Supplementary Material for "Integral Transform Methods in Goodness-of-Fit Testing, II: The Wishart Distributions"

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S.10 Bessel functions and Hankel transforms of matrix argument

Proof of Lemma 1. Since $|\mathrm{etr}(2\mathrm{i}V'Q)| \le 1$ then it follows from (11) and (12) that

$$\begin{aligned} |A_\nu(V'V)| &\le \frac{1}{\pi^{m^2/2}\Gamma_m(\nu+\frac{1}{2})} \int_{Q'Q<I_m} (\det(I_m - Q'Q))^{\nu-\frac{1}{2}m} \,\mathrm{d}Q \\ &= A_\nu(0) \\ &= \frac{1}{\Gamma_m(\nu+\frac{1}{2}(m+1))}. \qquad \square \end{aligned}$$

Proof of Lemma 2. By (13) and (22),

$$\begin{aligned} |L_\kappa^{(\gamma)}(Z)| &\le \mathrm{etr}(Z) \int_{Y>0} \mathrm{etr}(-Y)(\det Y)^\gamma \; C_\kappa(Y) \; |A_\gamma(ZY)| \,\mathrm{d}Y \\ &\le \frac{1}{\Gamma_m(\gamma+\frac{1}{2}(m+1))} \,\mathrm{etr}(Z) \int_{Y>0} \mathrm{etr}(-Y)(\det Y)^\gamma \; C_\kappa(Y)\,\mathrm{d}Y. \end{aligned}$$

Applying (9) to evaluate the latter integral, we obtain (23).

To establish (24), we substitute $Z = vI_m$ into (21), obtaining

$$\begin{aligned} &\int_{Y>0} \mathrm{etr}(-vY)(\det Y)^\gamma L_\kappa^{(\gamma)}(Y)\,\mathrm{d}Y \\ &\qquad = [\gamma+\tfrac{1}{2}(m+1)]_\kappa \; \Gamma_m(\gamma+\tfrac{1}{2}(m+1)) \; (v-1)^{|\kappa|} \; v^{-[m(\gamma+(m+1)/2)+|\kappa|]} \; C_\kappa(I_m). \end{aligned}$$

Differentiating both sides of the latter equation with respect to v and simplifying the outcome, we obtain the stated result. □

Proof of Lemma 4. By (25),

$$A_\nu(T,Z) = \int_{O(m)} A_\nu(HTH'Z)\,\mathrm{d}H.$$

It is straightforward to verify that the conditions given by Burkill and Burkill (2002, p. 289, Theorem 8.72) for interchanging derivatives and integrals are satisfied; therefore,

$$\nabla_Z A_\nu(T,Z) = \int_{O(m)} \nabla_Z A_\nu(HTH'Z)\,\mathrm{d}H. \tag{S.1}$$

Setting $M = HTH'$ and $Y = M^{1/2}ZM^{1/2}$, we have $Z = M^{-1/2}YM^{-1/2}$. By Maass (1971, p. 64), $\nabla_Z = M^{1/2}\nabla_Y M^{1/2}$; therefore,

$$\begin{aligned} \nabla_Z A_\nu(MZ) &= \nabla_Z A_\nu(M^{1/2}ZM^{1/2}) \\ &= M^{1/2}\nabla_Y M^{1/2} A_\nu(Y) = M^{1/2}\,\nabla_Y A_\nu(Y)\,M^{1/2}, \end{aligned} \tag{S.2}$$

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since $A_\nu(Y)$ is scalar-valued. Combining (S.1) and (S.2), we obtain (30). □

Proof of Lemma 5. Denote by $\nabla_Y \otimes Y^{1/2}$ the Kronecker product of the gradient ∇_Y acting on the matrix $Y^{1/2}$, and let $V_{ij} := (\nabla_Y \otimes Y^{1/2})_{ij}$ be the (i,j)th block matrix in that Kronecker product. Since the trace is a linear operator, we have

$$\begin{aligned}\nabla_Y(\operatorname{tr} QY^{1/2}) &= \Big(\tfrac{1}{2}(1+\delta_{ij})\frac{\partial}{\partial y_{ij}}\operatorname{tr} QY^{1/2}\Big)\\ &= \Big(\operatorname{tr} Q\Big(\tfrac{1}{2}(1+\delta_{ij})\frac{\partial}{\partial y_{ij}}Y^{1/2}\Big)\Big) = \Big(\operatorname{tr}\,[Q(\nabla_Y \otimes Y^{1/2})_{ij}]\Big), \qquad \text{(S.3)}\end{aligned}$$

By the Cauchy-Schwarz inequality, and the fact that $QQ' < I_m$ implies $\operatorname{tr}(QQ') \le m$, we obtain

$$\begin{aligned}\|\nabla_Y(\operatorname{tr} QY^{1/2})\|_F^2 &= \sum_i \sum_j \Big[\operatorname{tr}(QV_{ij})\Big]^2\\ &\le \sum_i \sum_j \operatorname{tr}(QQ')\operatorname{tr}(V_{ij}^2)\\ &\le m\sum_i \sum_j \operatorname{tr}(V_{ij}^2) = m\|(\nabla_Y \otimes Y^{1/2})\|_F^2. \qquad \text{(S.4)}\end{aligned}$$

Since all norms on a finite-dimensional space are equivalent, there exists a constant $c > 0$ such that $\|\nabla_Y \otimes Y^{1/2}\|_F \le 2c\,|||\nabla_Y \otimes Y^{1/2}|||$. By Del Moral and Niclas (2018, p. 262, Eq. (6)),

$$|||\nabla_Y \otimes Y^{1/2}||| \le 2^{-1}(\lambda_{\min}(Y))^{-1/2}.$$

Hence, $\|(\nabla_Y \otimes Y^{1/2})\|_F \le c(\lambda_{\min}(Y))^{-1/2}$, so we obtain

$$\|(\nabla_Y \otimes Y^{1/2})\|_F^2 = \sum_i \sum_j \operatorname{tr}(V_{ij}^2) \le c^2\,(\lambda_{\min}(Y))^{-1}. \qquad \text{(S.5)}$$

Combining (S.4) and (S.5), we obtain (31). □

Proof of Lemma 6. By Eq. (30),

$$\nabla_Z A_\nu(T,Z) = \int_{O(m)} M^{1/2}\nabla_Z A_\nu(Y)M^{1/2}\,\mathrm{d}H,$$

where $M := HTH'$ and $Y := M^{1/2}ZM^{1/2}$. By Minkowski's inequality for integrals,

$$\begin{aligned}\|\nabla_Z A_\nu(T,Z)\|_F &\le \int_{O(m)} \|M^{1/2}[\nabla_Y A_\nu(Y)]M^{1/2}\|_F\,\mathrm{d}H\\ &\le \int_{O(m)} \|M\|_F \cdot \|\nabla_Y A_\nu(Y)\|_F\,\mathrm{d}H, \qquad \text{(S.6)}\end{aligned}$$

since the Frobenius norm is sub-multiplicative.

By Herz's generalization, (12), of the Poisson integral,

$$A_\nu(Y) = c_1 \int_{Q'Q<I_m} \operatorname{etr}(2\mathrm{i}\,Y^{1/2}Q)(\det(I_m - Q'Q))^{\alpha-\frac{1}{2}(2m+1)}\,\mathrm{d}Q,$$

where $c_1 > 0$. Therefore,

$$\nabla_Y A_\nu(Y) = 2\mathrm{i}\,c_1 \int_{Q'Q<I_m} \operatorname{etr}(2\mathrm{i}\,Y^{1/2}Q)(\det(I_m - Q'Q))^{\alpha-\frac{1}{2}(2m+1)}\,\nabla_Y(\operatorname{tr} QY^{1/2})\,\mathrm{d}Q.$$

Applying Minkowski's inequality and then using (31) to bound the integrand, we obtain

$$\begin{aligned}\|\nabla_Y A_\nu(Y)\|_F &\le 2c_1 \int_{Q'Q<I_m} (\det(I_m - Q'Q))^{\alpha-\frac{1}{2}(2m+1)}\,\|\nabla_Y(\operatorname{tr} QY^{1/2})\|_F\,\mathrm{d}Q\\ &\le 2c_1 c(\lambda_{\min}(Y))^{-1/2} \int_{Q'Q<I_m} (\det(I_m - Q'Q))^{\alpha-\frac{1}{2}(2m+1)}\,\mathrm{d}Q\\ &= C(\lambda_{\min}(Y))^{-1/2}. \qquad \text{(S.7)}\end{aligned}$$

Combining (S.6) and (S.7), we obtain

$$\|\nabla_Z A_\nu(T,Z)\|_F \le C \int_{O(m)} \|M\|_F \cdot (\lambda_{\min}(Y))^{-1/2}\, \mathrm{d}H.$$

For $H \in O(m)$, $\|M\|_F = \|HTH'\|_F = \|T\|_F$ and

$$\begin{aligned}\lambda_{\min}(Y) &= \lambda_{\min}(M^{1/2} Z M^{1/2}) = \lambda_{\min}(MZ) \\ &= \lambda_{\min}(HTH'Z) \ge \lambda_{\min}(HTH')\lambda_{\min}(Z) = \lambda_{\min}(T)\lambda_{\min}(Z).\end{aligned}$$

Hence,

$$\begin{aligned}\|\nabla_Z A_\nu(T,Z)\|_F &\le C\,\|T\|_F (\lambda_{\min}(T))^{-1/2}\,(\lambda_{\min}(Z))^{-1/2} \int_{O(m)} \mathrm{d}H \\ &= C\,\|T\|_F (\lambda_{\min}(T))^{-1/2}\,(\lambda_{\min}(Z))^{-1/2},\end{aligned}$$

which completes the proof. □

Proof of Proposition 1. By (30),

$$\nabla_{Z_1} A_\nu(T,Z_1) - \nabla_{Z_2} A_\nu(T,Z_2) = \int_{O(m)} HT^{1/2}H' \Big[\nabla_{Y_1} A_\nu(Y_1) - \nabla_{Y_2} A_\nu(Y_2)\Big] HT^{1/2}H'\, \mathrm{d}H, \quad \text{(S.8)}$$

where $Y_j := M^{1/2} Z_j M^{1/2}$, $j = 1,2$, and $M := HTH'$. Applying (12) and interchanging derivatives and integrals, we obtain

$$\begin{aligned}&\nabla_{Y_1} A_\nu(Y_1) - \nabla_{Y_2} A_\nu(Y_2) \\ &\quad = 2\,\mathrm{i}\,c_1 \int_{Q'Q<I_m} \Big[\mathrm{etr}(2\,\mathrm{i}\,Y_1^{1/2}Q)\nabla_{Y_1}(\mathrm{tr}\,QY_1^{1/2}) - \mathrm{etr}(2\,\mathrm{i}\,Y_2^{1/2}Q)\nabla_{Y_2}(\mathrm{tr}\,QY_2^{1/2})\Big]\, \mathrm{d}\mu(Q),\end{aligned}$$

where $\mathrm{d}\mu(Q) := (\det(I_m - Q'Q))^{\alpha - \frac{1}{2}(2m+1)}\, \mathrm{d}Q$. Therefore,

$$\begin{aligned}&\Big\|\nabla_{Y_1} A_\nu(Y_1) - \nabla_{Y_2} A_\nu(Y_2)\Big\|_F \\ &\quad \le 2c_1 \int_{Q'Q<I_m} \Big\|\mathrm{etr}(2\,\mathrm{i}\,Y_1^{1/2}Q)\nabla_{Y_1}(\mathrm{tr}\,QY_1^{1/2}) - \mathrm{etr}(2\,\mathrm{i}\,Y_2^{1/2}Q)\nabla_{Y_2}(\mathrm{tr}\,QY_2^{1/2})\Big\|_F\, \mathrm{d}\mu(Q).\end{aligned}$$

