Supplementary Material to "Fixed point characterizations of continuous univariate probability distributions and their applications"

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Note that except for Equations (1.1) and (3.1) below, all numbers of equations, theorems, etc. used in the following supplementary material refer to the numbers in the manuscript to which this supplementary material belongs.

1 Proof of Lemma 1

If $X \sim p\mathcal{L}^1$, any $f \in \mathcal{F}_p$ satisfies

$$\begin{aligned}
\mathbb{E}\left[f'(X) + \frac{p'(X)}{p(X)}\, f(X) \right] &= \int_{S(p,f)} \big(f \cdot p\big)'(x)\,\mathrm{d}x \\
&= \int_L^{y_1^f} \big(f \cdot p\big)'(x)\,\mathrm{d}x + \sum_{\ell=1}^{m} \int_{y_\ell^f}^{y_{\ell+1}^f} \big(f \cdot p\big)'(x)\,\mathrm{d}x \\
&\quad + \int_{y_{m+1}^f}^{R} \big(f \cdot p\big)'(x)\,\mathrm{d}x \\
&= \lim_{x \nearrow R} f(x)\,p(x) - \lim_{x \searrow L} f(x)\,p(x) \\
&\quad + \sum_{\ell=1}^{m+1} \left(\lim_{x \nearrow y_\ell^f} f(x)\,p(x) - \lim_{x \searrow y_\ell^f} f(x)\,p(x) \right) \\
&= 0.
\end{aligned}$$

For the converse, fix $t \in S(p) \setminus \operatorname{disc}(X)$ and define $f_t^p : (L, R) \to \mathbb{R}$ through

$$f_t^p(x) = \frac{1}{p(x)} \int_L^x \Big(\mathbb{1}_{(L,t]}(s) - P(t) \Big) p(s) \, \mathrm{d}s.$$

The function f_t^p is continuous, and

$$\lim_{x \nearrow R} f_t^p(x) \, p(x) = \int_L^R \Big(\mathbb{1}_{(L,t]}(s) - P(t) \Big) p(s) \, \mathrm{d}s = P(t) - P(t) = 0.$$

Noting that $f_t^p(x) = \frac{1}{p(x)} \, P(x) \big(1 - P(t) \big)$ for $x < t$, we also have $\lim_{x \searrow L} f_t^p(x) \, p(x) = 0$. With this representation of $f_t^p(x)$ for $x < t$, as well as with $f_t^p(x) = \frac{1}{p(x)} \big(1 - P(x) \big) P(t)$, for $x > t$, we see that f_t^p is differentiable on $S(p) \setminus \{t\} = S(p, f_t^p)$ with

$$f_t^{p\,\prime}(x) = - \frac{p'(x)}{p(x)} \, f_t^p(x) + \mathbb{1}_{(L,t]}(x) - P(t), \quad x \in S(p) \setminus \{t\}. \tag{1.1}$$

We get with condition (C2)

$$\sup_{x \in S(p) \setminus \{t\}} \left| \frac{p'(x)}{p(x)} \, f_t^p(x) \right| \leq 2 \sup_{x \in S(p)} \left| \frac{p'(x) \min\{P(x), \, 1 - P(x)\}}{p^2(x)} \right| = 2 \sup_{x \in S(p)} \kappa_p(x) < \infty,$$

and, by (1.1),

$$\sup_{x \in S(p) \setminus \{t\}} \left| f_t^{p\,\prime}(x) \right| \leq \sup_{x \in S(p) \setminus \{t\}} \left| \frac{p'(x)}{p(x)} \, f_t^p(x) \right| + 2 \leq 2 \sup_{x \in S(p)} \kappa_p(x) + 2 < \infty.$$

Thus $f_t^p \in \mathcal{F}_p$. The assumption in the converse implication and (1.1) yield

$$0 = \mathbb{E} \left[f_t^{p\,\prime}(X) + \frac{p'(X)}{p(X)} \, f_t^p(X) \right] = \mathbb{P}(X \leq t) - P(t).$$

