The reproducing kernel Hilbert space approach in nonparametric regression problems with correlated observations

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Supplementary materials

In this section, we shall omit the index n in $t_{i,n}$ when there is no ambiguity.

Proof of Proposition 1.

It is known that (see for instance, Su and Cambanis (1993) page 88) if $R(s,t) = \int_0^{\min(s,t)} u^\beta \, du$ then for any functions u and v and for any sampling design T_n we have,

$$u_{|T_n}{}' R_{|T_n}^{-1} v_{|T_n} = \frac{u(t_1)v(t_1)}{t_1^{\beta+1}} + \sum_{k=1}^{n-1} \frac{(u(t_{k+1}) - u(t_k))(v(t_{k+1}) - v(t_k))}{t_{k+1}^{\beta+1} - t_k^{\beta+1}}.$$

Replacing $u = f_{x,h}$ and $v = \overline{Y}$ we have,

$$\hat{g}_n^{pro}(x) = \frac{f_{x,h}(t_1)\overline{Y}(t_1)}{t_1^{\beta+1}} + \sum_{i=1}^{n-1} \frac{(f_{x,h}(t_{i+1}) - f_{x,h}(t_i))(\overline{Y}(t_{i+1}) - \overline{Y}(t_i))}{t_{i+1}^{\beta+1} - t_i^{\beta+1}}. \quad (1)$$

Recall that $R(s,t) = \frac{1}{\beta+1} \min(s,t)^{\beta+1}$ and,

$$f_{x,h}(t_i) = \int_0^1 R(s,t_i)\varphi_{x,h}(s) \, ds = \frac{1}{\beta+1}\Big(\int_0^{t_i} s^{\beta+1} \, \varphi_{x,h}(s) \, ds + t_i^{\beta+1} \int_{t_i}^1 \varphi_{x,h}(s) \, ds \Big).$$

Thus,

$$\begin{aligned} f_{x,h}(t_{i+1}) - f_{x,h}(t_i) &= \frac{1}{\beta+1}\Big(\int_0^{t_{i+1}} s^{\beta+1}\varphi_{x,h}(s) \, ds + t_{i+1}^{\beta+1} \int_{t_{i+1}}^1 \varphi_{x,h}(s) \, ds \\ &- \int_0^{t_i} s^{\beta+1}\varphi_{x,h}(s) \, ds - t_i^{\beta+1} \int_{t_i}^1 \varphi_{x,h}(s) \, ds + t_{i+1}^{\beta+1} \int_{t_i}^1 \varphi_{x,h}(s) \, ds - t_{i+1}^{\beta+1} \int_{t_i}^1 \varphi_{x,h}(s) \, ds \Big) \\ &= \frac{1}{\beta+1}\Big(\int_{t_i}^{t_{i+1}} (s^{\beta+1} - t_{i+1}^{\beta+1})\varphi_{x,h}(s) \, ds + (t_{i+1}^{\beta+1} - t_i^{\beta+1}) \int_{t_i}^1 \varphi_{x,h}(s) \, ds \Big). \end{aligned}$$

Thus,

$$\hat{g}_n^{pro}(x) = \frac{f_{x,h}(t_1)\overline{Y}(t_1)}{t_1^{\beta+1}} + \frac{1}{\beta+1}\Big(\sum_{i=1}^{n-1}(\overline{Y}(t_{i+1}) - \overline{Y}(t_i))\int_{t_i}^{1}\varphi_{x,h}(s)\ ds$$
$$+\sum_{i=1}^{n-1}\frac{\overline{Y}(t_{i+1}) - \overline{Y}(t_i)}{t_{i+1}^{\beta+1} - t_i^{\beta+1}}\int_{t_i}^{t_{i+1}}(s^{\beta+1} - t_{i+1}^{\beta+1})\varphi_{x,h}(s)\ ds\Big)$$
$$= \frac{f_{x,h}(t_1)\overline{Y}(t_1)}{t_1^{\beta+1}} + \frac{1}{\beta+1}\Big(\sum_{i=2}^{n-1}\overline{Y}(t_i)\int_{t_{i-1}}^{t_i}\varphi_{x,h}(s)\ ds - \overline{Y}(t_1)\int_{t_1}^{1}\varphi_{x,h}(s)\ ds$$
$$+\overline{Y}(t_n)\int_{t_{n-1}}^{1}\varphi_{x,h}(s)\ ds + \sum_{i=1}^{n-1}\frac{\overline{Y}(t_{i+1}) - \overline{Y}(t_i)}{t_{i+1}^{\beta+1} - t_i^{\beta+1}}\int_{t_i}^{t_{i+1}}(s^{\beta+1} - t_{i+1}^{\beta+1})\varphi_{x,h}(s)\ ds\Big).$$

Letting $t_0 = \overline{Y}(t_0) = 0$ we have,

$$\frac{f_{x,h}(t_1)\overline{Y}(t_1)}{t_1^{\beta+1}} = \frac{1}{\beta+1}\Big(\frac{\overline{Y}(t_1)}{t_1^{\beta+1}}\int_0^{t_1}s^{\beta+1}\varphi_{x,h}(s)\ ds + \overline{Y}(t_1)\int_{t_1}^{1}\varphi_{x,h}(s)\ ds\Big)$$
$$= \frac{1}{\beta+1}\Big(\frac{\overline{Y}(t_1) - \overline{Y}(t_0)}{t_1^{\beta+1} - t_0^{\beta+1}}\int_0^{t_1}(s^{\beta+1} - t_1^{\beta+1})\varphi_{x,h}(s)\ ds$$
$$+\overline{Y}(t_1)\int_0^{t_1}\varphi_{x,h}(s)\ ds + \overline{Y}(t_1)\int_{t_1}^{1}\varphi_{x,h}(s)\ ds\Big).$$

Finally,

$$\hat{g}_n^{pro}(x) = \frac{1}{\beta+1}\Big(\sum_{i=2}^{n-1}\overline{Y}(t_i)\int_{t_{i-1}}^{t_i}\varphi_{x,h}(s)\ ds - \overline{Y}(t_1)\int_{t_1}^{1}\varphi_{x,h}(s)\ ds + \overline{Y}(t_n)\int_{t_{n-1}}^{1}\varphi_{x,h}(s)\ ds$$
$$+\overline{Y}(t_n)\int_{t_{n-1}}^{t_n}\varphi_{x,h}(s)\ ds - \overline{Y}(t_n)\int_{t_{n-1}}^{t_n}\varphi_{x,h}(s)\ ds$$
$$+\sum_{i=1}^{n-1}\frac{\overline{Y}(t_{i+1}) - \overline{Y}(t_i)}{t_{i+1}^{\beta+1} - t_i^{\beta+1}}\int_{t_i}^{t_{i+1}}(s^{\beta+1} - t_{i+1}^{\beta+1})\varphi_{x,h}(s)\ ds$$
$$+\frac{\overline{Y}(t_1) - \overline{Y}(t_0)}{t_1^{\beta+1} - t_0^{\beta+1}}\int_0^{t_1}(s^{\beta+1} - t_1^{\beta+1})\varphi_{x,h}(s)\ ds + \overline{Y}(t_1)\int_0^{t_1}\varphi_{x,h}(s)\ ds + \overline{Y}(t_1)\int_{t_1}^{1}\varphi_{x,h}(s)\ ds\Big)$$
$$= \frac{1}{\beta+1}\Big(\sum_{i=1}^{n+1}\overline{Y}(t_i)\int_{t_{i-1}}^{t_i}\varphi_{x,h}(s)\ ds + \sum_{i=0}^{n-1}\frac{\overline{Y}(t_{i+1}) - \overline{Y}(t_i)}{t_{i+1}^{\beta+1} - t_i^{\beta+1}}\int_{t_i}^{t_{i+1}}(s^{\beta+1} - t_{i+1}^{\beta+1})\varphi_{x,h}(s)\ ds\Big),$$

where $t_{n+1} = 1$ and $\overline{Y}(t_{n+1}) = \overline{Y}(t_n)$. This concludes the proof of Proposition 1. □

Proof of Proposition 2.

It is known (see Anderson (1960) page 210) that for every functions u and v and for every design T_n we have,

$$u'_{|T_n}R^{-1}_{|T_n}v_{|T_n} = \frac{u(t_1)v(t_1)}{1 - e^{-2(t_2-t_1)}} + \frac{u(t_n)v(t_n)}{1 - e^{-2(t_n-t_{n-1})}} + \sum_{i=2}^{n-1}\frac{u(t_i)v(t_i)(1 - e^{-2(t_{i+1}-t_{i-1})})}{(1 - e^{-2(t_{i+1}-t_i)})(1 - e^{-2(t_i-t_{i-1})})}$$
$$-\sum_{i=1}^{n-1}\frac{u(t_i)v(t_{i+1}) + u(t_{i+1})v(t_i)}{1 - e^{-2(t_{i+1}-t_i)}}e^{-(t_{i+1}-t_i)}.$$

Taking $u = f_{x,h}$ and $v = \overline{Y}$ we get,

$$\begin{aligned}\hat{g}_n^{pro}(x) &= \frac{f_{x,h}(t_1)\overline{Y}(t_1)}{1-e^{-2(t_2-t_1)}} + \frac{f_{x,h}(t_n)\overline{Y}(t_n)}{1-e^{-2(t_n-t_{n-1})}} + \sum_{i=2}^{n-1} \frac{f_{x,h}(t_i)\overline{Y}(t_i)(1-e^{-2(t_{i+1}-t_{i-1})})}{(1-e^{-2(t_{i+1}-t_i)})(1-e^{-2(t_i-t_{i-1})})} \\ &\quad - \sum_{i=1}^{n-1} \frac{f_{x,h}(t_i)\overline{Y}(t_{i+1}) + f_{x,h}(t_{i+1})\overline{Y}(t_i)}{1-e^{-2(t_{i+1}-t_i)}} e^{-(t_{i+1}-t_i)} \\ &\triangleq \frac{f_{x,h}(t_1)\overline{Y}(t_1)}{1-e^{-2(t_2-t_1)}} + \frac{f_{x,h}(t_n)\overline{Y}(t_n)}{1-e^{-2(t_n-t_{n-1})}} + A. \end{aligned} \tag{2}$$

Note that,

$$1-e^{-2(t_{i+1}-t_{i-1})} = (1-e^{-2(t_{i+1}-t_i)}) + (1-e^{-2(t_i-t_{i-1})}) - (1-e^{-2(t_i-t_{i-1})})(1-e^{-2(t_{i+1}-t_i)}).$$

Thus,

$$\begin{aligned} A &= \sum_{i=2}^{n-1} \frac{f_{x,h}(t_i)\overline{Y}(t_i)}{1-e^{-2(t_i-t_{i-1})}} + \sum_{i=2}^{n-1} \frac{f_{x,h}(t_i)\overline{Y}(t_i)}{1-e^{-2(t_{i+1}-t_i)}} - \sum_{i=2}^{n-1} f_{x,h}(t_i)\overline{Y}(t_i) \\ &\quad - \sum_{i=2}^{n} \frac{f_{x,h}(t_{i-1})\overline{Y}(t_i)}{1-e^{-2(t_i-t_{i-1})}} e^{-(t_i-t_{i-1})} - \sum_{i=1}^{n-1} \frac{f_{x,h}(t_{i+1})\overline{Y}(t_i)}{1-e^{-2(t_{i+1}-t_i)}} e^{-(t_{i+1}-t_i)} \\ &= \sum_{i=2}^{n-1} \frac{\overline{Y}(t_i)}{1-e^{-2(t_i-t_{i-1})}} \Big(f_{x,h}(t_i) - f_{x,h}(t_{i-1}) e^{-(t_i-t_{i-1})} \Big) - \frac{f_{x,h}(t_{n-1})\overline{Y}(t_n)}{1-e^{-2(t_n-t_{n-1})}} e^{-(t_n-t_{n-1})} \\ &\quad + \sum_{i=2}^{n-1} \frac{\overline{Y}(t_i)}{1-e^{-2(t_{i+1}-t_i)}} \Big(f_{x,h}(t_i) - f_{x,h}(t_{i+1}) e^{-(t_{i+1}-t_i)} \Big) - \frac{f_{x,h}(t_2)\overline{Y}(t_1)}{1-e^{-2(t_2-t_1)}} e^{-(t_2-t_1)} \\ &\quad - \sum_{i=2}^{n-1} f_{x,h}(t_i)\overline{Y}(t_i) \end{aligned} \tag{3}$$

Simple calculations yield,

$$\begin{aligned} & f_{x,h}(t_i) - f_{x,h}(t_{i-1}) e^{-(t_i-t_{i-1})} = \\ & e^{-t_i} \int_{t_{i-1}}^{t_i} e^{s} \varphi_{x,h}(s)\, ds - e^{t_i} \int_{t_{i-1}}^{t_i} e^{-s} \varphi_{x,h}(s)\, ds + e^{t_i}(1-e^{-2(t_i-t_{i-1})}) \int_{t_{i-1}}^{1} e^{-s} \varphi_{x,h}(s)\, ds. \end{aligned} \tag{4}$$

In the same way we have,

$$\begin{aligned} & f_{x,h}(t_i) - f_{x,h}(t_{i+1}) e^{-(t_{i+1}-t_i)} = \\ & e^{t_i} \int_{t_i}^{t_{i+1}} e^{-s} \varphi_{x,h}(s)\, ds - e^{-t_i} \int_{t_i}^{t_{i+1}} e^{s} \varphi_{x,h}(s)\, ds + e^{-t_i}(1-e^{-2(t_{i+1}-t_i)}) \int_{0}^{t_{i+1}} e^{s} \varphi_{x,h}(s)\, ds. \end{aligned} \tag{5}$$

It is easy to verify that,

$$\sum_{i=2}^{n-1} f_{x,h}(t_i)\overline{Y}(t_i) = \sum_{i=2}^{n-1} \overline{Y}(t_i) e^{-t_i} \int_0^{t_i} e^{s} \varphi_{x,h}(s)\, ds + \sum_{i=2}^{n-1} \overline{Y}(t_i) e^{t_i} \int_{t_i}^{1} e^{-s} \varphi_{x,h}(s)\, ds. \tag{6}$$

We obtain using Equations (3), (4), (5) and (6),

$$\begin{aligned}
A &= \sum_{i=2}^{n-1} \overline{Y}(t_i)e^{t_i} \int_{t_{i-1}}^{1} e^{-s}\varphi_{x,h}(s)\,ds + \sum_{i=2}^{n-1} \frac{\overline{Y}(t_i)e^{-t_i}}{1-e^{-2(t_i-t_{i-1})}} \int_{t_{i-1}}^{t_i} e^{s}\varphi_{x,h}(s)\,ds \\
&- \sum_{i=2}^{n-1} \frac{\overline{Y}(t_i)e^{t_i}}{1-e^{-2(t_i-t_{i-1})}} \int_{t_{i-1}}^{t_i} e^{-s}\varphi_{x,h}(s)\,ds + \sum_{i=2}^{n-1} \overline{Y}(t_i)e^{-t_i} \int_{0}^{t_{i+1}} e^{s}\varphi_{x,h}(s)\,ds \\
&+ \sum_{i=2}^{n-1} \frac{\overline{Y}(t_i)e^{t_i}}{1-e^{-2(t_{i+1}-t_i)}} \int_{t_i}^{t_{i+1}} e^{-s}\varphi_{x,h}(s)\,ds - \sum_{i=2}^{n-1} \frac{\overline{Y}(t_i)e^{-t_i}}{1-e^{-2(t_{i+1}-t_i)}} \int_{t_i}^{t_{i+1}} e^{s}\varphi_{x,h}(s)\,ds \\
&- \frac{f_{x,h}(t_2)\overline{Y}(t_1)}{1-e^{-2(t_2-t_1)}} e^{-(t_2-t_1)} - \frac{f_{x,h}(t_{n-1})\overline{Y}(t_n)}{1-e^{-2(t_n-t_{n-1})}} e^{-(t_n-t_{n-1})} \\
&- \sum_{i=2}^{n-1} \overline{Y}(t_i)e^{-t_i} \int_{0}^{t_i} e^{s}\varphi_{x,h}(s)\,ds - \sum_{i=2}^{n-1} \overline{Y}(t_i)e^{t_i} \int_{t_i}^{1} e^{-s}\varphi_{x,h}(s)\,ds.
\end{aligned}$$

Replacing this expression of A in (2) gives,

$$\begin{aligned}
\hat{g}_n^{pro}(x) &= \sum_{i=2}^{n-1} \overline{Y}(t_i) \int_{t_{i-1}}^{t_{i+1}} e^{|t_i-s|}\varphi_{x,h}(s)\,ds + \sum_{i=2}^{n-2} \frac{\overline{Y}(t_{i+1})e^{-t_{i+1}} - \overline{Y}(t_i)e^{-t_i}}{1-e^{-2(t_{i+1}-t_i)}} \int_{t_i}^{t_{i+1}} e^{s}\varphi_{x,h}(s)\,ds \\
&- \sum_{i=2}^{n-2} \frac{\overline{Y}(t_{i+1})e^{t_{i+1}} - \overline{Y}(t_i)e^{t_i}}{1-e^{-2(t_{i+1}-t_i)}} \int_{t_i}^{t_{i+1}} e^{-s}\varphi_{x,h}(s)\,ds + \frac{\overline{Y}(t_2)e^{-t_2}}{1-e^{-2(t_2-t_1)}} \int_{t_1}^{t_2} e^{s}\varphi_{x,h}(s)\,ds \\
&- \frac{\overline{Y}(t_{n-1})e^{-t_{n-1}}}{1-e^{-2(t_n-t_{n-1})}} \int_{t_{n-1}}^{t_n} e^{s}\varphi_{x,h}(s)\,ds - \frac{\overline{Y}(t_2)e^{t_2}}{1-e^{-2(t_2-t_1)}} \int_{t_1}^{t_2} e^{-s}\varphi_{x,h}(s)\,ds \\
&+ \frac{\overline{Y}(t_{n-1})e^{t_{n-1}}}{1-e^{-2(t_n-t_{n-1})}} \int_{t_{n-1}}^{t_n} e^{-s}\varphi_{x,h}(s)\,ds + \frac{f_{x,h}(t_1)\overline{Y}(t_1)}{1-e^{-2(t_2-t_1)}} - \frac{f_{x,h}(t_2)\overline{Y}(t_1)}{1-e^{-2(t_2-t_1)}} e^{-(t_2-t_1)} \\
&+ \frac{f_{x,h}(t_n)\overline{Y}(t_n)}{1-e^{-2(t_n-t_{n-1})}} - \frac{f_{x,h}(t_{n-1})\overline{Y}(t_n)}{1-e^{-2(t_n-t_{n-1})}} e^{-(t_n-t_{n-1})}. \qquad (7)
\end{aligned}$$

Note that Equation (5) yields,

$$\begin{aligned}
\frac{\overline{Y}(t_1)}{1-e^{-2(t_2-t_1)}} \big(f_{x,h}(t_1) - f_{x,h}(t_2)e^{-(t_2-t_1)}\big) &= \frac{\overline{Y}(t_1)e^{t_1}}{1-e^{-2(t_2-t_1)}} \int_{t_1}^{t_2} e^{-s}\varphi_{x,h}(s)\,ds \\
&- \frac{\overline{Y}(t_1)e^{-t_1}}{1-e^{-2(t_2-t_1)}} \int_{t_1}^{t_2} e^{s}\varphi_{x,h}(s)\,ds + \overline{Y}(t_1)e^{-t_1} \int_{t_1}^{t_2} e^{s}\varphi_{x,h}(s)\,ds. \qquad (8)
\end{aligned}$$

Similarly, Equation (4) yields,

$$\begin{aligned}
\frac{\overline{Y}(t_n)}{1-e^{-2(t_n-t_{n-1})}} \big(f_{x,h}(t_n) - f_{x,h}(t_{n-1})e^{-(t_n-t_{n-1})}\big) &= \frac{\overline{Y}(t_n)e^{-t_n}}{1-e^{-2(t_n-t_{n-1})}} \int_{t_{n-1}}^{t_n} e^{s}\varphi_{x,h}(s)\,ds \\
&- \frac{\overline{Y}(t_n)e^{t_n}}{1-e^{-2(t_n-t_{n-1})}} \int_{t_{n-1}}^{t_n} e^{-s}\varphi_{x,h}(s)\,ds + \overline{Y}(t_n)e^{t_n} \int_{t_{n-1}}^{1} e^{-s}\varphi_{x,h}(s)\,ds. \qquad (9)
\end{aligned}$$