Let $\theta_j := 2\,\mathrm{tr}(Y_j^{1/2}Q)$ and $N_j := \nabla_{Y_j}(\mathrm{tr}\,QY_j^{1/2})$, $j = 1,2$; then we observe that

$$\begin{aligned}\|e^{\mathrm{i}\theta_1}N_1 - e^{\mathrm{i}\theta_2}N_2\|_F &= \|e^{\mathrm{i}\theta_1}(N_1 - N_2) + (e^{\mathrm{i}\theta_1} - e^{\mathrm{i}\theta_2})N_2\|_F \\ &\le \|N_1 - N_2\|_F + |e^{\mathrm{i}\theta_1} - e^{\mathrm{i}\theta_2}| \cdot \|N_2\|_F,\end{aligned}$$

since $|e^{\mathrm{i}\theta_1}| = 1$. Also, using the identity $|e^{\mathrm{i}\theta_1} - e^{\mathrm{i}\theta_2}|^2 = 4\sin^2\big(\frac{1}{2}(\theta_1 - \theta_2)\big)$, we find that

$$\begin{aligned}&\Big\|\nabla_{Y_1} A_\nu(Y_1) - \nabla_{Y_2} A_\nu(Y_2)\Big\|_F \\ &\quad \le 2c_1 \int_{Q'Q<I_m} \Big[\,\Big\|\nabla_{Y_1}(\mathrm{tr}\,QY_1^{1/2}) - \nabla_{Y_2}(\mathrm{tr}\,QY_2^{1/2})\Big\|_F \\ &\qquad\qquad + 2\,|\sin(\mathrm{tr}(Y_1^{1/2} - Y_2^{1/2})Q)| \cdot \|\nabla_{Y_2}(\mathrm{tr}\,QY_2^{1/2})\|_F\Big]\, \mathrm{d}\mu(Q). \qquad \text{(S.9)}\end{aligned}$$

By applying the same argument as at (S.3), we obtain

$$\nabla_{Y_1}(\mathrm{tr}\,QY_1^{1/2}) - \nabla_{Y_2}(\mathrm{tr}\,QY_2^{1/2}) = \Big(\mathrm{tr}\,[Q(\nabla_{Y_1} \otimes Y_1^{1/2})_{ij}]\Big) - \Big(\mathrm{tr}\,[Q(\nabla_{Y_2} \otimes Y_2^{1/2})_{ij}]\Big);$$

so, by the Cauchy-Schwarz inequality and the fact that $Q'Q < I_m$ implies $\mathrm{tr}\,(QQ') \le m$, we obtain

$$\begin{aligned}\Big\|\nabla_{Y_1}(\mathrm{tr}\,QY_1^{1/2}) - \nabla_{Y_2}(\mathrm{tr}\,QY_2^{1/2})\Big\|_F^2 &= \sum_i \sum_j \Big(\mathrm{tr}\,(Q[(\nabla_{Y_1} \otimes Y_1^{1/2})_{ij} - (\nabla_{Y_2} \otimes Y_2^{1/2})_{ij}])\Big)^2 \\ &\le \sum_i \sum_j \mathrm{tr}\,(QQ')\,\mathrm{tr}\,\Big((\nabla_{Y_1} \otimes Y_1^{1/2})_{ij} - (\nabla_{Y_2} \otimes Y_2^{1/2})_{ij}\Big)^2 \\ &\le m \sum_i \sum_j \mathrm{tr}\,\Big((\nabla_{Y_1} \otimes Y_1^{1/2})_{ij} - (\nabla_{Y_2} \otimes Y_2^{1/2})_{ij}\Big)^2 \\ &= m \Big\|(\nabla_{Y_1} \otimes Y_1^{1/2}) - (\nabla_{Y_2} \otimes Y_2^{1/2})\Big\|_F^2. \end{aligned} \tag{S.10}$$

Since the norms $\|\cdot\|_F$ and $|||\cdot|||$ are equivalent, there exists $c > 0$ such that

$$\begin{aligned}\Big\|(\nabla_{Y_1} \otimes Y_1^{1/2}) - (\nabla_{Y_2} \otimes Y_2^{1/2})\Big\|_F &\le c \Big|\Big|\Big|(\nabla_{Y_1} \otimes Y_1^{1/2}) - (\nabla_{Y_2} \otimes Y_2^{1/2})\Big|\Big|\Big| \\ &\equiv c \sup_{\|K\|_F=1} \Big\|\Big((\nabla_{Y_1} \otimes Y_1^{1/2}) - (\nabla_{Y_2} \otimes Y_2^{1/2})\Big) \cdot K\Big\|_F. \end{aligned} \tag{S.11}$$

By a result of Del Moral and Niclas (2018, Theorem 1.1, Eq. (4)),

$$\begin{aligned}&\Big((\nabla_{Y_1} \otimes Y_1^{1/2}) - (\nabla_{Y_2} \otimes Y_2^{1/2})\Big) \cdot K \\ &\qquad = \int_0^\infty \Big[\exp(-tY_1^{1/2}) K \exp(-tY_1^{1/2}) - \exp(-tY_2^{1/2}) K \exp(-tY_2^{1/2})\Big]\, \mathrm{d}t,\end{aligned}$$

where $\exp(M) = \sum_{j=0}^\infty M^j/j!$ is the matrix exponential function. Therefore,

$$\begin{aligned}&\Big\|\Big((\nabla_{Y_1} \otimes Y_1^{1/2}) - (\nabla_{Y_2} \otimes Y_2^{1/2})\Big) \cdot K\Big\|_F \\ &\qquad \le \int_0^\infty \Big\|\exp(-tY_1^{1/2}) K \exp(-tY_1^{1/2}) - \exp(-tY_2^{1/2}) K \exp(-tY_2^{1/2})\Big\|_F\, \mathrm{d}t.\end{aligned}$$

For any $m \times m$ matrices M_1 and M_2, and for any K such that $\|K\|_F = 1$,

$$\begin{aligned}\|M_1 K M_1 - M_2 K M_2\|_F &= \|M_1 K (M_1 - M_2) + (M_1 - M_2) K M_2\|_F \\ &\le \|M_1\|_F \|M_1 - M_2\|_F + \|M_1 - M_2\|_F \|M_2\|_F \\ &= (\|M_1\|_F + \|M_2\|_F)\ \|M_1 - M_2\|_F.\end{aligned}$$

Now setting $M_j = \exp(-tY_j^{1/2})$, $j = 1, 2$, we obtain

$$\begin{aligned}&\Big\|\exp(-tY_1^{1/2}) K \exp(-tY_1^{1/2}) - \exp(-tY_2^{1/2}) K \exp(-tY_2^{1/2})\Big\|_F \\ &\qquad \le \Big(\Big\|\exp(-tY_1^{1/2})\Big\|_F + \Big\|\exp(-tY_2^{1/2})\Big\|_F\Big)\ \Big\|\exp(-tY_1^{1/2}) - \exp(-tY_2^{1/2})\Big\|_F.\end{aligned}$$

Therefore,

$$\begin{aligned}&\Big\|\Big((\nabla_{Y_1} \otimes Y_1^{1/2}) - (\nabla_{Y_2} \otimes Y_2^{1/2})\Big) \cdot K\Big\|_F \\ &\quad \le \int_0^\infty \Big(\Big\|\exp(-tY_1^{1/2})\Big\|_F + \Big\|\exp(-tY_2^{1/2})\Big\|_F\Big) \Big\|\exp(-tY_1^{1/2}) - \exp(-tY_2^{1/2})\Big\|_F\, \mathrm{d}t. \end{aligned} \tag{S.12}$$

For any $m \times m$ positive-definite matrix Y and for $t \ge 0$,

$$\mathrm{tr}\,(\exp(-2tY)) = \sum_{i=1}^m \exp(-2t\lambda_i(Y)) \le m \exp(-2t\lambda_{\min}(Y));$$

hence, for $t \ge 0$, and $j = 1, 2$,

$$\Big\|\exp(-tY_j^{1/2})\Big\|_F = \Big[\mathrm{tr}\,(\exp(-2tY_j^{1/2}))\Big]^{1/2} \le m^{1/2} \exp(-t\lambda_{\min}(Y_j^{1/2})). \tag{S.13}$$

Therefore, for $\|K\|_F = 1$, the right-hand side of (S.12) is bounded above by

$$m^{1/2} \int_0^\infty \Big[\exp(-t\lambda_{\min}(Y_1^{1/2})) + \exp(-t\lambda_{\min}(Y_2^{1/2}))\Big] \Big\| \exp(-tY_1^{1/2}) - \exp(-tY_2^{1/2}) \Big\|_F \, \mathrm{d}t.$$

Define $X(t) := \exp(-tY_1^{1/2})$, $Y(t) := \exp(-tY_2^{1/2})$, and $\psi(t) := X(t) - Y(t)$, $t \ge 0$. Notice that

$$X'(t) = -Y_1^{1/2} \exp(-tY_1^{1/2}) = -Y_1^{1/2} X(t)$$
$$Y'(t) = -Y_2^{1/2} \exp(-tY_2^{1/2}) = -Y_2^{1/2} Y(t),$$

with $X(0) = Y(0) = I_m$. Then $\psi(t)$ satisfies the inhomogeneous differential equation

$$\psi'(t) = -Y_1^{1/2} X(t) + Y_2^{1/2} Y(t) = -Y_2^{1/2} \psi(t) - (Y_1^{1/2} - Y_2^{1/2}) X(t),$$

with boundary condition $\psi(0) = 0$. By following the approach of Kågström (1977, Section 4), we find that the solution of this differential equation is

$$\psi(t) = -\int_0^t \exp(-(t-s)Y_2^{1/2})(Y_1^{1/2} - Y_2^{1/2}) \exp(-sY_1^{1/2}) \, \mathrm{d}s.$$

By Minkowski's inequality and the sub-multiplicative property of the Frobenius norm,

$$\|\psi(t)\|_F \le \int_0^t \Big\| \exp(-(t-s)Y_2^{1/2}) \Big\|_F \cdot \Big\| Y_1^{1/2} - Y_2^{1/2} \Big\|_F \cdot \Big\| \exp(-sY_1^{1/2}) \Big\|_F \, \mathrm{d}s.$$

Using (S.13) to bound both exponential terms in this integrand, we find that

$$\|\psi(t)\|_F \le m \, \Big\| Y_1^{1/2} - Y_2^{1/2} \Big\|_F \int_0^t \exp(-(t-s)\lambda_{\min}(Y_2^{1/2})) \exp(-s\lambda_{\min}(Y_1^{1/2})) \, \mathrm{d}s.$$

Assuming that $\lambda_{\min}(Y_1^{1/2}) \ne \lambda_{\min}(Y_2^{1/2})$, we calculate the latter integral, obtaining

$$\psi(t) = m \, \Big\| Y_1^{1/2} - Y_2^{1/2} \Big\|_F \cdot \frac{\exp(-t\lambda_{\min}(Y_1^{1/2})) - \exp(-t\lambda_{\min}(Y_2^{1/2}))}{\lambda_{\min}(Y_2^{1/2}) - \lambda_{\min}(Y_1^{1/2})}. \tag{S.14}$$

Combining (S.10)-(S.14), we obtain

$$\begin{aligned} &\Big\| \nabla_{Y_1}(\operatorname{tr} QY_1^{1/2}) - \nabla_{Y_2}(\operatorname{tr} QY_2^{1/2}) \Big\|_F \\ &\quad \le c_2 \frac{\Big\| Y_1^{1/2} - Y_2^{1/2} \Big\|_F}{\lambda_{\min}(Y_2^{1/2}) - \lambda_{\min}(Y_1^{1/2})} \int_0^\infty \Big[\exp(-2t\lambda_{\min}(Y_1^{1/2})) - \exp(-2t\lambda_{\min}(Y_2^{1/2}))\Big] \, \mathrm{d}t \\ &\quad = c_2 \frac{\Big\| Y_1^{1/2} - Y_2^{1/2} \Big\|_F}{\lambda_{\min}(Y_1^{1/2}) \lambda_{\min}(Y_2^{1/2})}. \end{aligned}$$

By continuity, this result remains valid for $\lambda_{\min}(Y_1^{1/2}) = \lambda_{\min}(Y_2^{1/2})$.