Hence $\mathbb{P}(X \leq t) = P(t)$ for all $t \in S(p) \setminus \operatorname{disc}(X)$. As $S(p) \setminus \operatorname{disc}(X)$ is dense in (L, R) and $t \mapsto \mathbb{P}(X \leq t)$, $t \mapsto P(t)$ are right-continuous, the claim follows. □

2 Proof of Theorem 1

Let $X \sim p\mathcal{L}^1$. By (C1) and (C3) we may use the fundamental theorem of calculus to obtain

$$\begin{aligned} F_X(t) = P(t) &= \int_{-\infty}^t \left(p(y_1) + \sum_{\ell=1}^{k-1} \Big(p(y_{\ell+1}) - p(y_\ell) \Big) + p(s) - p(y_k) \right) \mathrm{d}s \\ &= \int_{-\infty}^t \left(\int_{-\infty}^{y_1} p'(x) \, \mathrm{d}x + \sum_{\ell=1}^{k-1} \int_{y_\ell}^{y_{\ell+1}} p'(x) \, \mathrm{d}x + \int_{y_k}^s p'(x) \, \mathrm{d}x \right) \mathrm{d}s, \end{aligned}$$

where $k = k(s)$ is the largest index in $\{1, \ldots, m\}$ for which still $y_k < s$ [for all $s \leq y_1$ the y_ℓ need not be taken into account as p is continuously differentiable on $(-\infty, s)$ in these cases]. Now, since X has density function p, we easily see [still using (C1)] that

$$\int_{y_\ell}^{y_{\ell+1}} p'(x) \, \mathrm{d}x = \mathbb{E} \left[\frac{p'(X)}{p(X)} \, \mathbb{1}\{y_\ell < X \leq y_{\ell+1}\} \right],$$

for $\ell \in \{1, \ldots, k-1\}$, and similar representations for the other integrals give

$$F_X(t) = \int_{-\infty}^{t} \mathbb{E}\left[\frac{p'(X)}{p(X)} \mathbb{1}\{X \leq s\}\right] \mathrm{d}s = \mathbb{E}\left[\frac{p'(X)}{p(X)} (t - X) \mathbb{1}\{X \leq t\}\right], \quad t \in \mathbb{R},$$

where, in the second equality, we used Fubini's theorem. That is admissible since Tonelli's theorem and (C3) imply for each $t \in \mathbb{R}$

$$\begin{aligned}\int_{-\infty}^{t} \mathbb{E}\left[\left|\frac{p'(X)}{p(X)}\right| \mathbb{1}\{X \leq s\}\right] \mathrm{d}s &= \mathbb{E}\left[\frac{|p'(X)|}{p(X)} (t - X) \mathbb{1}\{X \leq t\}\right] \\ &\leq \int_{S(p)} \big|p'(x)\big| \big(|t| + |x|\big) \,\mathrm{d}x \\ &< \infty.\end{aligned}$$

For the converse, assume that the distribution function of X is given through the explicit formula in terms of X as in the theorem. Putting

$$d_p^X(t) = \mathbb{E}\left[\frac{p'(X)}{p(X)} \mathbb{1}\{X \leq t\}\right], \quad t \in \mathbb{R},$$

condition (6) entails

$$\mathbb{E}\left[\int_{-\infty}^{t} \frac{|p'(X)|}{p(X)} \mathbb{1}\{X \leq s\} \,\mathrm{d}s\right] = \mathbb{E}\left[\frac{|p'(X)|}{p(X)} \big(t - X\big) \mathbb{1}\{X \leq t\}\right] < \infty$$

