm{tr}(\boldsymbol{\Sigma}_*^4)/n^2\}.\end{aligned} \tag{D.10}$$

As for $q_\star<2$ or $q_\star=q$, we obtain the result similarly. The proof of Proposition 2 is completed.

As for Lemma 2, by noting that

$$\mathrm{tr}(\boldsymbol{\Sigma}\boldsymbol{A}_j\boldsymbol{\Sigma}\boldsymbol{A}_{j'})\sum_{s=1}^{d}\boldsymbol{\gamma}_s^T\boldsymbol{A}_j\boldsymbol{\gamma}_s\boldsymbol{\gamma}_s^T\boldsymbol{A}_{j'}\boldsymbol{\gamma}_s \le \mathrm{tr}(\boldsymbol{\Sigma}\boldsymbol{A}_j\boldsymbol{\Sigma}\boldsymbol{A}_{j'})^2+\Big(\sum_{s=1}^{d}\boldsymbol{\gamma}_s^T\boldsymbol{A}_j\boldsymbol{\gamma}_s\boldsymbol{\gamma}_s^T\boldsymbol{A}_{j'}\boldsymbol{\gamma}_s\Big)^2$$

and (D.9), we can conclude the result.

D.5 Proofs of Lemma 3, Theorem 4 and Corollary 3

Assume (A-i) and (7). We first consider the proof of Lemma 3. From (20) we write that

$$U_n - \mathrm{tr}(\boldsymbol{\Sigma}_*^2) = B_n(1) + 2\sum_{i<j}^{n} \frac{\sum_{l=1}^{2} \mathrm{tr}\{(\boldsymbol{y}_{ij(l)}\boldsymbol{y}_{ij(l)}^T - \boldsymbol{\Sigma})\boldsymbol{\Sigma}_*\}}{n(n-1)}. \tag{D.11}$$

We note that as $m \to \infty$

$$\begin{aligned} \sum_{j,j'=1}^{q} \frac{(\sum_{s=1}^{d} \boldsymbol{\gamma}_s^T \boldsymbol{A}_j \boldsymbol{\gamma}_s \boldsymbol{\gamma}_s^T \boldsymbol{A}_{j'} \boldsymbol{\gamma}_s)^2}{r_j r_{j'} n^2} &\le \sum_{j,j'=1}^{q} \frac{\mathrm{tr}\{(\boldsymbol{\Sigma A}_j)^2\}\mathrm{tr}\{(\boldsymbol{\Sigma A}_{j'})^2\}}{r_j r_{j'} n^2} \\ &\le \mathrm{tr}(\boldsymbol{\Sigma}_*^2)^2/n^2 = o\{\mathrm{tr}(\boldsymbol{\Sigma}_*^2)^2\}; \\ \sum_{j,j'=1}^{q} \frac{\mathrm{tr}(\boldsymbol{\Sigma A}_j \boldsymbol{\Sigma A}_{j'})^2}{r_j r_{j'} n^2} &\le \mathrm{tr}(\boldsymbol{\Sigma}_*^2)^2/n^2 = o\{\mathrm{tr}(\boldsymbol{\Sigma}_*^2)^2\} \end{aligned} \tag{D.12}$$

because it holds that

$$\mathrm{tr}(\boldsymbol{\Sigma A}_j \boldsymbol{\Sigma A}_{j'})^2 \le \mathrm{tr}\{(\boldsymbol{\Sigma A}_j)^2\}\mathrm{tr}\{(\boldsymbol{\Sigma A}_{j'})^2\} \le \mathrm{tr}(\boldsymbol{\Sigma A}_j)^2 \mathrm{tr}(\boldsymbol{\Sigma A}_{j'})^2 = r_j^2 r_{j'}^2 \kappa_j^2 \kappa_{j'}^2$$

for all j, j'. From Lemma 2 it follows that

$$B_n(1) = o_P\{\mathrm{tr}(\boldsymbol{\Sigma}_*^2)\}. \tag{D.13}$$

For the second term in (D.11), we have that for $l = 1, 2$

$$\mathrm{Var}\Big(\sum_{i<j}^{n} \frac{\mathrm{tr}\{(\boldsymbol{y}_{ij(l)}\boldsymbol{y}_{ij(l)}^T - \boldsymbol{\Sigma})\boldsymbol{\Sigma}_*\}}{n(n-1)}\Big) = O[\mathrm{tr}\{(\boldsymbol{\Sigma\Sigma}_*)^2\}/n] = o\{\mathrm{tr}(\boldsymbol{\Sigma}_*^2)^2\}$$