Next, it follows from (S.9) that

$$\begin{aligned} \Big\| \nabla_{Y_1} A_\nu(Y_1) - \nabla_{Y_2} A_\nu(Y_2) \Big\|_F &\le c_3 \frac{\Big\| Y_1^{1/2} - Y_2^{1/2} \Big\|_F}{\lambda_{\min}(Y_1^{1/2}) \lambda_{\min}(Y_2^{1/2})} \\ &\quad + c_4 \int_{Q'Q < I_m} |\sin(\operatorname{tr}(Y_1^{1/2} - Y_2^{1/2})Q)| \cdot \|\nabla_{Y_2}(\operatorname{tr} QY_2^{1/2})\|_F \, \mathrm{d}\mu(Q). \end{aligned}$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned} |\sin(\operatorname{tr}(Y_1^{1/2} - Y_2^{1/2})Q)| &\le |\operatorname{tr}(Y_1^{1/2} - Y_2^{1/2})Q| \\ &\le \Big\| Y_1^{1/2} - Y_2^{1/2} \Big\|_F \cdot (\operatorname{tr}(QQ'))^{1/2} \le m^{1/2} \, \Big\| Y_1^{1/2} - Y_2^{1/2} \Big\|_F, \end{aligned}$$

and by (31), $\|\nabla_{Y_2}(\operatorname{tr} QY_2^{1/2})\|_F \le c\,(\lambda_{\min}(Y_2))^{-1/2}$. Hence, with $c_5 = m^{1/2} c_4 c \int_{Q'Q<I_m} \mathrm{d}\mu(Q) < \infty$, we have derived the inequality

$$\Big\|\nabla_{Y_1} A_\nu(Y_1) - \nabla_{Y_2} A_\nu(Y_2)\Big\|_F = \Big\|Y_1^{1/2} - Y_2^{1/2}\Big\|_F \left[\frac{c_3}{\lambda_{\min}(Y_1^{1/2})\lambda_{\min}(Y_2^{1/2})} + \frac{c_5}{\lambda_{\min}(Y_2^{1/2})}\right]. \quad \text{(S.15)}$$

By (S.8), Minkowski's inequality, and the sub-multiplicative property of the Frobenius norm,

$$\begin{aligned}
\Big\|\nabla_{Z_1} A_\nu(T,Z_1) - \nabla_{Z_2} A_\nu(T,Z_2)\Big\|_F &\le \int_{O(m)} \Big\|HT^{1/2}H'\Big[\nabla_{Y_1} A_\nu(Y_1) - \nabla_{Y_2} A_\nu(Y_2)\Big]HT^{1/2}H'\Big\|_F \,\mathrm{d}H \\
&= \int_{O(m)} \Big\|HTH'\Big[\nabla_{Y_1} A_\nu(Y_1) - \nabla_{Y_2} A_\nu(Y_2)\Big]\Big\|_F \,\mathrm{d}H \\
&\le \int_{O(m)} \|T\|_F \cdot \Big\|\nabla_{Y_1} A_\nu(Y_1) - \nabla_{Y_2} A_\nu(Y_2)\Big\|_F \,\mathrm{d}H.
\end{aligned}$$

Applying the bound (S.15), we find that

$$\begin{aligned}
&\Big\|\nabla_{Z_1} A_\nu(T,Z_1) - \nabla_{Z_2} A_\nu(T,Z_2)\Big\|_F \\
&\qquad \le \int_{O(m)} \|T\|_F \cdot \Big\|Y_1^{1/2} - Y_2^{1/2}\Big\|_F \left[\frac{c_3}{\lambda_{\min}(Y_1^{1/2})\lambda_{\min}(Y_2^{1/2})} + \frac{c_5}{\lambda_{\min}(Y_2^{1/2})}\right] \mathrm{d}H. \quad \text{(S.16)}
\end{aligned}$$

By a result of Wihler (2009, Eq. (3.2)),

$$\Big\|Y_1^{1/2} - Y_2^{1/2}\Big\|_F \le m^{1/4}\, \|Y_1 - Y_2\|_F^{1/2}. \quad \text{(S.17)}$$

Since $M = HTH'$, $Y_1 = M^{1/2} Z_1 M^{1/2}$, and $Y_2 = M^{1/2} Z_2 M^{1/2}$, then we have

$$\begin{aligned}
\|Y_1 - Y_2\|_F^{1/2} &= \Big\|HT^{1/2}H'(Z_1 - Z_2)HT^{1/2}H'\Big\|_F^{1/2} \\
&= \|HTH'(Z_1 - Z_2)\|_F^{1/2} \\
&\le \|T\|_F^{1/2} \cdot \|Z_1 - Z_2\|_F^{1/2}. \quad \text{(S.18)}
\end{aligned}$$

Also, for $j = 1, 2$,

$$\begin{aligned}
\lambda_{\min}(Y_j^{1/2}) &= (\lambda_{\min}(Y_j))^{1/2} = (\lambda_{\min}(HTH'Z_j))^{1/2} \\
&\ge (\lambda_{\min}(HTH'))^{1/2}(\lambda_{\min}(Z_j))^{1/2} = \lambda_{\min}(T^{1/2})\lambda_{\min}(Z_j^{1/2}). \quad \text{(S.19)}
\end{aligned}$$

Combining (S.16)-(S.19), and using the fact that $\mathrm{d}H$ is normalized, we obtain

$$\Big\|\nabla_{Z_1} A_\nu(T,Z_1) - \nabla_{Z_2} A_\nu(T,Z_2)\Big\|_F \le \frac{\|Z_1 - Z_2\|_F^{1/2}\ \|T\|_F^{3/2}}{\lambda_{\min}(T^{1/2})\lambda_{\min}(Z_2^{1/2})}\Big[\frac{C_1}{\lambda_{\min}(T^{1/2})\lambda_{\min}(Z_1^{1/2})} + C_2\Big],$$

which is the same as (33). □

Proof of Lemma 7. We will establish this result by the method of Laplace transforms. For $R > 0$, the Laplace transform of the function $(\det T)^\nu \operatorname{tr} g(T)$ is

$$\widehat{g}(R) := \int_{T>0} \operatorname{etr}(-TR)\,(\det T)^\nu \operatorname{tr} g(T)\,\mathrm{d}T. \quad \text{(S.20)}$$

We substitute (34) into this integral, interchange the trace and expectation, apply Fubini's theorem to interchange the expectation and the integral, and verify the validity of interchanging derivatives and integrals; then we obtain

$$\widehat{g}(R) = \alpha^{-1} \operatorname{tr} E\Big[X\, \nabla_Z \int_{T>0} \operatorname{etr}(-TR)\ (\det T)^\nu\ A_\nu(T,Z)\,\mathrm{d}T\Big|_{Z=\alpha^{-1}X}\Big].$$

Applying (25) to write $A_\nu(T,Z)$ as an average over $O(m)$, and reversing the order of integration, we obtain

$$\widehat{g}(R) = \alpha^{-1} \mathrm{tr}\ E\Big[X \nabla_Z \int_{O(m)} \int_{T>0} \mathrm{etr}(-TR)\ (\det T)^\nu A_\nu(HTH'Z)\, \mathrm{d}T\, \mathrm{d}H\Big|_{Z=\alpha^{-1}X}\Big]. \tag{S.21}$$

The inner integral with respect to T is precisely the Laplace transform (14); substituting the outcome of that calculation into (S.21), we obtain

$$\widehat{g}(R) = \alpha^{-1} (\det R)^{-\alpha}\ \mathrm{tr}\ \ E\left[X\ \nabla_Z \int_{O(m)} \mathrm{etr}(-H'ZHR^{-1})\, \mathrm{d}H\Big|_{Z=\alpha^{-1}X}\right].$$

Interchanging the gradient and the integral, and then the integral and the trace, noting that

$$\nabla_Z\, \mathrm{etr}(-H'ZHR^{-1})\Big|_{Z=\alpha^{-1}X} = (-HR^{-1}H')\, \mathrm{etr}(-\alpha^{-1}H'XHR^{-1}),$$

we find that

$$\widehat{g}(R) = (\det R)^{-\alpha}\, E \int_{O(m)} \mathrm{tr}\,(-\alpha^{-1}H'XHR^{-1})\, \mathrm{etr}(-\alpha^{-1}H'XHR^{-1})\, \mathrm{d}H \tag{S.22}$$

since the trace and the integral commute. Next, we have

$$\begin{aligned}\int_{O(m)} &\mathrm{tr}\,(-\alpha^{-1}H'XHR^{-1})\, \mathrm{etr}(-\alpha^{-1}H'XHR^{-1})\, \mathrm{d}H \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{O(m)} \exp(-t\ \mathrm{tr}\,(\alpha^{-1}H'XHR^{-1})\, \mathrm{d}H\Big|_{t=1},\end{aligned} \tag{S.23}$$

by interchanging integral and derivative. By Muirhead (1982, p. 279, Eq. (41)),

$$\int_{O(m)} \exp[-t\, \mathrm{tr}\,(\alpha^{-1}H'XHR^{-1})]\, \mathrm{d}H = \sum_{k=0}^{\infty} \frac{(-t\alpha^{-1})^k}{k!} \sum_{|\kappa|=k} \frac{C_\kappa(X) C_\kappa(R^{-1})}{C_\kappa(I_m)};$$

differentiating this series term-by-term and evaluating the outcome at $t = 1$, we find that (S.23) equals

$$\sum_{k=1}^{\infty} \frac{(-\alpha)^{-k}}{(k-1)!} \sum_{|\kappa|=k} C_\kappa(X) \frac{C_\kappa(R^{-1})}{C_\kappa(I_m)}. \tag{S.24}$$

By (9), $E\, C_\kappa(X) = [\alpha]_\kappa C_\kappa(I_m)$; therefore, by combining (S.22)-(S.24), we obtain

$$\widehat{g}(R) = -\alpha^{-1}\, (\det R)^{-\alpha} \sum_{k=1}^{\infty} \frac{(-\alpha^{-1})^{k-1}}{(k-1)!} \sum_{|\kappa|=k} [\alpha]_\kappa C_\kappa(R^{-1}). \tag{S.25}$$

It is also known from Muirhead (1982, p. 248) that

$$(\det(I_m + tR^{-1}))^{-\alpha} = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \sum_{|\kappa|=k} [\alpha]_\kappa C_\kappa(R^{-1}),$$

for $\|tR^{-1}\| < 1$, where $\|\cdot\|$ denotes the maximum of the absolute values of the eigenvalues of tR^{-1}. Differentiating this series term-by-term with respect to t, we obtain

$$-\frac{\mathrm{d}}{\mathrm{d}t}(\det(I_m + tR^{-1}))^{-\alpha} = \sum_{k=1}^{\infty} \frac{(-t)^{k-1}}{(k-1)!} \sum_{|\kappa|=k} [\alpha]_\kappa C_\kappa(R^{-1});$$

now setting $t = \alpha^{-1}$ and comparing the outcome with (S.25), we find that

$$\widehat{g}(R) = \alpha^{-1}\ \frac{\mathrm{d}}{\mathrm{d}t}(\det(R + tI_m))^{-\alpha}\Big|_{t=\alpha^{-1}}.$$

Therefore, by (10),

$$\begin{aligned}\widehat{g}(R) &= \frac{\alpha^{-1}}{\Gamma_m(\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_{T>0} \operatorname{etr}(-T(R+tI_m))\,(\det T)^{\nu}\,\mathrm{d}T\Big|_{t=\alpha^{-1}} \\ &= \frac{\alpha^{-1}}{\Gamma_m(\alpha)} \int_{T>0} \operatorname{etr}(-TR)\,(\det T)^{\nu}\,(-\operatorname{tr} T)\operatorname{etr}(-\alpha^{-1}T)\,\mathrm{d}T,\end{aligned}$$

evidently a Laplace transform. Comparing this expression with (S.20) then the conclusion follows from the uniqueness theorem for Laplace transforms. □

Proof of Lemma 8. By (13),

$$\Gamma_m(\nu+\tfrac{1}{2}(m+1))\,|A_\nu(TX)| \le 1$$

for all $T, X > 0$. Therefore, by the triangle inequality, $|\mathcal{H}_{X,\nu}(T)| \le E_X(1) = 1$.

Since $A_\nu(TX)$ is bounded and continuous in $T > 0$ for every fixed $X > 0$, then the integrand in (39) is bounded by the Lebesgue integrable function $f(X)$ for all $T, X > 0$. Therefore, the continuity of $\mathcal{H}_{X,\nu}(T)$ follows by Dominated Convergence. □

Proof of Theorem 3. Suppose that $X_n \xrightarrow{d} X$ then, by the Continuous Mapping Theorem for random vectors (Severini 2005, p. 336), $A_\nu(TX_n) \xrightarrow{d} A_\nu(TX)$ as $n \to \infty$, for all $T > 0$. By (13), $A_\nu(TX_n)$ is uniformly bounded for all $n \in \mathbb{N}$ and $T > 0$; thus, by the Dominated Convergence Theorem, $EA_\nu(TX_n) \to EA_\nu(TX)$ as $n \to \infty$, for all $T > 0$, and therefore (44) holds.