for every $t \in \mathbb{R}$. Thus, Fubini's theorem implies

$$\int_{-\infty}^{t} d_p^X(s) \,\mathrm{d}s = \int_{-\infty}^{t} \mathbb{E}\left[\frac{p'(X)}{p(X)} \mathbb{1}\{X \leq s\}\right] \mathrm{d}s = \mathbb{E}\left[\frac{p'(X)}{p(X)} \int_{-\infty}^{t} \mathbb{1}\{X \leq s\} \,\mathrm{d}s\right] = F_X(t)$$

for $t \in \mathbb{R}$. Since F_X is increasing and d_p^X is right-continuous, we conclude $d_p^X \geq 0$. Moreover, we infer

$$\int_{\mathbb{R}} d_p^X(s) \,\mathrm{d}s = \lim_{t \to \infty} \int_{-\infty}^{t} d_p^X(s) \,\mathrm{d}s = \lim_{t \to \infty} F_X(t) = 1,$$

for F_X is a distribution function. Hence, d_p^X is the density function of X. Using the first part of (6), dominated convergence gives

$$\mathbb{E}\left[\frac{p'(X)}{p(X)}\right] = \lim_{t \to \infty} \mathbb{E}\left[\frac{p'(X)}{p(X)} \mathbb{1}\{X \leq t\}\right] = \lim_{t \to \infty} d_p^X(t) = 0.$$

Therefore, we conclude that for each $f \in \mathcal{F}_p$

$$
\begin{aligned}
\mathbb{E}\big[f'(X)\big] &= \int_{S(p,f)} f'(s)\, d_p^X(s)\,\mathrm{d}s \\
&= \int_{-\infty}^{y_1^f} f'(s)\,\mathbb{E}\left[\frac{p'(X)}{p(X)}\,\mathbb{1}\{X \le s\}\right]\mathrm{d}s + \sum_{\ell=1}^{m} \int_{y_\ell^f}^{y_{\ell+1}^f} f'(s)\,\mathbb{E}\left[\frac{p'(X)}{p(X)}\,\mathbb{1}\{X \le s\}\right]\mathrm{d}s \\
&\quad + \int_{y_{m+1}^f}^{\infty} f'(s)\,\mathbb{E}\left[-\frac{p'(X)}{p(X)}\,\mathbb{1}\{X > s\}\right]\mathrm{d}s \\
&= \mathbb{E}\left[\frac{p'(X)}{p(X)}\Big(f(y_1^f) - f(X)\Big)\mathbb{1}\{X \le y_1^f\}\right] \\
&\quad + \sum_{\ell=1}^{m} \mathbb{E}\left[\frac{p'(X)}{p(X)}\Big(f(y_{\ell+1}^f) - f(X)\Big)\mathbb{1}\{y_\ell^f < X \le y_{\ell+1}^f\}\right] \\
&\quad + \sum_{\ell=1}^{m} \mathbb{E}\left[\frac{p'(X)}{p(X)}\Big(f(y_{\ell+1}^f) - f(y_\ell^f)\Big)\mathbb{1}\{X \le y_\ell^f\}\right] \\
&\quad + \mathbb{E}\left[-\frac{p'(X)}{p(X)}\Big(f(X) - f(y_{m+1}^f)\Big)\mathbb{1}\{X > y_{m+1}^f\}\right] \\
&= \mathbb{E}\left[-\frac{p'(X)}{p(X)}\,f(X)\right].
\end{aligned}
$$

In the third equality, Fubini's theorem is applicable since f' is bounded on $S(p,f)$ and we have assumption (6). Lemma 1 yields the claim. □