We obtain using (8) and (9) in (7),

$$\hat{g}_n^{pro}(x) = \sum_{i=2}^{n-1} \overline{Y}(t_i) \int_{t_{i-1}}^{t_{i+1}} e^{|t_i - s|} \varphi_{x,h}(s)\, ds + \sum_{i=2}^{n-2} \frac{\overline{Y}(t_{i+1})e^{-t_{i+1}} - \overline{Y}(t_i)e^{-t_i}}{1 - e^{-2(t_{i+1}-t_i)}} \int_{t_i}^{t_{i+1}} e^{s} \varphi_{x,h}(s)\, ds$$
$$- \sum_{i=2}^{n-2} \frac{\overline{Y}(t_{i+1})e^{t_{i+1}} - \overline{Y}(t_i)e^{t_i}}{1 - e^{-2(t_{i+1}-t_i)}} \int_{t_i}^{t_{i+1}} e^{-s} \varphi_{x,h}(s)\, ds + \frac{\overline{Y}(t_2)e^{-t_2}}{1 - e^{-2(t_2-t_1)}} \int_{t_1}^{t_2} e^{s} \varphi_{x,h}(s)\, ds$$
$$- \frac{\overline{Y}(t_{n-1})e^{-t_{n-1}}}{1 - e^{-2(t_n-t_{n-1})}} \int_{t_{n-1}}^{t_n} e^{s} \varphi_{x,h}(s)\, ds - \frac{\overline{Y}(t_2)e^{t_2}}{1 - e^{-2(t_2-t_1)}} \int_{t_1}^{t_2} e^{-s} \varphi_{x,h}(s)\, ds$$
$$+ \frac{\overline{Y}(t_{n-1})e^{t_{n-1}}}{1 - e^{-2(t_n-t_{n-1})}} \int_{t_{n-1}}^{t_n} e^{-s} \varphi_{x,h}(s)\, ds + \frac{\overline{Y}(t_1)e^{t_1}}{1 - e^{-2(t_2-t_1)}} \int_{t_1}^{t_2} e^{-s} \varphi_{x,h}(s)\, ds$$
$$- \frac{\overline{Y}(t_1)e^{-t_1}}{1 - e^{-2(t_2-t_1)}} \int_{t_1}^{t_2} e^{s} \varphi_{x,h}(s)\, ds + \overline{Y}(t_1)e^{-t_1} \int_{t_1}^{t_2} e^{s} \varphi_{x,h}(s)\, ds$$
$$+ \frac{\overline{Y}(t_n)e^{-t_n}}{1 - e^{-2(t_n-t_{n-1})}} \int_{t_{n-1}}^{t_n} e^{s} \varphi_{x,h}(s)\, ds - \frac{\overline{Y}(t_n)e^{t_n}}{1 - e^{-2(t_n-t_{n-1})}} \int_{t_{n-1}}^{t_n} e^{-s} \varphi_{x,h}(s)\, ds$$
$$+ \overline{Y}(t_n)e^{t_n} \int_{t_{n-1}}^{1} e^{-s} \varphi_{x,h}(s)\, ds$$

$$= \sum_{i=2}^{n-1} \overline{Y}(t_i) \int_{t_{i-1}}^{t_{i+1}} e^{|s - t_i|} \varphi_{x,h}(s)\, ds + \overline{Y}(t_1) \int_{0}^{t_2} e^{s-t_1} \varphi_{x,h}(s)\, ds + \overline{Y}(t_n) \int_{t_{n-1}}^{1} e^{t_n - s} \varphi_{x,h}(s)\, ds$$
$$- \sum_{i=1}^{n-1} \frac{e^{t_{i+1}}\overline{Y}(t_{i+1}) - e^{t_i}\overline{Y}(t_i)}{1 - e^{-2(t_{i+1}-t_i)}} \int_{t_i}^{t_{i+1}} e^{-s} \varphi_{x,h}(s)\, ds$$
$$+ \sum_{i=1}^{n-1} \frac{e^{-t_{i+1}}\overline{Y}(t_{i+1}) - e^{-t_i}\overline{Y}(t_i)}{1 - e^{-2(t_{i+1}-t_i)}} \int_{t_i}^{t_{i+1}} e^{s} \varphi_{x,h}(s)\, ds.$$

This concludes the proof of Proposition 2. □

Proof of Lemma 1.

Let $(u, v) \in [-1, 1]^2$. We first consider the triangle $\{-1 < u < v < 1\}$ which is further split into smaller triangles:

$$D_1 = \{0 < u < v < 1\},\ D_2 = \{-1 < u < 0 < v < 1\} \ \text{ and } \ D_3 = \{-1 < u < v < 0\}.$$

Let $b \in]0, 1[$. For $(u, v) \in D_1$, using Assumption (A), Taylor expansion of R around (x, x) gives,

$$R(x + bu, x + bv) = R(x, x + bv) + buR^{(1,0)}(x, x + bv) + \frac{1}{2}b^2u^2R^{(2,0)}(\varepsilon_x, x + bv)$$
$$= R(x, x) + bvR^{(0,1)}(x, \eta_x) + buR^{(1,0)}(x, x + bv) + \frac{1}{2}b^2u^2R^{(2,0)}(\varepsilon_x, x + bv),$$

where $x < \varepsilon_x < x + bu < x + bv$ and $x < \eta_x < x + bv$. Thus,

$$R(x + bu, x + bv) = R(x, x) + bvR^{(0,1)}(x, x^+) + buR^{(0,1)}(x, x^-) + o(b).$$

Now, for $(u,v) \in D_2$ we obtain in the same way,

$$R(x+bu, x+bv) = R(x, x+bv) + buR^{(1,0)}(x, x+bv) + \frac{1}{2}b^2u^2R^{(2,0)}(\varepsilon_x, x+bv)$$
$$= R(x,x) + bvR^{(0,1)}(x,\eta_x) + buR^{(1,0)}(x, x+bv) + \frac{1}{2}b^2u^2R^{(2,0)}(\varepsilon_x, x+bv),$$

where $x+bu < \varepsilon_x < x < x+bv$ and $x < \eta_x < x+bv$. Thus,

$$R(x+bu, x+bv) = R(x,x) + bvR^{(0,1)}(x,x^+) + buR^{(0,1)}(x,x^-) + o(b).$$

Finally, for $(u,v) \in D_3$ we get,

$$R(x+bu, x+bv) = R(x+bu, x) + bvR^{(0,1)}(x+bu, x) + \frac{1}{2}b^2v^2R^{(0,2)}(x+bu, \eta_x)$$
$$= R(x,x) + ubR^{(1,0)}(\varepsilon_x, x) + bvR^{(0,1)}(x+bu, \eta_x) + \frac{1}{2}b^2v^2R^{(0,2)}(x+bu, \eta_x),$$

where $x+hu < x+bv < \eta_x < x$ and $x+bu < \varepsilon_x < x$. Thus,

$$R(x+bu, x+bv) = R(x,x) + bvR^{(0,1)}(x,x^+) + buR^{(0,1)}(x,x^-) + o(b).$$

Hence for $v > u$ we have,

$$R(x+bu, x+bv) = R(x,x) + \frac{b}{2}\big(R^{(0,1)}(x,x^+) + R^{(0,1)}(x,x^-)\big)(u+v)$$
$$+ \frac{b}{2}\big(R^{(0,1)}(x,x^+) - R^{(0,1)}(x,x^-)\big)(v-u) + o(b).$$

Similarly, we obtain for the triangular $\{1 > u > v > -1\}$,

$$R(x+bu, x+bv) = R(x,x) + \frac{b}{2}\big(R^{(0,1)}(x,x^+) + R^{(0,1)}(x,x^-)\big)(u+v)$$
$$+ \frac{b}{2}\big(R^{(0,1)}(x,x^+) - R^{(0,1)}(x,x^-)\big)(u-v).$$

Thus, for $(u,v) \in [-1,1]^2$ we have,

$$R(x+bu, x+bv) = R(x,x) + \frac{b}{2}\big(R^{(0,1)}(x,x^+) + R^{(0,1)}(x,x^-)\big)(u+v)$$
$$+ \frac{b}{2}\big(R^{(0,1)}(x,x^+) - R^{(0,1)}(x,x^-)\big)|u-v|. \tag{10}$$

Consider now a function g, bounded and integrable on $[-1,1]$. The Dominated Convergence Theorem yields that $R(.,t) \times g$ is an integrable function for every $t \in [-1,1]$. Using (10) and putting,

$$\gamma(x) = \frac{1}{2}\big(R^{(0,1)}(x,x^+) + R^{(0,1)}(x,x^-)\big),$$

we obtain,

$$\iint_{[-1,1]^2} R(x+bu, x+bv)g(u)g(v)dudv = R(x,x)\left(\int_{-1}^{1} g(u)du\right)^2$$
$$+ 2\gamma(x)b\int_{-1}^{1} g(u)du\int_{-1}^{1} vg(v)dv - \frac{b}{2}\alpha(x)\iint_{[-1,1]^2} g(u)g(v)|u-v|dudv + o(b). \tag{11}$$

The left side of (11) is non-negative since the autocovariance function R is a non-negative definite function. Taking $g(u) = u1_{[-1,1]}(u)$ we obtain,

$$\int_{-1}^{1} g(u)du = 0 \quad \text{and} \iint_{[-1,1]^2} uv|u-v|dudv = -\frac{8}{15}.$$

Thus,

$$\frac{4}{15}\alpha(x) + o(b) \geq 0.$$

Taking b small enough concludes the proof of Lemma 1. □

Proof of Lemma 3.

The great lines of this proof are based on the work of Sacks and Ylvisaker (1966) (c.f. Lemma 3.2 there). Let $x, h \in]0,1[$ and put $g_n = P_{T_n} f_{x,h}$, it is shown by (90) in the Appendix that,

$$g_n(t_i) = \sum_{j=1}^{n} m_{x,h}(t_j) R(t_j, t_i) \quad \text{for all } i = 1, \cdots, n.$$

On the one hand, Assumption (A) yields that g_n is twice differentiable on $[0,1]$ except on T_n, but it has left and right derivatives. Thus, for every $i = 1, \ldots, n$ we have,

$$g_n'(t_i^-) = \sum_{j=1}^{n} m_{x,h}(t_j) R^{(0,1)}(t_j, t_i^-) \quad \text{and} \quad g_n'(t_i^+) = \sum_{j=1}^{n} m_{x,h}(t_j) R^{(0,1)}(t_j, t_i^+).$$

Since for $j \neq i$, $R^{(0,1)}(t_j, t_i^-) = R^{(0,1)}(t_j, t_i^+)$ then Assumption (B) yields,

$$g_n'(t_i^-) - g_n'(t_i^+) = \alpha(t_i) m_{x,h}(t_i). \tag{12}$$

On the other hand, Assumption (A) yields that $f_{x,h}$ (as defined by (2) in the paper) is twice differentiable on $]0,1[$, thus for $i = 1, \ldots, n-1$, Taylor expansion of $f_{x,h} - g_n$ around t_i gives,

$$f_{x,h}(t_{i+1}) - g_n(t_{i+1}) = (f_{x,h}(t_i) - g_n(t_i)) + d_i(f_{x,h}'(t_i) - g_n'(t_i^+)) + \frac{1}{2}d_i^2(f_{x,h}''(\sigma_i) - g_n''(\sigma_i)),$$

where $d_i = t_{i+1} - t_i$ and $\sigma_i \in]t_i, t_{i+1}[$. Recall that, for all $i = 1, \ldots, n$, $f_{x,h}(t_i) = g_n(t_i)$ (see the Appendix, Equation (88)). Thus,

$$f_{x,h}'(t_i) - g_n'(t_i^+) = -\frac{1}{2}d_i(f_{x,h}''(\sigma_i) - g_n''(\sigma_i)), \tag{13}$$

Similarly, for $i = 2, \ldots, n$, we have,

$$f_{x,h}'(t_i) - g_n'(t_i^-) = \frac{1}{2}d_{i-1}(f_{x,h}''(\theta_i) - g_n''(\theta_i)), \tag{14}$$

for some $\theta_i \in]t_{i-1}, t_i[$. We obtain subtracting (14) from (13) and using (12) for $i = 2, \ldots, n-1$,

$$\alpha(t_i) m_{x,h}(t_i) = -\frac{1}{2}d_i(f_{x,h}''(\sigma_i) - g_n''(\sigma_i)) - \frac{1}{2}d_{i-1}(f_{x,h}''(\theta_i) - g_n''(\theta_i)). \tag{15}$$

We shall now control the last expression. On the one hand we have,

$$f'_{x,h}(t) = \int_0^t R^{(0,1)}(s,t^+)\varphi_{x,h}(s)\,ds + \int_t^1 R^{(0,1)}(s,t^-)\varphi_{x,h}(s)\,ds, \tag{16}$$

and,

$$\begin{aligned} f''_{x,h}(t) = (R^{(0,1)}(t,t^+) - R^{(0,1)}(t,t^-))\varphi_{x,h}(t) + \int_0^1 R^{(0,2)}(s,t^+)\varphi_{x,h}(s)\,ds \\ - \alpha(t)\varphi_{x,h}(t) + \int_0^1 R^{(0,2)}(s,t^+)\varphi_{x,h}(s)\,ds. \end{aligned} \tag{17}$$

On the other hand we know, using (F3) in the Appendix, that every function in the RKHS(R), noted by $\mathcal{F}(\varepsilon)$, is continuous, hence Assumption (C) implies that $R^{(0,2)}(\cdot,t^+)$ is a continuous function on $[0,1]$ for every fixed $t \in [0,1]$. Thus,

$$R^{(0,2)}(t,t^+) = \lim_{s\downarrow t} R^{(0,2)}(s,t^+) = \lim_{s\downarrow t} R^{(0,2)}(s,t^-) = R^{(0,2)}(t,t^-),$$

from which we get that $R^{(0,2)}(t,t)$ exists. Hence for $i = 1,\dots,n$ we have,

$$g''_n(t_i^-) = g''_n(t_i^+) = \sum_{j=1}^n m_{x,h}(t_j) R^{(0,2)}(t_j,t_i). \tag{18}$$

In addition, it is shown by (F4) in the Appendix that for every $t \in [0,1]$,

$$f''_{x,h}(t) - g''_n(t) = -\alpha(t)\varphi_{x,h}(t) + \langle R^{(0,2)}(\cdot,t), f_{x,h} - g_n\rangle, \tag{19}$$

where $\langle\cdot,\cdot\rangle$ is the inner product on $\mathcal{F}(\varepsilon)$. Injecting (19) in (15) we obtain,

$$\begin{aligned} \alpha(t_i)m_{x,h}(t_i) = \frac{1}{2}d_i\alpha(\sigma_i)\varphi_{x,h}(\sigma_i) + \frac{1}{2}d_{i-1}\alpha(\theta_i)\varphi_{x,h}(\theta_i) - \frac{1}{2}d_i\langle R^{(0,2)}(\cdot,\sigma_i), f_{x,h} - g_n\rangle \\ - \frac{1}{2}d_{i-1}\langle R^{(0,2)}(\cdot,\theta_i), f_{x,h} - g_n\rangle. \end{aligned}$$

Using Assumption (B) we obtain for $i = 2,\dots,n-1$,

$$\begin{aligned} m_{x,h}(t_i) = \frac{1}{2}(d_i + d_{i-1})\varphi_{x,h}(t_i) \ + \frac{1}{2\alpha(t_i)}d_i\big(\alpha(\sigma_i)\varphi_{x,h}(\sigma_i) - \alpha(t_i)\varphi_{x,h}(t_i)\big) \\ + \frac{1}{2\alpha(t_i)}d_{i-1}\big(\alpha(\theta_i)\varphi_{x,h}(\theta_i) - \alpha(t_i)\varphi_{x,h}(t_i)\big) - \frac{1}{2\alpha(t_i)}d_i\langle R^{(0,2)}(\cdot,\sigma_i), f_{x,h} - g_n\rangle \\ - \frac{1}{2\alpha(t_i)}d_{i-1}\langle R^{(0,2)}(\cdot,\theta_i), f_{x,h} - g_n\rangle \\ \triangleq \frac{1}{2}(d_i + d_{i-1})\varphi_{x,h}(t_i) \ + A_i^{(1)} + A_i^{(2)} - A_i^{(3)} - A_i^{(4)}, \end{aligned} \tag{20}$$

Using the Cauchy-Schwartz inequality, Assumption (C) and Equation (35) (in the proof of Proposition 5 below) we obtain,

$$|A_i^{(3)} + A_i^{(4)}| \leq \sup_{0\leq t\leq 1} \frac{1}{2\alpha(t)}||R^{(0,2)}(.,t)||\frac{\sqrt{C}}{\sqrt{h}} \sup_{0\leq j\leq n} d_j^2 \triangleq \beta_{n,h}, \tag{21}$$

where C is a positive constant defined in Proposition 5 below.
Recall that $\varphi_{x,h}$ is of support $[x-h, x+h]$, thus for t_i such that $[t_{i-1}, t_{i+1}] \cap]x-h, x+h[= \emptyset$, $\varphi_{x,h}(t) = 0$ so that $A_i^{(1)} = 0$ and $A_i^{(2)} = 0$. For t_i such that $[t_{i-1}, t_{i+1}] \cap]x-h, x+h[\neq \emptyset$, let,

$$\alpha_{n,h} = \sup_{0 \leq i \leq n} \sup_{t_i \leq s, t \leq t_{i+1}} \frac{1}{2\alpha(t)} d_i |\alpha(s)\varphi_{x,h}(s) - \alpha(t)\varphi_{x,h}(t)|. \tag{22}$$

We obtain using (21) and (22) together with (20) for $i = 2, \ldots, n-1$,

$$m_{x,h}(t_i) = \begin{cases} \frac{1}{2}\varphi_{x,h}(t_i)(t_{i+1} - t_{i-1}) \ + \ O\big(\alpha_{n,h} + \beta_{n,h}\big) & if \ [t_{i-1}, t_{i+1}] \cap]x-h, x+h[\neq \emptyset \\ O\big(\beta_{n,h}\big) & otherwise. \end{cases}$$

After having obtained $m_{x,h}(t_i)$ for $i = 2, \ldots, n-1$, we are now able to obtain $m_{x,h}(t_1)$ and $m_{x,h}(t_n)$. We have for $i = 1, \ldots, n$,

$$R(t_1, t_i) m_{x,h}(t_1) + R(t_n, t_i) m_{x,h}(t_n) = f_{x,h}(t_i) - \sum_{j=2}^{n-1} m_{x,h}(t_j) R(t_j, t_i). \tag{23}$$

Recall that $N_{T_n} = \text{Card } I_{x,h} = \text{Card } \{i = 1, \cdots, n \ : \ [t_{i-1}, t_{i+1}] \cap]x-h, x+h[\neq \emptyset\}$ and that $t_{x,i}$ are the points of T_n for which $i \in I_{x,h}$. We have,

$$\sum_{j=2}^{n-1} m_{x,h}(t_j) R(t_j, t_i) = \sum_{j=1}^{N_{T_n}} m_{x,h}(t_{x,j}) R(t_{x,j}, t_i) + \sum_{j=2}^{n-1} 1_{\{j \notin I_{x,h}\}} m_{x,h}(t_j) R(t_j, t_i).$$

On the one hand, we have using (20) (where $A_{x,j}$ stands for A_j with t_j replaced by $t_{x,j}$),

$$\begin{aligned} \sum_{j=2}^{n-1} m_{x,h}(t_j) R(t_j, t_i) &= \frac{1}{2} \sum_{j=1}^{N_{T_n}} (d_{x,j} + d_{x,j-1}) \varphi_{x,h}(t_{x,j}) R(t_{x,j}, t_i) \\ &+ \sum_{j=1}^{N_{T_n}} (A_{x,j}^1 + A_{x,j}^2 - A_{x,j}^3 - A_{x,j}^4) R(t_{x,j}, t_i) - \sum_{j=2}^{n-1} 1_{\{j \notin I_{x,h}\}} (A_j^3 + A_j^4) R(t_j, t_i) \\ &= \frac{1}{2} \sum_{j=1}^{N_{T_n}} (d_{x,j} + d_{x,j-1}) \varphi_{x,h}(t_{x,j}) R(t_{x,j}, t_i) + \sum_{j=1}^{N_{T_n}} (A_{x,j}^1 + A_{x,j}^2) R(t_{x,j}, t_i) - \sum_{j=1}^{n} (A_j^3 + A_j^4) R(t_j, t_i). \end{aligned} \tag{24}$$

On the other hand,

$$\begin{aligned} f_{x,h}(t_i) &= \int_0^1 R(s, t_i) \varphi_{x,h}(s) \, ds = \int_{x-h}^{x+h} R(s, t_i) \varphi_{x,h}(s) \, ds = \frac{1}{2} \sum_{j=1}^{N_{T_n}} \int_{t_{x,j-1}}^{t_{x,j+1}} R(s, t_i) \varphi_{x,h}(s) \, ds \\ &= \frac{1}{2} \sum_{j=1}^{N_{T_n}} (d_{x,j} + d_{x,j-1}) R(t_{x,j}, t_i) \varphi_{x,h}(t_j) + \frac{1}{2} \sum_{j=1}^{N_{T_n}} \int_{t_{x,j-1}}^{t_{x,j+1}} (R(s, t_i) \varphi_{x,h}(s) - R(t_{x,j}, t_i) \varphi_{x,h}(t_{x,j})) \, ds. \end{aligned} \tag{25}$$

Inserting (24) and (25) in (23) we obtain for $i = 1, \ldots, n$,

$$\begin{aligned} R(t_1, t_i) m_{x,h}(t_1) + R(t_n, t_i) m_{x,h}(t_n) &= \frac{1}{2} \sum_{j=1}^{N_{T_n}} \int_{t_{x,j-1}}^{t_{x,j+1}} (R(s, t_i) \varphi_{x,h}(s) - R(t_{x,j}, t_i) \varphi_{x,h}(t_{x,j})) \, ds \\ &- \sum_{j=1}^{N_{T_n}} (A_{x,j}^1 + A_{x,j}^2) R(t_{x,j}, t_i) + \sum_{j=1}^{n} (A_j^3 + A_j^4) R(t_j, t_i) \triangleq \Phi_{x,h}(t_i). \end{aligned}$$

We then obtain the following linear system,

$$\begin{cases} R(t_1,t_1)m_{x,h}(t_1) + R(t_n,t_1)m_{x,h}(t_1) = \Phi_{x,h}(t_1). \\ \\ R(t_1,t_n)m_{x,h}(t_1) + R(t_n,t_n)m_{x,h}(t_n) = \Phi_{x,h}(t_n). \end{cases} \tag{26}$$

Solving (26) for $m_{x,h}(t_1)$ and $m_{x,h}(t_n)$ we obtain,

$$m_{x,h}(t_1) = \frac{R(t_n,t_n)\Phi_{x,h}(t_1) - R(t_1,t_n)\Phi_{x,h}(t_n)}{R(t_1,t_1)R(t_n,t_n) - R(t_1,t_n)^2}. \tag{27}$$

$$m_{x,h}(t_n) = \frac{R(t_1,t_1)\Phi_{x,h}(t_n) - R(t_1,t_n)\Phi_{x,h}(t_1)}{R(t_1,t_1)R(t_n,t_n) - R(t_1,t_n)^2}. \tag{28}$$

Finally, simple calculations yield,

$$m_{x,h}(t_1) = O\big(N_{T_n}\alpha_{n,h} + n\beta_{n,h}\big) \quad \text{and} \quad m_{x,h}(t_n) = O\big(N_{T_n}\alpha_{n,h} + n\beta_{n,h}\big).$$

This completes the proof of Lemma 3. □

Proof of Proposition 3.