Conversely, suppose that $Z \sim W_m(\nu + \frac{1}{2}(m+1), I_m)$ where Z is independent of the sequence $\{X_n, n \in \mathbb{N}\}$. Also, let Ψ_{X_n} be the Laplace transform of X_n. By Example 2, we have

$$\Psi_{X_n}(T) = E_Z[\mathcal{H}_n(T^{1/2} Z T^{1/2})],$$

for all $T > 0$. Further, by Lemma 8, $|\mathcal{H}_n(T^{1/2} Z T^{1/2})| \le 1$ for all $T > 0$. Thus, by the Dominated Convergence Theorem, as $n \to \infty$,

$$\Psi_{X_n}(T) \to E_Z[\mathcal{H}(T^{1/2} Z T^{1/2})] = \Psi(T),$$

for all $T > 0$. Since $\mathcal{H}$ is continuous at 0 and $\mathcal{H}(0) = 1$ then $\Psi(T)$ also is continuous at 0 and $\Psi(0) = 1$. By the continuity of multivariate Laplace transforms (Farrell 1985, p. 15), there is a $m \times m$ positive semi-definite random matrix X whose Laplace transform is Ψ, and $X_n \xrightarrow{d} X$. □

Proof of Theorem 4. The Hankel transform, $\mathcal{H}_{X,\nu}(T)$, of X is holomorphic (analytic) in T. Also, the hypergeometric function ${}_1F_1(\alpha; \nu + \frac{1}{2}(m+1); -T)$ is holomorphic in T. Since these two functions agree on the open neighborhood $\{T : 0 < T < \epsilon I_m\}$ then, by analytic continuation, they agree wherever they both are well-defined. Since they both are well-defined everywhere then we conclude that $\mathcal{H}_{X,\nu}(T) = {}_1F_1(\alpha; \nu + \frac{1}{2}(m+1); -T)$ for all $T > 0$. By Example 1 and Theorem 1, the uniqueness theorem for Hankel transforms, it follows that $X \sim W_m(\alpha, I_m)$. □

Proof of Theorem 5. By Eq. (25) and the definition of the orthogonally invariant Hankel transform (45), we have

$$\widetilde{\mathcal{H}}_{X,\nu}(T) = E_X E_H\, \Gamma_m(\nu + \tfrac{1}{2}(m+1)) A_\nu(HTH'X).$$

Since the distribution of $\widetilde{X}$ is orthogonally invariant, $X \stackrel{d}{=} HXH'$ for all $H \in O(m)$; therefore,

$$\widetilde{\mathcal{H}}_{X,\nu}(T) = E_X\, \Gamma_m(\nu + \tfrac{1}{2}(m+1)) A_\nu(TX) = \mathcal{H}_{X,\nu}(T),$$

$T > 0$, and similarly for Y. By applying Theorem 1, the Uniqueness Theorem for Hankel transforms, we deduce the stated result. □

S.11 The test statistic and its limiting null distribution

Proof of Lemma 10. By the definition (52) of the test statistic we have

$$\begin{aligned}
\boldsymbol{T}_n^2 &= n\int_{T>0}\Big(\frac{1}{n}\sum_{j=1}^{n}\Gamma_m(\alpha)A_\nu(T,Y_j)-\operatorname{etr}(-T/\alpha)\Big)^2\,\mathrm{d}P_0(T)\\
&= \int_{T>0}\Big(\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\Big[\Gamma_m(\alpha)A_\nu(T,Y_j)-\operatorname{etr}(-T/\alpha)\Big]\Big)^2\,\mathrm{d}P_0(T)\\
&= \int_{T>0}\mathcal{Z}_n^2(T)\,\mathrm{d}P_0(T)\\
&= ||\mathcal{Z}_n||_{L^2}^2<\infty. \quad \square
\end{aligned}$$

S.12 Eigenvalues and eigenfunctions of the covariance operator

Proof of Theorem 7. Recall from Muirhead (1982, p. 290) the *Poisson kernel*: For $r\in(0,1)$ and $X,Y>0$,

$$A_\nu\Big(-\frac{r}{(1-r)^2}X,Y\Big)=(1-r)^{m\alpha}\operatorname{etr}\Big(\frac{r}{1-r}(X+Y)\Big)\frac{1}{\Gamma_m(\alpha)}\sum_{k=0}^{\infty}\sum_{|\kappa|=k}\mathcal{L}_\kappa^{(\nu)}(X)\mathcal{L}_\kappa^{(\nu)}(Y)\;r^k. \tag{S.26}$$

In this expansion, set

$$r=b_\alpha^4=(1+\tfrac{1}{2}\alpha(1-\beta))^2, \tag{S.27}$$

so that $r\in(0,1)$. Note that $r^{1/2}=1+\frac{1}{2}\alpha(1-\beta)$ satisfies the quadratic equation

$$r-(\alpha+2)r^{1/2}+1=0$$

and also that this equation is equivalent to the identity

$$\frac{1-r}{\alpha r^{1/2}}=1+\frac{2}{\alpha}(1-r^{1/2}).$$

On the right-hand side of this identity, substitute for $r^{1/2}$ in terms of α and β to obtain

$$\frac{1-r}{\alpha r^{1/2}}=1+\frac{2}{\alpha}[1-(1+\tfrac{1}{2}\alpha(1-\beta))]=\beta. \tag{S.28}$$

In (S.26), also set

$$X=\frac{1-r}{\alpha r^{1/2}}S\equiv\beta S\quad\text{and}\quad Y=\frac{1-r}{\alpha r^{1/2}}T\equiv\beta T.$$

Then,

$$\frac{r(X+Y)}{1-r}=\frac{(r^{1/2}-1)(S+T)}{\alpha}+\frac{(S+T)}{\alpha}. \tag{S.29}$$

Applying (85), (86) and (S.27)-(S.29) to (S.26), and substituting the result in (81), we obtain for $S,T>0$, the pointwise convergent series expansion,

$$K_0(S,T)=\sum_{k=0}^{\infty}\sum_{|\kappa|=k}\rho_\kappa\,\mathfrak{L}_\kappa^{(\nu)}(S)\mathfrak{L}_\kappa^{(\nu)}(T). \tag{S.30}$$

By (20), the generalized Laguerre polynomials $\{\mathcal{L}_\kappa^{(\nu)}\}$ form an orthonormal system; then it is simple to verify that the system $\{\mathfrak{L}_\kappa^{(\nu)}\}$ also is orthonormal in L^2, for κ ranging over all partitions, i.e.,

$$\int_{S>0}\mathfrak{L}_\kappa^{(\nu)}(S)\mathfrak{L}_\sigma^{(\nu)}(S)\,\mathrm{d}P_0(S)=\begin{cases}1, & \kappa=\sigma\\ 0, & \kappa\neq\sigma\end{cases} \tag{S.31}$$

Now we verify that the series (S.30) converges in the separable tensor product Hilbert space $L^2 \otimes L^2 := L^2(P_0 \times P_0)$. By the Cauchy criterion, it suffices to prove that for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\int_{\mathcal{P}_+^{m\times m}\times\mathcal{P}_+^{m\times m}} \Big[\sum_{k=l_1}^{l_2}\sum_{|\kappa|=k} \rho_\kappa \mathfrak{L}_\kappa^{(\nu)}(S)\mathfrak{L}_\kappa^{(\nu)}(T)\Big]^2 \,\mathrm{d}(P_0\otimes P_0)(S,T) < \epsilon,$$

for all $l_1, l_2 \in \mathbb{N}$ such that $l_2 \geq l_1 \geq N$. By squaring the integrand, it suffices by Fubini's theorem to consider

$$\sum_{k=l_1}^{l_2} \int_{S>0}\int_{T>0} \Big[\sum_{|\kappa|=k} \rho_\kappa \mathfrak{L}_\kappa^{(\nu)}(S)\mathfrak{L}_\kappa^{(\nu)}(T)\Big]^2 \,\mathrm{d}P_0(T)\,\mathrm{d}P_0(S)$$
$$+2\sum_{k_1=l_1}^{l_2-1}\sum_{k_2=l_1+1}^{l_2} \int_{S>0}\int_{T>0} \Big[\sum_{|\kappa|=k_1} \rho_\kappa \mathfrak{L}_\kappa^{(\nu)}(S)\mathfrak{L}_\kappa^{(\nu)}(T)\Big]\Big[\sum_{|\kappa|=k_2} \rho_\kappa \mathfrak{L}_\kappa^{(\nu)}(S)\mathfrak{L}_\kappa^{(\nu)}(T)\Big] \,\mathrm{d}P_0(T)\;\mathrm{d}P_0(S).$$

Since the system $\{\mathfrak{L}_\kappa^{(\nu)}\}$ is orthonormal, the latter sum reduces to

$$\sum_{k=l_1}^{l_2}\sum_{|\kappa|=k}\rho_\kappa^2 = \alpha^{2m\alpha} b_\alpha^{4m\alpha}\sum_{k=l_1}^{l_2} b_\alpha^{8k}\sum_{|\kappa|=k} 1 = \alpha^{2m\alpha} b_\alpha^{4m\alpha}\sum_{k=l_1}^{l_2} b_\alpha^{8k} p_m(k),$$

where $p_m(k)$ represents the number of partitions of k into at most m parts. It is well-known that

$$\sum_{k=0}^{\infty} b_\alpha^{8k} p_m(k) = \prod_{k=1}^{m}(1-b_\alpha^{8k})^{-1}.$$

Therefore, $\sum_{k=0}^{\infty} b_\alpha^{8k} p_m(k)$ is a convergent series. Since every convergent series in any metric space is Cauchy, it follows that for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\sum_{k=l_1}^{l_2} b_\alpha^{8k} p_m(k) < \epsilon$, for all $l_1, l_2 \in \mathbb{N}$ such that $l_2 \geq l_1 \geq N$. Therefore, the series (S.30) is Cauchy in $L^2 \otimes L^2$ and hence,

$$\lim_{l\to\infty}\int_{\mathcal{P}_+^{m\times m}\times\mathcal{P}_+^{m\times m}} \Big[K_0(S,T) - \sum_{k=0}^{l}\sum_{|\kappa|=k} \rho_\kappa \mathfrak{L}_\kappa^{(\nu)}(S)\mathfrak{L}_\kappa^{(\nu)}(T)\Big]^2 \,\mathrm{d}(P_0\otimes P_0)(S,T) = 0.$$

By Fubini's theorem, the latter expression equals

$$\lim_{l\to\infty}\int_{S>0}\int_{T>0} \Big[K_0(S,T) - \sum_{k=0}^{l}\sum_{|\kappa|=k} \rho_\kappa \mathfrak{L}_\kappa^{(\nu)}(S)\mathfrak{L}_\kappa^{(\nu)}(T)\Big]^2 \,\mathrm{d}P_0(T)\,\mathrm{d}P_0(S) = 0. \tag{S.32}$$

By the orthonormality, (S.31), of the system $\{\mathfrak{L}_\kappa^{(\nu)}\}$ we obtain for $l \in \mathbb{N}$ and partitions σ with $|\sigma| \leq l$,

$$\int_{T>0}\sum_{k=0}^{l}\sum_{|\kappa|=k} \rho_\kappa \mathfrak{L}_\kappa^{(\nu)}(S)\mathfrak{L}_\kappa^{(\nu)}(T)\;\mathfrak{L}_\sigma^{(\nu)}(T)\,\mathrm{d}P_0(T) = \rho_\sigma \mathfrak{L}_\sigma^{(\nu)}(S). \tag{S.33}$$

By (83) and (S.33),

$$\int_{S>0}\Big|\mathcal{S}_0\mathfrak{L}_\sigma^{(\nu)}(S) - \rho_\sigma\mathfrak{L}_\sigma^{(\nu)}(S)\Big|\,\mathrm{d}P_0(S)$$
$$= \int_{S>0}\Big|\int_{T>0}\Big[K_0(S,T) - \sum_{k=0}^{l}\sum_{|\kappa|=k} \rho_\kappa \mathfrak{L}_\kappa^{(\nu)}(S)\mathfrak{L}_\kappa^{(\nu)}(T)\Big]\mathfrak{L}_\sigma^{(\nu)}(T)\,\mathrm{d}P_0(T)\Big|\,\mathrm{d}P_0(S).$$