from the fact that $\mathrm{tr}\{(\boldsymbol{\Sigma\Sigma}_*)^2\} \le \mathrm{tr}(\boldsymbol{\Sigma\Sigma}_*)^2 = \mathrm{tr}(\boldsymbol{\Sigma}_*^2)^2$. Then, from (D.11) and (D.13), it holds that

$$U_n/\mathrm{tr}(\boldsymbol{\Sigma}_*^2) = 1 + o_P(1). \tag{D.14}$$

We have that under (C-iv)

$$\begin{aligned} &\sum_{s=1}^{q_*} E\Big\{\Big(2\sum_{i<j}^{n} \frac{\boldsymbol{y}_{ij(1)}^T \boldsymbol{A}_s \boldsymbol{y}_{ij(1)} \boldsymbol{y}_{ij(2)}^T \boldsymbol{A}_s \boldsymbol{y}_{ij(2)}}{n(n-1)}\Big)^2\Big\} \\ &= O\Big(\sum_{s=1}^{q_*} \mathrm{tr}\{(\boldsymbol{\Sigma A}_s)^2\}^2\Big) = o\{\mathrm{tr}(\boldsymbol{\Sigma}_*^2)^2\}, \end{aligned} \tag{D.15}$$

so that $\widetilde{\Psi}_n = U_n^2 + o_P\{\mathrm{tr}(\boldsymbol{\Sigma}_*^2)^2\}$ from Markov's inequality. Then, from (D.14), we can conclude the result of Lemma 3.

Next, we consider the proof of Corollary 3. From Lemma 2, (D.12) and $\mathrm{tr}(\boldsymbol{\Sigma}_*^2) \le \mathrm{tr}(\boldsymbol{\Sigma}^2)$, it holds that $B_n(1)/\Delta = o_P(1)$ under (C-ii). Thus, from Theorem 1 and (21), it holds that $\widetilde{\Delta}_n/\Delta = 1 + o_P(1)$ under (C-ii). Similar to the proof of Corollary 1, we can conclude the result of Corollary 3.

For Theorem 4, from (24) and Theorem 3 in view of (D.4) and (D.5), we can conclude the result.

D.6 Proofs of Lemmas 4, 5 and Theorem 5

Assume (A-i), (7) and $r_1 = 1$. We first consider the proof of Lemma 4. Similar to (D.2), it holds that $\mathrm{tr}\{(\boldsymbol{\Omega}_2\boldsymbol{\Sigma}_0)^2\} \le \mathrm{tr}(\boldsymbol{\Omega}_2^4)^{1/2}\Delta$. Note that

$$\begin{aligned}\mathrm{tr}\{(\boldsymbol{\Sigma}\boldsymbol{A}_1\boldsymbol{\Sigma}_0\boldsymbol{A}_{(1)})^2\} &\le \|\boldsymbol{\Sigma}^{1/2}\boldsymbol{A}_1\boldsymbol{\Sigma}_0\boldsymbol{A}_{(1)}\boldsymbol{\Sigma}^{1/2}\|_F^2 = \|\boldsymbol{\Sigma}^{1/2}\boldsymbol{A}_1\boldsymbol{\Omega}_1\boldsymbol{\Omega}_2^{1/2}\|_F^2 \\ &\le \lambda_{\max}(\boldsymbol{\Omega}_2)\kappa_1\|\boldsymbol{\Omega}_1\|_F^2 \le \Delta\mathrm{tr}(\boldsymbol{\Omega}_2^4)^{1/4}\kappa_1\end{aligned}$$

because $\boldsymbol{A}_1\boldsymbol{\Sigma}_0\boldsymbol{A}_{(1)} = \boldsymbol{A}_1\boldsymbol{\Sigma}\boldsymbol{A}_{(1)}$ and $\Delta = 2\|\boldsymbol{\Omega}_1\|_F^2 + \|\boldsymbol{A}_{(1)}\boldsymbol{\Sigma}_0\boldsymbol{A}_{(1)}\|_F^2$, where $\lambda_{\max}(\boldsymbol{\Omega}_2)$ denotes the largest eigenvalue of $\boldsymbol{\Omega}_2$. By noting that $\boldsymbol{A}_1\boldsymbol{\Sigma}_0\boldsymbol{A}_1 = \boldsymbol{O}$ and $\mathrm{tr}\{(\boldsymbol{\Sigma}\boldsymbol{A}_{(1)}\boldsymbol{\Sigma}_0\boldsymbol{A}_{(1)})^2\} = \mathrm{tr}\{(\boldsymbol{\Omega}_2\boldsymbol{\Sigma}_0)^2\}$, from (D.2) we have that as $m \to \infty$