Recall that $N_{T_n} = \text{Card } I_{x,h} = \text{Card } \{i = 1,\cdots,n \ : \ [t_{i-1},t_{i+1}]\cap]x-h,x+h[\neq \emptyset\}$ and denote by $t_{x,i}$ the points of T_n for which $i \in I_{x,h}$, that is $T_n\cap]x-h,x+h[= \{t_{x,2},\cdots,t_{x,N_{T_n}-1}\}$. Since $\mathbb{E}(\overline{Y}(t_i)) = g(t_i)$ then,

$$\begin{aligned}\mathbb{E}(\hat{g}_n^{pro}(x)) &= \sum_{j=1}^{n} m_{x,h}(t_j)g(t_j) \\ &= \sum_{i=1}^{N_{T_n}} m_{x,h}(t_{x,i})g(t_{x,i}) + \sum_{j=2}^{n-1} 1_{\{i\notin I_{x,h}\}} m_{x,h}(t_j)g(t_j) + m_{x,h}(t_1)g(t_1) + m_{x,h}(t_n)g(t_n).\end{aligned}$$

Using the asymptotic approximation of $m_{x,h_{|T_n}}$ given in Lemma 3 we obtain,

$$E(\hat{g}_n^{pro}(x)) = \frac{1}{2}\sum_{i=1}^{N_{T_n}}(t_{x,i+1} - t_{x,i-1})\varphi_{x,h}(t_{x,i})g(t_{x,i}) + O\big(N_{T_n}\alpha_{n,h} + n\beta_{n,h}\big), \tag{29}$$

For $x \in [0,1]$ let,

$$I_h(x) = \int_{x-h}^{x+h} \varphi_{x,h}(t)g(t)\ dt = \frac{1}{2}\sum_{i=1}^{N_{T_n}} \int_{t_{x,i-1}}^{t_{x,i+1}} \varphi_{x,h}(t)g(t)\ dt,$$

and write,

$$\mathbb{E}(\hat{g}_n^{pro}(x)) = \mathbb{E}(\hat{g}_n^{pro}(x)) - I_h(x) + I_h(x) = \Delta_{x,h} + I_h(x) + O\big(N_{T_n}\alpha_{n,h} + n\beta_{n,h}\big), \tag{30}$$

where,

$$\Delta_{x,h} = \frac{1}{2}\sum_{i=1}^{N_{T_n}} \int_{t_{x,i-1}}^{t_{x,i+1}} \Big(\varphi_{x,h}(t_{x,i})g(t_{x,i}) - \varphi_{x,h}(t)g(t)\Big)\ dt.$$

We first control $\Delta_{x,h}$. We have,

$$\Delta_{x,h} = \frac{1}{2}\sum_{i=1}^{N_{T_n}} \varphi_{x,h}(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i+1}} (g(t_{x,i}) - g(t))\ dt + \frac{1}{2}\sum_{i=1}^{N_{T_n}} \int_{t_{x,i-1}}^{t_{x,i+1}} g(t)(\varphi_{x,h}(t_{x,i}) - \varphi_{x,h}(t))\ dt.$$

Since $\varphi_{x,h}$ is in C^1 and g is in C^2 then Taylor expansions of $\varphi_{x,h}$ and g give,

$$g(t) = g(t_{x,i}) + (t - t_{x,i})g'(t_{x,i}) + \frac{1}{2}(t - t_{x,i})^2 g''(\theta_{x,i}),$$

and,

$$\varphi_{x,h}(t) = \varphi_{x,h}(t_{x,i}) + (t - t_{x,i})\varphi'_{x,h}(\eta_{x,i}),$$

for some $\theta_{x,i}$ and $\eta_{x,i}$ between t and $t_{x,i}$. Thus,

$$\Delta_{x,h} = -\frac{1}{2}\sum_{i=1}^{N_{T_n}} \varphi_{x,h}(t_{x,i})g'(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i+1}} (t - t_{x,i})\ dt - \frac{1}{4}\sum_{i=1}^{N_{T_n}} \varphi_{x,h}(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i+1}} g''(\theta_{x,i})(t - t_{x,i})^2\ dt$$
$$-\frac{1}{2}\sum_{i=1}^{N_{T_n}} g(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i+1}} \varphi'_{x,h}(\eta_{x,i})(t - t_{x,i})\ dt - \frac{1}{2}\sum_{i=1}^{N_{T_n}} g'(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i+1}} \varphi'_{x,h}(\eta_{x,i})(t - t_{x,i})^2\ dt$$
$$-\frac{1}{4}\sum_{i=1}^{N_{T_n}} \int_{t_{x,i-1}}^{t_{x,i+1}} g''(\theta_{x,i})\varphi'_{x,h}(\eta_{x,i})(t - t_{x,i})^3\ dt.$$

Recall that g' and g'' are both bounded and that,

$$\sup_{0\le t\le 1} |\varphi_{x,h}(t)| < \frac{c}{h} \quad \text{and} \quad \sup_{0\le t\le 1} |\varphi'_{x,h}(t)| < \frac{c'}{h^2}, \tag{31}$$

for appropriate positive constants c and c'. Using this we obtain,

$$\frac{1}{4}\sum_{i=1}^{N_{T_n}} \varphi_{x,h}(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i+1}} g''(\theta_{x,i})(t - t_{x,i})^2\ dt = O\Big(\frac{N_{T_n}}{h} \sup_{0\le j\le 1} d_{j,n}^3\Big)$$
$$\frac{1}{2}\sum_{i=1}^{N_{T_n}} g'(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i+1}} \varphi'_{x,h}(\eta_{x,i})(t - t_{x,i})^2\ dt = O\Big(\frac{N_{T_n}}{h^2} \sup_{0\le j\le 1} d_{j,n}^3\Big)$$
$$\frac{1}{4}\sum_{i=1}^{N_{T_n}} \int_{t_{x,i-1}}^{t_{x,i+1}} g''(\theta_{x,i})\varphi'_{x,h}(\eta_{x,i})(t - t_{x,i})^2\ dt = O\Big(\frac{N_{T_n}}{h^2} \sup_{0\le j\le 1} d_{j,n}^3\Big).$$

Thus,

$$\Delta_{x,h} = -\frac{1}{2}\sum_{i=1}^{N_{T_n}} \varphi_{x,h}(t_{x,i})g'(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i+1}} (t - t_{x,i})dt - \frac{1}{2}\sum_{i=1}^{N_{T_n}} g(t_{x,i})\varphi'_{x,h}(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i+1}} (t - t_{x,i})\ dt$$
$$-\frac{1}{2}\sum_{i=1}^{N_{T_n}} g(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i+1}} (t - t_{x,i})\Big(\varphi'_{x,h}(\eta_{x,i}) - \varphi'_{x,h}(t_{x,i})\Big)\ dt + O\Big(\frac{N_{T_n}}{h^2} \sup_{0\le j\le 1} d_{j,n}^3\Big).$$

Since $\varphi'_{x,h}$ is Lipschitz then,

$$\sum_{i=1}^{N_{T_n}} g(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i+1}} (t - t_{x,i})\Big(\varphi'_{x,h}(\eta_{x,i}) - \varphi'_{x,h}(t_{x,i})\Big)\ dt = O\Big(\frac{N_{T_n}}{h^3} \sup_{0\le j\le 1} d_{j,n}^3\Big).$$

Thus,

$$\Delta_{x,h} = -\frac{1}{2}\sum_{i=1}^{N_{T_n}} \varphi_{x,h}(t_{x,i})g'(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i+1}} (t-t_{x,i})dt - \frac{1}{2}\sum_{i=1}^{N_{T_n}} g(t_{x,i})\varphi'_{x,h}(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i+1}} (t-t_{x,i})\ dt$$
$$+ O\left(\frac{N_{T_n}}{h^3} \sup_{0\le j\le 1} d_{j,n}^3\right).$$

Basic integration gives,

$$\Delta_{x,h} = -\frac{1}{4}\sum_{i=1}^{N_{T_n}} \varphi_{x,h}(t_{x,i})g'(t_{x,i})(d_{x,i}^2 - d_{x,i-1}^2) - \frac{1}{4}\sum_{i=1}^{N_{T_n}} g(t_{x,i})\varphi'_{x,h}(t_{x,i})(d_{x,i}^2 - d_{x,i-1}^2)$$
$$+ O\left(\frac{N_{T_n}}{h^3} \sup_{0\le j\le 1} d_{j,n}^3\right).$$

We shall show that,

$$A \triangleq \sum_{i=1}^{N_{T_n}} \varphi_{x,h}(t_{x,i})g'(t_{x,i})(d_{x,i}^2 - d_{x,i-1}^2) = O\left(\frac{N_{T_n}}{h^2} \sup_{0\le j\le 1} d_{j,n}^3\right),$$
$$B \triangleq \sum_{i=1}^{N_{T_n}} g(t_{x,i})\varphi'_{x,h}(t_{x,i})(d_{x,i}^2 - d_{x,i-1}^2) = O\left(\frac{N_{T_n}}{h^3} \sup_{0\le j\le 1} d_{j,n}^3\right).$$

Starting with the term A. Recall that, since φ is of support $[x-h, x+h]$ and $t_{x,1}, t_{x,N_{T_n}-1} \notin]x-h, x+h[$, then $\varphi_{x,h}(t_{x,N_{T_n}}) = \varphi_{x,h}(t_{x,1}) = 0$ thus,

$$\begin{aligned} A &= \sum_{i=2}^{N_{T_n}-1} \varphi_{x,h}(t_{x,i})g'(t_{x,i})d_{x,i}^2 - \sum_{i=1}^{N_{T_n}-2} \varphi_{x,h}(t_{x,i+1})g'(t_{x,i+1})d_{x,i}^2 \\ &= \sum_{i=2}^{N_{T_n}-2} \big(\varphi_{x,h}(t_{x,i})g'(t_{x,i}) - \varphi_{x,h}(t_{x,i+1})g'(t_{x,i+1})\big)d_{x,i}^2 + \Big(\varphi_{x,h}(t_{x,N_{T_n}-1})g'(t_{x,N_{T_n}-1})d_{x,N_{T_n}-1}^2 \\ &\qquad - \varphi_{x,h}(t_{x,2})g'(t_{x,2})d_{x,1}^2\Big) \\ &\triangleq A_1 + A_2. \end{aligned}$$

On the one hand, Taylor expansions of $\varphi_{x,h}$ around $t_{x,N_{T_n}}$ and $t_{x,1}$ yield,

$$\varphi_{x,h}(t_{x,N_{T_n}-1}) = (t_{x,N_{T_n}-1} - t_{x,N_{T_n}})\varphi'_{x,h}(\gamma_{x,N_{T_n}}),$$
$$\varphi_{x,h}(t_{x,2}) = (t_{x,2} - t_{x,1})\varphi'_{x,h}(\gamma_{x,1}),$$

for some $\gamma_{x,N_{T_n}} \in]t_{x,N_{T_n}-1}, t_{x,N_{T_n}}[$ and some $\gamma_{x,1} \in]t_{x,1}, t_{x,2}[$. Using (31) and the fact that g' is bounded we obtain,

$$A_2 = O\left(\frac{1}{h^2} \sup_{0\le j\le 1} d_{j,n}^3\right).$$

On the other hand we have,

$$\begin{aligned} A_1 &= \sum_{i=2}^{N_{T_n}-2} \big(\varphi_{x,h}(t_{x,i})g'(t_{x,i}) - \varphi_{x,h}(t_{x,i+1})g'(t_{x,i+1})\big)d_{x,i}^2 \\ &= \sum_{i=2}^{N_{T_n}-2} \varphi_{x,h}(t_{x,i})\big(g'(t_{x,i}) - g'(t_{x,i+1})\big)d_{x,i}^2 + \sum_{i=2}^{N_{T_n}-2} g'(t_{x,i+1})\big(\varphi_{x,h}(t_{x,i}) - \varphi_{x,h}(t_{x,i+1})\big)d_{x,i}^2. \end{aligned}$$

Since $\varphi_{x,h}$ is in C^1 and g is in C^2 then using (31), we obtain,

$$A_1 = O\Big(\frac{N_{T_n}}{h^2} \sup_{0\le j\le 1} d_{j,n}^3\Big).$$

In a similar way and from Assumption (D), we obtain,

$$B = O\Big(\frac{N_{T_n}}{h^3} \sup_{0\le j\le 1} d_{j,n}^3\Big).$$

Hence,

$$\Delta_{x,h} = O\Big(\frac{N_{T_n}}{h^3} \sup_{0\le j\le 1} d_{j,n}^3\Big).$$

Thus using (30),

$$\mathbb{E}(\hat{g}_n^{pro}(x)) = I_h(x) + O(N_{T_n}\alpha_{n,h} + n\beta_{n,h}) + O\Big(\frac{N_{T_n}}{h^3} \sup_{0\le j\le 1} d_{j,n}^3\Big).$$

The control of $I_h(x)$ is classical and it can bee seen from Gasser and Müller (1984) that,

$$I_h(x) = g(x) + \frac{1}{2}h^2 g''(x) \int_{-1}^{1} t^2 K(t)\ dt + o(h^2). \tag{32}$$

Finally,

$$\mathbb{E}(\hat{g}_n^{pro}(x)) - g(x) = \frac{1}{2}h^2 g''(x) \int_{-1}^{1} t^2 K(t)\ dt + o(h^2) + O\Big(\frac{N_{T_n}}{h^3} \sup_{0\le j\le 1} d_{j,n}^3 + N_{T_n}\alpha_{n,h} + n\beta_{n,h}\Big).$$

This concludes the proof of Proposition 3. □

Proof of Proposition 4.

Let $t_0 = 0,\ t_{n+1} = 1$ and set $\overline{Y}(t_0) = 0$ and $\overline{Y}(t_{n+1}) = \overline{Y}(t_n)$. Recall that,

$$\hat{g}_n^{pro}(x) = \sum_{i=1}^{n+1} \overline{Y}(t_i) \int_{t_{i-1}}^{t_i} \varphi_{x,h}(s) ds + \sum_{i=0}^{n} \frac{\overline{Y}(t_{i+1}) - \overline{Y}(t_i)}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} (s - t_{i+1})\varphi_{x,h}(s) ds.$$

Since $\mathbb{E}\ (\overline{Y}(t_i)) = g(t_i)$ then,

$$\mathbb{E}\ (\hat{g}_n^{pro}(x)) = \sum_{i=1}^{n+1} g(t_i) \int_{t_{i-1}}^{t_i} \varphi_{x,h}(s) ds + \sum_{i=0}^{n} \frac{g(t_{i+1}) - g(t_i)}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} (s - t_{i+1})\varphi_{x,h}(s) ds.$$

Recall that $N_{T_n} = \text{Card}\ I_{x,h} = \{i = 1, \cdots, n\ :\ [t_{i-1,n}, t_{i+1,n}] \cap]x-h, x+h[\neq \emptyset\}$ and denote by $t_{x,i}$ the points of T_n for which $i \in I_{x,h}$. Using the support of $\varphi_{x,h}$ we obtain,

$$\mathbb{E}\ (\hat{g}_n^{pro}(x)) = \sum_{i=1}^{N_{T_n}} g(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i}} \varphi_{x,h}(s) ds + \sum_{i=1}^{N_{T_n}} \frac{g(t_{x,i+1}) - g(t_{x,i})}{t_{x,i+1} - t_{x,i}} \int_{t_{x,i}}^{t_{x,i+1}} (s - t_{x,i+1})\varphi_{x,h}(s) ds.$$

Let $d_{x,i} = t_{x,i+1} - t_{x,i}$. Since g is in C^2 and $\varphi_{x,h}$ is in C^1 then Taylor expansions of g around $t_{x,i}$ and of $\varphi_{x,h}$ around $t_{x,i+1}$ yield,

$$g(t_{x,i+1}) = g(t_{x,i}) + d_{x,i}\ g'(t_{x,i}) + \frac{1}{2}d_{x,i}^2\ g''(\theta_{x,i}),$$
$$\varphi_{x,h}(s) = \varphi_{x,h}(t_{x,i+1}) + (s - t_{x,i+1})\varphi'_{x,h}(s_i).$$

for some $\theta_{x,i} \in]t_{x,i}, t_{x,i+1}[$ and some $s_i \in]s, t_{x,i+1}[$. Recall that, using the support of φ, $\varphi_{x,h}(t_{x,1}) = \varphi_{x,h}(t_{x,N_{T_n}}) = 0$ thus,

$$\begin{aligned}\mathbb{E}\ (\hat{g}_n^{pro}(x)) &= \sum_{i=1}^{N_{T_n}} g(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i}} \varphi_{x,h}(s)ds\ + \sum_{i=1}^{N_{T_n}-2} g'(t_{x,i})\varphi_{x,h}(t_{x,i+1}) \int_{t_{x,i}}^{t_{x,i+1}} (s - t_{x,i+1})ds \\ &+ \sum_{i=1}^{N_{T_n}} g'(t_{x,i}) \int_{t_{x,i}}^{t_{x,i+1}} (s - t_{x,i+1})^2\varphi'_{x,h}(s_i)ds + \frac{1}{2}\sum_{i=1}^{N_{T_n}} \varphi_{x,h}(t_{x,i+1})g''(\theta_{x,i})d_{x,i} \int_{t_{x,i}}^{t_{x,i+1}} (s - t_{x,i+1})ds \\ &+ \frac{1}{2}\sum_{i=1}^{N_{T_n}} g''(\theta_{x,i})d_{x,i} \int_{t_{x,i}}^{t_{x,i+1}} (s - t_{x,i+1})^2\varphi'_{x,h}(s_i)ds.\end{aligned}$$

Recall that g' and g'' are bounded, Lemma 2 yields $N_{T_n} = O(nh)$ and $d_{x,i} = O(\frac{1}{n})$ and using (31) we obtain,

$$\sum_{i=1}^{N_{T_n}} g'(t_{x,i}) \int_{t_{x,i}}^{t_{x,i+1}} (s - t_{x,i+1})^2\varphi'_{x,h}(s_i)ds = O\left(\frac{1}{n^2h}\right).$$
$$\frac{1}{2}\sum_{i=1}^{N_{T_n}} \varphi_{x,h}(t_{x,i+1})g''(\theta_{x,i})d_{x,i} \int_{t_{x,i}}^{t_{x,i+1}} (s - t_{x,i+1})ds = O\left(\frac{1}{n^2}\right).$$
$$\frac{1}{2}\sum_{i=1}^{N_{T_n}} g''(\theta_{x,i})d_{x,i} \int_{t_{x,i}}^{t_{x,i+1}} (s - t_{x,i+1})^2\varphi'_{x,h}(s_i)ds = O\left(\frac{1}{n^3h}\right).$$

It follows that by simple integration,

$$\begin{aligned}\mathbb{E}\ (\hat{g}_n^{pro}(x)) &= \sum_{i=1}^{N_{T_n}} g(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i}} \varphi_{x,h}(s)ds - \frac{1}{2} \sum_{i=1}^{N_{T_n}-2} g'(t_{x,i})\varphi_{x,h}(t_{x,i+1})d_{x,i}^2 + O\left(\frac{1}{n^2h}\right) \\ &= \sum_{i=1}^{N_{T_n}} \int_{t_{x,i-1}}^{t_{x,i}} \varphi_{x,h}(s)g(s)\ ds + \sum_{i=1}^{N_{T_n}} \int_{t_{x,i-1}}^{t_{x,i}} \varphi_{x,h}(s)(g(t_{x,i}) - g(s))\ ds \\ &\quad - \frac{1}{2} \sum_{i=1}^{N_{T_n}-2} g'(t_{x,i})\varphi_{x,h}(t_{x,i+1})d_{x,i}^2 + O\left(\frac{1}{n^2h}\right).\end{aligned}$$