By the Cauchy-Schwarz inequality, this latter expression is bounded by

$$\Big(\int_{S>0}\int_{T>0}\Big|K_0(S,T) - \sum_{k=0}^{l}\sum_{|\kappa|=k} \rho_\kappa \mathfrak{L}_\kappa^{(\nu)}(S)\mathfrak{L}_\kappa^{(\nu)}(T)\Big|^2 \,\mathrm{d}P_0(T)\;\mathrm{d}P_0(S)\Big)^{1/2}$$
$$\times\Big(\int_{S>0}\int_{T>0}\Big|\mathfrak{L}_\sigma^{(\nu)}(T)\Big|^2 \,\mathrm{d}P_0(T)\;\mathrm{d}P_0(S)\Big)^{1/2}. \tag{S.34}$$

By the orthonormality property (S.31) and the fact that P_0 is a probability distribution, the second term in (S.34) equals 1; therefore,

$$\int_{S>0} \Big|\mathcal{S}_0 \mathfrak{L}_\sigma^{(\nu)}(S) - \rho_\sigma \mathfrak{L}_\sigma^{(\nu)}(S)\Big| \, \mathrm{d}P_0(S) \\ \le \Big(\int_{S>0}\int_{T>0} \Big|K_0(S,T) - \sum_{k=0}^{l}\sum_{|\kappa|=k} \rho_\kappa \mathfrak{L}_\kappa^{(\nu)}(S)\mathfrak{L}_\kappa^{(\nu)}(T)\Big|^2 \, \mathrm{d}P_0(T)\, \mathrm{d}P_0(S)\Big)^{1/2}. \quad \text{(S.35)}$$

Since l is arbitrary, we let $l \to \infty$; by (S.32), the right-hand side of (S.35) converges to 0, hence

$$\int_{S>0} \Big|\mathcal{S}_0 \mathfrak{L}_\sigma^{(\nu)}(S) - \rho_\sigma \mathfrak{L}_\sigma^{(\nu)}(S)\Big| \, \mathrm{d}P_0(S) = 0,$$

which proves that $\mathcal{S}_0 \mathfrak{L}_\sigma^{(\nu)}(S) = \rho_\sigma \mathfrak{L}_\sigma^{(\nu)}(S)$, for P_0-almost every S. Therefore, ρ_κ is an eigenvalue of $\mathcal{S}_0$ with corresponding eigenfunction $\mathfrak{L}_\kappa^{(\nu)}$.

Since the kernel $K_0(S,T)$ is symmetric in (S,T), it follows that $\mathcal{S}_0$ is symmetric. To show that $\mathcal{S}_0$ is positive, we observe that for $f \in L^2$,

$$\begin{aligned}\langle \mathcal{S}_0 f, f\rangle_{L^2} &= \int_{S>0} \mathcal{S}_0 f(S)\overline{f(S)}\, \mathrm{d}P_0(S) \\ &= \int_{S>0} \Big[\int_{T>0} K_0(S,T)\; f(T)\, \mathrm{d}P_0(T)\Big]\overline{f(S)}\, \mathrm{d}P_0(S).\end{aligned}$$

Substituting for $K_0(S,T)$ from (82), we obtain

$$\langle \mathcal{S}_0 f, f\rangle_{L^2} = \int_{S>0}\Big[\int_{T>0}\Big(\int_{X>0} [\Gamma_m(\alpha)]^2 A_\nu(S,\alpha^{-1}X) \\ \times A_\nu(T,\alpha^{-1}X)\, \mathrm{d}P_0(X)\Big) f(T)\, \mathrm{d}P_0(T)\Big]\overline{f(S)}\, \mathrm{d}P_0(S).$$

Applying Fubini's theorem to reverse the order of the integration, we find that the inner integrals with respect to S and T are complex conjugates of each other; therefore,

$$\langle \mathcal{S}_0 f, f\rangle_{L^2} = \int_{X>0} \Big|\int_{S>0} \Gamma_m(\alpha) A_\nu(S,\alpha^{-1}X) f(S)\, \mathrm{d}P_0(S)\Big|^2 \, \mathrm{d}P_0(X), \quad \text{(S.36)}$$

which is positive. Thus, $\mathcal{S}_0$ is positive.

Next, we prove that $\mathcal{S}_0$ is of trace-class. For $f \in L^2$, $S > 0$, it again follows by (82) and Fubini's theorem that

$$\begin{aligned}\mathcal{S}_0 f(S) &= \int_{T>0} K_0(S,T)\; f(T)\, \mathrm{d}P_0(T) \\ &= \int_{X>0}\int_{T>0} [\Gamma_m(\alpha)]^2 A_\nu(T,\alpha^{-1}X) f(T)\, \mathrm{d}P_0(T) A_\nu(S,\alpha^{-1}X)\, \mathrm{d}P_0(X). \quad \text{(S.37)}\end{aligned}$$

Denote by $\mathcal{T}_0 : L^2 \to L^2$ the integral operator,

$$\mathcal{T}_0 f(T) = \int_{X>0} \Gamma_m(\alpha)\; A_\nu(T,\alpha^{-1}X) f(X)\, \mathrm{d}P_0(X),$$

$T > 0$. By (26), $|\Gamma_m(\alpha)\; A_\nu(T,\alpha^{-1}X)| \le 1$ and therefore

$$\big|\Gamma_m(\alpha)\; A_\nu(T,\alpha^{-1}X)\big|^2_{L^2\otimes L^2} < \infty,$$

for $T, X > 0$. By Young (1998, p. 93), it follows that $\mathcal{T}_0$ is a Hilbert-Schmidt operator. Now, we can write (S.37) as

$$\mathcal{S}_0 f(S) = \int_{X>0} \mathcal{T}_0 f(X)\, \big[\Gamma_m(\alpha) A_\nu(S,\alpha^{-1}X)\big]\; \mathrm{d}P_0(X) = \mathcal{T}_0(\mathcal{T}_0 f)(S),$$

$S > 0$, which proves that $\mathcal{S}_0$ is of trace-class.

To complete the proof, we now show that the set $\{\mathcal{L}_\kappa^{(\nu)}\}$ is complete. It suffices to show that if $f \in L^2$ and $\langle f, \mathcal{L}_\kappa^{(\nu)}\rangle_{L^2} = 0$ for all κ then $f = 0$ P_0-almost everywhere. First, we note that

$$\begin{aligned}
&\int_{S>0} \Big| (\mathcal{S}_0 f)(S)\overline{f(S)} - \sum_{k=0}^{l} \sum_{|\kappa|=k} \rho_\kappa \langle f, \mathcal{L}_\kappa^{(\nu)}\rangle_{L^2}\ \mathcal{L}_\kappa^{(\nu)}(S)\overline{f(S)}\Big| \,\mathrm{d}P_0(S) \\
&= \int_{S>0} \Big| \int_{T>0} \Big[K_0(S,T) - \sum_{k=0}^{l} \sum_{|\kappa|=k} \rho_\kappa \mathcal{L}_\kappa^{(\nu)}(S)\ \mathcal{L}_\kappa^{(\nu)}(T)\Big] f(T)\overline{f(S)} \,\mathrm{d}P_0(T)\Big| \,\mathrm{d}P_0(S) \\
&\le \Big(\int_{S>0}\int_{T>0} \Big| K_0(S,T) - \sum_{k=0}^{l} \sum_{|\kappa|=k} \rho_\kappa \mathcal{L}_\kappa^{(\nu)}(S)\ \mathcal{L}_\kappa^{(\nu)}(T)\Big|^2 \,\mathrm{d}P_0(T)\,\mathrm{d}P_0(S)\Big)^{1/2} \\
&\qquad \times \Big(\int_{S>0}\int_{T>0} |f(S)|^2\ |f(T)|^2 \,\mathrm{d}P_0(T)\,\mathrm{d}P_0(S)\Big)^{1/2},
\end{aligned} \tag{S.38}$$

by the Cauchy-Schwarz inequality. Since $f \in L^2$, the second term on the right-hand side of (S.38) is finite. Taking the limit on both sides of (S.38) as $l \to \infty$ and applying (S.32), we obtain

$$\lim_{l\to\infty} \int_{S>0} \Big| (\mathcal{S}_0 f)(S)\overline{f(S)} - \sum_{k=0}^{l} \sum_{|\kappa|=k} \rho_\kappa \langle f, \mathcal{L}_\kappa^{(\nu)}\rangle_{L^2}\ \mathcal{L}_\kappa^{(\nu)}(S)\overline{f(S)}\Big| \,\mathrm{d}P_0(S) = 0. \tag{S.39}$$

Since $\langle f, \mathcal{L}_\kappa^{(\nu)}\rangle_{L^2} = 0$ for all partitions κ then (S.39) reduces to

$$\langle \mathcal{S}_0 f, f\rangle_{L^2} = \int_{S>0} (\mathcal{S}_0 f)(S)\overline{f(S)} \,\mathrm{d}P_0(S) = 0.$$

Therefore, by (S.36), we obtain for P_0-almost every X,

$$\int_{S>0} \Gamma_m(\alpha) A_\nu(S, \alpha^{-1}X) f(S) \,\mathrm{d}P_0(S) = 0. \tag{S.40}$$

Since the function $\Gamma_m(\alpha)A_\nu(S,\alpha^{-1}X)$ is continuous for all $X > 0$ and fixed $S > 0$ and by (26), $|\Gamma_m(\alpha)A_\nu(S,\alpha^{-1}X)| \le 1$, for $X, S > 0$, then by the Dominated Convergence Theorem, the integral on the left-hand side of (S.40) is a continuous function of X. If two continuous functions are equal P_0-almost everywhere then they are equal everywhere; hence (S.40) holds for all $X > 0$.

Henceforth, without loss of generality, we assume that f is real-valued. Denote by f^+ and f^- the positive and negative parts of f, respectively. Then, $f = f^+ - f^-$, f^+ and f^- are nonnegative; and since $f \in L^2$ then by the Cauchy-Schwarz inequality, f^+ and f^- are P_0-integrable. By (S.40),

$$\int_{S>0} \Gamma_m(\alpha) A_\nu(S,\alpha^{-1}X) f^+(S)\,\mathrm{d}P_0(S) = \int_{S>0} \Gamma_m(\alpha) A_\nu(S,\alpha^{-1}X) f^-(S)\,\mathrm{d}P_0(S),$$

$X > 0$. By Theorem 5, the Uniqueness Theorem for orthogonally invariant Hankel transforms, we notice that there are only two possible cases. Either

$$\int_{S>0} f^+(S)\,\mathrm{d}P_0(S) = \int_{S>0} f^-(S)\,\mathrm{d}P_0(S) = 0,$$

or

$$\int_{S>0} f^+(S)\,\mathrm{d}P_0(S) = \int_{S>0} f^-(S)\,\mathrm{d}P_0(S) = C > 0.$$

For the first case, we have $f^+ = f^- = 0$ and so $f = 0$ P_0-almost everywhere. As for the second case, we have

$$\int_{S>0} A_\nu(S,\alpha^{-1}X) C^{-1} f^+(S)\,\mathrm{d}P_0(S) = \int_{S>0} A_\nu(S,\alpha^{-1}X) C^{-1} f^-(S)\,\mathrm{d}P_0(S),$$

$X > 0$. By the Uniqueness Theorem for orthogonally invariant Hankel transforms, we obtain $f^+ = f^-$ and hence $f = 0$ P_0-almost everywhere. This proves that the orthonormal set $\{\mathcal{L}_\kappa^{(\nu)}\}$ is complete, and therefore it forms a basis in the separable Hilbert space L^2. $\square$

Proof of Theorem 9. Since the set $\{\mathfrak{L}_\kappa^{(\nu)}\}$, for κ ranging over all partitions, is an orthonormal basis for L^2, the eigenfunction $\phi \in L^2$ corresponding to an eigenvalue δ can be written as

$$\phi = \sum_{k=0}^{\infty} \sum_{|\kappa|=k} \langle \phi, \mathfrak{L}_\kappa^{(\nu)} \rangle_{L^2} \; \mathfrak{L}_\kappa^{(\nu)}.$$