$$\mathrm{Var}\Big(\sum_{j=1}^{n}\frac{\mathrm{tr}\{(\boldsymbol{\Sigma}-\boldsymbol{\Sigma}_*)(\boldsymbol{x}_i\boldsymbol{x}_i^T-\boldsymbol{\Sigma})\}}{n}\Big) = O\Big(\frac{\Delta\mathrm{tr}(\boldsymbol{\Omega}_2^4)^{1/4}\{\mathrm{tr}(\boldsymbol{\Omega}_2^4)^{1/4}+\kappa_1\}}{n}\Big). \tag{D.16}$$

Similar to (D.6) and (D.7), from (D.16) we have that

$$\begin{aligned}\mathrm{Var}\{\acute{\Delta}_n - \acute{\Delta}_n(1)\} = L\{1+o(1)\} &+ O\Big(\frac{\Delta\mathrm{tr}(\boldsymbol{\Omega}_2^4)^{1/4}\{\mathrm{tr}(\boldsymbol{\Omega}_2^4)^{1/4}+\kappa_1\}}{n}\Big) \\ &+ O\Big(\frac{\mathrm{tr}(\boldsymbol{\Omega}_2^4)^{1/2}\{\kappa_1^2+\mathrm{tr}(\boldsymbol{\Omega}_2^4)^{1/2}\}}{n^2}\Big).\end{aligned}$$

Then, similar to (D.8) and (D.10), we have that

$$\mathrm{Var}\{\widetilde{\Delta}_n + B_n(2)\} = \mathrm{Var}\{\acute{\Delta}_n - \acute{\Delta}_n(1)\}\{1+o(1)\}.$$

It concludes the first result of Lemma 4. Note that

$$\frac{\Delta\mathrm{tr}(\boldsymbol{\Omega}^4)^{1/4}\{\mathrm{tr}(\boldsymbol{\Omega}^4)^{1/4}+\kappa_1\}}{n} = o(L) \text{ and } \frac{\mathrm{tr}(\boldsymbol{\Omega}_2^4)^{1/2}\{\kappa_1^2+\mathrm{tr}(\boldsymbol{\Omega}_2^4)^{1/2}\}}{n^2} = o(L)$$

under (C-i') and (C-iii'). Thus, from the first result of Lemma 4, we can conclude the second result of Lemma 4.

For Lemma 5 and Theorem 5, from (A.3), it concludes the result of Lemma 5. From (26) and Lemma 5, we can conclude the result of Theorem 5.

D.7 Proofs of Lemma 6, Theorem 6 and Corollary 4

Assume (A-i), (7) and $r_1 = 1$. We first consider the proof of Lemma 6. Let

$$\xi_s = 2\sum_{i<j}^{n}\boldsymbol{y}_{ij(1)}^T\boldsymbol{A}_s\boldsymbol{y}_{ij(1)}\boldsymbol{y}_{ij(2)}^T\boldsymbol{A}_s\boldsymbol{y}_{ij(2)}/\{r_s n(n-1)\}$$

for $s = 1, \dots, q$. We write that

$$U_n^2 - \xi_1^2 = 2\xi_1\sum_{s=2}^{q}\xi_s + \Big(\sum_{s=2}^{q}\xi_s\Big)^2. \tag{D.17}$$

Similar to (D.11) to (D.14), we have that as $m \to \infty$

$$\sum_{s=2}^{q} \xi_s = \Big(\sum_{s=2}^{q} r_s \kappa_s^2\Big)\{1 + o_P(1)\}.$$

Note that $\mathrm{Var}(\xi_1) = O(\kappa_1^4/n)$, so that $\xi_1 = \kappa_1^2\{1 + o_P(1)\}$. Thus, from (D.17) it holds that

$$U_n^2 - \xi_1^2 = \Psi_1\{1 + o_P(1)\}. \tag{D.18}$$

We note that when $q_\star \geq 2$,

$$\sum_{s=2}^{q_\star} \mathrm{tr}\{(\boldsymbol{\Sigma}\boldsymbol{A}_s)^4\} = \sum_{s=2}^{q_\star} \kappa_s^4 = o(\Psi_1) \tag{D.19}$$

under (C-v) because $\mathrm{tr}(\boldsymbol{\Sigma}^2) \geq (\sum_{s=2}^{q} r_s\kappa_s^2)^2 + \kappa_1^2$ and $\sum_{s=2}^{q_\star} \kappa_s^4 \leq (\sum_{s=2}^{q} r_s\kappa_s^2)^2$. Similar to (D.15), from (D.18) and (D.19), we can conclude the result of Lemma 6.