On the one hand, we have,

$$\sum_{i=1}^{N_{T_n}} \int_{t_{x,i-1}}^{t_{x,i}} \varphi_{x,h}(s)g(s)\ ds = \int_{x-h}^{x+h} \varphi_{x,h}(s)g(s)\ ds.$$

On the other hand, Taylor expansion of g and $\varphi_{x,h}$ arround $t_{x,i}$ yield,

$$g(t_{x,i}) = g(s) + (t_{x,i} - s)g'(t_{x,i}) - \frac{1}{2}(t_{x,i} - s)^2 g''(s'_i),$$
$$\varphi_{x,h}(s) = \varphi_{x,h}(t_{x,i}) + (s - t_{x,i})\varphi'_{x,h}(s''_i).$$

for some s_i' and s_i'' in $]s, t_{x,i}[$. Thus,

$$\begin{aligned}
\mathbb{E}\left(\hat{g}_n^{pro}(x)\right) &= \int_{x-h}^{x+h} \varphi_{x,h}(s)g(s)\ ds + \sum_{i=2}^{N_{T_n}-1} g'(t_{x,i})\varphi_{x,h}(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i}} (t_{x,i}-s)\ ds \\
&- \sum_{i=1}^{N_{T_n}} g'(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i}} (t_{x,i}-s)^2 \varphi_{x,h}'(s_i'')\ ds - \frac{1}{2}\sum_{i=1}^{N_{T_n}} \varphi_{x,h}(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i}} g''(s_i')(t_{x,i}-s)^2\ ds \\
&+ \frac{1}{2}\sum_{i=1}^{N_{T_n}} \int_{t_{x,i-1}}^{t_{x,i}} g''(s_i')\varphi_{x,h}'(s_i'')(t_{x,i}-s)^3\ ds - \frac{1}{2}\sum_{i=1}^{N_{T_n}-2} g'(t_{x,i})\varphi_{x,h}(t_{x,i+1})d_{x,i}^2 \\
&+ O\left(\frac{1}{n^2h}\right).
\end{aligned}$$

Using the boundedness of g' and g'' in addition to Lemma 2 and Equation (31), we obtain,

$$\begin{aligned}
&\sum_{i=1}^{N_{T_n}} g'(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i}} (t_{x,i}-s)^2 \varphi_{x,h}'(s_i'')\ ds = O\left(\frac{1}{n^2h}\right). \\
&\frac{1}{2}\sum_{i=1}^{N_{T_n}} \varphi_{x,h}(t_{x,i}) \int_{t_{x,i-1}}^{t_{x,i}} g''(s_i')(t_{x,i}-s)^2\ ds = O\left(\frac{1}{n^2}\right). \\
&\frac{1}{2}\sum_{i=1}^{N_{T_n}} \int_{t_{x,i-1}}^{t_{x,i}} g''(s_i')\varphi_{x,h}'(s_i'')(t_{x,i}-s)^3\ ds = O\left(\frac{1}{n^3h}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbb{E}\left(\hat{g}_n^{pro}(x)\right) &= \int_{x-h}^{x+h} \varphi_{x,h}(s)g(s)\ ds + \frac{1}{2}\sum_{i=2}^{N_{T_n}-2} g'(t_{x,i})\varphi_{x,h}(t_{x,i})d_{x,i-1}^2 \\
&\quad - \frac{1}{2}\sum_{i=1}^{N_{T_n}-2} g'(t_{x,i})\varphi_{x,h}(t_{x,i+1})d_{x,i}^2 + O\left(\frac{1}{n^2h}\right) \\
&= \int_{x-h}^{x+h} \varphi_{x,h}(s)g(s)\ ds + \frac{1}{2}\sum_{i=1}^{N_{T_n}-2} \Big(g'(t_{x,i+1}) - g'(t_{x,i})\Big)\varphi_{x,h}(t_{x,i+1})d_{x,i}^2 + O\left(\frac{1}{n^2h}\right).
\end{aligned}$$

Since g' is Lipschitz, then we have,

$$\mathbb{E}\left(\hat{g}_n^{pro}(x)\right) = \int_{x-h}^{x+h} \varphi_{x,h}(s)\ g(s)\ ds + O\left(\frac{1}{n^2h}\right). \tag{33}$$

Finally, from (32) we obtain,

$$\mathbb{E}\left(\hat{g}_n^{pro}(x)\right) - g(x) = \frac{1}{2}h^2 g''(x) \int_{-1}^{1} t^2 K(t)dt + o(h^2) + O\left(\frac{1}{n^2h}\right).$$

This concludes the proof of Proposition 4. □

Proof of Proposition 5.

The great lines of this proof are based on Sacks and Ylvisaker (1966). From the definition of the orthogonal projection (see the Appendix) and using the Pythagore theorem we obtain,

$$m\Big(\frac{\sigma_{x,h}^2}{m} - \mathrm{Var} g_n^{pro}(x)\Big) = ||f_{x,h}||^2 - ||P_{|T_n} f_{x,h}||^2 = ||f_{x,h} - P_{|T_n} f_{x,h}||^2, \tag{34}$$

where $P_{|T_n} f_{x,h}$ is the orthogonal projection of $f_{x,h}$ on the subspace of $\mathcal{F}(\varepsilon)$ spanned by $\{R(\cdot, t_i), t_i \in T_n\}$, denoted here by V_{T_n}. We shall then prove that,

$$||f_{x,h} - P_{|T_n} f_{x,h}||^2 \leq \frac{C}{h} \sup_{0\leq j\leq n} d_{j,n}^2. \tag{35}$$

Recall that $N_{T_n} = \mathrm{Card}\, I_{x,h} = \mathrm{Card}\, I_{x,h} = \{i = 1, \cdots, n \ : \ [t_{i-1,n}, t_{i+1,n}] \cap]x-h, x+h[\neq \emptyset\}$ and denote by $t_{x,i}$ the points of T_n for which $i \in I_{x,h}$. Let $g_n := g_{n,x} = \sum_{i=1}^n \gamma_{x,i} R(\cdot, t_{x,i})$ with $\gamma_{x,i} = 0$ for every $i \notin I_{x,h}$. It is clear that $g_n \in V_{T_n}$ and thus from the definition of the orthogonal projection we have,

$$||f_{x,h} - P_{|T_n} f_{x,h}||^2 \leq ||f_{x,h} - g_n||^2.$$

Now using (F1) in the Appendix and the support of $\varphi_{x,h}$ we obtain,

$$\begin{aligned} ||f_{x,h} - g_n||^2 &= \int_0^1 (f_{x,h}(t) - g_n(t))\varphi_{x,h}(t)\, dt - \sum_{i=1}^n (f_{x,h}(t_i) - g_n(t_i))\gamma_{x,i} \\ &= \int_{x-h}^{x+h} (f_{x,h}(t) - g_n(t))\varphi_{x,h}(t)\, dt - \sum_{i=1}^{N_{T_n}} (f_{x,h}(t_{x,i}) - g_n(t_{x,i}))\gamma_{x,i} \end{aligned} \tag{36}$$

In what follows, we distinguish between three cases according to the location of $t_{x,1}$ and $t_{x,N_{T_n}}$ in the interval $[x-h, x+h]$.

First case. Suppose first that $t_{x,1} = x - h$ and $t_{x,N_{T_n}} = x + h$ and take,

$$\gamma_{x,i} = \int_{t_{x,i}}^{t_{x,i+1}} \varphi_{x,h}(t)\, dt \quad \text{for} \quad i = 1, \ldots, N_{T_n} - 1. \tag{37}$$

we have in this case,

$$||f_{x,h} - g_n||^2 = \sum_{i=1}^{N_{T_n}} \int_{t_{x,i}}^{t_{x,i+1}} \Big((f_{x,h}(t) - g_n(t)) - (f_{x,h}(t_{x,i}) - g_n(t_{x,i})) \Big) \varphi_{x,h}(t)\, dt. \tag{38}$$

Assumption (A) yields that $f_{x,h}$ is twice differentiable on $[0, 1]$, while g_n is twice differentiable everywhere except on T_n, but it has left and right derivatives. Taylor expansion of $f_{x,h} - g_n$ around $t_{x,i}$ for $i = 1, \cdots, N_{T_n} - 1$ and $t \in]t_{x,i}, t_{x,i+1}[$ gives,

$$\begin{aligned} f_{x,h}(t) - g_n(t) &= (f_{x,h}(t_{x,i}) - g_n(t_{x,i})) + (t - t_{x,i})(f'_{x,h}(t_{x,i}) - g'_n(t_{x,i}^+)) \\ &\quad + \frac{1}{2}(t - t_{x,i})^2 (f''_{x,h}(\theta_{x,t}) - g''_n(\theta_{x,t}^+)), \end{aligned} \tag{39}$$

for some $\theta_{x,t} \in]t_{x,i}, t[$. On the one hand, we have,

$$g_n'(t_{x,i}^+) = \sum_{j=1}^{N_{T_n}-1} R^{(0,1)}(t_{x,j}, t_{x,i}^+)\gamma_{x,j}. \tag{40}$$

On the other hand, using (16) we obtain,

$$\begin{aligned} f_{x,h}'(t_{x,i}) &= \int_{x-h}^{x+h} R^{(0,1)}(s, t_{x,i}^+)\varphi_{x,h}(s)\, ds = \sum_{j=1}^{N_{T_n}-1} \int_{t_{x,j}}^{t_{x,j+1}} R^{(0,1)}(s, t_{x,i}^+)\varphi_{x,h}(s)\, ds \\ &= \sum_{\substack{j=1 \\ j\neq i}}^{N_{T_n}-1} \int_{t_{x,j}}^{t_{x,j+1}} R^{(0,1)}(s, t_{x,i}^+)\varphi_{x,h}(s)\, ds \;+\; \int_{t_{x,i}}^{t_{x,i+1}} R^{(0,1)}(s, t_{x,i}^+)\varphi_{x,h}(s)\, ds. \end{aligned} \tag{41}$$

When $j \neq i$ we have,

$$\int_{t_{x,j}}^{t_{x,j+1}} R^{(0,1)}(s, t_{x,i}^+)\varphi_{x,h}(s)\, ds = R^{(0,1)}(t_{x,j}, t_{x,i})\gamma_{x,j} + \int_{t_{x,j}}^{t_{x,j+1}} (s - t_{x,j})R^{(1,1)}(\delta_{s,j}, t_{x,i})\varphi_{x,h}(s)\, ds, \tag{42}$$

for some $\delta_{s,j} \in]t_{x,j}, s[$, while for $j = i$ we have,

$$\begin{aligned} \int_{t_{x,i}}^{t_{x,i+1}} R^{(0,1)}(s, t_{x,i}^+)\varphi_{x,h}(s)\, ds &= \int_{t_{x,i}}^{t_{x,i+1}} R^{(0,1)}(s, t_{x,i}^-)\varphi_{x,h}(s)\, ds \\ &= R^{(0,1)}(t_{x,i}, t_{x,i}^-)\gamma_{x,i} + \int_{t_{x,i}}^{t_{x,i+1}} (s - t_{x,i})R^{(1,1)}(\delta_{s,i}^+, t_{x,i}^-)\varphi_{x,h}(s)\, ds. \end{aligned} \tag{43}$$

Collecting (40), (41), (42) and (43) we obtain,

$$\begin{aligned} f_{x,h}'(t_{x,i}) - g_n'(t_{x,i}^+) = &\sum_{\substack{j=1 \\ j\neq i}}^{N_{T_n}-1} R^{(0,1)}(t_{x,j}, t_{x,i})\gamma_{x,j} + \sum_{\substack{j=1 \\ j\neq i}}^{N_{T_n}-1} \int_{t_{x,j}}^{t_{x,j+1}} (s - t_{x,j})R^{(1,1)}(\delta_{s,j}, t_{x,i})\varphi_{x,h}(s)\, ds \\ &+ R^{(0,1)}(t_{x,i}, t_{x,i}^-)\gamma_{x,i} + \int_{t_{x,i}}^{t_{x,i+1}} R^{(1,1)}(\delta_{s,i}^+, t_{x,i}^-)\varphi_{x,h}(s)\, ds - \sum_{j=1}^{N_{T_n}-1} R^{(0,1)}(t_{x,j}, t_{x,i}^+)\gamma_{x,j} \\ = &\ \alpha(t_{x,i})\gamma_{x,i} + \sum_{j=1}^{N_{T_n}-1} \int_{t_{x,j}}^{t_{x,j+1}} (s - t_{x,j})R^{(1,1)}(\delta_{s,j}^+, t_{x,i}^-)\varphi_{x,h}(s)\, ds. \end{aligned}$$

It is easy to see that,

$$\begin{aligned} |f_{x,h}'(t_{x,i}) - g_n'(t_{x,i}^+)| &\leq \alpha_1\gamma_{x,i} + \frac{K_\infty}{h} R_1 \sum_{j=1}^{N_{T_n}-1} \int_{t_{x,j}}^{t_{x,j+1}} (s - t_{x,j})\, ds \\ &\leq \frac{K_\infty}{h}\alpha_1 d_{x,i} + \frac{K_\infty}{2h} R_1 \sum_{j=1}^{N_{T_n}-1} d_{x,j}^2. \end{aligned} \tag{44}$$

We deduce from (17) that for all $\theta_{x,t} \in]t_{x,i}, t_{x,i+1}[$ we have,

$$|f_{x,h}''(\theta_{x,t})| \leq \frac{K_\infty}{h}\alpha_1 + \frac{K_\infty}{h} R_2 \times 2h = \frac{K_\infty}{h}\alpha_1 + 2K_\infty R_2.$$

In addition, for $\theta_{x,t} \in]t_{x,i}, t_{x,i+1}[$ we have,

$$|g_n''(\theta_{x,t}^+)| = \left| \sum_{j=1}^{N_{T_n}-1} R^{(0,2)}(t_{x,j}, \theta_{x,t}^+)\gamma_{x,j} \right| \leq \frac{K_\infty}{h} R_2 \sum_{j=1}^{N_{T_n}-1} d_{x,j} = \frac{K_\infty}{h} R_2 \times 2h = 2K_\infty R_2,$$

Thus,

$$|f_{x,h}''(\theta_{x,t}) - g_n''(\theta_{x,t}^+)| \leq \frac{K_\infty}{h}\alpha_1 + 4K_\infty R_2. \tag{45}$$

Equations (39), (44) and (45) yield that for $i = 1, \cdots, N_{T_n} - 1$,

$$\begin{aligned}
&\left| \int_{t_{x,i}}^{t_{x,i+1}} \left[(f_{x,h}(t) - g_n(t)) - (f_{x,h}(t_{x,i}) - g_n(t_{x,i})) \right] \varphi_{x,h}(t)\, dt \right| \\
&\leq \int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i}) |f_{x,h}'(t_{x,i}) - g_n'(t_{x,i}^+)| |\varphi_{x,h}(t)|\, dt \\
&\quad + \frac{1}{2} \int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})^2 |f_{x,h}''(\theta_{x,t}) - g_n''(\theta_{x,t}^+)| |\varphi_{x,h}(t)|\, dt \\
&\leq \left(\frac{K_\infty}{h}\alpha_1 d_{x,i} + \frac{K_\infty}{2h} R_1 \sum_{j=1}^{N_{T_n}-1} d_{x,j}^2 \right) \int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i}) |\varphi_{x,h}(t)|\, dt \\
&\quad + \frac{1}{2} \left(\frac{K_\infty}{h}\alpha_1 + 4K_\infty R_2 \right) \int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})^2 |\varphi_{x,h}(t)|\, dt \\
&\leq \left(\frac{K_\infty}{h}\alpha_1 d_{x,i} + \frac{K_\infty}{2h} R_1 \sum_{j=1}^{N_{T_n}-1} d_{x,j}^2 \right) \frac{K_\infty}{2h} d_{x,i}^2 + \frac{1}{2} \left(\frac{K_\infty}{h}\alpha_1 + 4K_\infty R_2 \right) \frac{K_\infty}{3h} d_{x,i}^3 \\
&\leq \frac{K_\infty^2}{4h^2} R_1 d_{x,i}^2 \sum_{j=1}^{N_{T_n}-1} d_{x,j}^2 + \frac{2K_\infty^2}{3h} \Big(\frac{\alpha_1}{h} + R_2 \Big) d_{x,i}^3.
\end{aligned} \tag{46}$$

Injecting this inequality in (38) yields,

$$\begin{aligned}
||f_{x,h} - P_{|T_n} f_{x,h}||^2 &\leq \frac{K_\infty^2}{4h^2} R_1 \left(\sum_{i=1}^{N_{T_n}-1} d_{x,i}^2 \right)^2 + \frac{2K_\infty^2}{3h} \Big(\frac{\alpha_1}{h} + R_2 \Big) \sum_{i=1}^{N_{T_n}-1} d_{x,i}^3 \\
&\leq \frac{K_\infty^2}{4h^2} R_1 \sup_{1 \leq i \leq n} d_{i,n}^2 \left(\sum_{i=1}^{N_{T_n}-1} d_{x,i} \right)^2 + \frac{2K_\infty^2}{3h} \big(\frac{\alpha_1}{h} + R_2 \big) \sup_{1 \leq i \leq n} d_{i,n}^2 \sum_{i=1}^{N_{T_n}-1} d_{x,i}.
\end{aligned}$$

Since $\sum_{i=1}^{N_{T_n}-1} d_{x,i} = 2h$ then,

$$||f_{x,h} - P_{|T_n} f_{x,h}||^2 \leq \left(\frac{4}{3h}\alpha_1 + R_1 + \frac{4}{3} R_2 \right) K_\infty^2 \sup_{1 \leq i \leq n} d_{i,n}^2$$

Finally, since $h < 1$ then,

$$||f_{x,h} - P_{|T_n} f_{x,h}||^2 \leq \left(\frac{4}{3}\alpha_1 + R_1 + \frac{4}{3} R_2 \right) K_\infty^2 \frac{1}{h} \sup_{1 \leq i \leq n} d_{i,n}^2.$$

Proposition 5 is then proved for the first case.