We restrict ourselves temporarily to eigenfunctions for which this series is pointwise convergent. Substituting this series into the equation $\mathcal{S}\phi = \delta\phi$, we obtain

$$\int_{T>0} K(S,T) \sum_{k=0}^{\infty} \sum_{|\kappa|=k} \langle \phi, \mathfrak{L}_\kappa^{(\nu)} \rangle_{L^2} \mathfrak{L}_\kappa^{(\nu)}(T) \, \mathrm{d}P_0(T) = \delta \sum_{k=0}^{\infty} \sum_{|\kappa|=k} \langle \phi, \mathfrak{L}_\kappa^{(\nu)} \rangle_{L^2} \mathfrak{L}_\kappa^{(\nu)}(S). \tag{S.41}$$

Substituting the covariance function $K(S,T)$ in the left-hand side of (S.41), writing K in terms of K_0, and assuming that we can interchange the order of integration and summation, we obtain

$$\sum_{k=0}^{\infty} \sum_{|\kappa|=k} \langle \phi, \mathfrak{L}_\kappa^{(\nu)} \rangle_{L^2} \int_{T>0} \Big[K_0(S,T) - \operatorname{etr}(-\alpha^{-1}(S+T))(\alpha^{-3} m^{-1} (\operatorname{tr} S)(\operatorname{tr} T) + 1)\Big] \mathfrak{L}_\kappa^{(\nu)}(T) \, \mathrm{d}P_0(T)$$
$$= \delta \sum_{k=0}^{\infty} \sum_{|\kappa|=k} \langle \phi, \mathfrak{L}_\kappa^{(\nu)} \rangle_{L^2} \mathfrak{L}_\kappa^{(\nu)}(S). \tag{S.42}$$

By Theorem 7,

$$\int_{T>0} K_0(S,T) \mathfrak{L}_\kappa^{(\nu)}(T) \, \mathrm{d}P_0(T) = \rho_\kappa \mathfrak{L}_\kappa^{(\nu)}(S).$$

On writing $\mathfrak{L}_\kappa^{(\nu)}$ in terms of $L_\kappa^{(\nu)}$, the generalized Laguerre polynomial, applying (21) for the Laplace transform of $L_\kappa^{(\nu)}$, and using (S.27) and (S.28), we obtain

$$\begin{aligned} \langle \operatorname{etr}(-\alpha^{-1}T), \mathfrak{L}_\kappa^{(\nu)} \rangle_{L^2} &:= \int_{T>0} \operatorname{etr}(-\alpha^{-1}T) \mathfrak{L}_\kappa^{(\nu)}(T) \, \mathrm{d}P_0(T) \\ &= \Big(\frac{C_\kappa(I_m) \; [\alpha]_\kappa}{|\kappa|!} \Big)^{1/2} \; \beta^{m\alpha/2} \rho_\kappa. \end{aligned} \tag{S.43}$$

Again writing $\mathfrak{L}_\kappa^{(\nu)}$ in terms of $L_\kappa^{(\nu)}$, applying (24), and using (S.27) and (S.28), we obtain

$$\begin{aligned} \langle \operatorname{etr}(-\alpha^{-1}T)(\operatorname{tr} T), \mathfrak{L}_\kappa^{(\nu)} \rangle_{L^2} &:= \int_{T>0} \operatorname{etr}(-\alpha^{-1}T)(\operatorname{tr} T) \mathfrak{L}_\kappa^{(\nu)}(T) \, \mathrm{d}P_0(T) \\ &= \Big(\frac{C_\kappa(I_m) \; [\alpha]_\kappa}{|\kappa! \, |} \Big)^{1/2} \; \alpha^2 \beta^{m\alpha/2} \rho_\kappa (m b_\alpha^2 - |\kappa| \beta). \end{aligned} \tag{S.44}$$

Therefore, (S.42) reduces to

$$\sum_{k=0}^{\infty} \sum_{|\kappa|=k} \rho_\kappa \langle \phi, \mathfrak{L}_\kappa^{(\nu)} \rangle_{L^2} \Big[\mathfrak{L}_\kappa^{(\nu)}(S)$$
$$- \operatorname{etr}(-\alpha^{-1}S) \Big(\frac{C_\kappa(I_m) \; [\alpha]_\kappa}{|\kappa! \, |} \Big)^{1/2} \beta^{m\alpha/2} (\alpha^{-1} m^{-1} (\operatorname{tr} S)(m b_\alpha^2 - |\kappa| \beta) + 1) \Big]$$
$$= \delta \sum_{k=0}^{\infty} \sum_{|\kappa|=k} \langle \phi, \mathfrak{L}_\kappa^{(\nu)} \rangle_{L^2} \mathfrak{L}_\kappa^{(\nu)}(S).$$

By applying (S.43), we obtain the Fourier-Laguerre expansion of $\operatorname{etr}(-\alpha^{-1}S)$ with respect to the orthonormal basis $\{\mathfrak{L}_\kappa^{(\nu)}\}$; indeed,

$$\begin{aligned} \operatorname{etr}(-\alpha^{-1}S) &= \sum_{k=0}^{\infty} \sum_{|\kappa|=k} \langle \operatorname{etr}(-\alpha^{-1}S), \mathfrak{L}_\kappa^{(\nu)} \rangle_{L^2} \; \mathfrak{L}_\kappa^{(\nu)}(S) \\ &= \beta^{m\alpha/2} \sum_{k=0}^{\infty} \sum_{|\kappa|=k} \Big(\frac{C_\kappa(I_m) \; [\alpha]_\kappa}{|\kappa|!} \Big)^{1/2} \; \rho_\kappa \; \mathfrak{L}_\kappa^{(\nu)}(S). \end{aligned}$$

Similarly, by applying (S.44), we have

$$\begin{aligned}\operatorname{etr}(-\alpha^{-1}S)(\operatorname{tr} S) &= \sum_{k=0}^{\infty}\sum_{|\kappa|=k} \langle \operatorname{etr}(-\alpha^{-1}S)(\operatorname{tr} S), \mathfrak{L}_\kappa^{(\nu)}\rangle_{L^2}\ \mathfrak{L}_\kappa^{(\nu)}(S) \\ &= \alpha^2\beta^{m\alpha/2}\sum_{k=0}^{\infty}\sum_{|\kappa|=k}\Big(\frac{C_\kappa(I_m)\ [\alpha]_\kappa}{|\kappa|!}\Big)^{1/2}\ \rho_\kappa(mb_\alpha^2-|\kappa|\beta)\ \mathfrak{L}_\kappa^{(\nu)}(S).\end{aligned}$$

Let

$$\begin{aligned}C_1 &:= \int_{T>0}\operatorname{etr}(-\alpha^{-1}T)\phi(T)\,\mathrm{d}P_0(T) \\ &= \beta^{m\alpha/2}\sum_{k=0}^{\infty}\sum_{|\kappa|=k}\langle\phi,\mathfrak{L}_\kappa^{(\nu)}\rangle_{L^2}\ \Big(\frac{C_\kappa(I_m)\ [\alpha]_\kappa}{|\kappa|!}\Big)^{1/2}\ \rho_\kappa,\end{aligned} \tag{S.45}$$

and

$$\begin{aligned}C_2 &:= \int_{T>0}\operatorname{etr}(-\alpha^{-1}T)(\operatorname{tr} T)\phi(T)\,\mathrm{d}P_0(T) \\ &= \alpha^2\beta^{m\alpha/2}\sum_{k=0}^{\infty}\sum_{|\kappa|=k}\langle\phi,\mathfrak{L}_\kappa^{(\nu)}\rangle_{L^2}\ \Big(\frac{C_\kappa(I_m)\ [\alpha]_\kappa}{|\kappa|!}\Big)^{1/2}\ \rho_\kappa(mb_\alpha^2-|\kappa|\beta).\end{aligned} \tag{S.46}$$

Combining (S.42)-(S.46), we find that (S.41) reduces to

$$\begin{aligned}&\delta\sum_{k=0}^{\infty}\sum_{|\kappa|=k}\langle\phi,\mathfrak{L}_\kappa^{(\nu)}\rangle_{L^2}\mathfrak{L}_\kappa^{(\nu)}(S) \\ &\quad= \sum_{k=0}^{\infty}\sum_{|\kappa|=k}\rho_\kappa\Big[\langle\phi,\mathfrak{L}_\kappa^{(\nu)}\rangle_{L^2} - \beta^{m\alpha/2}\Big(\frac{C_\kappa(I_m)\ [\alpha]_\kappa}{|\kappa|!}\Big)^{1/2}(C_1+C_2\alpha^{-1}(b_\alpha^2-m^{-1}|\kappa|\beta))\Big]\ \mathfrak{L}_\kappa^{(\nu)}(S),\end{aligned}$$

and by comparing the coefficients of $\mathfrak{L}_\kappa^{(\nu)}(S)$, we obtain

$$\delta\langle\phi,\mathfrak{L}_\kappa^{(\nu)}\rangle_{L^2} = \rho_\kappa\Big[\langle\phi,\mathfrak{L}_\kappa^{(\nu)}\rangle_{L^2} - \beta^{m\alpha/2}\Big(\frac{C_\kappa(I_m)\ [\alpha]_\kappa}{|\kappa|!}\Big)^{1/2}(C_1+C_2\alpha^{-1}(b_\alpha^2-m^{-1}|\kappa|\beta))\Big],$$

for all partitions κ. Since we have assumed that $\delta\neq\rho_\kappa$ for any κ then we can solve the equation for $\langle\phi,\mathfrak{L}_\kappa^{(\nu)}\rangle_{L^2}$ to obtain

$$\langle\phi,\mathfrak{L}_\kappa^{(\nu)}\rangle_{L^2} = \beta^{m\alpha/2}\frac{\rho_\kappa}{\rho_\kappa-\delta}\Big(\frac{C_\kappa(I_m)\ [\alpha]_\kappa}{|\kappa|!}\Big)^{1/2}(C_1+C_2\alpha^{-1}(b_\alpha^2-m^{-1}|\kappa|\beta)). \tag{S.47}$$

Substituting (S.47) into (S.45), and applying Lemma 6, we get

$$\begin{aligned}C_1 &= C_1\beta^{m\alpha}\ \sum_{k=0}^{\infty}\sum_{|\kappa|=k}\frac{C_\kappa(I_m)\ [\alpha]_\kappa}{|\kappa|!\,(\rho_\kappa-\delta)}\rho_\kappa^2 + C_2\alpha^{-1}\beta^{m\alpha}\ \sum_{k=0}^{\infty}\sum_{|\kappa|=k}\frac{C_\kappa(I_m)\ [\alpha]_\kappa}{|\kappa|!\,(\rho_\kappa-\delta)}\rho_\kappa^2\ (b_\alpha^2-m^{-1}|\kappa|\beta) \\ &= C_1(1-m^{-1}A(\delta)) + C_2\alpha^{-3}m^{-1}D(\delta);\end{aligned}$$

therefore,

$$\alpha^3C_1A(\delta) = C_2D(\delta). \tag{S.48}$$

Similarly, by substituting (S.47) into (S.46) and applying Lemma 6, we get

$$\begin{aligned}C_2 &= C_1\ \alpha^2\beta^{m\alpha}m\sum_{k=0}^{\infty}\sum_{|\kappa|=k}\frac{C_\kappa(I_m)\ [\alpha]_\kappa}{|\kappa|!\,(\rho_\kappa-\delta)}\rho_\kappa^2\ (b_\alpha^2-m^{-1}|\kappa|\beta) \\ &\qquad + C_2\ \alpha\beta^{m\alpha}m\sum_{k=0}^{\infty}\sum_{|\kappa|=k}\frac{C_\kappa(I_m)\ [\alpha]_\kappa}{|\kappa|!\,(\rho_\kappa-\delta)}\rho_\kappa^2\ (b_\alpha^2-m^{-1}|\kappa|\beta)^2 \\ &= C_1D(\delta)+C_2(1-B(\delta));\end{aligned}$$

hence

$$C_2 B(\delta) = C_1 D(\delta). \tag{S.49}$$

Suppose $C_1 = C_2 = 0$; then it follows from (S.47) that $\langle \phi, \mathfrak{L}_\kappa^{(\nu)} \rangle_{L^2} = 0$ for all partitions κ, which implies that $\phi = 0$, which is a contradiction since ϕ is a non-trivial eigenfunction. Hence, C_1 and C_2 cannot be both equal to 0. Combining (S.48) and (S.49), and using the fact that C_1 and C_2 are not both 0, it is straightforward to deduce that $\alpha^3 A(\delta) B(\delta) = D^2(\delta)$. Therefore, if δ is a positive eigenvalue of $\mathcal{S}$ then it is a positive root of the function $G(\delta) = \alpha^3 A(\delta) B(\delta) - D^2(\delta)$.