3 Proof of Theorem 3

Let $X \sim p\mathcal{L}^1$. By Theorem 2, we have

$$
F_X(t) = P(t) = \int_L^t \mathbb{E}\left[-\frac{p'(X)}{p(X)}\,\mathbb{1}\{X > s\}\right]\mathrm{d}s = \mathbb{E}\left[-\frac{p'(X)}{p(X)}\Big(\min\{X,t\} - L\Big)\right], \quad t > L,
$$

where Fubini's theorem is applicable since Tonelli's theorem and (C3) imply

$$
\begin{aligned}
\int_L^\infty \mathbb{E}\left[\left|\frac{p'(X)}{p(X)}\right|\,\mathbb{1}\{X > s\}\right]\mathrm{d}s &= \mathbb{E}\left[\frac{|p'(X)|}{p(X)}\,\big(X - L\big)\right] \\
&\le \int_{S(p)} |x|\,\big|p'(x)\big|\,\mathrm{d}x + |L| \int_{S(p)} \big|p'(x)\big|\,\mathrm{d}x \\
&< \infty.
\end{aligned}
$$

For the converse implication, we put

$$
d_p^X(s) = \mathbb{E}\left[-\frac{p'(X)}{p(X)}\,\mathbb{1}\{X > s\}\right], \quad s > L,
$$

and notice that the integrability conditions on X imply

$$
\mathbb{E}\left[\int_L^\infty \frac{|p'(X)|}{p(X)}\,\mathbb{1}\{X > s\}\,\mathrm{d}s\right] \le \mathbb{E}\left|\frac{p'(X)}{p(X)}\,X\right| + |L| \cdot \mathbb{E}\left|\frac{p'(X)}{p(X)}\right| < \infty. \tag{3.1}
$$

Thus, Fubini's theorem gives

$$\int_L^t d_p^X(s)\,\mathrm{d}s = \mathbb{E}\left[-\frac{p'(X)}{p(X)}\int_L^t \mathbb{1}\{X > s\}\,\mathrm{d}s\right] = F_X(t), \quad t > L.$$

Since d_p^X is integrable by (3.1), dominated convergence implies that F_X is continuous. Moreover, Lebesgue's differentiation theorem [see Theorem 3.21 from Folland (1999), with nicely shrinking sets $E_h = (t, t+h)$, $h > 0$] implies

$$d_p^X(t) = \lim_{h \searrow 0} \frac{1}{h}\int_t^{t+h} d_p^X(s)\,\mathrm{d}s = \lim_{h \searrow 0} \frac{F_X(t+h) - F_X(t)}{h} \geq 0$$

for $\mathcal{L}^1$-a.e. $t > L$, where we used that F_X is increasing. Finally,

$$\int_L^\infty d_p^X(s)\,\mathrm{d}s = \lim_{t \to \infty} \int_L^t d_p^X(s)\,\mathrm{d}s = \lim_{t \to \infty} F_X(t) = 1.$$

We conclude that d_p^X is the density function of X. The claim follows immediately from Theorem 2. □

Remark. Note that we could have proven the theorem with the same argument we used in Theorem 1, since the first integrability condition on X ensures that d_p^X is left-continuous. However, in Remark 6 we extended the argument of Remark 5 dropping that first integrability condition in the case $L = 0$. Then we can no longer conclude the left-continuity, so we had to use the different argument via Lebesgue's differentiation theorem.

4 Proof of Lemma 2

The necessity part follows with a simple rewriting of the density function, as before. For the converse implication, assume that X is as in the statement of the lemma, and that

$$d_p^X(t) = \mathbb{E}\left[-\frac{p'(X)}{p(X)}\,\mathbb{1}\{X > t\}\right] + \lim_{x \nearrow R} p(x), \quad L < t < R,$$

is the density function of X. Since we assume both (C4) and (C5), we have by Remark 3 for any $f \in \mathcal{F}_p$ [note that f is continuous on (L, R)]

$$\begin{aligned}
\int_{S(p,f)} f'(x)\,\mathrm{d}x &= \int_L^{y_1^f} f'(x)\,\mathrm{d}x + \sum_{\ell=1}^m \int_{y_\ell^f}^{y_{\ell+1}^f} f'(x)\,\mathrm{d}x + \int_{y_{m+1}^f}^R f'(x)\,\mathrm{d}x \\
&= \lim_{x \nearrow y_1^f} f(x) - \lim_{x \searrow L} f(x) + \sum_{\ell=1}^m \left(\lim_{x \nearrow y_{\ell+1}^f} f(x) - \lim_{x \searrow y_\ell^f} f(x) \right) \\
&\quad + \lim_{x \nearrow R} f(x) - \lim_{x \searrow y_{m+1}^f} f(x) \\
&= 0,
\end{aligned}$$