Second case. Consider now the case where $t_{x,1} < x-h$ and $t_{x,N_{T_n}} > x+h$. For $i = 2, \dots, N_{T_n} - 2$ set,

$$\gamma_{x,i} = \int_{t_{x,i}}^{t_{x,i+1}} \varphi_{x,h}(t)\ dt,\ \gamma_{x,1} = \int_{x-h}^{t_{x,2}} \varphi_{x,h}(t)\ dt,\ \gamma_{x,N_{T_n}-1} = \int_{t_{x,N_{T_n}-1}}^{x+h} \varphi_{x,h}(t)\ dt \text{ and } \gamma_{x,N_{T_n}} = 0. \tag{47}$$

Using this we obtain,

$$\begin{aligned} ||f_{x,h} - g_n||^2 = &\int_{x-h}^{t_{x,2}} \Big((f_{x,h}(t) - g_n(t)) - (f_{x,h}(t_{x,1}) - g_n(t_{x,1}))\Big)\varphi_{x,h}(t)\ dt \\ &+ \sum_{i=2}^{N_{T_n}} \int_{t_{x,i}}^{t_{x,i+1}} \Big((f_{x,h}(t) - g_n(t)) - (f_{x,h}(t_{x,i}) - g_n(t_{x,i}))\Big)\varphi_{x,h}(t)\ dt \\ &+ \int_{t_{x,N_{T_n}}}^{x+h} \Big((f_{x,h}(t) - g_n(t)) - (f_{x,h}(t_{x,N_{T_n}}) - g_n(t_{x,N_{T_n}}))\Big)\varphi_{x,h}(t)\ dt. \end{aligned} \tag{48}$$

We first control the first term of (48). Let,

$$A_{x,h}^{(1)} = \int_{x-h}^{t_{x,2}} \Big((f_{x,h}(t) - g_n(t)) - (f_{x,h}(t_{x,1}) - g_n(t_{x,1}))\Big)\varphi_{x,h}(t)\ dt.$$

For $t \in]x-h, t_{x,2}[$ we have,

$$\begin{aligned} f_{x,h}(t) - g_n(t) = &(f_{x,h}(t_{x,1}) - g_n(t_{x,1})) + (t - t_{x,1})(f'_{x,h}(t_{x,1}) - g'_n(t_{x,1}^+)) \\ &+ \frac{1}{2}(t - t_{x,1})^2(f''_{x,h}(\theta_{x,1}) - g''_n(\theta_{x,1}^+)), \end{aligned} \tag{49}$$

for some $\theta_{x,1} \in]x-h, t[$. Equation (16) yields,

$$\begin{aligned} f'_{x,h}(t_{x,1}) &= \int_{x-h}^{x+h} R^{(0,1)}(s, t_{x,1}^+)\varphi_{x,h}(s)\ ds = \sum_{j=1}^{N_{T_n}-1} \int_{t_{x,j}}^{t_{x,j+1}} R^{(0,1)}(s, t_{x,1}^+)\varphi_{x,h}(s)\ ds \\ &= \int_{x-h}^{t_{x,2}} R^{(0,1)}(s, t_{x,1}^-)\varphi_{x,h}(s)\ ds \ + \sum_{j=2}^{N_{T_n}-1} \int_{t_{x,j}}^{t_{x,j+1}} R^{(0,1)}(s, t_{x,1}^+)\varphi_{x,h}(s)\ ds \\ &= R^{(0,1)}(t_{x,1}, t_{x,1}^-)\gamma_{x,1} + \int_{x-h}^{t_{x,2}} (s - t_{x,1})R^{(1,1)}(\delta_{s,1}^+, t_{x,1}^-)\varphi_{x,h}(s)\ ds \\ &\quad + \sum_{j=2}^{N_{T_n}-1} R^{(0,1)}(t_{x,j}, t_{x,1})\gamma_{x,j} \ + \sum_{j=2}^{N_{T_n}-1} \int_{t_{x,j}}^{t_{x,j+1}} (s - t_{x,j})R^{(1,1)}(\delta_{s,j}, t_{x,1}^+)\varphi_{x,h}(s)\ ds. \end{aligned} \tag{50}$$

Recall that,

$$g'_n(t_{x,1}^+) = R^{(0,1)}(t_{x,1}, t_{x,1}^+)\gamma_{x,1} + \sum_{j=2}^{N_{T_n}-1} R^{(0,1)}(t_{x,j}, t_{x,1})\gamma_{x,j}. \tag{51}$$

Equations (50) and (51) give,

$$\begin{aligned} f'_{x,h}(t_{x,1}) - g'_n(t_{x,1}^+) = &\alpha(t_{x,1})\gamma_{x,1} + \sum_{j=2}^{N_{T_n}-1} \int_{t_{x,j}}^{t_{x,j+1}} (s - t_{x,j})R^{(1,1)}(\delta_{s,j}, t_{x,1}^+)\varphi_{x,h}(s)\ ds \\ &+ \int_{x-h}^{t_{x,2}} (s - t_{x,1})R^{(1,1)}(\delta_{s,1}^+, t_{x,1}^-)\varphi_{x,h}(s)\ ds. \end{aligned}$$

Note that $t_{x,2} - (x-h) \leq \sup_{1\leq i\leq n} d_{i,n}$. We obtain,

$$\begin{aligned}|f'_{x,h}(t_{x,1}) - g'_n(t_{x,1}^-)| &\leq \frac{K_\infty}{h}\alpha_1 \sup_{1\leq i\leq n} d_{i,n} + \frac{K_\infty}{2h}R_1 \sum_{j=2}^{N_{T_n}-1} d_{x,j}^2 + \frac{K_\infty}{2h}R_1 \sup_{1\leq i\leq n} d_{i,n}^2 \\ &\leq \frac{K_\infty}{h}\alpha_1 \sup_{1\leq i\leq n} d_{i,n} + K_\infty R_1 \sup_{1\leq i\leq n} d_{i,n} + \frac{K_\infty}{2h}R_1 \sup_{1\leq i\leq n} d_{i,n}^2 \\ &\leq K_\infty\Big(\frac{\alpha_1}{h} + \frac{3}{2}R_1\Big) \sup_{1\leq i\leq n} d_{i,n} \end{aligned} \tag{52}$$

By (45) we have,

$$|f''_{x,h}(\theta_{x,t}) - g''_n(\theta_{x,t}^-)| \leq \frac{K_\infty}{h}\alpha_1 + 4K_\infty R_2. \tag{53}$$

Equations (49), (52) and (53) yield,

$$\begin{aligned}|A_{x,h}^{(1)}| \leq\ & |f'_{x,h}(t_{x,1}) - g'_n(t_{x,1}^+)| \int_{x-h}^{t_{x,2}} (t - t_{x,1})|\varphi_{x,h}(t)|\, dt \\ &+ \frac{1}{2}\int_{x-h}^{t_{x,2}} (t-t_{x,1})^2 |f''_{x,h}(\theta_{x,1}) - g''_n(\theta_{x,1}^+)||\varphi_{x,h}(t)|\, dt \\ &\leq \bigg(K_\infty\Big(\frac{\alpha_1}{h} + \frac{3}{2}R_1\Big) \sup_{1\leq i\leq n} d_{i,n}\bigg)\frac{K_\infty}{2h} \sup_{1\leq i\leq n} d_{i,n}^2 + \Big(\frac{K_\infty}{h}\alpha_1 + 4K_\infty R_2\Big)\frac{K_\infty}{6h} \sup_{1\leq i\leq n} d_{i,n}^3 \\ &\leq \bigg(\frac{2}{3}\alpha_1 + \frac{3}{4}R_1 + \frac{2}{3}R_2\bigg)\frac{K_\infty^2}{h^2} \sup_{1\leq i\leq n} d_{i,n}^3. \end{aligned} \tag{54}$$

Similarly we obtain,

$$A_{x,h}^{(2)} \triangleq \int_{t_{x,N_{T_n}}}^{x+h} \bigg((f_{x,h}(t) - g_n(t)) - (f_{x,h}(t_{x,N_{T_n}}) - g_n(t_{x,N_{T_n}}))\bigg)\varphi_{x,h}(t)\, dt$$

$$|A_{x,h}^{(2)}| \leq \bigg(\frac{2}{3}\alpha_1 + \frac{3}{4}R_1 + \frac{2}{3}R_2\bigg)\frac{K_\infty^2}{h^2} \sup_{1\leq i\leq n} d_{i,n}^3. \tag{55}$$

Thus,

$$|A_{x,h}^{(1)} + A_{x,h}^{(2)}| \leq \bigg(\frac{4}{3}\alpha_1 + \frac{3}{2}R_1 + \frac{4}{3}R_2\bigg)\frac{K_\infty^2}{h^2} \sup_{1\leq i\leq n} d_{i,n}^3.$$

For $i = 2, \ldots, N_{T_n} - 2$, similar calculations as those leading to (46) give,

$$\begin{aligned}&\bigg|\int_{t_{x,i}}^{t_{x,i+1}} \big((f_{x,h}(t) - g_n(t)) - (f_{x,h}(t_{x,i}) - g_n(t_{x,i}))\big)\varphi_{x,h}(t)\, dt\bigg| \\ &\leq \frac{K_\infty^2}{4h^2}R_1 d_{x,i}^2 \sum_{j=1}^{N_{T_n}} d_{x,j}^2 + \frac{2K_\infty^2}{3h}(\frac{\alpha_1}{h} + R_2)d_{x,i}^3.\end{aligned}$$

Thus,

$$\begin{aligned}&\bigg|\sum_{i=2}^{N_{T_n}-2}\int_{t_{x,i}}^{t_{x,i+1}} \Big((f_{x,h}(t) - g_n(t)) - (f_{x,h}(t_{x,i}) - g_n(t_{x,i}))\Big)\varphi_{x,h}(t)\, dt\bigg| \\ &\leq \bigg(\frac{4}{3}\alpha_1 + R_1 + \frac{4}{3}R_2\bigg)\frac{K_\infty^2}{h} \sup_{1\leq i\leq n} d_{i,n}^2. \end{aligned} \tag{56}$$

Then, Equations (54), (55) and (56) yield,

$$\begin{aligned}||f_{x,h} - P_{T_n} f_{x,h}||^2 &\leq \left(\frac{4}{3}\alpha_1 + \frac{3}{2}R_1 + \frac{4}{3}R_2\right)\frac{K_\infty^2}{h}\sup_{1\leq i\leq n} d_{i,n}^2 + \left(\frac{4}{3}\alpha_1 + R_1 + \frac{4}{3}R_2\right)\frac{K_\infty^2}{h^2}\sup_{1\leq i\leq n} d_{i,n}^3 \\ &= \left(\frac{8}{3}\alpha_1 + \frac{5}{2}R_1 + \frac{8}{3}R_2\right)\frac{K_\infty^2}{h}\sup_{1\leq i\leq n} d_{i,n}^2\end{aligned}$$

Third case. Suppose now that $t_{x,1} = x - h$ and $t_{x,N_{T_n}} > x + h$ (respectively $t_{x,1} < x - h$ and $t_{x,N_{T_n}} = x + h$). Let $T_{n-1} = T_n - \{x - h\}$ (respectively $T_{n-1} = T_n - \{x + h\}$). Since $P_{T_{n-1}} f_{x,h} \in V_{T_n}$ we obtain,

$$||f_{x,h} - P_{T_n} f_{x,h}||^2 \leq ||f_{x,h} - P_{T_{n-1}} f_{x,h}||^2,$$

we can then apply the result of the second case to the right side of the previous inequality. The proof of Proposition 5 is complete. □

Proof of Proposition 6.

The great lines of this proof are based on the work of Sacks and Ylvisaker (1966). Keeping Equation (34) in mind we deduce that Equation (11) (in the paper) is equivalent to,

$$\underline{\lim}_{n\to\infty} \frac{N_{T_n}^2}{h} ||f_{x,h} - P_{|T_n} f_{x,h}||^2 \geq \frac{1}{12}\alpha(x)\left\{\int_{-1}^{1} K^{2/3}(t)dt\right\}^3. \tag{57}$$

We shall take the same notation as in the previous proof. Let $g_n = P_{|T_n} f_{x,h}$, it is shown by Equation (90) in the Appendix that:

$$g_n(t_i) = f_{x,h}(t_i) = \sum_{j=1}^{n} R(t_j, t_i) m_{x,h}(t_j), \quad \text{for } i = 1, \cdots, n.$$

We have from (F1) in the Appendix that,

$$\begin{aligned}||f_{x,h} - g_n||^2 &= \int_0^1 (f_{x,h}(t) - g_n(t))\varphi_{x,h}(t)\, dt - \sum_{i=1}^{n} m_{x,h}(t_i)(f_{x,h}(t_i) - g_n(t_i)) \\ &= \int_{x-h}^{x+h} (f_{x,h}(t) - g_n(t))\varphi_{x,h}(t) dt.\end{aligned}$$

Suppose first that $t_{x,1} = x - h$ and $t_{x,N_{T_n}} = x + h$, then the last equalities give,

$$||f_{x,h} - g_n||^2 = \sum_{i=1}^{N_{T_n}-1} \int_{t_{x,i}}^{t_{x,i+1}} (f_{x,h}(t) - g_n(t))\varphi_{x,h}(t) dt. \tag{58}$$

Under Assumptions (A) and (B), the function $f_{x,h}$ is twice differentiable at every $t \in [0, 1]$ and g_n is twice differentiable at every $t \in [0, 1]$ except on T_n, however, it has left and right derivatives. We expand $(f_{x,h} - g_n)$ in a Taylor series around $t_{x,i}$ for $t \in]t_{x,i}, t_{x,i+1}[$ up to order 2 we obtain,

$$\begin{aligned}f_{x,h}(t) - g_n(t) &= (f_{x,h}(t_{x,i}) - g_n(t_{x,i})) + (t - t_{x,i})(f'_{x,h}(t_{x,i}) - g'_n(t^+_{x,i})) \\ &\quad + \frac{1}{2}(t - t_{x,i})^2 (f''_{x,h}(\sigma_{x,t}) - g''_n(\sigma^+_{x,t})),\end{aligned}$$

for some $\sigma_{x,t} \in]t_{x,i}, t[$. Since $g_n(t_{x,i}) = f_{x,h}(t_{x,i})$ then,

$$f_{x,h}(t) - g_n(t) = (t - t_{x,i})(f'_{x,h}(t_{x,i}) - g'_n(t^+_{x,i})) + \frac{1}{2}(t - t_{x,i})^2(f''_{x,h}(\sigma_{x,t}) - g''_n(\sigma^+_{x,t})), \quad (59)$$

On the one hand, we have for $i \in 1, \ldots, N_{T_n} - 1$,

$$f_{x,h}(t_{x,i+1}) - g_n(t_{x,i+1}) = d_{x,i}(f'_{x,h}(t_{x,i}) - g'_n(t^+_{x,i})) + \frac{1}{2}d^2_{x,i}(f''_{x,h}(\sigma_{x,i}) - g''_n(\sigma^+_{x,i})).$$

for some $\sigma_{x,i} \in]t_{x,i}, t_{x,i+1}[$. Thus,

$$f'_{x,h}(t_{x,i}) - g'_n(t^+_{x,i}) = -\frac{1}{2}d_{x,i}(f''_{x,h}(\sigma_{x,i}) - g''_n(\sigma^+_{x,i})). \quad (60)$$

On the other hand, it is shown by (F4) in the Appendix that,

$$f''_{x,h}(t) - g''_n(t^+) = -\alpha(t)\varphi_{x,h}(t) + \langle R^{(0,2)}(\cdot, t^+), f_{x,h} - g_n\rangle. \quad (61)$$

Injecting (60) and (61) in (59) gives,

$$\begin{aligned}
f_{x,h}(t) - g_n(t) &= -\frac{1}{2}(t - t_{x,i})d_{x,i}(f''_{x,h}(\sigma_{x,i}) - g''_n(\sigma^+_{x,i})) + \frac{1}{2}(t - t_{x_i})^2(f''_{x,h}(\sigma_{x,i}) - g''_n(\sigma^+_{x,i})) \\
&= \frac{1}{2}d_{x,i}(t - t_{x,i})\alpha(\sigma_{x,i})\varphi_{x,h}(\sigma_{x,i}) - \frac{1}{2}(t - t_{x,i})^2\alpha(\sigma_{x,t})\varphi_{x,h}(\sigma_{x,t}) \\
&\quad - \frac{1}{2}d_{x,i}(t - t_{x,i})\langle R^{(0,2)}(\cdot, \sigma^+_{x,i}), f_{x,h} - g_n\rangle + \frac{1}{2}(t - t_{x,i})^2\langle R^{(0,2)}(\cdot, \sigma^+_{x,t}), f_{x,h} - g_n\rangle.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_{t_{x,i}}^{t_{x,i+1}} (f_{x,h}(t) - g_n(t))\varphi_{x,h}(t)\, dt = \\
&\frac{1}{2}d_{x,i}\alpha(\sigma_{x,i})\varphi_{x,h}(\sigma_{x,i}) \int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})\varphi_{x,h}(t)\, dt - \frac{1}{2}\int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})^2\alpha(\sigma_{x,t})\varphi_{x,h}(\sigma_{x,t})\varphi_{x,h}(t)\, dt \\
&\quad - \frac{1}{2}d_{x,i}\langle R^{(0,2)}(\cdot, \sigma^+_{x,i}), f_{x,h} - g_n\rangle \int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})\varphi_{x,h}(t)\, dt \\
&\quad + \frac{1}{2}\int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})^2\langle R^{(0,2)}(\cdot, \sigma^+_{x,t}), f_{x,h} - g_n\rangle\varphi_{x,h}(t)\, dt \\
&= \frac{1}{4}d^3_{x,i}\alpha(\sigma_{x,i})\varphi^2_{x,h}(\sigma_{x,i}) - \frac{1}{6}d^3_{x,i}\alpha(\sigma_{x,i})\varphi^2_{x,h}(\sigma_{x,i}) \\
&\quad + \frac{1}{2}d_{x,i}\alpha(\sigma_{x_i})\varphi_{x,h}(\sigma_{x,i}) \int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})[\varphi_{x,h}(t) - \varphi_{x,h}(\sigma_{x,i})]\, dt \\
&\quad - \frac{1}{2}\alpha(\sigma_{x,i})\varphi_{x,h}(\sigma_{x,i}) \int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})^2[\varphi_{x,h}(t) - \varphi_{x,h}(\sigma_{x,i})]\, dt \\
&\quad - \frac{1}{2}\int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})^2[\alpha(\sigma_{x,t})\varphi_{x,h}(\sigma_{x,t}) - \alpha(\sigma_{x,i})\varphi_{x,h}(\sigma_{x,i})]\varphi_{x,h}(t)\, dt \\
&\quad - \frac{1}{2}d_{x,i}\langle R^{(0,2)}(\cdot, \sigma^+_{x,i}), f_{x,h} - g_n\rangle \int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})\varphi_{x,h}(t)\, dt \\
&\quad + \frac{1}{2}\int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})^2\langle R^{(0,2)}(\cdot, \sigma^+_{x,t}), f_{x,h} - g_n\rangle\varphi_{x,h}(t)\, dt \\
&= \frac{1}{12}d^3_{x,i}\alpha(\sigma_{x,i})\varphi^2_{x,h}(\sigma_{x,i}) + A^{(1)}_{x,i} - A^{(2)}_{x,i} - A^{(3)}_{x,i} - A^{(4)}_{x,i} + A^{(5)}_{x,i}, \quad (62)
\end{aligned}$$

where, $A_{x,i}^{(1)} = \frac{1}{2}d_{x,i}\alpha(\sigma_{x,i})\varphi_{x,h}(\sigma_{x,i})\int_{t_{x,i}}^{t_{x,i+1}}(t-t_{x,i})[\varphi_{x,h}(t)-\varphi_{x,h}(\sigma_{x,i})]\,dt.$

$$A_{x,i}^{(2)} = \frac{1}{2}\alpha(\sigma_{x,i})\varphi_{x,h}(\sigma_{x,i})\int_{t_{x,i}}^{t_{x,i+1}}(t-t_{x,i})^2[\varphi_{x,h}(t)-\varphi_{x,h}(\sigma_{x,i})]\,dt.$$

$$A_{x,i}^{(3)} = \frac{1}{2}\int_{t_{x,i}}^{t_{x,i+1}}(t-t_{x,i})^2[\alpha(\sigma_{x,t})\varphi_{x,h}(\sigma_{x,t})-\alpha(\sigma_{x,i})\varphi_{x,h}(\sigma_{x,i})]\varphi_{x,h}(t)\,dt.$$

$$A_{x,i}^{(4)} = \frac{1}{2}d_{x,i}\langle R^{(0,2)}(\cdot,\sigma_{x,i}^{+}), f_{x,h}-g_n\rangle\int_{t_{x,i}}^{t_{x,i+1}}(t-t_{x,i})\varphi_{x,h}(t)\,dt.$$

$$A_{x,i}^{(5)} = \frac{1}{2}\int_{t_{x,i}}^{t_{x,i+1}}(t-t_{x,i})^2\langle R^{(0,2)}(\cdot,\sigma_{x,t}^{+}), f_{x,h}-g_n\rangle\varphi_{x,h}(t)\,dt.$$

We shall now control these quantities. Let,

$$B_{x,i}^{(1)} = \sup_{t_{x,i}<s,t<t_{x,i+1}}|\varphi_{x,h}(t)-\varphi_{x,h}(s)| \text{ and } B_{x,i}^{(2)} = \sup_{t_{x,i}<s,t<t_{x,i+1}}|\alpha(t)\varphi_{x,h}(t)-\alpha(s)\varphi_{x,h}(s)|.$$

Since α and $\varphi_{x,h}$ are Lipschitz then,

$$\sup_{0\le i\le n} B_{x,i}^{(1)} = O\left(\frac{1}{h^2}\sup_{0\le j\le n} d_{j,n}\right) \quad \text{and} \quad \sup_{0\le i\le n} B_{x,i}^{(2)} = O\left(\frac{1}{h^2}\sup_{0\le j\le n} d_{j,n}\right). \tag{63}$$

Elementary calculations show that,

$$|A_{x,i}^{(1)}| \le \frac{a_1}{h}B_{x,i}^{(1)}d_{x,i}^3, \qquad |A_{x,i}^{(2)}| \le \frac{a_2}{h}B_{x,i}^{(1)}d_{x,i}^3 \quad \text{and} \quad |A_{x,i}^{(3)}| \le \frac{a_3}{h}B_{x,i}^{(2)}d_{x,i}^3, \tag{64}$$

for appropriate constants a_1, a_2 and a_3. We obtain from the Cauchy-Schwartz inequality, Assumption (C) and Proposition 5 that,

$$|A_{x,i}^{(4)}| + |A_{x,i}^{(5)}| \le \frac{a_4}{h}d_{x,i}^3||f_{x,h}-g_n|| \le \frac{1}{h}d_{x,i}^3\underbrace{a_4\sqrt{\frac{C}{h}}}_{a_h}\sup_{0\le j\le n} d_{j,n}, \tag{65}$$

for an appropriate constant a_4 (C is defined in Proposition 5). Thus,

$$\begin{aligned}
&\int_{t_{x,i}}^{t_{x,i+1}}(f_{x,h}(t)-g_n(t))\varphi_{x,h}(t)\,dt \\
&= \frac{1}{12}d_{x,i}^3\alpha(\sigma_{x,i})\varphi_{x,h}^2(\sigma_{x,i}) + A_{x,i}^{(1)} - A_{x,i}^{(2)} - A_{x,i}^{(3)} - A_{x,i}^{(4)} + A_{x,i}^{(5)} \\
&\ge \frac{1}{12}d_{x,i}^3\alpha(\sigma_{x,i})\varphi_{x,h}^2(\sigma_{x,i}) - d_{x,i}^3\left(\frac{a_1}{h}B_{x,i}^{(1)} + \frac{a_2}{h}B_{x,i}^{(2)} + \frac{a_h}{h}\sup_{0\le j\le n} d_{j,n}\right).
\end{aligned} \tag{66}$$