Conversely, suppose that δ is a positive root of $G(\delta)$ with $\delta \neq \rho_\kappa$ for any partition κ. Define

$$\gamma_\kappa := \beta^{m\alpha/2} \Big(\frac{C_\kappa(I_m) \, [\alpha]_\kappa}{|\kappa|!} \Big)^{1/2} \frac{\rho_\kappa}{\rho_\kappa - \delta} \Big(C_1 + C_2 \alpha^{-1} (b_\alpha^2 - m^{-1} |\kappa| \beta) \Big), \tag{S.50}$$

where C_1 and C_2 are real constants that are not both equal to 0 and which satisfy (S.48) and (S.49). That such constants exist can be shown by following a case-by-case argument similar to Taherizadeh (2009, p. 48).

Now define, for $S > 0$, the function

$$\tilde{\phi}(S) = \sum_{k=0}^{\infty} \sum_{|\kappa|=k} \gamma_\kappa \; \mathfrak{L}_\kappa^{(\nu)}(S). \tag{S.51}$$

By applying the ratio test, we obtain $\sum_{k=0}^{\infty} \sum_{|\kappa|=k} \gamma_\kappa^2 < \infty$; therefore $\tilde{\phi} \in L^2$.

We also verify that the series (S.51) converges pointwise. By (19) and (86),

$$|\mathfrak{L}_\kappa^{(\nu)}(S)| = \beta^{m\alpha/2} \; \operatorname{etr}((1-\beta) S/2) \; (|\kappa|! \; C_\kappa(I_m) \; [\alpha]_\kappa)^{-1/2} \; |L_\kappa^{\nu}(\beta S)|,$$

$S > 0$. By inequality (23),

$$|L_\kappa^{(\nu)}(\beta S)| \leq \operatorname{etr}(\beta S) \; C_\kappa(I_m) \; [\alpha]_\kappa,$$

$S > 0$. Therefore,

$$|\mathfrak{L}_\kappa^{(\nu)}(S)| \leq \; \beta^{m\alpha/2} \; \operatorname{etr}((1+\beta) S/2) \; \Big(\frac{C_\kappa(I_m) \; [\alpha]_\kappa}{|\kappa|!} \Big)^{1/2}. \tag{S.52}$$

Thus, to establish the pointwise convergence of the series (S.51), we need to show that

$$\sum_{k=0}^{\infty} \sum_{|\kappa|=k} \Big(\frac{C_\kappa(I_m) \; [\alpha]_\kappa}{|\kappa|!} \Big)^{1/2} \; |\gamma_\kappa| < \infty. \tag{S.53}$$

The convergence of the above series follows from the ratio test.

Next, we justify the interchange of summation and integration in our calculations. By a corollary to Theorem 16.7 in Billingsley (1979, p. 224), we need to verify that

$$\sum_{k=0}^{\infty} \sum_{|\kappa|=k} |\gamma_\kappa| \int_{S>0} K(S,T) \; |\mathfrak{L}_\kappa^{(\nu)}(T)| \, \mathrm{d}P_0(T) < \infty. \tag{S.54}$$

First, we find a bound for $K_0(S,T)$. By (26), $|\Gamma_m(\alpha) A_\nu(-\alpha^{-2} S, T)| \leq 1$, $S, T > 0$. Thus, by (81),

$$0 \leq K_0(S,T) \leq \operatorname{etr}(-\alpha^{-1}(S+T)) \tag{S.55}$$

By the triangle inequality and by (S.55), we have

$$\begin{aligned} 0 \leq K(S,T) &\leq K_0(S,T) + \operatorname{etr}(-\alpha^{-1}(S+T)) (\alpha^{-3} m^{-1} (\operatorname{tr} S)(\operatorname{tr} T) + 1) \\ &\leq \operatorname{etr}(-\alpha^{-1}(S+T)) \; (2 + \alpha^{-3} m^{-1} (\operatorname{tr} S)(\operatorname{tr} T)). \end{aligned}$$

Thus, to prove (S.54), we need to establish that

$$\sum_{k=0}^{\infty} \sum_{|\kappa|=k} |\gamma_\kappa| \int_{T>0} \operatorname{etr}(-\alpha^{-1} T) \, (2 + \alpha^{-3} m^{-1} (\operatorname{tr} S)(\operatorname{tr} T)) \; |\mathfrak{L}_\kappa^{(\nu)}(T)| \, \mathrm{d}P_0(T) < \infty.$$

By applying the bound (S.52), we see that it suffices to prove that

$$\sum_{k=0}^{\infty}\sum_{|\kappa|=k}\Big(\frac{C_\kappa(I_m)\ [\alpha]_\kappa}{|\kappa|!}\Big)^{1/2}\ |\gamma_\kappa|\int_{T>0}\operatorname{etr}(-\alpha^{-1}T)\ \operatorname{etr}((1+\beta)T/2)\,\mathrm{d}P_0(T)<\infty,$$

and

$$\sum_{k=0}^{\infty}\sum_{|\kappa|=k}\Big(\frac{C_\kappa(I_m)\ [\alpha]_\kappa}{|\kappa|!}\Big)^{1/2}\ |\gamma_\kappa|\int_{T>0}(\operatorname{tr}T)\operatorname{etr}(-\alpha^{-1}T)\ \operatorname{etr}((1+\beta)T/2)\,\mathrm{d}P_0(T)<\infty.$$

As these integrals are finite, the convergence of both series follows from (S.53).

To calculate $\mathcal{S}\tilde{\phi}(S)$ from (S.51), we follow the same steps as before to obtain

$$\begin{aligned}\mathcal{S}\tilde{\phi}(S)&=\int_{S>0}K(S,T)\sum_{k=0}^{\infty}\sum_{|\kappa|=k}\gamma_\kappa\mathfrak{L}_\kappa^{(\nu)}(T)\,\mathrm{d}P_0(T)\\&=\sum_{k=0}^{\infty}\sum_{|\kappa|=k}\rho_\kappa\gamma_\kappa\mathfrak{L}_\kappa^{(\nu)}(S)-C_1\beta^{m\alpha/2}\sum_{k=0}^{\infty}\sum_{|\kappa|=k}\Big(\frac{C_\kappa(I_m)\ [\alpha]_\kappa}{|\kappa|!}\Big)^{1/2}\ \rho_\kappa\mathfrak{L}_\kappa^{(\nu)}(S)\\&\qquad-C_2\alpha^{-1}\beta^{m\alpha/2}\sum_{k=0}^{\infty}\sum_{|\kappa|=k}\Big(\frac{C_\kappa(I_m)\ [\alpha]_\kappa}{|\kappa|!}\Big)^{1/2}\ \rho_\kappa(b_\alpha^2-m^{-1}|\kappa|\beta)\mathfrak{L}_\kappa^{(\nu)}(S).\end{aligned}$$

By the definition (S.50) of γ_κ, and noting that

$$\frac{\rho_\kappa}{\rho_\kappa-\delta}-1=\frac{\delta}{\rho_\kappa-\delta},$$

we have

$$\begin{aligned}\mathcal{S}\tilde{\phi}(S)&=\beta^{m\alpha/2}\sum_{k=0}^{\infty}\sum_{|\kappa|=k}\Big[\frac{\rho_\kappa}{\rho_\kappa-\delta}-1\Big]\Big(\frac{C_\kappa(I_m)\ [\alpha]_\kappa}{|\kappa|!}\Big)^{1/2}\rho_\kappa\\&\qquad\times\Big(C_1+C_2\alpha^{-1}(b_\alpha^2-m^{-1}|\kappa|\beta)\Big)\mathfrak{L}_\kappa^{(\nu)}(S)\\&=\beta^{m\alpha/2}\delta\sum_{k=0}^{\infty}\sum_{|\kappa|=k}\frac{\rho_\kappa^2}{\rho_\kappa-\delta}\Big(\frac{C_\kappa(I_m)\ [\alpha]_\kappa}{|\kappa|!}\Big)^{1/2}\Big(C_1+C_2\alpha^{-1}(b_\alpha^2-m^{-1}|\kappa|\beta)\Big)\mathfrak{L}_\kappa^{(\nu)}(S)\\&=\delta\sum_{k=0}^{\infty}\sum_{|\kappa|=k}\gamma_\kappa\mathfrak{L}_\kappa^{(\nu)}(S)\\&=\delta\tilde{\phi}(S).\end{aligned}$$

Therefore, δ is an eigenvalue of $\mathcal{S}$ with corresponding eigenfunction $\tilde{\phi}$. □

S.13 The efficiency of the test

Proof of Theorem 14. For $T>0$ and $\theta\in\Theta$, consider the orthogonally invariant Hankel transform, $\mathcal{H}_{X_1,\theta}(T)=E_\theta[\Gamma_m(\alpha)A_\nu(T,\alpha^{-1}X_1)]$. We have

$$n^{-1/2}\boldsymbol{T}_n=\Big[\int_{T>0}\Big(\frac{1}{n}\sum_{j=1}^{n}\Gamma_m(\alpha)A_\nu(T,Z_j)-\operatorname{etr}(-\alpha^{-1}T)\Big)^2\,\mathrm{d}P_0(T)\Big]^{1/2}.$$

Adding and subtracting $\mathcal{H}_{X_1,\theta}(T)$ inside the squared term, and applying Minkowski's inequality, we obtain

$$\begin{aligned}n^{-1/2}\boldsymbol{T}_n\le\ &\Big[\int_{T>0}\Big(\frac{1}{n}\sum_{j=1}^{n}\Gamma_m(\alpha)A_\nu(T,Z_j)-\mathcal{H}_{X_1,\theta}(T)\Big)^2\,\mathrm{d}P_0(T)\Big]^{1/2}\\&+\Big[\int_{T>0}(\mathcal{H}_{X_1,\theta}(T)-\operatorname{etr}(-\alpha^{-1}T))^2\,\mathrm{d}P_0(T)\Big]^{1/2}.\end{aligned}\tag{S.56}$$

Now set

$$b(\theta) := \Big[\int_{T>0} \big(\mathcal{H}_{X_1,\theta}(T) - \operatorname{etr}(-\alpha^{-1}T)\big)^2 \,\mathrm{d}P_0(T)\Big]^{1/2}.$$

By adding and subtracting the term $\frac{1}{n}\sum_{j=1}^{n}\Gamma_m(\alpha)A_\nu(T,Z_j)$ inside the squared term, and then again applying Minkowski's inequality, we obtain

$$b(\theta) \le n^{-1/2}\boldsymbol{T}_n + \Big[\int_{T>0}\Big(\frac{1}{n}\sum_{j=1}^{n}\Gamma_m(\alpha)A_\nu(T,Z_j) - \mathcal{H}_{X_1,\theta}(T)\Big)^2 \,\mathrm{d}P_0(T)\Big]^{1/2}. \tag{S.57}$$

Combining (S.56) and (S.57), we find that

$$|n^{-1/2}\boldsymbol{T}_n - b(\theta)| \le \Big[\int_{T>0}\Big(\frac{1}{n}\sum_{j=1}^{n}\Gamma_m(\alpha)A_\nu(T,Z_j) - \mathcal{H}_{X_1,\theta}(T)\Big)^2 \,\mathrm{d}P_0(T)\Big]^{1/2}. \tag{S.58}$$

Further, by subtracting and adding the term $\frac{1}{n}\sum_{j=1}^{n}\Gamma_m(\alpha)A_\nu(T,\alpha^{-1}X_j)$ inside the squared term in (S.58), and then applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\Big(\frac{1}{n}\sum_{j=1}^{n}\Gamma_m(\alpha)A_\nu(T,Z_j) - \mathcal{H}_{X_1,\theta}(T)\Big)^2 &\le 2\Big[\frac{1}{n}\sum_{j=1}^{n}\Gamma_m(\alpha)\Big(A_\nu(T,Z_j) - A_\nu(T,\alpha^{-1}X_j)\Big)\Big]^2 \\ &\quad + 2\Big[\frac{1}{n}\sum_{j=1}^{n}\Gamma_m(\alpha)A_\nu(T,\alpha^{-1}X_j) - \mathcal{H}_{X_1,\theta}(T)\Big]^2. \end{aligned}\tag{S.59}$$