Next, we consider the proof of Corollary 4. From (26), it holds that $B_n(2)/\Delta = o_P(1)$ under (C-ii'). From Lemma 4 we have that $\mathrm{Var}\{\widetilde{\Delta}_n + B_n(2)\}/\Delta^2 = o(1)$ under (C-ii'). Thus, it follows that $\widetilde{\Delta}_n/\Delta = 1 + o_P(1)$ under (C-ii'). Note that $\mathrm{tr}(\boldsymbol{\Sigma}_*^2) - \kappa_1^2 + 2\|\boldsymbol{\Omega}_1\|_F^2 \leq \mathrm{tr}(\boldsymbol{\Sigma}_*^2) - \kappa_1^2 + \Delta = \mathrm{tr}(\boldsymbol{\Sigma}^2) - \kappa_1^2 = \mathrm{tr}(\boldsymbol{\Omega}_2^2) + 2\|\boldsymbol{\Omega}_1\|_F^2$. It holds that $\sum_{s=2}^{q} r_s\kappa_s^2 \leq \mathrm{tr}(\boldsymbol{\Omega}_2^2)$, so that $L_* = O(L)$. Then, similar to the proof of Corollary 1, from Lemma 6 we can conclude the result of Corollary 4.

For Theorem 6, similar to the proof of Theorem 3, from Theorem 5 and Lemma 6 we can conclude the result.

D.8 Proofs of Proposition 3 and Corollary 5

From (A.4) and (A.5) in Yata et al. (2018), we can conclude the result of Proposition 3. From Theorem 4 in view of (D.4)-(D.5), (28), Proposition 3 and Corollary 3, we can conclude the results of Corollary 5.

D.9 Proofs of Proposition 4 and Corollary 6

Assume $\boldsymbol{\Sigma}_* = \boldsymbol{\Sigma}_{\mathrm{D}}$. Note that

$$\mathrm{tr}(\boldsymbol{\Sigma}^2)/\mathrm{tr}(\boldsymbol{\Sigma}_{\mathrm{D}}^2) \in (0, \infty) \quad \text{as } p \to \infty \tag{D.20}$$

under (C-iii). We write that $\boldsymbol{h}_1 = (h_{11}, \ldots, h_{1p})^T$. Note that $\boldsymbol{h}_1^T(\boldsymbol{\Sigma} - \boldsymbol{\Sigma}_{\mathrm{D}})\boldsymbol{h}_1 = \lambda_1 - \sum_{j=1}^{p} h_{1j}^2\sigma_{jj}$. Also, note that $\Delta_{\mathrm{D}} \geq \{\boldsymbol{h}_1^T(\boldsymbol{\Sigma} - \boldsymbol{\Sigma}_{\mathrm{D}})\boldsymbol{h}_1\}^2$ and $\sum_{j=1}^{p} h_{1j}^2\sigma_{jj} = O(1)$ from the fact that $\sigma_{jj} = O(1)$ for all j. Then, if $\liminf_{p\to\infty} \lambda_1^2/\mathrm{tr}(\boldsymbol{\Sigma}_{\mathrm{D}}^2) > 0$, it holds that $\liminf_{p\to\infty} \Delta_{\mathrm{D}}/\mathrm{tr}(\boldsymbol{\Sigma}_{\mathrm{D}}^2) > 0$, so that (C-ii) holds. Then, from