Let,

$$\rho_{h,N_{T_n}} = \sup_{0\le i\le N_{T_n}}\left(\frac{a_1}{h}B_{x,i}^{(1)} + \frac{a_2}{h}B_{x,i}^{(2)} + \frac{a_h}{h}\sup_{0\le j\le n} d_{j,n}\right).$$

Equation (63) implies that for an appropriate constant c and c' we have,

$$|\rho_{h,N_{T_n}}| \le \left(\frac{c}{h^3}\sup_{0\le j\le n} d_{j,n} + \frac{c'}{h^{3/2}}\sup_{0\le j\le n} d_{j,n}\right).$$

Using (66) and (58) together with Equation (66) in (58) we obtain,

$$
\begin{aligned}
||f_{x,h}-g_n||^2 &\geq \sum_{i=1}^{N_{T_n}-1}\Big(\frac{1}{12}\alpha(\sigma_{x,i})\varphi_{x,h}^2(\sigma_{x,i})-\rho_{h,N_{T_n}}\Big)d_{x,i}^3\\
&\geq \frac{1}{12}\sum_{i=1}^{N_{T_n}-1}\alpha(\sigma_{x,i})\varphi_{x,h}^2(\sigma_{x,i})d_{x,i}^3-\frac{cN_{T_n}}{h^3}\sup_{0\leq j\leq n}d_{j,n}^4-\frac{c'N_{T_n}}{h^{3/2}}\sup_{0\leq j\leq n}d_{j,n}^4. \qquad (67)
\end{aligned}
$$

Then the Hölder's inequality gives,

$$
||f_{x,h}-g_n||^2 \geq \frac{1}{12(N_{T_n}-1)^2}\Big\{\sum_{j=1}^{N_{T_n}-1}[\alpha(\sigma_{x,i})\varphi_{x,h}^2(\sigma_{x,i})]^{\frac{1}{3}}d_{x,i}\Big\}^3-\frac{cN_{T_n}}{h^3}\sup_{0\leq j\leq n}d_{j,n}^4-\frac{c'N_{T_n}}{h^{3/2}}\sup_{0\leq j\leq n}d_{j,n}^4.
$$

We shall now control the first term of the right side of this inequality. We have,

$$
\Big\{\sum_{j=1}^{N_{T_n}-1}\Big(\alpha(\sigma_{x,i})\varphi_{x,h}^2(\sigma_{x,i})\Big)^{\frac{1}{3}}d_{x,i}\Big\}^3
$$

$$
\begin{aligned}
&=\Big\{\int_{x-h}^{x+h}\Big(\alpha(t)\varphi_{x,h}^2(t)\Big)^{\frac{1}{3}}\,dt-\sum_{j=1}^{N_{T_n}-1}\int_{t_{x,i}}^{t_{x,i+1}}\Big((\alpha(t)\varphi_{x,h}^2(t))^{\frac{1}{3}}-(\alpha(\sigma_{x,i})\varphi_{x,h}^2(\sigma_{x,i}))^{\frac{1}{3}}\Big)\Big\}^3\\
&=\Big\{\int_{x-h}^{x+h}\Big(\alpha(t)\varphi_{x,h}^2(t)\Big)^{\frac{1}{3}}\,dt\Big\}^3-\Big\{\sum_{j=1}^{N_{T_n}-1}\int_{t_{x,i}}^{t_{x,i+1}}\Big((\alpha(t)\varphi_{x,h}^2(t))^{\frac{1}{3}}-(\alpha(\sigma_{x,i})\varphi_{x,h}^2(\sigma_{x,i}))^{\frac{1}{3}}\Big)\Big\}^3\\
&\quad-3\Big\{\int_{x-h}^{x+h}\Big(\alpha(t)\varphi_{x,h}^2(t)\Big)^{\frac{1}{3}}\,dt\Big\}^2\Big\{\sum_{j=1}^{N_{T_n}-1}\int_{t_{x,i}}^{t_{x,i+1}}\Big((\alpha(t)\varphi_{x,h}^2(t))^{\frac{1}{3}}-(\alpha(\sigma_{x,i})\varphi_{x,h}^2(\sigma_{x,i}))^{\frac{1}{3}}\Big)\Big\}\\
&\quad+3\Big\{\int_{x-h}^{x+h}\Big(\alpha(t)\varphi_{x,h}^2(t)\Big)^{\frac{1}{3}}\,dt\Big\}\Big\{\sum_{j=1}^{N_{T_n}-1}\int_{t_{x,i}}^{t_{x,i+1}}\Big((\alpha(t)\varphi_{x,h}^2(t))^{\frac{1}{3}}-(\alpha(\sigma_{x,i})\varphi_{x,h}^2(\sigma_{x,i}))^{\frac{1}{3}}\Big)\Big\}^2\\
&\triangleq\Big\{\int_{x-h}^{x+h}\Big(\alpha(t)\varphi_{x,h}^2(t)\Big)^{\frac{1}{3}}\,dt\Big\}^3+B,
\end{aligned}
$$

We obtain using (31) and the fact that α is Lipschitz,

$$
B=O\Big(\Big(\frac{N_{T_n}}{h^{5/3}}\sup_{0\leq j\leq n}d_{j,n}^2\Big)^3\Big)+O\Big(\Big(\frac{N_{T_n}}{h^{5/3}}\sup_{0\leq j\leq n}d_{j,n}^2\Big)\,h^{2/3}\Big)+O\Big(\Big(\frac{N_{T_n}}{h^{5/3}}\sup_{0\leq j\leq n}d_{j,n}^2\Big)^2\,h^{1/3}\Big).
$$

Assumption (E) implies that for an appropriate constant c'' we have,

$$
|B|\leq\frac{c''N_{T_n}}{h}\sup_{0\leq j\leq n}d_{j,n}^2.
$$

Using the Riemann integrability of α and $\varphi_{x,h}$ we get,

$$||f_{x,h} - g_n||^2 \geq \frac{1}{12(N_{T_n}-1)^2}\Big\{ \int_{x-h}^{x+h} \Big(\alpha(t)\varphi_{x,h}^2(t)\Big)^{\frac{1}{3}} dt \Big\}^3 - \frac{c''}{N_{T_n} h} \sup_{0\leq j\leq n} d_{j,n}^2$$
$$- \frac{cN_{T_n}}{h^3} \sup_{0\leq j\leq n} d_{j,n}^4 - \frac{c' N_{T_n}}{h^{3/2}} \sup_{0\leq j\leq n} d_{j,n}^4$$
$$\geq \frac{1}{12N_{T_n}^2}\Big\{ \int_{x-h}^{x+h} \Big(\alpha(t)\varphi_{x,h}^2(t)\Big)^{\frac{1}{3}} dt \Big\}^3 - \frac{c''}{N_{T_n} h} \sup_{0\leq j\leq n} d_{j,n}^2 - \frac{cN_{T_n}}{h^3} \sup_{0\leq j\leq n} d_{j,n}^4 - \frac{c' N_{T_n}}{h^{3/2}} \sup_{0\leq j\leq n} d_{j,n}^4$$
$$= \frac{1}{12h^2N_{T_n}^2}\Big\{ \int_{x-h}^{x+h} \Big(\alpha(t)K^2(\frac{x-t}{h})\Big)^{\frac{1}{3}} dt \Big\}^3 - \frac{c''}{N_{T_n} h} \sup_{0\leq j\leq n} d_{j,n}^2$$
$$- \frac{cN_{T_n}}{h^3} \sup_{0\leq j\leq n} d_{j,n}^4 - \frac{c' N_{T_n}}{h^{3/2}} \sup_{0\leq j\leq n} d_{j,n}^4$$
$$= \frac{h}{12N_{T_n}^2}\Big\{ \int_{-1}^{1} \Big(\alpha(x-th)K^2(t)\Big)^{\frac{1}{3}} dt \Big\}^3 - \frac{c''}{N_{T_n} h} \sup_{0\leq j\leq n} d_{j,n}^2$$
$$- \frac{cN_{T_n}}{h^3} \sup_{0\leq j\leq n} d_{j,n}^4 - \frac{c' N_{T_n}}{h^{3/2}} \sup_{0\leq j\leq n} d_{j,n}^4.$$

Assumption (E) implies that,

$$\lim_{n\to\infty} \frac{1}{h^2} N_{T_n} \sup_{0\leq j\leq n} d_{j,n}^2 = 0, \quad \lim_{n\to\infty} \frac{1}{h^4} \sup_{0\leq j\leq n} d_{j,n}^4 N_{T_n}^3 = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{c'}{h^{3/2}} \sup_{0\leq j\leq n} d_{j,n}^4 N_{T_n} = 0.$$

Finally the continuity of α yields,

$$\underline{\lim}_{n\to\infty} \frac{N_{T_n}^2}{h} ||f_{x,h} - g_n||^2 \geq \frac{1}{12}\alpha(x)\Big\{ \int_{-1}^{1} K(t)^{\frac{2}{3}} dt \Big\}^3.$$

Inequality (57) is then proved for a sequence of designs containing $x-h$ and $x+h$. Consider now any sequence of designs $\{T_n, n \geq 1\}$ satisfying Assumption (E) we can adjoin the points $\{x-h, x+h\}$ to T_n (if they aren't present). Hence we form a sequence $\{S_n, n \geq 1\}$ with $S_n \in D_{n+2}$ and satisfying (57). We have,

$$||f_{x,h} - P_{|S_n} f_{x,h}||^2 \leq ||f_{x,h} - P_{|T_n} f_{x,h}||^2.$$

Then,

$$N_{S_n}^2 ||f_{x,h} - P_{|S_n} f_{x,h}||^2 \leq N_{S_n}^2 ||f_{x,h} - P_{|T_n} f_{x,h}||^2. \tag{68}$$

We know that $N_{S_n} \in \{N_{T_n}+1, N_{T_n}+2\}$, replacing N_{S_n} in the right term of (68) by $(N_{T_n}+2)$ (or $(N_{T_n}+1)$) gives,

$$\frac{N_{S_n}^2}{h} ||f_{x,h} - P_{|S_n} f_{x,h}||^2 - \frac{(4+2N_{T_n})}{h} ||f_{x,h} - P_{|T_n} f_{x,h}||^2 \leq \frac{N_{T_n}^2}{h} ||f_{x,h} - P_{|T_n} f_{x,h}||^2.$$

Assumption (E) and Equation (35) yield,

$$\underline{\lim}_{n\to\infty} \frac{(4+2N_{T_n})}{h} ||f_{x,h} - P_{|T_n} f_{x,h}||^2 = 0.$$

Hence, for any sequence $\{T_n, n \geq 1\}$ we have,

$$\underline{\lim}_{n\to\infty} \frac{N_{T_n}^2}{h} ||f_{x,h} - P_{|T_n} f_{x,h}||^2 \geq \frac{1}{12}\alpha(x)\Big\{ \int_{-1}^{1} K^{2/3}(t) dt \Big\}^3.$$

This completes the proof of Proposition 6. □

Proof of Proposition 7.

On the one hand, Proposition 5 yields that there exists a constant $c > 0$ such that,

$$0 \leq \frac{\sigma^2_{x,h}}{m} - \text{Var } \hat{g}^{pro}_n(x) \leq \frac{c}{mh} \sup_{0\leq j\leq n} d^2_{j,n}.$$

Lemma 2 implies that there exists a constant $c' > 0$ such that,

$$\sup_{0\leq j\leq n} d^2_{j,n} \leq \frac{c'}{n^2}.$$

Thus, for $n \geq 1$ we have,

$$0 \leq \frac{\sigma^2_{x,h}}{m} - \text{Var } \hat{g}^{pro}_n(x) \leq \frac{c'c}{mn^2h}.$$

Finally, taking $C = cc'$ we obtain,

$$\overline{\lim_{n\to\infty}}\ mn^2h\Big(\frac{\sigma^2_{x,h}}{m} - \text{Var } \hat{g}^{pro}_n(x)\Big) \leq C.$$

Inequality (12) (in the paper) is then proved. On the other hand, Proposition 6 yields,

$$\frac{mN^2_{T_n}}{h}\Big(\frac{\sigma^2_{x,h}}{m} - \text{Var } \hat{g}^{pro}_n(x)\Big) \geq \frac{1}{12}\alpha(x)\bigg\{\int_{-1}^{1} K^{2/3}(t)dt\bigg\}^3.$$

Lemma 2 implies that there exists a constant $c'' > 0$ such that,

$$N_{T_n} < c''nh,$$

which implies that,

$$c''mn^2h\ \Big(\frac{\sigma^2_{x,h}}{m} - \text{Var } \hat{g}^{pro}_n(x)\Big) \geq \frac{1}{12}\alpha(x)\bigg\{\int_{-1}^{1} K^{2/3}(t)dt\bigg\}^3.$$

Finally, taking $C' = \frac{1}{12c''}\alpha(x)\bigg\{\int_{-1}^{1} K^{2/3}(t)dt\bigg\}^3$ we obtain,

$$\underline{\lim_{n\to\infty}}\ mn^2h\Big(\frac{\sigma^2_{x,h}}{m} - \text{Var } \hat{g}^{pro}_n(x)\Big) \geq C'.$$

This concludes the proof of Proposition 7. □

Proof of Proposition 8.

The first part of this proof is the same as that of Proposition 6. Recall that,

$$m\Big(\frac{\sigma^2_{x,h}}{m} - \text{Var } \hat{g}^{pro}_n(x)\Big) = ||f_{x,h}||^2 - ||P_{|T_n}f_{x,h}||^2 = ||f_{x,h} - P_{|T_n}f_{x,h}||^2.$$

Using (58) and (62) we obtain,

$$\begin{aligned}
\text{Var } \hat{g}_n^{pro}(x) - \frac{\sigma_{x,h}^2}{m} &= -\frac{1}{m}||f_{x,h} - P_{|T_n} f_{x,h}||^2 \\
&= -\frac{1}{m}\sum_{i=1}^{N_{T_n}} \left(\frac{1}{12} d_{x,i}^3 \alpha(\sigma_{x,i}) \varphi_{x,h}^2(\sigma_{x,i}) + A_{x,i}^{(1)} - A_{x,i}^{(2)} - A_{x,i}^{(3)} - A_{x,i}^{(4)} + A_{x,i}^{(5)} \right), \qquad (69)
\end{aligned}$$

for some $\sigma_{x,i} \in]t_{x,i}, t_{x,i+1}[$ and some $\sigma_{x,t} \in]t_{x,i}, t[$, where,

$$\begin{aligned}
A_{x,i}^{(1)} &= \frac{1}{2} d_{x,i} \alpha(\sigma_{x,i}) \varphi_{x,h}(\sigma_{x,i}) \int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})[\varphi_{x,h}(t) - \varphi_{x,h}(\sigma_{x,i})]\, dt. \\
A_{x,i}^{(2)} &= \frac{1}{2} \alpha(\sigma_{x,i}) \varphi_{x,h}(\sigma_{x,i}) \int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})^2 [\varphi_{x,h}(t) - \varphi_{x,h}(\sigma_{x,i})]\, dt. \\
A_{x,i}^{(3)} &= \frac{1}{2} \int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})^2 [\alpha(\sigma_{x,t}) \varphi_{x,h}(\sigma_{x,t}) - \alpha(\sigma_{x,i}) \varphi_{x,h}(\sigma_{x,i})] \varphi_{x,h}(t)\, dt. \\
A_{x,i}^{(4)} &= \frac{1}{2} d_{x,i} \langle R^{(0,2)}(\cdot, \sigma_{x,i}^+), f_{x,h} - g_n \rangle \int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i}) \varphi_{x,h}(t)\, dt. \\
A_{x,i}^{(5)} &= \frac{1}{2} \int_{t_{x,i}}^{t_{x,i+1}} (t - t_{x,i})^2 \langle R^{(0,2)}(\cdot, \sigma_{x,t}^+), f_{x,h} - g_n \rangle \varphi_{x,h}(t)\, dt.
\end{aligned}$$

From the definition of the regular sequence of designs (see Definition 2) and the mean value theorem we have for $i = 1, \cdots, N_{T_n}$,

$$d_{x,i} = t_{x,i+1} - t_{x,i} = F^{-1}\left(\frac{i+1}{n}\right) - F^{-1}\left(\frac{i}{n}\right) = \frac{1}{n f(t_{x,i}^*)},$$

where $t_{x,i}^* \in]t_{x,i}, t_{x,i+1}[$. Using this together with (69) we obtain,

$$\begin{aligned}
\text{Var } \hat{g}_n^{pro}(x) - \frac{\sigma_{x,h}^2}{m} = &-\frac{1}{12mn^2} \sum_{i=1}^{N_{T_n}} d_{x,i} \frac{1}{f^2(t_{x,i}^*)} \alpha(\sigma_{x,i}) \varphi_{x,h}^2(\sigma_{x,i}) \\
&- \frac{1}{m} \sum_{i=1}^{N_{T_n}} \left(A_{x,i}^{(1)} - A_{x,i}^{(2)} - A_{x,i}^{(3)} - A_{x,i}^{(4)} + A_{x,i}^{(5)} \right).
\end{aligned}$$

Lemma 2 yields that $N_{T_n} = O(nh)$. Using (64), (65) and (63) we obtain,

$$A_{x,i}^{(1)} = O\left(\frac{1}{n^4 h^3}\right), \quad A_{x,i}^{(2)} = O\left(\frac{1}{n^4 h^3}\right), \quad A_{x,i}^{(3)} = O\left(\frac{1}{n^4 h^3}\right) \text{ and } A_{x,i}^{(4)} + A_{x,i}^{(5)} = O\left(\frac{1}{n^4 h^{3/2}}\right).$$

Finally,

$$\text{Var } \hat{g}_n^{pro}(x) - \frac{\sigma_{x,h}^2}{m} = -\frac{1}{12mn^2} \sum_{i=1}^{N_{T_n}} d_{x,i} \frac{1}{f^2(t_{x,i}^*)} \alpha(\sigma_{x,i}) \varphi_{x,h}^2(\sigma_{x,i}) + O\left(\frac{1}{mn^3 h^2} + \frac{1}{mn^3 \sqrt{h}}\right).$$

Using a classical approximation of a sum by an integral (see for instance, Lemma 2 in Benelmadani et al. (2019b) and the fact that $0 < h < 1$ we obtain,

$$\text{Var } \hat{g}_n^{pro}(x) - \frac{\sigma_{x,h}^2}{m} = -\frac{1}{12mn^2} \int_{x-h}^{x+h} \frac{\alpha(t)}{f^2(t)} \varphi_{x,h}^2(t)\, dt + O\left(\frac{1}{mn^3 h^2}\right).$$

This concludes the proof of Proposition 8. □

Proof of Theorem 2.

First, note that since α and f are Lipschitz functions then the asymptotic expression of the integral in (14) (in the paper) is:

$$\begin{aligned}\frac{1}{mn^2}\int_{x-h}^{x+h}\frac{\alpha(t)}{f^2(t)}\varphi_{x,h}^2(t)dt &= \frac{1}{mn^2h}\int_{-1}^{1}\frac{\alpha(x-th)}{f^2(x-th)}K^2(t)\,dt\\ &= \frac{1}{mn^2h}\Big(\frac{\alpha(x)}{f^2(x)}\int_{-1}^{1}K^2(t)\,dt+\int_{-1}^{1}\Big(\frac{\alpha(x-th)}{f^2(x-th)}-\frac{\alpha(x)}{f^2(x)}\Big)K^2(t)\,dt\Big)\\ &= \frac{1}{mn^2h}\frac{\alpha(x)}{f^2(x)}\int_{-1}^{1}K^2(t)\,dt+O\Big(\frac{1}{mn^2}\Big).\end{aligned}$$

This last equality together with Proposition 8 and Proposition 4 concludes the proof of Theorem 2. □

Proof of Corollary 1.