where the integral exists by the boundedness of f' and the fact that $S(p, f) \subset S(p) \subset (L, R)$ which is a bounded interval. Using this fact, the proof is concluded via Lemma 1 with a similar calculation as in previous proofs. $\square$

5 Goodness-of-fit tests for the Gamma distribution

In Betsch and Ebner (2019a), the authors establish the result of Corollary 3 for the special case of the Gamma distribution and examine the corresponding goodness-of-fit statistic. Denote by $p_\vartheta(x) = \frac{\lambda^{-k}}{\Gamma(k)}\, x^{k-1}\, e^{-x/\lambda}$, $x > 0$, where $\vartheta = (k, \lambda) \in (0, \infty)^2 = \Theta$, the density function of the Gamma distribution with shape parameter k and scale parameter λ. Let X be a positive random variable with $\mathbb{E}X < \infty$. To reflect the scale invariance of the class of Gamma distributions, choose the scaling function $s(x; \vartheta) = x/\lambda$. Apparently, $X \sim p_\vartheta \mathcal{L}^1$ if, and only if, $s(X; \vartheta) \sim p_{\vartheta^*} \mathcal{L}^1$, where $\vartheta^* = (k, 1) \in (0, \infty) \times \{1\} = \Theta^*$, and

$$\mathbb{E}\left|\frac{p'_{\vartheta^*}\big(s(X;\vartheta)\big)}{p_{\vartheta^*}\big(s(X;\vartheta)\big)}\, s(X;\vartheta)\right| \leq |k-1| + \lambda^{-1}\mathbb{E}X < \infty.$$

By Example 3, X follows a Gamma law with parameter vector $\vartheta = (k, \lambda)$ if, and only if,

$$F_{X/\lambda}(t) = F_{s(X;\vartheta)}(t) = \mathbb{E}\left[\left(-\frac{k-1}{s(X;\vartheta)} + 1\right) \min\big\{s(X;\vartheta), t\big\}\right], \quad t > 0.$$

To construct the goodness-of-fit test, let $X_1, \ldots, X_n$ be iid. copies of X and consider a consistent, scale equivariant estimator $\widehat{\lambda}_n = \widehat{\lambda}_n(X_1, \ldots, X_n)$ of λ as well as a consistent, scale invariant estimator $\widehat{k}_n = \widehat{k}_n(X_1, \ldots, X_n)$ of k. We set $Y_{n,j} = s(X_j; \widehat{k}_n, \widehat{\lambda}_n) = X_j/\widehat{\lambda}_n$, for each $j = 1, \ldots, n$. Naturally, $\widehat{\lambda}_n^* = \widehat{\lambda}_n(Y_{n,1}, \ldots, Y_{n,n}) = 1$ and $\widehat{k}_n^* = \widehat{k}_n(Y_{n,1}, \ldots, Y_{n,n}) = \widehat{k}_n(X_1, \ldots, X_n) = \widehat{k}_n$ are consistent estimators of $\lambda^* = 1$ and $k^* = k$. In accordance with our general consideration in Section 7, we take

$$\widehat{T}_n(t) = \frac{1}{n}\sum_{j=1}^{n}\left(-\frac{\widehat{k}_n - 1}{Y_{n,j}} + 1\right)\min\{Y_{n,j}, t\}, \quad t > 0.$$