(D.20), under (C-iii) it follows that $\lambda_1^2/\mathrm{tr}(\boldsymbol{\Sigma}^2)\to 0$ as $p\to\infty$. Thus, (C-iii) implies (C-i). On the other hand, we have that

$$\sum_{j,j'=1}^{q}\frac{(\sum_{s=1}^{d}\boldsymbol{\gamma}_s^T\boldsymbol{A}_j\boldsymbol{\gamma}_s\boldsymbol{\gamma}_s^T\boldsymbol{A}_{j'}\boldsymbol{\gamma}_s)^2}{r_jr_{j'}}\le \max_{j=1,\dots,p}\sigma_{jj}^2\sum_{j,j'=1}^{q}\sum_{s=1}^{d}\boldsymbol{\gamma}_s^T\boldsymbol{A}_j\boldsymbol{\gamma}_s\boldsymbol{\gamma}_s^T\boldsymbol{A}_{j'}\boldsymbol{\gamma}_s$$
$$\le \max_{j=1,\dots,p}\sigma_{jj}^2\mathrm{tr}(\boldsymbol{\Sigma}^2)$$
$$\text{and}\ \sum_{j,j'=1}^{q}\frac{\mathrm{tr}(\boldsymbol{\Sigma}\boldsymbol{A}_j\boldsymbol{\Sigma}\boldsymbol{A}_{j'})^2}{r_jr_{j'}}=\sum_{j,j'=1}^{p}\sigma_{jj'}^4\le\mathrm{tr}(\boldsymbol{\Sigma}^4)\qquad\text{(D.21)}$$

because $\sum_{s=1}^{d}\boldsymbol{\gamma}_s^T\boldsymbol{A}_j\boldsymbol{\gamma}_s\boldsymbol{\gamma}_s^T\boldsymbol{A}_{j'}\boldsymbol{\gamma}_s\le\mathrm{tr}(\boldsymbol{\Sigma}\boldsymbol{A}_j)\mathrm{tr}(\boldsymbol{\Sigma}\boldsymbol{A}_{j'})=\sigma_{ii}\sigma_{jj}$. From (D.20) we note that $\max_{j=1,\dots,p}\sigma_{jj}^2\mathrm{tr}(\boldsymbol{\Sigma}^2)=o\{\mathrm{tr}(\boldsymbol{\Sigma}_{\mathrm{D}}^2)^2\}$ and $\mathrm{tr}(\boldsymbol{\Sigma}^4)=o\{\mathrm{tr}(\boldsymbol{\Sigma}_{\mathrm{D}}^2)^2\}$ under (C-i) and (C-iii). Thus, (C-iii) implies (C-iv). It concludes the results of Proposition 4.

Next, we consider the proof of Corollary 6. From Theorem 1, Lemma 2, (21) and (D.21), we have that $\widetilde{\Delta}_n/\Delta=1+o_P(1)$ under (A-i) and (C-ii). Then, from Theorem 4 in view of (D.4)-(D.5), Proposition 4 and Corollary 3, we can conclude the results of Corollary 6.

D.10 Proof of Corollary 7

Note that (C-v') implies (C-v). Then, from Theorem 6 and Corollary 4, we can conclude the results.

D.11 Proofs of Propositions A1 and A2

Let $\boldsymbol{H}_a=(\boldsymbol{a}_1,\dots\boldsymbol{a}_p)$, $\boldsymbol{x}_{j,H}=\boldsymbol{H}_a^T\boldsymbol{x}_{ij}=(x_{j1,H},\dots,x_{jp,H})^T$ $(j=1,\dots,n)$ and $\boldsymbol{\Gamma}_H=\boldsymbol{H}_a^T\boldsymbol{\Gamma}$. We write that $\boldsymbol{x}_{j,H}=\boldsymbol{\Gamma}_H\boldsymbol{w}_j$. Then, for $x_{j1,H}$ and $(x_{j2,H},\dots,x_{jp,H})^T$, from Theorem 3.2 in Yata and Aoshima (2016), we can conclude the result of Proposition A1.

For Proposition A2, let $\boldsymbol{x}_{j,A}=\boldsymbol{A}_{(1)}\boldsymbol{x}_j$ $(j=1,\dots,n)$. Note that $\mathrm{Var}(\boldsymbol{x}_{j,A})=\boldsymbol{\Omega}_2$. Then, from Lemma 1 we can conclude the result of Proposition A2.

D.12 Proofs of Propositions B1 and B2

From Lemma 3.3 in Yata and Aoshima (2016), we have that as $m\to\infty$

$$Q_1/\kappa_1^2=1+o_P(1)\ \text{ and }\ Q_2/\mathrm{tr}(\boldsymbol{\Omega}_2^2)=1+o_P(1)$$

under (A-i). Then, from Corollary 4.1 in Yata and Aoshima (2016), we can conclude the result of Proposition B2. From Theorem 4 in view of (D.4)-(D.5), Propositions A1 and B2, we can conclude the result of Proposition B1.

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