Let $I_1=\int_0^1 R(x,x)w(x)\,dx$ and put,

$$\Psi(h,m)=-\frac{C_Kh}{2m}\int_0^1\alpha(x)w(x)\,dx+\frac{1}{4}h^4B^2\int_0^1[g''(x)]^2w(x)\,dx.$$

We have from Theorem 1,

$$\text{IMSE}(h)=\frac{I}{m}+\Psi(h,m)+o\Big(h^4+\frac{h}{m}\Big)+O\Big(\frac{1}{mn^2h}+\frac{h}{n}+\frac{1}{n^2h^2}\Big),$$

Let h^* be as defined by (16) (in the paper). It is clear that $h^*=\underset{0<h<1}{\text{argmin}}\ \Psi(h,m)$ so that $\Psi(h,m)\geq\Psi(h^*,m)$ for every $0<h<1$. Let $h_{n,m}$ be as defined in Corollary 1. We have,

$$\frac{\text{IMSE}(h^*)}{\text{IMSE}(h_{n,m})}=\frac{\frac{I_1}{m}+\Psi(h^*,m)+o\Big(h^{*4}+\frac{h^*}{m}\Big)+O\Big(\frac{1}{mn^2h^*}+\frac{h^*}{n}+\frac{1}{n^2h^{*2}}\Big)}{\frac{I_1}{m}+\Psi(h_{n,m},m)+o\Big(h_{n,m}^4+\frac{h_{n,m}}{m}\Big)+O\Big(\frac{1}{mn^2h_{n,m}}+\frac{h_{n,m}}{n}+\frac{1}{n^2h_{n,m}^2}\Big)}$$

$$\leq\frac{I_1+m\Psi(h_{n,m},m)+o\Big(mh^{*4}+h^*\Big)+O\Big(\frac{1}{n^2h^*}+\frac{mh^*}{n}+\frac{m}{n^2h^2}\Big)}{I_1+m\Psi(h_{n,m},m)+o\Big(mh_{n,m}^4+h_{n,m}\Big)+O\Big(\frac{1}{n^2h_{n,m}}+\frac{mh_{n,m}}{n}+\frac{m}{n^2h_{n,m}^2}\Big)}.$$

We have, using the definition of h^*, $mh_{n,m}^3=O(1)$, $\lim_{n,m\to\infty}h_{n,m}=0$ and using the assumption $\frac{m}{n}=O(1)$ as $n,m\to\infty$ we know that $m\Psi(h_{n,m},m)=O(h_{n,m})$. Thus,

$$\overline{\lim_{n,m\to\infty}}\frac{\text{IMSE}(h^*)}{\text{IMSE}(h_{n,m})}\leq 1.$$

This concludes the proof of Corollary 1. □

Proof of Theorem 3.

Let $x \in]0,1[$ be fixed. We have the following decomposition,

$$\sqrt{m}\Big(\hat{g}_{n,m}^{pro}(x) - g(x)\Big) = \sqrt{m}\Big(\hat{g}_{n,m}^{pro}(x) - \mathbb{E}\big(\hat{g}_{n,m}^{pro}(x)\big)\Big) + \sqrt{m}\ \Big(\mathbb{E}(\hat{g}_{n,m}^{pro}(x)) - g(x)\Big). \tag{70}$$

Since $\lim\limits_{n,m\to\infty} \sqrt{m}h = 0$, $\frac{n}{m} = O(1)$ as $n,m \to \infty$ and $\lim\limits_{n,m\to\infty} nh^2 = \infty$ then Remark 6 implies that,

$$\lim_{n,m\to\infty} \sqrt{m}\ \Big(\mathbb{E}(\hat{g}_{n,m}^{pro}(x)) - g(x)\Big) = 0. \tag{71}$$

Consider now the first term of the right side of (70). Since $\overline{Y}(t_{x,i}) - \mathbb{E}(\overline{Y}(t_{x,i})) = \overline{\varepsilon}(t_{x,i})$, we have, as done by Fraiman and Pérez Iribarren (1991),

$$\sqrt{m}\Big(\hat{g}_{n,m}^{pro}(x) - \mathbb{E}\big(\hat{g}_{n,m}^{pro}(x)\big)\Big) = \frac{1}{\sqrt{m}}\sum_{j=1}^{m}\sum_{i=1}^{n} m_{x,h}(t_i)\varepsilon_j(t_i)$$

$$= \frac{1}{\sqrt{m}}\sum_{j=1}^{m}\sum_{i=1}^{n} m_{x,h}(t_i)\big(\varepsilon_j(t_i) - \varepsilon_j(x)\big) + \Big(\sum_{i=1}^{n} m_{x,h}(t_i)\Big)\Big(\frac{1}{\sqrt{m}}\sum_{j=1}^{m}\varepsilon_j(x)\Big). \tag{72}$$

We start by controlling the second term of this last equation. Using Lemma 3 together with Lemma 2 we obtain,

$$m_{x,h}(t_{i,n}) = \begin{cases} \frac{1}{2}\varphi_{x,h}(t_{i,n})(t_{i+1,n} - t_{i-1,n}) + O\Big(\frac{1}{n^2h^2} + \frac{1}{n^2\sqrt{h}}\Big) & if\ \ i \notin \{1,n\} \text{ and} \\ & [t_{i-1,n}, t_{i+1,n}] \cap [x-h, x+h] \neq \emptyset, \\ O\Big(\frac{1}{n^2h^2} + \frac{1}{n^2\sqrt{h}}\Big) & \text{if}\ \ i \in \{1,n\}, \\ O\Big(\frac{1}{n^2\sqrt{h}}\Big) & otherwise. \end{cases}$$

Recall that $N_{T_n} = \text{Card } I_{x,h} = \text{Card }\{i = 1,\cdots,n\ :\ [t_{i-1},t_{i+1}]\cap]x-h,x+h[\neq\emptyset\}$ and denote by $t_{x,i}$ the points of T_n for which $i \in I_{x,h}$, Lemma 2 yields that $N_{T_n} = O(nh)$. Thus,

$$\sum_{i=1}^{n} m_{x,h}(t_i) = \frac{1}{2}\sum_{i=2}^{N_{T_n}-1} \varphi_{x,h}(t_{x,i})(t_{x,i+1} - t_{x,i-1}) + O\Big(\frac{1}{nh}\Big).$$

Since $\lim\limits_{n,m\to\infty} nh = +\infty$, then using the Riemann integrability of K, we obtain,

$$\lim_{n,m\to\infty}\sum_{i=1}^{n} m_{x,h}(t_i) = \frac{1}{2}\lim_{n,m\to\infty}\sum_{i=2}^{N_{T_n}-1} \varphi_{x,h}(t_{x,i})(t_{x,i+1} - t_{x,i-1}) = \int_{-1}^{1} K(t)\ dt = 1.$$

The Central Limit Theorem for i.i.d. variables yields,

$$\frac{1}{\sqrt{m}}\sum_{j=1}^{m}\varepsilon_j(x) \xrightarrow[m\to\infty]{\mathscr{D}} Z \quad where\ \ Z \sim \mathcal{N}(0, R(x,x)).$$

We shall prove now that the first term of Equation (72) tends to 0 in probability as n, m tends to infinity. Let,

$$A_{m,n}(x) = \frac{1}{\sqrt{m}} \sum_{j=1}^{m} \sum_{i=1}^{n} m_{x,h}(t_i)\big(\varepsilon_j(t_i) - \varepsilon_j(x)\big) \triangleq \frac{1}{\sqrt{m}} \sum_{j=1}^{m} T_{n,j}(x).$$

From the Chebyshev inequality, it suffices to prove that $\lim_{n,m\to\infty} \mathbb{E}(A_{m,n}^2(x)) = 0$. We have for $j \neq l$, $\mathbb{E}(\varepsilon_j(x)\varepsilon_l(y)) = 0$ then $\mathbb{E}(T_{n,j}(x)T_{n,l}(x)) = 0$. Hence,

$$\mathbb{E}(A_{m,n}^2(x)) = \frac{1}{m} \sum_{j=1}^{m} \sum_{l=1}^{m} \mathbb{E}(T_{n,j}(x)T_{n,l}(x)) = \frac{1}{m} \sum_{j=1}^{m} \mathbb{E}(T_{n,j}^2(x)).$$

We have,

$$\begin{aligned}
\mathbb{E}(T_{n,j}^2(x)) &= \sum_{i=1}^{n} \sum_{k=1}^{n} m_{x,h}(t_i) m_{x,h}(t_k) \mathbb{E}\Big(\big(\varepsilon_j(t_i) - \varepsilon_j(x)\big)\big(\varepsilon_j(t_k) - \varepsilon_j(x)\big)\Big) \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} m_{x,h}(t_i) m_{x,h}(t_k) \Big(R(t_i, t_k) - R(t_i, x) - R(x, t_k) + R(x, x)\Big).
\end{aligned}$$

Note that $\mathbb{E}(T_{n,j}^2(x))$ does not depend on j hence,

$$\begin{aligned}
\mathbb{E}(A_{m,n}^2(x)) &= \sum_{i=1}^{n} \sum_{k=1}^{n} m_{x,h}(t_i) m_{x,h}(t_k) \Big(R(t_i, t_k) - R(t_i, x) - R(x, t_k) + R(x, x)\Big) \\
&\triangleq B_{n,1}(x) - B_{n,2}(x) - B_{n,3}(x) + B_{n,4}(x).
\end{aligned} \tag{73}$$

Using Lemma 3 and the approximation of a sum by an integral (see, for instance, Lemma 2 in Benelmadani et al. (2019b)) we obtain,

$$B_{n,1}(x) = \int_{x-h}^{x+h} \int_{x-h}^{x+h} \varphi_{x,h}(s)\varphi_{x,h}(t) R(s,t)\ ds\ dt + O\Big(\frac{1}{nh}\Big) = \sigma_{x,h}^2 + O\Big(\frac{1}{nh}\Big).$$

Using Equation (15) (in the paper) we obtain,

$$B_{n,1}(x) = R(x,x) - \frac{1}{2}\alpha(x) C_K h + o(h) + O\Big(\frac{1}{nh}\Big).$$

where $C_K = \int_{-1}^{1}\int_{-1}^{1} |u-v| K(u)K(v) du dv$. Since $\lim_{n,m\to\infty} h = 0$ and $\lim_{n,m\to\infty} nh = \infty$ then,

$$\lim_{n\to\infty} B_{n,1}(x) = R(x,x). \tag{74}$$

Consider now the term $B_{n,2}(x)$. We obtain using Lemma 3 and the approximation of a sum by an integral,

$$\begin{aligned}
B_{n,2}(x) &= \int_{x-h}^{x+h} \int_{x-h}^{x+h} \varphi_{x,h}(s)\varphi_{x,h}(t) R(s,x)\ ds\ dt + O\Big(\frac{1}{nh}\Big) \\
&= \int_{x-h}^{x+h} \varphi_{x,h}(s) R(s,x)\ ds + O\Big(\frac{1}{nh}\Big) \\
&= \int_{-1}^{1} K(s) R(x-hs, x)\ ds + O\Big(\frac{1}{nh}\Big) \\
&= \int_{-1}^{0} K(s) R(x-hs, x)\ ds + \int_{0}^{1} K(s) R(x-hs, x)\ ds + O\Big(\frac{1}{nh}\Big).
\end{aligned}$$

For $s \in]-1,0[$, Taylor expansion of $R(\cdot,x)$ around x yields,

$$R(s,x) = R(x-sh,x) - shR^{(1,0)}(x^+,x) + o(h).$$

Similarly for $s \in]0,1[$ we obtain,

$$R(x-sh,x) = R(x,x) - shR^{(1,0)}(x^-,x) + o(h).$$

Thus,

$$B_{n,2}(x) = R(x,x) - hR^{(1,0)}(x^+,x)\int_{-1}^{0} s\ k(s)\ ds - hR^{(1,0)}(x^-,x)\int_{0}^{1} s\ k(s)\ ds + o(h) + O\Big(\frac{1}{nh}\Big).$$

Hence,

$$\lim_{n\to\infty} B_{n,2}(x) = R(x,x). \tag{75}$$

Similarly,

$$\lim_{n\to\infty} B_{n,3}(x) = R(x,x). \tag{76}$$

It is easy to verify that,

$$\lim_{n\to\infty} B_{n,4}(x) = R(x,x). \tag{77}$$

Inserting (74), (75), (76) and (77) in (73) yields,

$$\lim_{n,m\to\infty} \mathbb{E}(A^2_{m,n}(x)) = 0.$$

This concludes the proof of Theorem 3. □

Proof of Theorem 5.

Let $x \in]0,1[$. On the one hand, we have from Proposition 8 and Remark 7,

$$\text{Var}\ \hat{g}_n^{pro}(x) = \frac{\sigma^2_{x,h}}{m} - \frac{A}{12mn^2h}\frac{\alpha(x)}{f^2(x)} + O\Big(\frac{1}{mn^3h^2} + \frac{1}{mn^2}\Big), \tag{78}$$

where $A = \int_{-1}^{1} K^2(t)\ dt$. On the other hand, it can be seen in Benelmadani (2019a) that,

$$\text{Var}\ \hat{g}_n^{GM}(x) = \frac{\sigma^2_{x,h}}{m} + O\Big(\frac{1}{mn^2} + \frac{1}{mn^3h^2}\Big). \tag{79}$$

Equations (78) and (79) then yield,

$$mn^2h\ \Big(\text{Var}\ \hat{g}_n^{GM} - \text{Var}\ \hat{g}_n^{pro}\Big) = \frac{A}{12}\frac{\alpha(x)}{f^2(x)} + O\Big(h + \frac{1}{nh}\Big).$$

Recall that $\alpha(x) > 0$ and that $\frac{1}{f(x)} > 0$. Since $h \to 0$ and $nh \to \infty$ as $n,m \to \infty$ we obtain,

$$\lim_{n,m\to\infty} mn^2h\ \Big(\text{Var}\ \hat{g}_n^{GM}(x) - \text{Var}\ \hat{g}_n^{pro}(x)\Big) = \frac{A}{12}\frac{\alpha(x)}{f^2(x)} > 0.$$

This concludes the proof of Theorem 5. □

Proof of Theorem 6.

We have from the proof of Proposition 4 (Equation (33)) for any $x \in]0,1[$,

$$\mathbb{E}(\hat{g}^{pro}_{n,m}(x)) - g(x) = I_h(x) - g(x) + O\Big(\frac{1}{n^2h}\Big), \tag{80}$$

where,

$$I_h(x) = \int_{x-h}^{x+h} \varphi_{x,h}(s) g(s) \; ds.$$

Hence, using (78) and (80) we get for a positive density measure w,

$$\begin{aligned}\text{IMSE}(\hat{g}^{pro}_n) = \frac{1}{m}\int_0^1 \sigma^2_{x,h} \; w(x) \; dx - \frac{A}{12mn^2h}\int_0^1 \frac{\alpha(x)}{f^2(x)} \; w(x) \; dx + \int_0^1 \big(I_h(x) - g(x)\big)^2 \; w(x) \; dx \\ + O\Big(\frac{1}{n^4h^2} + \frac{h}{n^2} + \frac{1}{mn^3h^2} + \frac{1}{mn^2}\Big).\end{aligned} \tag{81}$$

It can be seen in Benelmadani (2019a) that,

$$\mathbb{E}(\hat{g}^{GM}_{n,m}(x)) - g(x) = I_h(x) - g(x) + O\Big(\frac{1}{n^2h}\Big). \tag{82}$$

Using (79) and (82) yield,

$$\begin{aligned}\text{IMSE}(\hat{g}^{GM}_n) = \frac{1}{m}\int_0^1 \sigma^2_{x,h} \; w(x) \; dx + \int_0^1 \big(I_h(x) - g(x)\big)^2 \; w(x) \; dx \\ + O\Big(\frac{1}{n^4h^2} + \frac{h}{n^2} + \frac{1}{mn^2} + \frac{1}{mn^3h^2}\Big).\end{aligned} \tag{83}$$

Then, Equations (81) and (83) yield,

$$mn^2h\,\Big(\text{IMSE }(\hat{g}^{GM}_n) - \text{IMSE }(\hat{g}^{pro}_n)\Big) = \frac{A}{12}\int_0^1 \frac{\alpha(x)}{f^2(x)} \; w(x) \; dx + O\Big(\frac{m}{n^2h} + mh^2 + h + \frac{1}{nh}\Big).$$

Since $\frac{m}{n} = O(1)$ and $mh^2 \to 0$ as $n, m \to \infty$ we obtain,

$$\lim_{n,m\to\infty} mn^2h\,\Big(\text{IMSE }(\hat{g}^{GM}_n) - \text{IMSE }(\hat{g}^{pro}_n)\Big) = \frac{A}{12}\int_0^1 \frac{\alpha(x)}{f^2(x)} \; w(x) \; dx > 0.$$

This concludes the proof of Theorem 6. □

Appendix

Let $\varepsilon = (\varepsilon(t))_{t\in[0,1]}$ be a centered and a second order process of autocovariance R, such that R is invertible when restricted to any finite set on $[0,1]$. Let $L(\varepsilon(t), t \in [0,1])$ be the set of all random variables which maybe be written as a linear combinations of $\varepsilon(t)$ for $t \in [0,1]$, i.e., the set of random variables of the form $\sum_{i=1}^{l} \alpha_i \varepsilon(t_i)$ for some positive integer l and some constants α_i, $t_i \in [0,1]$ for $i = 1, \cdots, l$. Let also $L_2(\varepsilon)$ be the Hilbert space of all square integrable random variables in the linear manifold $L(\varepsilon(t), t \in [0,1])$, together with all random variables U that are limits in $\mathbb{L}^2$ of a sequence of random variables U_n in $L(\varepsilon(t), t \in [0,1])$, i.e, U is such that,

$$\exists\,(U_n)_{n\geq 0} \in L(\varepsilon(t), t \in [0,1]) : \;\; \lim_{n\to\infty} \mathbb{E}((U_n - U)^2) = 0.$$

Denote by $\mathcal{F}(\varepsilon)$ the family of functions g on $[0,1]$ defined by,

$$\mathcal{F}(\varepsilon) = \{g : [0,1] \to \mathbb{R} \text{ with } g(\cdot) = \mathbb{E}(U\varepsilon(\cdot)) \text{ where } U \in L_2(\varepsilon)\},$$

We note here that for every $g \in \mathcal{F}(\varepsilon)$, the associated U is unique. It is easy to verify that $\mathcal{F}(\varepsilon)$ is a Hilbert space equipped with the norm $|| \; ||$ defined for $g \in \mathcal{F}(\varepsilon)$ by,

$$||g||^2 = \mathbb{E}(U^2).$$

In fact, let $g \in \mathcal{F}(\varepsilon)$, i.e, $g(\cdot) = \mathbb{E}(U\varepsilon(\cdot))$ for some $U \in L_2(\epsilon)$. We have,

- $||g|| = \sqrt{\mathbb{E}(U^2)} \geq 0$.
- $||g|| = \sqrt{\mathbb{E}(U^2)} = 0 \Rightarrow U = 0$ a.s. $\Rightarrow g = 0$.
- For $g \in \mathcal{F}(\varepsilon)$, i.e, $f(\cdot) = \mathbb{E}(V\varepsilon(\cdot))$ some $V \in L_2(\epsilon)$. We have,

$$\begin{aligned} ||g+f||^2 &= \mathbb{E}((U+V)^2) = \mathbb{E}(U^2) + \mathbb{E}(V^2) + 2\mathbb{E}(UV) \\ &\leq \mathbb{E}(U^2) + \mathbb{E}(V^2) + 2\sqrt{\mathbb{E}(U^2)}\sqrt{\mathbb{E}(V^2)} = \left(\sqrt{\mathbb{E}(U^2)} + \sqrt{\mathbb{E}(V^2)}\right)^2. \end{aligned}$$

  Thus, $||g+f|| \leq \sqrt{\mathbb{E}(U^2)} + \sqrt{\mathbb{E}(V^2)} = ||g|| + ||f||$.

We now prove the completeness of $\mathcal{F}(\varepsilon)$. For this let $g_n(\cdot) = E(U_n\varepsilon(\cdot))$ be a Cauchy sequence in $\mathcal{F}(\varepsilon)$, i.e.,

$$\lim_{n,m\to\infty} ||g_n - g_m||^2 = 0.$$

From the definition of the norm $|| \; ||$ we obtain,

$$\lim_{n,m\to\infty} \mathbb{E}((U_n - U_m)^2) = \lim_{n,m\to\infty} ||g_n - g_m||^2 = 0.$$

This yields that $(U_n)_{n\geq 1}$ is a Cauchy sequence in $L_2(\varepsilon)$, which is a Hilbert space as proven by Parzen (1959) (see page 8 there). Thus it exists $U \in L_2(\varepsilon)$ such that,

$$\lim_{n\to\infty} \mathbb{E}((U_n - U)^2) = 0.$$

Taking $g(\cdot) = \mathbb{E}(U\varepsilon(\cdot))$, which is clearly an element of $\mathcal{F}(\varepsilon)$ gives,

$$\lim_{n\to\infty} ||g_n - g||^2 = \lim_{n\to\infty} \mathbb{E}((U_n - U)^2) = 0.$$

This concludes the proof of completness of $\mathcal{F}(\varepsilon)$.

The Hilbert space $\mathcal{F}(\varepsilon)$ can easily be identified as the Reproducing Kernel Hilbert Space associated to a reproducing kernel R (with $R(s,t) = \mathbb{E}(\varepsilon(s)\varepsilon(t))$), which is defined as follows.

Definition 1. Parzen (1959) A Hilbert space H is said to be a Reproducing Kernel Hilbert Space associated to a reproducing kernel (or function) R (RKHS(R)), if its members are functions on some set T, and if there is a kernel R on $T \times T$ having the following two properties:

$$\begin{cases} R(\cdot,t) \in H & \text{for all } \; t \in T, \\ \langle g, R(\cdot,t)\rangle = g(t) & \text{for all } \; t \in T \; \text{ and } \; g \in H, \end{cases} \tag{84}$$

where $\langle \cdot,\cdot \rangle$ is the inner (or scalar) product in H.