Betsch and Ebner (2019a) considered the function $\widehat{T}_n$ and the empirical distribution function of $Y_{n,1}, \ldots, Y_{n,n}$, $\widehat{F}_n$, as random elements of the Hilbert space $L^2\big((0,\infty), \mathcal{B}^1_{>0}, w(t)\,\mathrm{d}t\big)$, where w is an appropriate weight function. They obtained the statistic

$$G_n = n\int_0^\infty \left|\widehat{T}_n(t) - \widehat{F}_n(t)\right|^2 w(t)\,\mathrm{d}t,$$

derived the limit distribution under the hypothesis using the Hilbert space central limit theorem, and gave a proof of the consistency of this test procedure against fixed alternatives with existing expectation. Moreover, they explained how to implement the test using a parametric bootstrap and showed in a Monte Carlo simulation study that the test excels classical procedures and keeps up with the best Gamma tests proposed so far. Contributions like Henze et al (2012), Plubin and Siripanich (2017), and Villaseñor and González-Estrada (2015) indicate that testing fit to the Gamma distribution is also a topic of ongoing research.

The characterization of the exponential distribution via the mean residual life function is a special case of Corollary 3 (cf. Example 4), and thus the corresponding test for exponentiality is to be seen as a special case of the test for the Gamma distribution at hand. Baringhaus and Henze (2000) used the characterization, which was known in a different disguise already, to construct the associated test for exponentiality in the sense described above. They showed that the limit distribution under the hypothesis coincides with the limiting null distribution of the classical Cramér-von Mises statistic when testing for uniformity over the unit interval. Furthermore, they proved the consistency of the test procedure against any fixed alternative distribution. The test has already been included in the extensive comparative simulation study conducted by Allison et al (2017). Adding a tuning parameter to the weight function leads to the test statistic proposed by Baringhaus and Henze (2008). The recent papers by Cuparić et al (2019), Jovanović et al (2015), Nikitin (2017), Noughabi (2015), Torabi et al (2018), Volkova and Nikitin (2015), and Zardasht et al (2015) show that tests for exponentiality are still of importance to the research community.

6 Goodness-of-fit tests for normality

The goodness-of-fit tests for normality proposed by Betsch and Ebner (2019b) are also included in our framework (cf. Example 1). To fix notation, we write $p_\vartheta(x)$ for the normal distribution density with mean-variance-parameter vector $\vartheta = (\mu, \sigma^2) \in \mathbb{R} \times (0, \infty) = \Theta$. Consider a real-valued random variable X with $\mathbb{E}X^2 < \infty$. Taking into account the invariance under linear transformations of the class of normal distributions, Betsch and Ebner (2019b) used the scaling function $s(x;\vartheta) = (x - \mu)/\sigma$. With this choice, $X \sim p_\vartheta \mathcal{L}^1$ if, and only if, $s(X;\vartheta) \sim p_{\vartheta^*} \mathcal{L}^1$, where $\vartheta^* = (0, 1)$, i.e. if $s(X;\vartheta)$ follows the standard Gaussian law. Furthermore, we have

$$\mathbb{E}\left|\frac{p'_{\vartheta^*}\big(s(X;\vartheta)\big)}{p_{\vartheta^*}\big(s(X;\vartheta)\big)}\right| = \mathbb{E}\big|s(X;\vartheta)\big| \le \frac{1}{\sigma}\Big(\mathbb{E}|X| + |\mu|\Big) < \infty$$

and

$$\mathbb{E}\left|\frac{p'_{\vartheta^*}\big(s(X;\vartheta)\big)}{p_{\vartheta^*}\big(s(X;\vartheta)\big)}\, s(X;\vartheta)\right| = \mathbb{E}\big(s(X;\vartheta)\big)^2 \le \frac{1}{\sigma^2}\Big(\mathbb{E}X^2 + 2|\mu|\,\mathbb{E}|X| + \mu^2\Big) < \infty.$$