Next, by (27) in Lemma 3,

$$\frac{1}{n}\sum_{j=1}^{n}\Gamma_m(\alpha)|A_\nu(T,Z_j) - A_\nu(T,\alpha^{-1}X_j)| \le 2m^{3/4}\,\|T\|_F^{1/2}\ \frac{1}{n}\sum_{j=1}^{n}\|Z_j - \alpha^{-1}X_j\|_F^{1/2}. \tag{S.60}$$

Since

$$\begin{aligned} Z_j - \alpha^{-1}X_j &= X_j^{1/2}\bar{X}_n^{-1}X_j^{1/2} - \alpha^{-1}X_j^{1/2}X_j^{1/2} \\ &= \alpha^{-1}X_j^{1/2}\bar{X}_n^{-1/2}(\alpha I_m - \bar{X}_n)\bar{X}_n^{-1/2}X_j^{1/2}, \end{aligned}$$

and since the trace is invariant under cyclic permutations and $\|\cdot\|_F$ is sub-multiplicative,

$$\begin{aligned} \frac{1}{n}\sum_{j=1}^{n}\|Z_j - \alpha^{-1}X_j\|_F^{1/2} &= \alpha^{-1}\frac{1}{n}\sum_{j=1}^{n}\|X_j^{1/2}\bar{X}_n^{-1/2}(\alpha I_m - \bar{X}_n)\bar{X}_n^{-1/2}X_j^{1/2}\|_F^{1/2} \\ &= \alpha^{-1}\frac{1}{n}\sum_{j=1}^{n}\|\bar{X}_n^{-1/2}X_j\bar{X}_n^{-1/2}(\alpha I_m - \bar{X}_n)\|_F^{1/2} \\ &\le \alpha^{-1}\frac{1}{n}\sum_{j=1}^{n}\|\bar{X}_n^{-1/2}X_j\bar{X}_n^{-1/2}\|_F^{1/2}\ \|\alpha I_m - \bar{X}_n\|_F^{1/2}. \end{aligned}$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned} \frac{1}{n}\sum_{j=1}^{n}\|\bar{X}_n^{-1/2}X_j\bar{X}_n^{-1/2}\|_F^{1/2} &\le \frac{1}{\sqrt{n}}\ \Big(\sum_{j=1}^{n}\|\bar{X}_n^{-1/2}X_j\bar{X}_n^{-1/2}\|_F\Big)^{1/2} \\ &= \frac{1}{\sqrt{n}}\ \Big(\sum_{j=1}^{n}[\operatorname{tr}(\bar{X}_n^{-1/2}X_j\bar{X}_n^{-1/2})^2]^{1/2}\Big)^{1/2}. \end{aligned}$$

Since $\bar{X}_n^{-1/2}X_j\bar{X}_n^{-1/2}$ is positive definite then $\operatorname{tr}(\bar{X}_n^{-1/2}X_j\bar{X}_n^{-1/2})^2 \le (\operatorname{tr}\bar{X}_n^{-1/2}X_j\bar{X}_n^{-1/2})^2$, so,

$$\begin{aligned} \frac{1}{n}\sum_{j=1}^{n}\|\bar{X}_n^{-1/2}X_j\bar{X}_n^{-1/2}\|_F^{1/2} &\le n^{-1/2}\ \Big(\sum_{j=1}^{n}\operatorname{tr}\bar{X}_n^{-1/2}X_j\bar{X}_n^{-1/2}\Big)^{1/2} \\ &= n^{-1/2}\ \Big(\operatorname{tr}\bar{X}_n^{-1/2}n\bar{X}_n\bar{X}_n^{-1/2}\Big)^{1/2} = n^{-1/2}\ (n\operatorname{tr}I_m)^{1/2} = m^{1/2}. \end{aligned}$$

Therefore,

$$\frac{1}{n}\sum_{j=1}^{n}\|Z_j-\alpha^{-1}X_j\|_F^{1/2}\le \alpha^{-1}m^{1/2}\ \|\alpha I_m-\bar{X}_n\|_F^{1/2},$$

and by (S.60), we obtain

$$\frac{1}{n}\sum_{j=1}^{n}\Gamma_m(\alpha)|A_\nu(T,Z_j)-A_\nu(T,\alpha^{-1}X_j)|\le 2\alpha^{-1}m^{5/4}\ \|T\|_F^{1/2}\ \|\alpha I_m-\bar{X}_n\|_F^{1/2}. \tag{S.61}$$

By (S.58), Markov's inequality, and Fubini's theorem,

$$\begin{aligned}P(|n^{-1/2}\boldsymbol{T}_n-b(\theta)|\le \epsilon b(\theta))&\\ \ge 1-\frac{1}{\epsilon^2 b^2(\theta)}\int_{T>0}E_\theta\Big(\frac{1}{n}\sum_{j=1}^{n}\Gamma_m(\alpha)A_\nu(T,Z_j)-\mathcal{H}_{X_1,\theta}(T)\Big)^2\,\mathrm{d}P_0(T).\end{aligned} \tag{S.62}$$

By (S.59) and (S.61), we see that (S.62) is greater than or equal to

$$\begin{aligned}1-\frac{1}{\epsilon^2 b^2(\theta)}\Big[8\alpha^{-2}m^{5/2}\Big(\int_{T>0}\|T\|_F\,\mathrm{d}P_0(T)\Big)\ E_\theta\|\alpha I_m-\bar{X}_n\|_F&\\ +2\int_0^\infty E_\theta\Big(\frac{1}{n}\sum_{j=1}^{n}\Gamma_m(\alpha)A_\nu(T,\alpha^{-1}X_j)-\mathcal{H}_{X_1,\theta}(T)\Big)^2\,\mathrm{d}P_0(T)\Big].\end{aligned}$$

In the proof of Theorem 10, we showed that $\tilde{C}:=\int_{T>0}\|T\|_F\,\mathrm{d}P_0(T)<\infty$. Further, by (26),

$$E_\theta\Big(\frac{1}{n}\sum_{j=1}^{n}\Gamma_m(\alpha)A_\nu(T,\alpha^{-1}X_j)-\mathcal{H}_{X_1,\theta}(T)\Big)^2=n^{-1}\,\mathrm{Var}_\theta\,(\Gamma_m(\alpha)A_\nu(T,\alpha^{-1}X_1))\le n^{-1};$$

therefore

$$P(|n^{-1/2}\boldsymbol{T}_n-b(\theta)|\le \epsilon b(\theta))\ge 1-\frac{1}{\epsilon^2 b^2(\theta)}\Big[8\alpha^{-2}m^{5/2}\ \tilde{C}\ E_\theta\|\alpha I_m-\bar{X}_n\|_F+\frac{2}{n}\Big]. \tag{S.63}$$

Next, we write

$$\|\alpha I_m-\bar{X}_n\|_F=(\mathrm{tr}\,(\alpha I_m-\bar{X}_n)^2)^{1/2}=\frac{1}{n}\Big(\mathrm{tr}\,\Big(\sum_{j=1}^{n}(X_j-\alpha I_m)\Big)^2\Big)^{1/2},$$

and expand the sum. By the Cauchy-Schwarz inequality and the i.i.d. property of $X_1,\dots,X_n$,

$$\begin{aligned}E_\theta\|\alpha I_m-\bar{X}_n\|_F&\le\frac{1}{n}\Big[E_\theta\Big(\mathrm{tr}\,\Big(\sum_{j=1}^{n}(X_j-\alpha I_m)\Big)^2\Big)\Big]^{1/2}\\ &=\frac{1}{n}\Big[E_0\Big(\mathrm{tr}\,\Big(\sum_{j=1}^{n}(X_j-\alpha I_m)\Big)^2\cdot\prod_{j=1}^{n}(1+\theta h_\theta(X_j))\Big)\Big]^{1/2}.\end{aligned}$$

Squaring the above sum and using the fact that $X_1,\dots,X_n$ are i.i.d., we obtain

$$\begin{aligned}E_\theta\|\alpha I_m-\bar{X}_n\|_F\le n^{-1/2}\Big[E_0\Big(\mathrm{tr}\,[(X_1-\alpha I_m)^2]\cdot\prod_{j=1}^{n}(1+\theta h_\theta(X_j))\Big)\Big]^{1/2}&\\ +\Big(\frac{n-1}{n}\Big)^{1/2}\Big[E_0\Big(\mathrm{tr}\,[(X_1-\alpha I_m)(X_2-\alpha I_m)]\cdot\prod_{j=1}^{n}(1+\theta h_\theta(X_j))\Big)\Big]^{1/2}.\end{aligned} \tag{S.64}$$

Since $E_0 h_\theta(X)=0$ and, by (107), $E_0 X h_\theta(X)=0$ for $\theta\in\Theta_1$, then $E_0(1+\theta h_\theta(X_1))=1$ and

$$E_0\Big(\mathrm{tr}\,[(X_1-\alpha I_m)]\cdot(1+\theta h_\theta(X_1))\Big)=\mathrm{tr}\,E_0\Big((X_1-\alpha I_m)(1+\theta h_\theta(X_1))\Big)=0.$$

Thus, the first term in the right-hand side of (S.64) equals

$$n^{-1/2}\Big[E_0\Big(\operatorname{tr}[(X_1-\alpha I_m)^2]\cdot(1+\theta h_\theta(X_1))\Big)\Big]^{1/2}$$

and the second term equals 0. Further, by applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} E_0\Big(\operatorname{tr}[(X_1-\alpha I_m)^2]\cdot(1+\theta h_\theta(X_1))\Big) &= E_0(\operatorname{tr}[(X_1-\alpha I_m)^2])+\theta E_0(\operatorname{tr}[(X_1-\alpha I_m)^2]\cdot h_\theta(X_1)) \\ &\le \Big[E_0(\operatorname{tr}[(X_1-\alpha I_m)^2])^2\Big]^{1/2}\Big[1+|\theta|(E_0(h_\theta^2(X_1)))^{1/2}\Big]. \end{aligned}$$

To show that $E_0(\operatorname{tr}[(X_1-\alpha I_m)^2])^2$ is finite, we write

$$\operatorname{tr}[(X_1-\alpha I_m)^2]=\operatorname{tr}(X_1^2-2\alpha X_1+\alpha^2 I_m)=\operatorname{tr}X_1^2-2\alpha\operatorname{tr}X_1+\alpha^2 m,$$

and since $(a+b+c)^2\le 3(a^2+b^2+c^2)$, for $a,b,c\in\mathbb{R}$, it suffices to show that $E_0(\operatorname{tr}X_1^2)^2<\infty$ and $E_0(\operatorname{tr}X_1)^2<\infty$. However, by (9),

$$E_0(\operatorname{tr}X_1^2)^2\le E_0(\operatorname{tr}X_1)^4=\sum_{|\kappa|=4}E_0(C_\kappa(X_1))<\infty,$$

and similarly, $E_0(\operatorname{tr}X_1)^2<\infty$. By assumption (106), we find that there exists $\theta^*\in(0,\eta)$ such that

$$\bar\sigma^2:=\sup_{\theta\in(-\theta^*,\theta^*)}E_0\Big(\operatorname{tr}[(X_1-\alpha I_m)^2]\cdot(1+\theta h_\theta(X_1))\Big)<\infty.$$

Therefore, (S.63) can be written as

$$\begin{aligned} P(|n^{-1/2}\boldsymbol{T}_n-b(\theta)|\le\epsilon b(\theta)) &\ge 1-\frac{1}{n^{1/2}\epsilon^2 b^2(\theta)}\Big[8\alpha^{-2}m^{5/2}\,\tilde C\bar\sigma+\frac{2}{n^{1/2}}\Big] \\ &\ge 1-\frac{8\alpha^{-2}m^{5/2}\,\tilde C\bar\sigma+2}{n^{1/2}\epsilon^2 b^2(\theta)}, \end{aligned}$$

for all $\theta\in(-\theta^*,\theta^*)$. Setting $C=(8\alpha^{-2}m^{5/2}\,\tilde C\bar\sigma+2)/\epsilon^2\gamma$ then we obtain

$$P(|n^{-1/2}\boldsymbol{T}_n-b(\theta)|\le\epsilon b(\theta))\ge 1-\frac{\gamma C}{n^{1/2}b^2(\theta)}>1-\gamma.$$

for all $\theta\in(-\theta^*,\theta^*)$ and $n^{1/2}>C/b^2(\theta)$. The proof now is complete. □