To prove this, we need to verify the properties given in (84). For $t \in [0,1]$ we have,

$$R(s,t) = \mathbb{E}(\varepsilon(s)\varepsilon(t)) \quad \text{for all} \quad s \in [0,1].$$

Since $\varepsilon(s) \in L_2(\varepsilon)$ then $R(\cdot,t) \in \mathcal{F}(\varepsilon)$ for any fixed $t \in [0,1]$. Now let $g \in \mathcal{F}(\varepsilon)$, i.e.,

$$g(\cdot) = \mathbb{E}(U\varepsilon(\cdot)) \ \text{ for some } \ U \in L_2(\varepsilon).$$

Then,

$$\begin{aligned}\langle g, R(\cdot,t)\rangle &= \frac{1}{2}\big(||g||^2 + ||R(\cdot,t)||^2 - ||g - R(\cdot,t)||^2\big) = \frac{1}{2}\big(\mathbb{E}(U^2) + \mathbb{E}(\varepsilon(t)^2) - \mathbb{E}((U-\varepsilon(t))^2)\big)\\ &= \frac{1}{2}\mathbb{E}(2U\varepsilon(t)) = g(t).\end{aligned}$$

These properties together with the following theorem yield that $\mathcal{F}(\varepsilon)$ is the RKHS(R).

Theorem 7 (E. H. Moor)**.** *Aronszajn (1944) A symmetric non-negative Kernel R generates a unique Hilbert space.*

In the sequel, we take R to be continuous on $[0,1]^2$ and we shall consider the function of interest given by (2) (in the paper). More generally, we consider the function f, defined for a continuous function φ and $t \in [0,1]$, by

$$f(t) = \int_0^1 R(s,t)\varphi(s)\, ds. \tag{85}$$

Lemma 5. *We have $f \in \mathcal{F}(\varepsilon)$, i.e., there exists $X \in L_2(\varepsilon)$ with,*

$$f(\cdot) = \mathbb{E}(X\varepsilon(\cdot)). \tag{86}$$

In addition,

$$\|f\|^2 = \mathbb{E}(X^2) = \int_0^1\int_0^1 R(s,t)\varphi(s)\varphi(t)\, dt\, ds.$$

Proof. Define, for a suitable partition $(x_{i,n})_{i=1,\cdots,n}$ of $[0,1]$,

$$X_n = \sum_{i=1}^{n-1}(x_{i+1,n} - x_{i,n})\varphi(x_{i,n})\varepsilon(x_{i,n}) \in L_2(\varepsilon),$$

such that for any $t \in [0,1]$,

$$f(t) = \lim_{n\to\infty}\sum_{i=1}^{n-1}(x_{i+1,n} - x_{i,n})\varphi(x_{i,n})R(x_{i,n},t) = \lim_{n\to\infty}\mathbb{E}(X_n\varepsilon(t)).$$

We shall prove that $(X_n)_n$ converges to a certain element of $\mathbb{L}^2$, i.e.,

$$\exists\ X \in \mathbb{L}^2 \ :\ \lim_{n\to\infty}\mathbb{E}\big((X_n - X)^2\big) = 0, \tag{87}$$

and by the definition of $L_2(\varepsilon)$ the limit in (87) proves that X is an element of $L_2(\varepsilon)$. Now the proof (87) is immediate, in fact it is easy to check that (X_n) id a Cauchy sequence in

$\mathbb{L}^2$. By the completeness of $\mathbb{L}^2$, we deduce (87). In addition we have, $\lim_{n\to\infty}\mathbb{E}(X_n\varepsilon(t)) = \mathbb{E}(X\varepsilon(t))$, this is due to the following inequality,

$$\Big|\mathbb{E}(X_n\varepsilon(t)) - \mathbb{E}(X\varepsilon(t))\Big| \leq \mathbb{E}\Big|(X_n - X)\varepsilon(t)\Big| \leq \sqrt{\mathbb{E}((X_n - X)^2)}\sqrt{\mathbb{E}(\varepsilon(t)^2)},$$

and the fact that $\lim_{n\to\infty}\mathbb{E}((X_n-X)^2) = 0$ and $\mathbb{E}(\varepsilon(t)^2) < \infty$. The proof of (86) is concluded. Finally,

$$\begin{aligned}\mathbb{E}(X^2) = \lim_{n\to\infty}\mathbb{E}(X_n^2) &= \lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_{i+1,n} - x_{i,n})(x_{j+1,n} - x_{j,n})\varphi(x_{i,n})\varphi(x_{j,n})R(x_{i,n}, x_{j,n}) \\ &= \int_0^1\int_0^1 \varphi(t)\varphi(t)R(s,t)\ ds\ dt.\end{aligned}$$

This concludes the proof of Lemma 5. □

Now let $T_n = (t_1, t_2, \cdots, t_n)$ with $0 \leq t_1 < t_2 < \cdots < t_n \leq 1$ and let V_{T_n} be the subspace of $\mathcal{F}(\varepsilon)$ spanned by the functions $R(\cdot, t)$ for $t \in T_n$, i.e.,

$$V_{T_n} = \{g : [0,1] \to \mathbb{R} \ \text{ with } \ g(\cdot) = \mathbb{E}(U\varepsilon(\cdot)) \ \text{ where } \ U \in L(\varepsilon(t), t \in T_n)\}.$$

Our task is to prove that if $R_{|T_n} = (R(t_i, t_j)_{1\leq i,j\leq n})$ is a non-singular matrix then V_{T_n} is a closed subspace of $\mathcal{F}(\varepsilon)$. For this let, $(g_m)_{m\geq 1}$ be a sequence in V_{T_n} converging to $g \in \mathcal{F}(\varepsilon)$. We shall prove that $g \in V_{T_n}$. Note that,

$$g_m(t) = \mathbb{E}(U_m\varepsilon(t)) \quad \text{with} \quad U_m = \sum_{i=1}^{n} a_{i,m}\varepsilon(t_i), \quad \text{where} \ \ (a_{i,m})_{m\geq 1} \in \mathbb{R}.$$

Since g_m converges in $\mathcal{F}(\varepsilon)$ then it is a Cauchy sequence, i.e.,

$$\lim_{m_1,m_2\to\infty} ||g_{m_1} - g_{m_2}||^2 = 0.$$

By the definition of the norm on $\mathcal{F}(\varepsilon)$ we have,

$$\begin{aligned}||g_{m_1} - g_{m_2}||^2 &= \mathbb{E}((U_{m_1} - U_{m_2})^2) = \mathbb{E}\Big(\big(\sum_{i=1}^{n}(a_{i,m_1} - a_{i,m_2})\varepsilon(t_i)\big)^2\Big) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}(a_{i,m_1} - a_{i,m_2})(a_{j,m_1} - a_{j,m_2})R(t_i, t_j) = A'_{m_1,m_2}R_{|T_n}A_{m_1,m_2},\end{aligned}$$

where $A'_{m_1,m_2} = (a_{1,m_1} - a_{1,m_2}, \cdots, a_{n,m_1} - a_{n,m_2})'$. Thus,

$$\lim_{m_1,m_2\to\infty} A'_{m_1,m_2}R_{|T_n}A_{m_1,m_2} = 0.$$

Since $R_{|T_n}$ is a symmetric positive matrix, we obtain,

$$\lim_{m_1,m_2\to\infty} A'_{m_1,m_2} = \lim_{m_1,m_2\to\infty}(a_{1,m_1} - a_{1,m_2}, \cdots, a_{n,m_1} - a_{n,m_2})' = (0, \dots, 0)',$$

which yields that $(a_{i,m})_m$ is a Cauchy sequence on $\mathbb{R}$ for all $i = 1, \cdots, n$. Taking $a_i = \lim_{m\to\infty} a_{i,m}$ we obtain by the uniqueness of the limit,

$$g(\cdot) = \mathbb{E}(U\varepsilon(\cdot)) \ \text{ with } \ U = \sum_{i=1}^{n} a_i\varepsilon(t_i),$$

which yields that $g \in V_{T_n}$. Hence V_{T_n} is closed. $\square$

Since V_{T_n} is a closed subspace in the Hilbert space $\mathcal{F}(\varepsilon)$, one can define the orthogonal projection operator from $\mathcal{F}(\varepsilon)$ to V_{T_n} which we note by $P_{|T_n}$, i.e., for every $f \in \mathcal{F}(\varepsilon)$,

$$P_{|T_n} f = \underset{g\ \in V_{T_n}}{\operatorname{argmin}}\ ||f - g||.$$

Par definition of $P_{|T_n}$, we have for any $g \in V_{T_n}$

$$\langle P_{|T_n} f - f, g \rangle = 0.$$

Now, for $t_i \in T_n$, $R(\cdot, t_i) \in V_{T_n}$. Hence, for every $i = 1, \dots, n$.

$$\langle P_{|T_n} f - f, R(\cdot, t_i) \rangle = 0 \ \text{ or equivalently } \ \langle P_{|T_n} f, R(\cdot, t_i) \rangle = \langle f, R(\cdot, t_i) \rangle.$$

The last equality, together with (84), gives that,

$$P_{|T_n} f(\cdot) = f(\cdot) \ \text{ on } \ T_n. \ \square \tag{88}$$

Supplementary facts

(F1) Let f be defined by (85). We shall prove that if $g \in V_{T_n}$, i.e., if $g(\cdot) = \sum_{j=1}^n a_j R(t_j, \cdot)$ for some $a_i \in \mathbb{R}$, then

$$||f - g||^2 = \int_0^1 \varphi(s)(f(s) - g(s))\, ds - \sum_{i=1}^n a_i (f(t_i) - g(t_i)).$$

In fact,

$$||f - g||^2 = \langle f - g, f - g \rangle = \langle f, f - g \rangle - \langle g, f - g \rangle$$

On the one hand, note that $f - g \in \mathcal{F}(\varepsilon)$ and by using (84) we obtain,

$$\langle g, f - g \rangle = \sum_{i=1}^n a_i \langle R(t_i, \cdot), f - g \rangle = \sum_{i=1}^n a_i (f(t_i) - g(t_i)). \tag{89}$$

On the another hand, Lemma 5 and its proof yield that $f(\cdot) = \mathbb{E}(X\varepsilon(\cdot))$ where $X \in L_2(\varepsilon)$ and that,

$$\lim_{l \to \infty} \mathbb{E}(X_l - X)^2 = 0 \ \text{ where } \quad X_l = \sum_{j=1}^{l-1} (x_{j+1,l} - x_{j,l}) \varphi_{x,h}(x_{j,l}) \varepsilon(x_{j,l}),$$

where $(x_{j,l})_{j=1,\cdots,l}$ is a suitable partition of $[0,1]$. Let $F_l(\cdot) = \mathbb{E}(X_l \varepsilon(\cdot))$ which is an element of $\mathcal{F}(\varepsilon)$. Clearly,

$$\langle f, f - g \rangle = \langle f - F_l, f - g \rangle + \langle F_l, f - g \rangle.$$

We have,

$$|\langle f - F_l, f - g \rangle| \le ||f - F_l||\ ||f - g|| \le \sqrt{\mathbb{E}((X_l - X)^2)}||f - g||.$$

Thus $\lim_{l\to\infty}\langle f - F_l, f-g\rangle = 0$. In addition,

$$\langle F_l, f-g\rangle = \Big\langle \sum_{j=1}^{l-1}(x_{j+1,l}-x_{j,l})\varphi(x_{j,l})R(x_{j,l},\cdot), f-g\Big\rangle$$
$$= \sum_{j=1}^{l-1}(x_{j+1,l}-x_{j,l})\varphi(x_{j,l})\langle R(x_{j,l},\cdot), f-g\rangle = \sum_{j=1}^{l-1}(x_{j+1,l}-x_{j,l})\varphi(x_{j,l})(f(x_{j,l})-g(x_{j,l})).$$

Hence,

$$\lim_{l\to\infty}\langle F_l, f-g\rangle = \int_0^1 \varphi(t)(f(t)-g(t))\,dt.$$

Finally,

$$\langle f, f-g\rangle = \int_0^1 \varphi(t)(f(t)-g(t))\,dt. \ \square$$

(F2) For $x\in[0,1]$, let $f_{x,h}$ be defined by (2) (in the paper). We shall prove that,

$$m\mathrm{Var}(\hat{g}_n^{pro}(x)) = ||P_{|T_n}f_{x,h}||^2.$$

In fact, by the definition of the projection operator $P_{|T_n}$, we have $P_{|T_n}f_{x,h}\in V_{T_n}$ and for $t\in[0,1]$,

$$P_{|T_n}f_{x,h}(t) = \sum_{i=1}^n a_iR(t_i,t) = \mathbb{E}(\sum_{i=1}^n a_i\epsilon(t_i)\epsilon(t)) \ \text{ for some } \ a_i\in\mathbb{R} \text{ for } i=1,\cdots,n,$$

and then,

$$||P_{|T_n}f_{x,h}||^2 = \mathbb{E}\left(\sum_{i=1}^n a_i\epsilon(t_i)\right)^2 = \sum_{i=1}^n a_i\sum_{j=1}^n a_jR(t_i,t_j) = \sum_{i=1}^n a_iP_{|T_n}f_{x,h}(t_i).$$

Recall that $m_{x,h}{}'_{|T_n} = f_{x,h}{}'_{|T_n}R_{|T_n}$ and using (88) we obtain,

$$P_{|T_n}f_{x,h}(t_i) = f_{x,h}(t_i) = \sum_{j=1}^n m_{x,h}(t_j)R(t_i,t_j). \tag{90}$$

We have then, using (90),

$$||P_{|T_n}f_{x,h}||^2 = \sum_{i=1}^n a_i\sum_{j=1}^n m_{x,h}(t_j)R(t_i,t_j) = \sum_{j=1}^n m_{x,h}(t_j)\sum_{i=1}^n a_iR(t_i,t_j)$$
$$= \sum_{j=1}^n m_{x,h}(t_j)\sum_{i=1}^n m_{x,h}(t_i)R(t_i,t_j) = m\mathrm{Var}(\hat{g}_n^{pro}(x)). \ \square$$

(F3) We shall now prove prove that every function in $\mathcal{F}(\varepsilon)$ is continuous on $[0,1]$. In fact let $g\in\mathcal{F}(\varepsilon)$, i.e.,

$$g(\cdot) = \mathbb{E}(U\varepsilon(\cdot)) \ \text{ for some } \ U\in L_2(\varepsilon).$$

For $s,t \in [0,1]$, (84) and Cauchy-Swartz inequality yields,

$$\begin{aligned}|g(t)-g(s)| &= |\langle R(\cdot,t),g\rangle - \langle R(\cdot,s),g\rangle| = |\langle R(\cdot,t)-R(\cdot,s),g\rangle| \\ &\leq ||R(\cdot,t)-R(\cdot,s)||\ ||g|| = ||R(\cdot,t)-R(\cdot,s)||\ \sqrt{\mathbb{E}(U^2)}.\end{aligned}$$

Since ε is of second order process then $\mathbb{E}(U^2) < \infty$ and since R is continuous on $[0,1]^2$ we obtain,

$$\lim_{s\to t} ||R(\cdot,t)-R(\cdot,s)||^2 = \lim_{s\to t} (R(t,t)+R(s,s)-2R(s,t)) = 0,$$

which yields that $\lim\limits_{s\to t} |g(t)-g(s)| = 0$. Hence g is continuous. □

(F4) Suppose that R verifies Assumptions $(A),(B)$ and (C). Let f be defined by (85). We shall prove that if $g \in V_{T_n}$, i.e., $g(\cdot) = \sum_{j=1}^n a_j R(t_j,\cdot)$ with $(a_i)_i \in \mathbb{R}$ then,

$$f''(t) - g''(t^+) = -\alpha(t)\varphi(t) + \langle R^{(0,2)}(\cdot,t^+), f-g\rangle.$$

In fact, we have, as in Equation (17),

$$f''(t) = -\alpha(t)\varphi(t) + \int_0^1 R^{(0,2)}(s,t^+)\varphi(s)\, ds.$$

In addition, we have clearly

$$g''(t^+) = \sum_{j=1}^n a_j R^{(0,2)}(t_j,t^+).$$

Thus,

$$f''(t) - g''(t^+) = -\alpha(t)\varphi(t) + \int_0^1 R^{(0,2)}(s,t^+)\varphi(s)\, ds - \sum_{j=1}^n a_j R^{(0,2)}(t_j,t^+).$$

We have,

$$\langle R^{(0,2)}(\cdot,t^+), f-g\rangle = \langle R^{(0,2)}(\cdot,t^+), f\rangle - \langle R^{(0,2)}(\cdot,t^+), g\rangle$$

On the one hand, since by Assumption (C), $R^{(0,2)}(\cdot,t^+)$ is in $\mathcal{F}(\varepsilon)$ then (84) yields,

$$\langle R^{(0,2)}(\cdot,t^+), g\rangle = \sum_{j=1}^n a_j \langle R^{(0,2)}(\cdot,t^+), R(\cdot,t_j)\rangle = \sum_{j=1}^n a_j R^{(0,2)}(t_j,t^+). \tag{91}$$

On the other hand, from Lemma 5 we have $f(\cdot) = \mathbb{E}(X\varepsilon(\cdot))$ where $X \in L_2(\varepsilon)$ and,

$$\lim_{l\to\infty} \mathbb{E}(X_l - X)^2 = 0 \quad \text{with} \quad X_l = \sum_{j=1}^{l-1} (x_{j+1,l} - x_{j,l})\varphi(x_{j,l})\varepsilon(x_{j,l}),$$

where $(x_{j,l})_{j=1,\cdots,l}$ is a suitable partition of $[0,1]$. Let $F_l(\cdot) = \mathbb{E}(X_l\varepsilon(\cdot)) \in \mathcal{F}(\varepsilon)$, we have,

$$\langle R^{(0,2)}(\cdot,t^+), f\rangle = \langle R^{(0,2)}(\cdot,t^+), f - F_l\rangle + \langle R^{(0,2)}(\cdot,t^+), F_l\rangle, \tag{92}$$

and,

$$|\langle R^{(0,2)}(\cdot,t^+), f-F_l\rangle| \leq ||R^{(0,2)}(\cdot,t^+)||\ ||f-F_l|| = ||R^{(0,2)}(\cdot,t^+)||\sqrt{\mathbb{E}((X_l-X)^2)}.$$

The last bound together with Assumption (C) gives $\lim\limits_{l\to\infty}|\langle R^{(0,2)}(\cdot,t^+), f-F_l\rangle| = 0$, in addition,

$$\begin{aligned}\langle R^{(0,2)}(\cdot,t^+), F_l\rangle &= \sum_{j=1}^{l-1}(x_{j+1,l}-x_{j,l})\varphi(x_{j,l})\langle R^{(0,2)}(\cdot,t^+), \varepsilon(x_{j,l})\rangle \\ &= \sum_{j=1}^{l-1}(x_{j+1,l}-x_{j,l})\varphi(x_{j,l})R^{(0,2)}(x_{j,l},t^+).\end{aligned}$$

Thus,

$$\lim_{l\to\infty}\langle R^{(0,2)}(\cdot,t^+), F_l\rangle = \int_0^1 \varphi(s)R^{(0,2)}(s,t^+)\ ds. \tag{93}$$

Finally, using (91), (92) and (93) yield,

$$\langle R^{(0,2)}(\cdot,t^+), f-g\rangle = \int_0^1 \varphi(s)R^{(0,2)}(s,t^+)\ ds - \sum_{j=1}^{n} a_j R^{(0,2)}(t_j,t^+).\ \square$$

References

[1] Anderson, T W. (1960). Some Stochastic Process Models For Intelligence Test Scores. *Mathematical Methods in Social Sciences.* Stanford University Press, Stanford, California. 205-220.

[2] Aronszajn, N. (1944). La Théorie des Noyaux Reproduisants et ses Applications. *Proceedings of the Cambridge Philosophical Society.* 39, 133-153.

[3] Benelmadani, D. (2019a). *Contribution à la Régression non Paramétrique avec un Processus Erreur d'Autocovariance Générale et Application en Pharmacocinétique*, Ph.D. 2019, Grenoble Alpes University, France.

[4] Benelmadani, D., Benhenni, K., Louhichi, S. (2019b). Trapezoidal Rule and Sampling Designs for the Nonparametric Estimation of the Regression Function in Models with Correlated Errors. `https://arxiv.org/abs/1806.04896`

[5] Fraiman, R., Pérez Iribarren, G. (1991). Nonparametric Regression in Models with Weak Error's Structure. *Journal of Multivariate Analysis.* 37, 180-196.

[6] Gasser, T., Müller, H-G. (1984). Estimating Regression Function and Their Derivatives by the Kernel Method. *Scandinavian Journal of Statistics.* 11, 171-185.

[7] Parzen, E. (1959). Statistical Inference on Time Series by Hilbert Space Methods. Department of Statistics. Stanford University, Stanford, California. Technical Report. 23.

[8] Sacks, J., Ylvisaker, D. (1966). Designs for Regression Problems with Correlated Errors. *The Annals of Mathematical Statistics.* 37, 66-89.

[9] Su, Y., Cambanis, S. (1993). Sampling Designs for Estimation of a Random Process. *Stochastic Processes and their Applications.* 46, 47-89.