As a consequence, Example 1 states that X follows a normal distribution with parameter vector $\vartheta = (\mu, \sigma^2)$ if, and only if,

$$F_{s(X;\vartheta)}(t) = \mathbb{E}\Big[s(X;\vartheta)\,\big(s(X;\vartheta) - t\big)\,\mathbb{1}\{s(X;\vartheta) \leq t\}\Big], \quad t \in \mathbb{R}.$$

For iid. copies $X_1, \ldots, X_n$ of X, we consider the sample mean $\overline{X}_n$ and sample variance $S_n^2 = n^{-1}\sum_{j=1}^{n}(X_j - \overline{X}_n)^2$ as consistent estimators of μ and σ^2. We put

$$Y_{n,j} = s(X_j; \overline{X}_n, S_n^2) = (X_j - \overline{X}_n)/S_n, \quad j = 1, \ldots, n,$$

and notice that $\widehat{\vartheta}_n^* = \big(\overline{X}_n^*, S_n^{2*}\big) = (0,1)$. Thus, we take

$$\widehat{T}_n(t) = \frac{1}{n}\sum_{j=1}^{n} Y_{n,j}\,(Y_{n,j} - t)\,\mathbb{1}\{Y_{n,j} \leq t\}, \quad t \in \mathbb{R}.$$

It remains to compare $\widehat{T}_n$ with the empirical distribution function $\widehat{F}_n$ of $Y_{n,1}, \ldots, Y_{n,n}$ by an appropriate measure of deviation. In particular, Betsch and Ebner (2019b) considered $\widehat{T}_n$ and $\widehat{F}_n$ as random elements in the Hilbert space $L^2\big(\mathbb{R},\, \mathcal{B}^1,\, w(t)\,\mathrm{d}t\big)$, where w is a suitable weight function, and chose as a metric the one induced by the Hilbert space norm. In accordance with our general considerations at the beginning of Section 7, their statistic has the form

$$G_n = n\int_{\mathbb{R}}\Big|\widehat{T}_n(t) - \widehat{F}_n(t)\Big|^2\, w(t)\,\mathrm{d}t.$$

Besides specifying weight functions for which the statistic has an explicit formula, Betsch and Ebner (2019b) used the central limit theorem for random elements in separable Hilbert spaces to derive the limit distributions under the hypothesis $\mathbf{H_0}$ in (8). Furthermore, they established the consistency of the test procedures against fixed alternatives with existing second moment, and showed in a Monte Carlo simulation study that these tests are serious competitors to established procedures. The problem of testing for normality is still of interest in research, as evidenced by Henze and Jiménez-Gamero (2019), Henze et al (2019), Henze and Koch (2017), and numerous preprints.

7 Classical goodness-of-fit procedures

We consider the uniform distribution on the unit interval, $p(t) = \mathbb{1}_{(0,1)}(t)$, $t \in \mathbb{R}$. According to Example 12, our characterization results for the uniform distribution reduce to the fact the law is determined uniquely by its distribution function $F(t) = t$, $0 < t < 1$. Thus, in line with the general construction in Section 7, we obtain the statistics

$$K_n = \sqrt{n}\sup_{0<t<1}\Big|\widehat{F}_n(t) - F(t)\Big| \qquad \text{and} \qquad \omega_n^2 = n\int_0^1\Big|\widehat{F}_n(t) - F(t)\Big|^2\,\mathrm{d}F(t)$$

for testing the uniformity hypothesis. Here, $\widehat{F}_n$ is the empirical distribution function of $X_1, \ldots, X_n$, which are iid. copies of a random variable X with values in $(0,1)$. Thus we have recovered in this special case the classical Kolmogorov-Smirnov and Cramér-von Mises statistics. Using a weight function in the integral statistic, we may also obtain the one from Anderson and Darling. For an account of the historical development of these classical procedures, a synoptic derivation of their limit distribution and an explanation on how to extend these tests to situations where the null hypothesis includes a whole (parametric) family of continuous distributions, as well as for further references, we recommend del Barrio et al (2000).

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